On Hopf Cyclicity of Planar Systems

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Submitted by C. T. Leondes

Received April 14, 1999

We investigate the maximal number of limit cycles which appear under perturbations in Hopf bifurcations by using the first-order Melnikov function with multiple parameters. © 2000 Academic Press

Key Words: limit cycle; Hopf bifurcation; cyclicity.

1. INTRODUCTION AND MAIN RESULTS

Consider a plane system of the form

\[ \dot{x} = f(x) + \epsilon g(x, \epsilon, a), \] (1.1)

where \( x \in \mathbb{R}^2, \epsilon \in \mathbb{R}, a \in D \subseteq \mathbb{R}^n \) with \( D \) bounded, \( n \geq 1 \), and \( f, g \) are \( C^\infty \) functions. Suppose the unperturbed system

\[ \dot{x} = f(x) \] (1.2)

has a family of periodic orbits \( \{L_h: h \in J\} \) with \( J \) an open interval. We may suppose \( J = (0, h_0) \) and the limit \( \lim_{h \to 0} L_h = L_0 \) exists and is an elementary center. If the limit \( \lim_{h \to 0} L_h = L_{h_0} \) exists finitely, it is usually a separatrix cycle. For Eq. (1.1) the problem is to find the maximal number of limit cycles in a given neighborhood of the union \( \bigcup_{0 \leq h \leq h_0} L_h \).

Recall that we say, Eq. (1.1) has cyclicity \( k \) at \( L_h \) with \( h \in [0, h_0] \) for all \( a \in D \) and \( \epsilon \) small if Eq. (1.1) has at most \( k \) limit cycles near \( L_h \) for all \( a \in D \) and \( \epsilon \) small and \( k \) limit cycles can appear for some \( a \in D \) and \( \epsilon \)

\(^1\) Research supported by the National Science Foundation of China Grant 19531070.
small. Then the above problem can be divided into three problems as follows:

(i) For each $h \in (0, h_0)$, find the cyclicity at $L_h$.
(ii) Find the cyclicity at $L_{h_0}$.
(iii) Find the cyclicity at $L_0$ (Hopf bifurcation).

In studying these problems, we use a return map of the form

$$P(h, \epsilon, a) - h = M(h, a) \epsilon + M_2(h, a) \epsilon^2 + \cdots,$$

where

$$M(h, a) = \oint_{L_h} f(x) \wedge g(x, 0, a) \, dt$$

(1.3)

if the unperturbed system (1.2) is Hamiltonian. Theoretically, problem (i) is easy since the function $P$ is of class $C^3$ in $h \in (0, h_0)$. See [18]. Of course, it is difficult in practice. The problem (ii) is most difficult since each of $M$ and $P$ is singular at $h = h_0$ (even not $C^3$); see [5, 6, 8, 9, 17]. Note that $P$ is also singular at $h = 0$ (not $C^2$ in general) by [7]. The problem (iii) is also difficult; see [4, 10, 11]. And from [4–11, 17, 18], we see that the techniques to attack the three problems are very different.

For Hopf bifurcations, as we knew, Bautin [1] proved that any quadratic system has at most three limit cycles near a singular point of focus or center type. For some systems of special form, the problem of cyclicity in Hopf bifurcations was studied in Blows and Lloyd [2], Cima, Gasull, and Manosas [3], Lins, De Melo, and Pugh [12], and Lloyd and Lynch [13]. Petrov [16], Mardesic [15], and Li and Zhang [11] studied the following polynomial system with a single parameter,

$$\dot{x} = y + \epsilon P_n(x, y), \quad \dot{y} = -(x + x^2) + \epsilon Q_n(x, y).$$

More precisely, Petrov obtained the number of zeros of the first-order Melnikov function $M(h)$ on the interval $(0, 1/6)$. Mardesic gave the maximal number of limit cycles for $\epsilon$ small if $M(h) \neq 0$. However, in discussing Hopf bifurcations, the author [15, pp. 524–525] used $P \in C^3$ at $h = 0$, which is not true by [7]. Li and Zhang studied the maximal number of limit cycles under the condition that $M(h) = 0$ and $M_1(h) \neq 0$.

We note that there is a difference between perturbations of single parameter and multiple parameters. In fact, in some cases one cannot obtain the maximal number of limit cycles if one only uses Melnikov functions $M_1$, $M_2$, and so on. In other words, the maximal number of zeros
of all nontrivial Melnikov functions may be less than the maximal number of limit cycles. An example of this kind is given at the end of the paper.

In this paper, we consider the problem of cyclicity in the Hopf bifurcation for general system (1.1) with multiple parameters. Without loss of generality, we suppose that Eq. (1.2) has an elementary center at the origin, and that

$$f(0) = g(0, \epsilon, a) = 0$$

for all $(\epsilon, a) \in R \times R^n$. Suppose Eq. (1.2) has a $C^\infty$ first integral $I(x)$ and an associated integral factor $\mu(x)$ defined in a neighborhood $U$ of the origin such that

$$I(x) = A|x|^2 + O(|x|^3), \quad A > 0 \text{ for } x \in U,$$

$$\mu(x)f(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}DI(x) \quad \text{for } x \in U. \quad (1.5)$$

Then we can suppose that the periodic orbits $L_h$ are level curves of $I(x)$. The following theorem was obtained in [4, 10].

**Theorem 1.1.** Let $\mu(x) \equiv 1$. We have

(i) If system (1.1) is analytic, and $M(h, a_0) \neq 0$ for some $a_0 \in D$, then there exists an integer $k \geq 0$ such that

$$M(h, a_0) = B_k h^{k+1} + O(h^{k+3/2}), \quad B_k \neq 0 \text{ for } 0 < h \ll 1. \quad (1.6)$$

(ii) If Eq. (1.6) holds, there exist a constant $\epsilon_0 > 0$ and a neighborhood $V$ of the origin such that Eq. (1.1) has at most $k$ limit cycles in $V$ for all $0 < |\epsilon| + |a - a_0| < \epsilon_0$.

When the parameter $a$ does not appear, a similar theorem was also given in [11].

It was proved in [7] that $M(h, a)$ is at least $C^2$ in $h \geq 0$ at $h = 0$, but $M_2(h, a)$ is generally not $C^2$ at $h = 0$. When $\mu \neq 1$, we introduce a function as follows (denote it also by $M$),

$$M(h, a) = \hat{\Phi}_{\int_{\Phi(x) = h} DI(x)g(x, 0, a) dt}. \quad (1.7)$$

As before, we also call this $M$ the first-order Melnikov function of system (1.1). Obviously, Eqs. (1.3) and (1.7) coincide in the case $\mu(x) = 1$. 
Our main results are the following.

**THEOREM 1.2.** Suppose the $C^\infty$ system (1.1) satisfies Eqs. (1.4) and (1.5). Then

(i) The function $M$ given by Eq. (1.7) is of class $C^\infty$ in $h \geq 0$ at $h = 0$. If the functions $f$ and $g$ in Eq. (1.1) are analytic in $x$, then $M$ is also analytic in $h$ at $h = 0$.

(ii) If there exist a compact set $D_0 \subset D$ and a function $B_k(a) \neq 0$ for $a \in D_0$ such that

$$M(h, a) = B_k(a) h^{k+1} + O(h^{k+2}), \quad 0 < h \ll 1,$$

then there exist $\epsilon_0 > 0$ and an open set $U(D_0) \supset D_0$, and a neighborhood $V$ of the origin such that Eq. (1.1) has at most $k$ limit cycles in $V$ for $0 < |\epsilon| < \epsilon_0$ and $a \in U(D_0)$.

**THEOREM 1.3.** Suppose

$$M(h, a) = b_0(a) h + b_1(a) h^2 + \cdots + b_k(a) h^{k+1} + O(h^{k+2}),$$

$$0 < h \ll 1,$$

and

$$b_j(a_{i_0}, \ldots, a_{k+1}, \ldots, a_n) = 0, \quad j = 0, \ldots, k - 1,$$

$$\det \left( \frac{\partial (b_0, \ldots, b_{k-1})}{\partial (a_1, \ldots, a_k)} \right)_{(a_{i_0}, \ldots, a_{k+1}, \ldots, a_n)} \neq 0, \quad n \geq k \geq 1,$$

where $(a_{i_0}, \ldots, a_{k+1}, \ldots, a_n) \in D$ with $(a_{i_0}, \ldots, a_{k+1})$ constant. If there exists a vector valued function $\phi(\epsilon, a_{k+1}, \ldots, a_n) \in \mathbb{R}^k$ such that Eq. (1.1) has a center at the origin for $(a_1, \ldots, a_n) = \phi(\epsilon, a_{k+1}, \ldots, a_n)$, then Eq. (1.1) has at most $k - 1$ limit cycles near the origin for $a \in D$ and $|\epsilon| + \sum_{i=1}^k |a_i - a_{i_0}|$ sufficiently small, and $k - 1$ limit cycles can appear for some $(\epsilon, a)$. In other words, (1.1) has cyclicity $k - 1$ at the origin for $a \in D$ and $|\epsilon| + \sum_{i=1}^k |a_i - a_{i_0}|$ sufficiently small.

We remark that if Eqs. (1.6) and (1.10) hold, then the cyclicity of Eq. (1.1) at the origin is $k$. Also, under the conditions of Theorem 1.3, we have

$$M(h, a_0) = 0, \quad M(h, a) \neq 0.$$

This case has not been considered before for Hopf bifurcations.

This paper is organized as follows. In Section 2, we prove Theorems 1.2 and 1.3. In Section 3, we use the two theorems to discuss some $C^\infty$ systems.
of certain form and some polynomial Lienard systems. New and interesting results are obtained.

2. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.2. Let \( x(t, c) \) denote the solution of Eq. (1.2) with the initial value \( x(0, c) = (c, 0) \). From Eq. (1.4), we have

\[
I(c, 0) = c^2(A + S(c)), \quad S(0) = 0.
\]

Let \( r(c) = c\sqrt{A + S(c)} \). Then \( r \) is \( C^\infty \) in \( c \). Since \( I(x) \) is a first integral of (1.2), we have

\[
I(x(t, c)) = r^2(c) \quad \text{for all } t \in \mathbb{R}.
\]

Noting that \( r'(0) > 0 \), the function \( r = r(c) \) has a unique inverse \( c = c(r) \). Let \( x(t, c(r)) = u(t, r) \). The equation \( I(x) = r^2 \) defines a closed curve \( L_r \), which is a periodic orbit of Eq. (1.2) having period \( T(r) \). Then we have

\[
I(u(t, r)) = r^2, \quad u(T(r), r) = (c, 0).
\]

Obviously, \( u(t, r) \) and \( T(r) \) are \( C^\infty \) functions and \( T(0) > 0 \). Let \( G(\theta, r) = u(T(\theta)/2\pi, r) \). The function \( G \) is \( 2\pi \)-periodic in \( \theta \). By [10, Lemma 5.14] or [14, Lemma 4.4.3], the transformation \( x = G(\theta, r) \) carries Eq. (1.1) into the system

\[
\begin{align*}
\dot{\theta} &= \frac{2\pi}{T(r)} \left[ 1 + \frac{\epsilon G_r \wedge g(G, \epsilon, a)}{G_r \wedge f(G)} \right], \\
\dot{r} &= \frac{\epsilon}{2r} Di(G)g(G, \epsilon, a).
\end{align*}
\]  

\tag{2.1}

\]

Differentiating the equality \( I(G) = r^2 \) in \( r \) yields that \( Di(G)G_r = 2r \). Then using Eq. (1.5), we have \( G_r \wedge f(G) = -2r/\mu \). Hence, noting that \( g(G, \epsilon, a) = O(r) \), the right-hand-side functions of Eq. (2.1) are \( C^\infty \) in \( r \). From Eq. (2.1), we obtain the following \( C^\infty \) \( 2\pi \)-periodic equation

\[
\frac{dr}{d\theta} = \epsilon R(\theta, r, \epsilon, a),
\]

\tag{2.2}

with

\[
R(\theta, 0, \epsilon, a) = 0, \quad R(\theta, r, 0, a) = \frac{T(r)}{2r\pi} Di(G)g(G, 0, a).
\]

\tag{2.3}
It is clear that system (1.1) has a limit cycle near the origin if and only if the cylinder equation (2.2) has two $2\pi$-periodic solutions near $r = 0$: one is positive, the other is negative.

Let $P(r, \epsilon, a)$ denote the Poincaré map of Eq. (2.2). We can write

$$P(r, \epsilon, a) = r + \epsilon r F(r, \epsilon, a),$$

where

$$r F(r, 0, a) = \int_{0}^{2\pi} R(\theta, r, 0, a) d\theta \equiv R_0(r, a).$$

Note that since $\dot{r} > 0$ for $\epsilon$ small, the origin is stable (resp., unstable) for Eq. (1.1) if and only if the zero solution of Eq. (2.2) is stable (resp., unstable). We have

$$r P(r, \epsilon, a) - r = \epsilon r^2 F(r, \epsilon, a) \geq 0 \quad \text{(or} \leq 0) \quad \text{for all } r \text{ small}$$

for fixed $(\epsilon, a)$. It follows that if

$$R_0(r, a) = b^a_m(r^m + O(r^{m+1})), \quad b^a_m \neq 0, \quad (2.6)$$

then $m$ is odd; $m = 2k + 1, k \geq 0$. Then by Eq. (2.4) and Rolle’s theorem, it is easy to verify that there exist $\epsilon_0 > 0$ and a neighborhood $U$ of $D_0$ such that, for $0 < |\epsilon| < \epsilon_0$, $a \in U$, the function $F$ has at most $2k$ roots in $r$ if Eq. (2.6) is satisfied for $a \in D_0$. This implies that Eq. (1.1) has at most $k$ limit cycles near the origin for $0 < |\epsilon| < \epsilon_0$, $a \in U$, if Eq. (2.6) is satisfied for $a \in D_0$.

Next, we establish a relation between the functions $R_0$ and $M$ defined by Eq. (1.7). By Eqs. (2.3) and (1.5), we have

$$R_0(r, a) = \frac{1}{2} \int_{r f(x) = r^2} DI(x) g(x, 0, a) \, dx$$

$$= \frac{1}{2} \int_{r f(x) = r^2} \mu(x) f(x) \wedge g(x, 0, a) \, dx$$

$$= -\frac{1}{2} \int_{r f(x) = r^2} \mu g(x, 0, a) \wedge dx,$$

which yields that $R_0(r, a)$ is odd in $r$. Let $M^*(r, a) = r R_0(r, a)$. Then $M^*$ is even in $r$ and it holds that

$$M^*(r, a) = \sum_{i=1}^{j} A_i r^{2i} + N(r)$$
for any $j \geq 1$, where $N$ is even and $C^\infty$ in $r$ with $N^{(i)}(0) = 0$, $i = 0, \ldots, 2j$. Let $\tilde{N}(h) = N(\sqrt{h})$, $h \geq 0$. By induction, we can show that $\tilde{N}^{(i)}(0) = 0$, $i = 0, \ldots, j$. In fact, first we can write

$$N(r) = r^{2j}N_0(r), \quad N_0 \in C^\infty, \quad N_0(0) = 0.$$  

It follows that

$$\tilde{N}(h) = h^{j}N_0(\sqrt{h}) = h^{j}N_0^+(\sqrt{h}).$$

By differentiating the above equality in $h$, we have

$$\tilde{N}'(h) = h^{j-1}\left[jN_0(\sqrt{h}) + \frac{1}{2}\sqrt{h}N_0'(\sqrt{h})\right] = h^{j-1}N_0^+(\sqrt{h}),$$

with $N_0^+ \in C^\infty$ and $N_0^+(0) = 0$. Suppose, for $0 < k < j$, there exists $N_k^+ \in C^\infty$ with $N_k^+(0) = 0$ such that

$$\tilde{N}^{(k)}(h) = h^{j-k}N_k^+(\sqrt{h}).$$

Differentiating the above in $h$ gives that

$$\tilde{N}^{(k+1)}(h) = h^{j-k-1}\left[(j-k)N_k^+(\sqrt{h}) + \frac{1}{2}\sqrt{h}(N_k^+)'(\sqrt{h})\right]$$

$$= h^{j-k-1}N_{k+1}^+(\sqrt{h}).$$

It is obvious that $N_{k+1}^+ \in C^\infty$ and $N_{k+1}^+(0) = 0$. Then by induction, we have

$$\tilde{N}^{(i)}(h) = h^{j-i}N_i^+(\sqrt{h}), \quad N_i^+ \in C^\infty, \quad N_i^+(0) = 0, \quad 0 \leq i \leq j.$$ 

Hence, $\tilde{N} \in C^j$ for $h \geq 0$ small. Let

$$\tilde{M}(h, a) = \sum_{i=1}^{j} A_i h^i + \tilde{N}(h).$$

Then $\tilde{M} \in C^j$ for $h \geq 0$ small. Since $j$ is arbitrary, it follows that $\tilde{M} \in C^\infty$ for $h \geq 0$ small. Now from Eqs. (2.7) and (1.7), it is evident that $M(h, a) = M^\star(\sqrt{h}, a) = \sqrt{h} R_0(\sqrt{h}, a) = M(h, a)$.

Hence, Eq. (1.8) holds if and only if Eq. (2.6) holds with $m = 2k + 1$. Also, from the above discussion, the function $M(h, a)$ is analytic in $h$ at $h = 0$ if the right-hand-side functions of Eq. (1.1) are analytic. This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3.** Since $M(h, a) = rR_0(r, a)$ for $r = \sqrt{h}$, $h \geq 0$, from Eq. (1.9) we have

$$R_0(r, a) = r\left[\sum_{j=0}^{k} b_j(a) r^{2j} + O(r^{2k+2})\right].$$
Note that the Poincaré map $P(r, \epsilon, a)$ in (2.4) is $C^\infty$ in $r$. We can write

$$F(r, \epsilon, a) = \sum_{j=0}^{2k-1} C_j(\epsilon, a)r^j + r^{2k}Q(r, \epsilon, a),$$

(2.8)

where $Q$ is $C^\infty$ in $r$, and

$$C_{2j}(0, a) = b_j(a), \quad C_{2j+1}(0, a) = 0, \quad j = 0, \ldots, k - 1.$$ 

By Eq. (1.10), the equations $b_j = C_{2j}(\epsilon, a)$, $j = 0, \ldots, k - 1$, have a vector valued solution of the form

$$(a_1, \ldots, a_k) = \overline{\phi}(\epsilon, b_0, \ldots, b_{k-1}, a_{k+1}, \ldots, a_n)$$

(2.9)

for $a \in D$ and $|\epsilon| + \sum_{i=1}^{k} |a_i - a_{i0}|$ sufficiently small. Since Eq. (1.1) has a center at the origin for $(a_1, \ldots, a_k) = \phi(\epsilon, a_{k+1}, \ldots, a_n)$, we have $P(r, \epsilon, a) - r = 0$ and especially $C_{2j}(\epsilon, a) = 0$, $j = 0, \ldots, k - 1$, in this case. The implicit function theorem implies that

$$\overline{\phi}(\epsilon, b_0, \ldots, b_{k-1}, a_{k+1}, \ldots, a_n) = \phi(\epsilon, a_{k+1}, \ldots, a_n) \Rightarrow b_j = 0, \quad j = 0, \ldots, k - 1.$$ 

(2.10)

Substituting Eq. (2.9) into Eq. (2.8) yields that

$$F = \sum_{j=0}^{k-1} [b_j r^{2j} + C_{2j+1}(\epsilon, \overline{\phi}, a_{k+1}, \ldots, a_n) r^{2j+1}]$$

$$+ r^{2k}Q(r, \epsilon, \overline{\phi}, a_{k+1}, \ldots, a_n)$$

$$= F^\ast(r, \epsilon, \mu),$$

(2.11)

where $\mu = (b_0, \ldots, b_{k-1}, a_{k+1}, \ldots, a_n)$. Then from Eq. (2.10) and our assumption, we have that $F^\ast(r, \epsilon, \mu) = 0$ for $b_j = 0$, $j = 0, \ldots, k - 1$. Hence we can write

$$C_{2j+1}(\epsilon, \overline{\phi}, a_{k+1}, \ldots, a_n) = \epsilon \sum_{i=0}^{k-1} b_i A_{ij}(\epsilon, \mu),$$

(2.12)

$$Q(r, \epsilon, \overline{\phi}, a_{k+1}, \ldots, a_n) = \sum_{i=0}^{k-1} b_i Q_i(r, \epsilon, \mu).$$

Furthermore, it follows from Eqs. (2.5) and (2.11) that

$$C_{2j+1}(\epsilon, \overline{\phi}, a_{k+1}, \ldots, a_n) = 0 \quad \text{as} \quad b_i = 0, \quad i = 0, \ldots, j, \quad j = 0, \ldots, k - 1.$$
Thus, from Eq. (2.12), we have
\[ A_{ij} = 0 \quad \text{for } j + 1 \leq i \leq k - 1, \quad j = 0, \ldots, k - 2. \] (2.13)

Substituting Eqs. (2.12) and (2.13) into Eq. (2.11), we obtain
\[ F^*(r, \epsilon, \mu) = \sum_{j=0}^{k-1} b_j r^{2j} P_j(r, \epsilon, \mu), \] (2.14)
where
\[ P_j = 1 + \epsilon \sum_{i=j}^{k-1} A_i r^{2(i-j)+1} + r^{2(k-j)} Q_j, \quad 0 \leq j \leq k - 1. \]

From Eq. (2.11), it is easy to see that \( F^* \) can have \( k - 1 \) positive roots in \( r \) for some \((\epsilon, b_0, \ldots, b_{k-1})\). From Eq. (2.14) and by the technique of Bautin [1], we can prove that \( F^* \) has at most \( k - 1 \) roots in \( r > 0 \). For the sake of simplicity, let us prove the conclusion for \( k = 3 \). In this case, we have
\[ F^* = b_0 P_0 + b_1 r^2 P_1 + b_2 r^4 P_2 = P_0 \left[ b_0 + b_1 r^2 P_{11} + b_2 r^4 P_{12} \right] = P_0 P^*_1, \]
\[ \frac{\partial P^*_i}{\partial r} = 2rP_{2i} \left[ b_1 + 2b_2 r^2 P_{22} \right], \quad P_{ij} = 1 + O(|\epsilon| + r^2), \quad i, j = 1, 2, \]
\[ \frac{\partial}{\partial r} (b_1 + 2b_2 r^2 P_{22}) = 4b_2 r P_{23}, \quad P_{23} = 1 + O(|\epsilon| + r^2). \]

Hence, \( \partial P^*_i / \partial r \) has at most one root in \( r > 0 \), and therefore, \( F^* \) has at most two roots in \( r > 0 \). This completes the proof of Theorem 1.3.

3. APPLICATIONS

Let us first consider a nonlinear system of Lienard type
\[ \dot{x} = p(y) - \epsilon \left[ F_\epsilon(x) + F_0(x, a) \right], \quad \dot{y} = -g(x), \] (3.1)
where \( p, g, \) and \( F_\epsilon \) are \( C^\infty \) functions and satisfy
\[ p(0) = F_\epsilon(0) = 0, \quad p'(0) > 0, \quad g'(0) > 0, \quad g(-x) = -g(x), \quad F_\epsilon(-x) = F_\epsilon(x), \] (3.2)
and
\[
F_0(x, a) = \sum_{i=0}^{k} a_i x^{2i+1}, \quad a = (a_0, \ldots, a_k), \quad k \geq 1. \quad (3.3)
\]

We have

**Theorem 3.1.** For any given \(B > 0\), there exist \(\epsilon^* > 0\) and a neighborhood \(V\) of the origin such that Eq. (3.1) has at most \(k\) limit cycles in \(V\) for all \(0 < |\epsilon| < \epsilon^*, |a_i| \leq B, i = 1, \ldots, k\). Moreover, \(k\) limit cycles can appear. Hence, Eq. (3.1) has cyclicity \(k\) at the origin.

**Proof.** For \(\epsilon = 0\), Eq. (3.1) has a Hamiltonian function as follows
\[
I(x, y) = \int_0^x g(u) \, du + \int_0^y p(u) \, du = G(x) + P(y).
\]

Then by Eq. (1.7), we have
\[
M(h, a) = \oint_{I=h} \left( F_e(x) + F_0(x, a) \right) \, dy.
\]

By Eqs. (3.2) and (3.3), we know that if \(a = 0\), orbits of Eq. (3.1) are symmetric with respect to the \(y\)-axis, and therefore the origin is a center point. It implies that
\[
\oint_{I=h} F_e(x) \, dy = 0.
\]

Hence,
\[
M(h, a) = \oint_{I=h} F_0(x, a) \, dy = \sum_{i=0}^{k} a_i N_i(h),
\]

where
\[
N_i(h) = \oint_{I=h} x^{2i+1} \, dy, \quad 0 \leq i \leq k.
\]

By Green’s formula and then letting
\[
x = \sqrt{\frac{2}{g'(0)}} \, r \cos \theta, \quad y = \sqrt{\frac{2}{p'(0)}} \, r \sin \theta,
\]
we have

\[ N_i(h) = -\int \int_{I \leq h} (2i + 1)x^{2i} \, dx \, dy \]

\[ = - (2i + 1) \left( \frac{2}{g'(0)} \right)^i \int_0^{2\pi} \frac{2}{\sqrt{g''(0)p'(0)}} \int_0^r \cos^{2i} \theta \, d\theta \]

\[ \times \int \sqrt{r + O(h^i)^{i+1}} \, dr \]

\[ = N_{i0}h^{i+1} + O(h^{i+2}), \quad (3.4) \]

where

\[ N_{i0} = - \frac{2i + 1}{(i + 1)^{i+1}} \left( \frac{2}{g'(0)} \right)^i \int_0^{2\pi} \cos^{2i} \theta \, d\theta, \]

\[ 0 \leq i \leq k. \]

Thus, we obtain

\[ M(h, a) = b_0(a)h + b_1(a)h^2 + \cdots + b_k(a)h^{k+1} + O(h^{k+2}), \quad (3.5) \]

where \( b_j(a) = a_jN_{i0} + O(a_0 + \cdots + a_{j-1}) \) with that \( b_j(0) = 0, \ 0 \leq j \leq k, \) and

\[ \det \frac{\partial (b_0, \ldots, b_k)}{\partial (a_0, \ldots, a_k)} = N_{i0}N_{i0} \ldots N_{k0} \neq 0. \]

By Theorem 1.3, there exists \( \epsilon_0 > 0 \) such that, for \( 0 < |\epsilon| < \epsilon_0, \ a \in U(\epsilon_0) = \{ a \in R^{k+1}; \ |a_j| < \epsilon_0, \ j = 0, \ldots, k \}, \) Eq. (3.1) has at most \( k \) limit cycles near the origin, and \( k \) limit cycles can appear for some \( (\epsilon, a) \).

Now for given \( B > \epsilon_0, \) let \( a \in CLU(B) - U(\epsilon_0) = W. \) Note that \( b_j(a) = 0, \ j = 0, \ldots, k, \) if and only if \( a = 0. \) For each \( a \in W, \) we have \( (b_0(a), \ldots, b_k(a)) \neq 0. \) Hence, there exist \( k + 1 \) compact sets \( W_0, \ldots, W_k \) such that

\[ W = \bigcup_{j=0}^k W'_j \quad \text{and} \quad b_j(a) \neq 0 \ \text{for} \ a \in W'_j. \]

By Theorem 1.2, for each \( 0 \leq j \leq k, \) there exists \( \epsilon_{j+1} > 0 \) such that for \( 0 < |\epsilon| < \epsilon_{j+1}, \ a \in W'_j, \) Eq. (3.1) has at most \( j \) limit cycles near the origin.

Let \( \epsilon_0^* = \min(\epsilon_{j+1}; \ j = 0, \ldots, k). \) It follows that Eq. (3.1) has at most \( k \) limit cycles near the origin for \( 0 < |\epsilon| < \epsilon_0^*, \ a \in W. \)
Now it is clear that the proof follows by choosing $\epsilon^* = \min(\epsilon_0, \epsilon_0^*)$. 

Next, we consider a polynomial Lienard system of the form
\[
\dot{x} = y - \epsilon \sum_{i=1}^{n} a_i x^i, \quad \dot{y} = -(x + x^3).
\] (3.6)

We prove that

**Theorem 3.2.** For given $B > 0$, there exists $\epsilon_0 > 0$ such that, for $m = 1, 2, 3$, the following hold.

(i) If $n = 3m - 1$ or $n = 3m$, then cyclicity of Eq. (3.6) at the origin is $2m - 1$ for $0 < |\epsilon| < \epsilon_0$, $|a_i| \leq B$, $i = 1, \ldots, n$.

(ii) If $n = 3m + 1$, then cyclicity of Eq. (3.6) at the origin is $2m$ for $0 < |\epsilon| < \epsilon_0$, $|a_i| \leq B$, $i = 1, \ldots, n$.

Before proving Theorem 3.2, we first give a lemma which can be proved by [10, Chap. 8, Theorem 8.11] or by a theorem in [18, Chap. 5].

**Lemma 3.1.** Consider the $C^1$ system
\[
\dot{x} = y - F(x), \quad \dot{y} = -g(x),
\] (3.7)
where $F(0) = 0$, $g(0) = 0$, $g'(0) > 0$. If there exists a continuous function $C: R \to R$ such that $F(x) = C(G(x))$, $G(x) = \int_0^x g(u) du$, then Eq. (3.7) has a center at the origin.

**Proof of Theorem 3.2.** For $n = 2$ or 3, the conclusion follows from [14, Theorem 3.3.7]. So, we suppose $n \geq 4$. Let
\[
I(x, y) = \frac{1}{2} y^2 + \frac{1}{2} x^2 + \frac{1}{5} x^3.
\]
Then, as before, we have
\[
M(h, a) = \sum_{i=1}^{n} a_i M_i(h),
\] (3.8)
where
\[
M_i(h) = \oint_{\gamma_{h}^{i}} x^i dy.
\]
By Eqs. (3.4) and (3.5), we have
\[
M_{2i+1}(h) = N_i(h) = N_{i0} h^{i+1} + O(h^{i+2}),
\]
\[
N_{i0} = -\frac{2^i (2i + 1)}{i + 1} \int_0^{2\pi} \cos^{2i} \theta d\theta.
\] (3.9)
Note that
\[ x^2 = 2h - y^2 - \frac{2}{3}x^3, \quad 2h - y^2 = x^2 + \frac{2}{3}x^3, \]
along \( l = h \). We have
\[
M_2(h) = \int_{l=h} \left(2h - y^2 - \frac{2}{3}x^3\right) \, dy = -\frac{2}{3}M_3(h), \quad (3.10)
\]
\[
M_4(h) = \int_{l=h} \left(2h - y^2 - \frac{2}{3}x^3\right)^2 \, dy
= \int_{l=h} \left[\left(2h - y^2\right)^2 - \frac{4}{3}x^3(2h - y^2) + \frac{4}{9}x^6\right] \, dy
= \int_{l=h} \left[-\frac{4}{3}x^3(x^2 + \frac{2}{3}x^3) + \frac{4}{9}x^6\right] \, dy
= -\frac{4}{3}M_5(h) - \frac{4}{9}M_6(h). \quad (3.11)
\]
In the same way, we have the following formulas
\[
M_6(h) = -2M_7(h) - \frac{8}{3}M_8(h) - \frac{8}{7}M_9(h),
M_8(h) = -\frac{8}{7}M_9(h) - \frac{3}{5}M_{10}(h) - \frac{32}{7}M_{11}(h) - \frac{16}{7}M_{12}(h),
M_{10}(h) = -\frac{10}{7}M_{11}(h) - \frac{80}{9}M_{12}(h) - \frac{320}{29}M_{13}(h)
- \frac{560}{7}M_{14}(h) - \frac{192}{23}M_{15}(h), \quad (3.12)
\]
In general, we have
\[
M_{2k}(h) = -c_{k1}M_{2k+1}(h) - c_{k2}M_{2k+2}(h) - \cdots - c_{kK}M_{3k}(h),
\]
with \( c_{k1} = \frac{2k}{3}, \ c_{kj} > 0, \ j = 2, \ldots, k \). In particular,
\[
M_{12}(h) = -4M_{13}(h) + O(M_{15}(h)), \quad M_{14}(h) = O(M_{15}(h)).
\]
By Eqs. (3.11) and (3.12), we can obtain
\[
M_{10} = -\frac{10}{7}M_{11} + \frac{60}{27}M_{13} + O(M_{15}),
M_8 = -\frac{8}{7}M_9 + \frac{348}{27}M_{11} - \frac{5056}{81}M_{13} + O(M_{15}),
M_6 = -2M_7 + \frac{88}{27}M_9 - \frac{832}{81}M_{11} + \frac{20024}{243}M_{13} + O(M_{15}),
M_4 = -\frac{4}{3}M_5 + \frac{8}{9}M_7 - \frac{352}{243}M_9 + \frac{4328}{729}M_{11} - \frac{80996}{2187}M_{13} + O(M_{15}).
\]
Inserting Eq. (3.10) and the above into Eq. (3.8) \((n \leq 10)\) gives that
\[
M = a_1 M_1 + (a_3 - \frac{7}{2} a_2) M_5 + (a_5 - \frac{4}{3} a_4) M_5 \\
+ (a_7 - 2 a_6 + \frac{8}{3} a_4) M_7 + (a_9 - \frac{8}{7} a_8 + \frac{88}{27} a_6 - \frac{352}{235} a_4) M_9 \\
+ (- \frac{10}{3} a_10 + \frac{586}{27} a_8 - \frac{338}{73} a_6 + \frac{838}{722} a_4) M_{11} \\
+ \left(\frac{540}{27} a_10 - \frac{5056}{81} a_8 + \frac{20224}{2343} a_6 - \frac{80896}{2357} a_4\right) M_{13} + O(M_{15}).
\] (3.13)

Now we continue the proof for each \(4 \leq n \leq 10\). Let
\[
F(x) = \sum_{i=1}^{n} a_i x^i, \quad G(x) = \frac{1}{2} x^2 + \frac{1}{3} x^3.
\]

For \(n = 4\), let \(b = (b_1, b_2, b_3) = (a_1, a_3 - \frac{7}{2} a_2, - \frac{4}{3} a_4)\). Then Eq. (3.13) becomes
\[
M = b_1 M_1 + b_2 M_5 + b_3 M_5 + O(M_7).
\]

When \(b = 0\), we have \(F(x) = 2a_2 G(x)\). Hence, by Lemma 3.1, Eq. (3.6) has a center at the origin if \(b = 0\). Note that we can write \(M\) as in the form of Eq. (3.5). By Theorem 1.3, there exists \(\epsilon_0 > 0\) such that, for \(0 < |\epsilon| < \epsilon_0, |b| < \epsilon_0\), Eq. (3.6) has cyclicity 2 at the origin. For \(\epsilon_0 \leq |b| \leq B\), similar to the proof of Theorem 3.1, we can show that there exists \(\epsilon_0^* > 0\) such that Eq. (3.6) has at most two limit cycles near the origin for \(0 < |\epsilon| < \epsilon_0^*, \epsilon_0 \leq |b| \leq B\). This ends the proof for \(n = 4\).

For \(n = 5\), let \(b = (b_1, b_2, b_3, b_4) = (a_1, a_3 - \frac{7}{2} a_2, a_5 - \frac{4}{3} a_4, - \frac{8}{9} a_4)\). Then,
\[
M = b_1 M_1 + b_2 M_5 + b_3 M_5 + b_4 M_7 + O(M_9).
\]

As before, Eq. (3.6) has a center at the origin when \(b = 0\). Hence, in the same way, we have that Eq. (3.6) has cyclicity 3 at the origin. For \(n = 6\), also let \(b = (b_1, b_2, b_3, b_4, b_5, b_6)\), with \(b_1, b_2, b_3\) being the same as above and \(b_4 = -2a_6 + \frac{8}{9} a_4\). In this case, when \(b = 0\), we have
\[
a_1 = 0, \quad a_3 = \frac{7}{2} a_2, \quad a_5 = \frac{4}{3} a_4, \quad a_6 = \frac{8}{9} a_4,
\]

which imply that
\[
F(x) = 2a_2 G(x) + 4a_4 G^2(x).
\]

Also by Lemma 3.1, and Theorems 1.2 and 1.3, the conclusion follows for \(n = 6\).

For \(n = 7, 8, 9\), the proof is just similar. We consider the case \(n = 10\). Let \(b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)\) such that Eq. (3.13) becomes
\[
M = \sum_{i=1}^{7} b_i M_{2i-1} + O(M_{15}).
\]
It is direct that if $b = 0$, then
\[
\begin{align*}
    a_1 &= 0, \quad a_3 = \frac{2}{7}, \quad a_5 = \frac{4}{7}a_4, \quad a_7 = 2(a_6 - \frac{4}{7}a_4), \\
    a_{10} &= 0, \quad a_8 = \frac{4}{7}(a_6 - \frac{4}{7}a_4), \quad a_9 = \frac{8}{7}(a_6 - \frac{4}{7}a_4),
\end{align*}
\]
which yield that
\[
F(x) = 2a_2G(x) + 4a_4G^2(x) + 8(a_6 - \frac{4}{7}a_4)G^3(x).
\]
Using Lemma 3.1 again, Eq. (3.6) has a center at the origin if $b = 0$. The rest is similar. The proof is completed.

We remark that it is possible to prove the conclusion of Theorem 3.2 for some more integers $m \geq 4$. In general, we have the following conjecture.

**Conjecture.** The conclusion of Theorem 3.2 is true for all integers $m \geq 1$.

Now we consider a $C^\infty$ system of the form
\[
\begin{align*}
    \dot{x} &= y + f_1(x, y) + g_1(x, y, a) + R_1(x, y, a), \\
    \dot{y} &= -x + f_2(x, y) + g_2(x, y, a) + R_2(x, y, a),
\end{align*}
\]
(3.14)
where $a = (a_0, \ldots, a_m)$, $f_i = O(|x|^2)$, $R_i = O(|a|^2)$, $R_i(0, 0, a) = 0$, and
\[
g_i(x, y, a) = \sum_{j=0}^{m} a_jp_{2j+1, i}(x, y), \quad i = 1, 2, \quad (3.15)
\]
and $p_{2j+1, i}$ are homogeneous polynomials of degree $2j + 1$. The form of Eq. (3.14) is more general than the system (1) discussed in [3, Theorem 1]. We have

**THEOREM 3.3.** Suppose Eq. (3.14) ($a = 0$) has a first integral of the form $I(x, y) = x^2 + y^2 + O(|x|^3)$. If
\[
I_j = \int_{x^2+y^2 \leq 1} \text{div}(p_{2j+1, 1}, p_{2j+1, 2}) \, dx \, dy \neq 0, \quad j = 0, \ldots, m,
\]
then there exists $\epsilon_0 > 0$ such that, for all $|a| \leq \epsilon_0$, (3.14) has cyclicity $m$ at the origin.

**Proof.** Let $\epsilon = |a|$, $b = a/\epsilon = (b_0, \ldots, b_m)$. Then Eq. (3.14) can be rewritten as
\[
\begin{align*}
    \dot{x} &= y + f_1(x, y) + \epsilon g_1(x, y, b) + O(\epsilon^2), \\
    \dot{y} &= -x + f_2(x, y) + \epsilon g_2(x, y, b) + O(\epsilon^2).
\end{align*}
\]
(3.16)
By Eqs. (1.7) and (3.15), the first-order Melnikov function of Eq. (3.16) is

$$M(h, b) = \sum_{j=0}^{m} b_j N_j(h),$$

where

$$N_j(h) = \oint_{l=h} \left( I_x p_{2j+1,1} + I_y p_{2j+1,2} \right) dt.$$

Let

$$N_j^*(h) = \oint_{l=h} \left[ (x - f_2) p_{2j+1,1} + (y + f_1) p_{2j+1,2} \right] dt.$$

By Green's formula, we have

$$N_j^*(h) = \oint_{l=h} p_{2j+1,2} dx - p_{2j+1,1} dy$$

$$= \int \int_{l \leq h} \text{div} (p_{2j+1,1}, p_{2j+1,2}) dx dy$$

$$= \int \int_{x^2 + y^2 \leq h} \text{div} (p_{2j+1,1}, p_{2j+1,2}) dx dy (1 + o(1))$$

$$= h^{j+1} I_j + O(h^{j+2}).$$

Hence,

$$N_j(h) = 2N_j^*(h)(1 + o(1)) = 2h^{j+1} I_j + O(h^{j+2}).$$

Then similar to Eq. (3.5), we have

$$M(h, b) = 2 \left[ c_0 h + c_1 h^2 + \cdots + c_m h^{m+1} + O(h^{m+2}) \right],$$

with that

$$c_0 = I_0 b_0,$$

$$c_1 = I_1 b_1 + (*) b_0,$$

$$\cdots$$

$$c_m = I_m b_m + (*) b_0 + \cdots + (*) b_{m-1},$$

where (*)'s denote constants independent of b. Note that |b| = 1. There exists a constant $\epsilon_0 > 0$ such that $\epsilon_0 \leq |c| = \sup_j |c_j| \leq \epsilon_0^{-1}$. Then as before, the conclusion follows from Theorem 1.2. This ends the proof.
Finally, we give an example to show a difference between single and multiple parameter perturbations when the condition (1.10) does not hold.

Consider
\[
\dot{x} = y - \epsilon [a_1 x + a_2 x^3 - a_3 x^5 - \epsilon a_3 x^7], \quad \dot{y} = -x. \tag{3.17}
\]

We claim that:

(i) For each fixed \(a = (a_1, a_2, a_3) \in \mathbb{R}^3\), there exists \(\epsilon^*(a) > 0\) such that system (3.17) has at most two limit cycles near the origin for all \(0 < |\epsilon| < \epsilon^*(a)\).

(ii) However, system (3.17) has cyclicity 3 at the origin for \(\epsilon\) small. This implies that \(\lim_{\epsilon \to 0} \inf \epsilon^*(a) = 0\).

In fact, it is easy to see that, for (3.17),
\[
M(h, a) = -\pi h [a_1 + \frac{3}{4} a_2 h - \frac{5}{2} a_3 h^2].
\]

Note that system (3.17) has no limit cycles for \(a = 0\). The claim (i) follows from Theorem 1.1 or 1.2.

For claim (ii), by Theorem 3.1, system (3.17) has at most three limit cycles near the origin for \(|\epsilon|\) small and \(a\) varying in a bounded set. So, it suffices to prove that three limit cycles can appear for some \(|\epsilon|\) small.

For this purpose, we first consider the following system
\[
\dot{x} = y - c_1 x^5 (c_2 - c_3 x^2), \quad \dot{y} = -x. \tag{3.18}
\]

Let \(x = r \cos \theta, y = r \sin \theta\). We can obtain from Eq. (3.18)
\[
\frac{dr}{d\theta} = \frac{c_1 r^5 \cos^6 \theta [c_2 - c_3 r^2 \cos^2 \theta]}{1 - c_1 r^4 \sin \theta \cos^5 \theta [c_2 - c_3 r^2 \cos^2 \theta]}. \tag{3.19}
\]

The solution of Eq. (3.19) with initial value at \(\theta = 0\) is of the form
\[
r(\theta, r_0, c) = r_0 + r_5(\theta, c) r_0^5 + r_7(\theta, c) r_0^7 + O(r_0^9),
\]

where
\[
r_5(\theta, c) = c_1 c_2 \int_0^\theta \cos^5 t dt = c_1 c_2 A(\theta),
\]
\[
r_7(\theta, c) = -c_1 c_3 \int_0^\theta \cos^7 t dt = -c_1 c_3 B(\theta).
\]

From Eq. (3.19), we know that \(r(\theta, r_0, c) = r_0\) if and only if \(c_1 = 0\) or \(c_2 = c_3 = 0\). Hence, we have
\[
r(\theta, r_0, c) = r_0 + c_1 \left[ c_2 A(\theta) r_0^5 - c_3 B(\theta) r_0^7 + r_0^9 O(|c_2, c_3|) \right]
\]
\[
= r_0 + c_1 r_0^5 \left[ c_2 \left( A(\theta) + O(r_0^9) \right) - c_3 \left( B(\theta) + O(r_0^9) \right) \right].
\]
Since $A(2\pi) = \frac{5}{3} \pi$, $B(2\pi) = \frac{35}{12} \pi$, there is a unique $r_0 = \sqrt{\frac{8c_2}{7c_3}} + O(c_3/c_2)$ such that $r(2\pi, r_0, c) = r_0$ for $0 < c_2 \ll c_3$. Thus, Eq. (3.18) has a limit cycle near the origin for $0 < c_2 \ll c_3$, and it is stable for $c_1 > 0$. Now letting $c_1 = \epsilon a_3$, $c_2 = a_3$, $c_3 = \epsilon$, it follows that when $a_1 = a_3 = 0$, $0 < a_1 \ll \epsilon$, system (3.17) has a stable small limit cycle $L$. Fix $(\epsilon, a_3)$ and vary $(a_1, a_2)$ such that $0 < -a_1 \ll a_2 \ll a_3$. Then the origin has changed its stability twice and two smaller limit cycles have been created from the origin. Therefore, (3.19) has three limit cycles near the origin for $0 < -a_1 \ll a_2 \ll a_3 \ll \epsilon$. Then claim (ii) follows.

ACKNOWLEDGMENTS

This paper was completed during the stay of the author in the Center of Dynamical System and Nonlinear Studies, Georgia Institute of Technology. He thanks Professor J. K. Hale for the invitation, hospitality, and support, and also for the helpful suggestions in the revision.

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