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# Global well-posedness and inviscid limit for the Korteweg–de Vries–Burgers equation

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## ABSTRACT

Considering the Cauchy problem for the Korteweg–de Vries–Burgers equation

$$u_t + u_{xxx} + \epsilon |\partial_x|^{2\alpha} u + (u^2)_x = 0, \quad u(0) = \phi,$$

where  $0 < \epsilon, \alpha \leq 1$  and  $u$  is a real-valued function, we show that it is globally well-posed in  $H^s$  ( $s > s_\alpha$ ), and uniformly globally well-posed in  $H^s$  ( $s > -3/4$ ) for all  $\epsilon \in (0, 1]$ . Moreover, we prove that for any  $T > 0$ , its solution converges in  $C([0, T]; H^s)$  to that of the KdV equation if  $\epsilon$  tends to 0.

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## 1. Introduction

In this paper, we study the Cauchy problem for the Korteweg–de Vries–Burgers (KdV–B) equation with fractional dissipation

$$u_t + u_{xxx} + \epsilon |\partial_x|^{2\alpha} u + (u^2)_x = 0, \quad u(0) = \phi, \quad (1.1)$$

where  $0 < \epsilon, \alpha \leq 1$ ,  $u$  is a real-valued function of  $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ . Eq. (1.1) has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur (cf. [9]). The global well-posedness of (1.1) and the generalized KdV–Burgers equation has been studied by many authors (see [7,8] and the reference therein).

In [7] Molinet and Ribaud studied Eq. (1.1) in the case  $\alpha = 1$  and showed that (1.1) is globally well-posed in  $H^s$  ( $s > -1$ ). The main tool used in [7] is an  $X^{s,b}$ -type space which contains the dissipative structure. Their result is sharp in the sense that the solution map of (1.1) fails to be  $C^2$  smooth at  $t = 0$

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if  $s < -1$ . In particular, one cannot get lower regularity simply using fixed-point machinery. Note that  $s = -1$  is lower than the critical index  $s = -3/4$  for the KdV equation and also lower than the critical index  $s = -1/2$  for the dissipative Burgers equation. The case  $0 < \alpha < 1$  was left open and it was conjectured in [7] that one can get that (1.1) is globally well-posed in  $H^s$  ( $s > s_c = (\alpha - 3)/2(2 - \alpha)$ ) by using the same strategy as  $\alpha = 1$ .

In the first part of this paper, we will study the global well-posedness of Eq. (1.1) by following some ideas in [7].<sup>1</sup> The main issue reduces to a bilinear estimate

$$\|\partial_x(uv)\|_{X^{-1/2+\delta,s,\alpha}} \leq C \|u\|_{X^{1/2,s,\alpha}} \|v\|_{X^{1/2,s,\alpha}}. \tag{1.2}$$

For the definition of  $X^{b,s,\alpha}$ , one can refer to (2.2) below. We will apply the  $[k; Z]$ -multiplier method in [10] to prove (1.2). We obtain a critical number

$$s_\alpha = \begin{cases} -3/4, & 0 < \alpha \leq 1/2, \\ -3/(5 - 2\alpha), & 1/2 < \alpha \leq 1. \end{cases} \tag{1.3}$$

The well-posedness in  $H^s(\mathbb{R})$ ,  $s > -3/4$ , for any  $\alpha > 0$  was first obtained by Molinet and Ribaud [8]. It is worthy to note that  $s_\alpha$  is strictly bigger than the conjectured number  $s_c$  for  $0 < \alpha < 1$ . We prove that (1.2) holds if and only if  $s > s_\alpha$ . So, it seems that  $s > s_\alpha$  is an essential limitation of this method in these resolution spaces.

In the second part of this paper, we study the inviscid limit behavior of (1.1) when  $\epsilon$  goes to 0. Formally, if  $\epsilon = 0$  then (1.1) reduces to the KdV equation

$$u_t + u_{xxx} + (u^2)_x = 0, \quad u(0) = \phi. \tag{1.4}$$

The local well-posedness of Eq. (1.4) in  $L^2$  was established by Bourgain [1] and the  $X^{b,s}$ -theory was discovered. This local solution is a global one by using the conservation of  $L^2$  norm. The optimal result on local well-posedness in  $H^s$  was obtained by Kenig, Ponce, Vega [5], where they developed the sharp bilinear estimates and obtained that (1.4) is locally well-posed for  $s > -3/4$ . The sharp result on global well-posedness in  $H^s$  was obtained in [2], it was shown that (1.4) is globally well-posed in  $H^s$  for  $s > -3/4$ , where a kind of modified energy method, so-called I-method, is introduced.

A natural question is whether the solution of (1.1) converges to that of (1.4) if  $\epsilon$  goes to 0. We will prove that the global solution of (1.1) converges to the solution of (1.4) as  $\epsilon \rightarrow 0$  in the natural space  $C([0, T], H^s)$  for  $-3/4 < s \leq 0$ . To achieve this, we need to control the solution uniformly in  $\epsilon$ , which is independent of the properties of dissipative term. We prove a uniform global well-posedness result using  $l^1$ -variant  $X^{b,s}$ -type space and the I-method. Notice that (1.1) is invariant under the following scaling for  $0 < \lambda \leq 1$

$$u(x, t) \rightarrow \lambda^2 u(\lambda x, \lambda^3 t), \quad \phi(x) \rightarrow \lambda^2 \phi(\lambda x), \quad \epsilon \rightarrow \lambda^{3-2\alpha} \epsilon. \tag{1.5}$$

Eq. (1.1) has less symmetries than the KdV equation (1.4) due to the dissipative term. Hence the proofs for the pointwise estimate of the multipliers in our argument are different from those in the KdV equation [2]. The basic idea is the same, and to exploit dedicated cancellation to remove the singularity in the denominator.

For the limit behavior, we need to study the difference equation between (1.1) and (1.4). We first treat the dissipative term as perturbation and then use the uniform Lipschitz continuity property of the solution map. Similar idea can be found in [14] for the inviscid limit of the complex Ginzburg–Landau equation. For  $T > 0$ , we denote  $S_T^\epsilon, S_T$  the solution map of (1.1), (1.4) respectively. Now we state our main results. The notations used in this paper can be found in Section 2.

<sup>1</sup> After the paper was finished, the authors have been informed that the same results in this part were also obtained by Stéphane Vento [13] using the similar method.

**Theorem 1.1.** Assume  $0 < \epsilon, \alpha \leq 1$ . Let  $s_\alpha$  be given in (1.3). Let  $\phi \in H^s(\mathbb{R})$ ,  $s > s_\alpha$ . For any  $T > 0$ , there exists a unique solution  $u_\epsilon$  of (1.1) in

$$Z_T = C([0, T], H^s) \cap X_T^{1/2, s, \alpha}. \tag{1.6}$$

Moreover, the solution map  $S_T^\epsilon : \phi \rightarrow u$  is smooth from  $H^s(\mathbb{R})$  to  $Z_T$  and  $u$  belongs to  $C((0, \infty), H^\infty(\mathbb{R}))$ .

Notice that the critical regularity for the fractional Burgers equation is  $s = 3/2 - 2\alpha$  in the sense of scaling. Thus if  $1/2 < \alpha \leq 1$  then  $s_\alpha$  is lower than the critical regularity for the KdV and also for the fractional Burgers equation. In the proof we need to exploit the properties of the dissipative term both in bilinear estimates and regularity for the solution. Therefore, the results in Theorem 1.1 depend on  $\epsilon > 0$ . For the uniform well-posedness, we have the following:

**Theorem 1.2.** Assume  $0 < \alpha \leq 1$  and  $-3/4 < s \leq 0$ . Let  $\phi \in H^s(\mathbb{R})$ . Then for any  $T > 0$ , the solution map  $S_T^\epsilon$  in Theorem 1.1 satisfies for all  $0 < \epsilon \leq 1$

$$\|S_T^\epsilon \phi\|_{F^s(T)} \lesssim C(T, \|u\|_{H^s}), \tag{1.7}$$

where  $F^s(T) \subset C([0, T]; H^s)$  which will be defined later and  $C(\cdot, \cdot)$  is a continuous function with  $C(\cdot, 0) = 0$ , and also satisfies that for all  $0 < \epsilon \leq 1$

$$\|S_T^\epsilon(\phi_1) - S_T^\epsilon(\phi_2)\|_{C([0, T], H^s)} \leq C(T, \|\phi_1\|_{H^s}, \|\phi_2\|_{H^s}) \|\phi_1 - \phi_2\|_{H^s}. \tag{1.8}$$

We also have the uniform persistence of regularity, following the standard argument. The similar conclusions in Theorem 1.2 also hold for the complex-valued equation (1.1) for a small  $T = T(\|u\|_{H^s}) > 0$ . Our final result is on the limit behavior.

**Theorem 1.3.** Assume  $0 < \alpha \leq 1$ . Let  $\phi \in H^s(\mathbb{R})$ ,  $-3/4 < s \leq 0$ . For any  $T > 0$ , then

$$\lim_{\epsilon \rightarrow 0^+} \|S_T^\epsilon(\phi) - S_T(\phi)\|_{C([0, T], H^s)} = 0. \tag{1.9}$$

**Remark 1.4.** We are only concerned with the limit in the same regularity space. There seems no convergence rate. This can be seen from the linear solution,

$$\|e^{-t\partial_x^3 - t\epsilon|\partial_x|^{2\alpha}} \phi - e^{-t\partial_x^3} \phi\|_{C([0, T], H^s)} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0, \tag{1.10}$$

but without any convergence rate. We believe that there is a convergence rate if we assume the initial data has higher regularity than the limit space. For example, we prove that

$$\|S_T^\epsilon(\phi_1) - S_T(\phi_2)\|_{C([0, T], L^2)} \lesssim \|\phi_1 - \phi_2\|_{L^2} + \epsilon^{1/2} C(T, \|\phi_1\|_{H^1}, \|\phi_2\|_{L^2}). \tag{1.11}$$

We only prove our results in the case  $s \leq 0$  and our method also works for  $s > 0$ . For the complex-valued equation (1.1), the limit behavior (1.9) holds for a small  $T = T(\|\phi\|_{H^s}) > 0$ .

The rest of the paper is organized as follows. In Section 2 we present some notations and Banach function spaces. The proof of Theorem 1.1 is given in Section 3. We present uniform LWP in Section 4 and prove Theorem 1.2 in Section 5. Theorem 1.3 is proved in Section 6.

### 2. Notation and definitions

For  $x, y \in \mathbb{R}$ ,  $x \sim y$  means that there exist  $C_1, C_2 > 0$  such that  $C_1|x| \leq |y| \leq C_2|x|$ . For  $f \in \mathcal{S}'$  we denote by  $\widehat{f}$  or  $\mathcal{F}(f)$  the Fourier transform of  $f$  for both spatial and time variables,

$$\widehat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x, t) dx dt.$$

We denote by  $\mathcal{F}_x$  the Fourier transform on spatial variable and if there is no confusion, we still write  $\mathcal{F} = \mathcal{F}_x$ . Let  $\mathbb{Z}$  and  $\mathbb{N}$  be the sets of integers and natural numbers, respectively.  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . For  $k \in \mathbb{Z}_+$  let

$$I_k = \{\xi: |\xi| \in [2^{k-1}, 2^{k+1}]\}, \quad k \geq 1; \quad I_0 = \{\xi: |\xi| \leq 2\}.$$

Let  $\eta_0 : \mathbb{R} \rightarrow [0, 1]$  denote an even smooth function supported in  $[-8/5, 8/5]$  and equal to 1 in  $[-5/4, 5/4]$ . For  $k \in \mathbb{N}$  let  $\eta_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$  and  $\eta_{\leq k} = \sum_{k'=0}^k \eta_{k'}$ . For  $k \in \mathbb{Z}$  let  $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$ . Roughly speaking,  $\{\chi_k\}_{k \in \mathbb{Z}}$  is the homogeneous decomposition function sequence and  $\{\eta_k\}_{k \in \mathbb{Z}_+}$  is the non-homogeneous decomposition function sequence to the frequency space.

For  $k \in \mathbb{Z}_+$  let  $P_k$  denote the operator on  $L^2(\mathbb{R})$  defined by

$$\widehat{P_k u}(\xi) = \eta_k(\xi) \widehat{u}(\xi).$$

By a slight abuse of notation we also define the operator  $P_k$  on  $L^2(\mathbb{R} \times \mathbb{R})$  by the formula  $\mathcal{F}(P_k u)(\xi, \tau) = \eta_k(\xi) \mathcal{F}(u)(\xi, \tau)$ . For  $l \in \mathbb{Z}$  let

$$P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.$$

We define the Lebesgue spaces  $L_T^q L_x^p$  and  $L_x^p L_T^q$  by the norms

$$\|f\|_{L_T^q L_x^p} = \|\|f\|_{L_x^p}\|_{L_T^q(0,T)}, \quad \|f\|_{L_x^p L_T^q} = \|\|f\|_{L_T^q(0,T)}\|_{L_x^p}. \tag{2.1}$$

We denote by  $W_0$  the semigroup associated with Airy equation

$$\mathcal{F}_x(W_0(t)\phi)(\xi) = \exp[i\xi^3 t] \widehat{\phi}(\xi), \quad \forall t \in \mathbb{R}, \phi \in \mathcal{S}'.$$

For  $0 < \epsilon \leq 1$  and  $0 < \alpha \leq 1$ , we denote by  $W_\epsilon^\alpha$  the semigroup associated with the free evolution of (1.1),

$$\mathcal{F}_x(W_\epsilon^\alpha(t)\phi)(\xi) = \exp[-\epsilon|\xi|^{2\alpha} t + i\xi^3 t] \widehat{\phi}(\xi), \quad \forall t \geq 0, \phi \in \mathcal{S}',$$

and we extend  $W_\epsilon^\alpha$  to a linear operator defined on the whole real axis by setting

$$\mathcal{F}_x(W_\epsilon^\alpha(t)\phi)(\xi) = \exp[-\epsilon|\xi|^{2\alpha} |t| + i\xi^3 t] \widehat{\phi}(\xi), \quad \forall t \in \mathbb{R}, \phi \in \mathcal{S}'.$$

To study the low regularity of (1.1), Molinet and Ribaud introduce the variant version of Bourgain's spaces with dissipation

$$\|u\|_{X^{b,s,\alpha}} = \|\langle i(\tau - \xi^3) + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \widehat{u}\|_{L^2(\mathbb{R}^2)}, \tag{2.2}$$

where  $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ . The standard  $X^{b,s}$  space for (1.4) used by Bourgain [1] and Kenig, Ponce, Vega [5] is defined by

$$\|u\|_{X^{b,s}} = \|(\tau - \xi^3)^b \langle \xi \rangle^s \widehat{u}\|_{L^2(\mathbb{R}^2)}.$$

The space  $X^{1/2,s,\alpha}$  turns out to be very useful to capture both dispersive and dissipative effect. From the technical level, the dissipation will give bounds below for the modulations. These bounds will weaken the frequency interaction for  $\alpha > 1/2$ , but will not for  $\alpha \leq 1/2$ .

In order to study the uniform global well-posedness for (1.1) and the limit behavior, we use an  $l^1$  Besov-type norm of  $X^{b,s}$ . For  $k \in \mathbb{Z}_+$  we define the dyadic  $X^{b,s}$ -type normed spaces  $X_k = X_k(\mathbb{R}^2)$ ,

$$X_k = \left\{ f \in L^2(\mathbb{R}^2): f(\xi, \tau) \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{X_k} = \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - \xi^3) \cdot f\|_{L^2} \right\}.$$

Structures of this kind of spaces were introduced, for instance, in [12], [4] and [3] for the BO equation. From the definition of  $X_k$ , we see that for any  $l \in \mathbb{Z}_+$  and  $f_k \in X_k$  (see also [4]),

$$\sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - \xi^3) \int |f_k(\xi, \tau')| 2^{-l} (1 + 2^{-l} |\tau - \tau'|)^{-4} d\tau'\|_{L^2} \lesssim \|f_k\|_{X_k}. \tag{2.3}$$

Hence for any  $l \in \mathbb{Z}_+$ ,  $t_0 \in \mathbb{R}$ ,  $f_k \in X_k$ , and  $\gamma \in \mathcal{S}(\mathbb{R})$ , then

$$\|\mathcal{F}[\gamma(2^l(t - t_0)) \cdot \mathcal{F}^{-1} f_k]\|_{X_k} \lesssim \|f_k\|_{X_k}. \tag{2.4}$$

For  $-3/4 < s \leq 0$ , we define the following spaces:

$$F^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^2): \|u\|_{F^s}^2 = \sum_{k \in \mathbb{Z}_+} 2^{2sk} \|\eta_k(\xi) \mathcal{F}(u)\|_{X_k}^2 < \infty \right\}, \tag{2.5}$$

$$N^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^2): \|u\|_{N^s}^2 = \sum_{k \in \mathbb{Z}_+} 2^{2sk} \|(i + \tau - \xi^3)^{-1} \eta_k(\xi) \mathcal{F}(u)\|_{X_k}^2 < \infty \right\}. \tag{2.6}$$

The space  $F^s$  is between  $X^{1/2,s}$  and  $X^{1/2+,s}$ . It can be embedded into  $C(\mathbb{R}; H^s)$  and into the Strichartz-type space, say  $L_t^p L_x^q$  as  $X^{1/2+,s}$ . On the other hand, it has the same scaling in time as  $X^{1/2,s}$ , which is crucial in the uniform linear estimate, see Section 4. That is the main reason for us applying  $F^s$ .

For  $T \geq 0$ , we define the time-localized spaces  $X_T^{b,s,\alpha}$ ,  $X_T^{b,s}$ ,  $F^s(T)$ , and  $N^s(T)$

$$\begin{aligned} \|u\|_{X_T^{b,s,\alpha}} &= \inf_{w \in X^{b,s,\alpha}} \{ \|w\|_{X^{b,s,\alpha}}, w(t) = u(t) \text{ on } [0, T] \}; \\ \|u\|_{X_T^{b,s}} &= \inf_{w \in X^{b,s}} \{ \|w\|_{X^{b,s}}, w(t) = u(t) \text{ on } [0, T] \}; \\ \|u\|_{F^s(T)} &= \inf_{w \in F^s} \{ \|w\|_{F^s}, w(t) = u(t) \text{ on } [0, T] \}; \\ \|u\|_{N^s(T)} &= \inf_{w \in N^s} \{ \|w\|_{N^s}, w(t) = u(t) \text{ on } [0, T] \}. \end{aligned} \tag{2.7}$$

As a conclusion of this section we prove that the norm on  $F^s$  controls some space-time norm as the norm  $X^{1/2+,s}$ . If applying to frequency dyadic localized function, we see that the norm  $F^s$  is almost the same as the norm  $X^{1/2+,s}$ . Fortunately, in application we usually encounter this case. See [11] for a survey on  $X^{s,b}$  space.

**Proposition 2.1.** *Let  $Y$  be a Banach space of functions on  $\mathbb{R} \times \mathbb{R}$  with the property that*

$$\|e^{it\tau_0} e^{-t\partial_x^3} f\|_Y \lesssim \|f\|_{H^s(\mathbb{R})}$$

*holds for all  $f \in H^s(\mathbb{R})$  and  $\tau_0 \in \mathbb{R}$ . Then we have the embedding*

$$\left( \sum_{k \in \mathbb{Z}_+} \|P_k u\|_Y^2 \right)^{1/2} \lesssim \|u\|_{F^s}. \tag{2.8}$$

**Proof.** In view of definition, it suffices to prove that if  $k \in \mathbb{Z}_+$

$$\|P_k u\|_Y \lesssim 2^{sk} \|\eta_k(\xi) \mathcal{F}u\|_{X_k}. \tag{2.9}$$

Indeed, we have

$$\begin{aligned} P_k u &= \int \eta_k(\xi) \mathcal{F}u(\xi, \tau) e^{ix\xi} e^{it\tau} d\xi d\tau \\ &= \sum_{j=0}^{\infty} \int \eta_j(\tau - \xi^3) \eta_k(\xi) \mathcal{F}u(\xi, \tau) e^{ix\xi} e^{it\tau} d\xi d\tau \\ &= \sum_{j=0}^{\infty} \int \eta_j(\tau) e^{it\tau} \int \eta_k(\xi) \mathcal{F}u(\xi, \tau + \xi^3) e^{ix\xi} e^{it\xi^3} d\xi d\tau. \end{aligned} \tag{2.10}$$

From the hypothesis on  $Y$ , we obtain

$$\begin{aligned} \|P_k u\|_Y &\lesssim \sum_{j=0}^{\infty} \int \eta_j(\tau) \left\| e^{it\tau} \int \eta_k(\xi) \mathcal{F}u(\xi, \tau + \xi^3) e^{ix\xi} e^{it\xi^3} d\xi \right\|_Y d\tau \\ &\lesssim 2^{sk} \|\eta_k(\xi) \mathcal{F}u\|_{X_k}, \end{aligned} \tag{2.11}$$

which completes the proof of the proposition.  $\square$

### 3. Global well-posedness for KdV–B equation

In this section, we prove a global well-posedness result for the KdV–Burgers equation by following the idea of Molinet and Ribaud [7]. Using Duhamel’s principle, we will mainly work on the integral formulation of the KdV–Burgers equation

$$u(t) = W_\epsilon^\alpha(t) \phi_1 - \frac{1}{2} \int_0^t W_\epsilon^\alpha(t - \tau) \partial_x(u^2(\tau)) d\tau, \quad t \geq 0. \tag{3.1}$$

We will apply a fixed point argument to solve the following truncated version

$$u(t) = \psi(t) \left[ W_\epsilon^\alpha(t) \phi_1 - \frac{\chi_{\mathbb{R}_+}(t)}{2} \int_0^t W_\epsilon^\alpha(t - \tau) \partial_x(\psi_T^2(\tau) u^2(\tau)) d\tau \right], \tag{3.2}$$

where  $t \in \mathbb{R}$  and  $\psi$  is a smooth time cutoff function satisfying

$$\psi \in C_0^\infty(\mathbb{R}), \quad \text{supp } \psi \subset [-2, 2], \quad \psi \equiv 1 \quad \text{on } [-1, 1], \tag{3.3}$$

and  $\psi_T(\cdot) = \psi(\cdot/T)$ . Indeed, if  $u$  solves (3.2) then  $u$  is a solution of (3.1) on  $[0, T]$ ,  $T \leq 1$ .

Theorem 1.1 can be proved by a slightly modified argument in [7] combined with the following bilinear estimate. See also [13].

**Proposition 3.1.** *Let  $s_\alpha$  be given by (1.3). Let  $s \in (s_\alpha, 0]$ ,  $0 < \delta \ll 1$ , then there exists  $C_{s,\alpha} > 0$  such that for any  $u, v \in \mathcal{S}$ ,*

$$\|\partial_x(uv)\|_{X^{-1/2+\delta,s,\alpha}} \leq C_{s,\alpha} \|u\|_{X^{1/2,s,\alpha}} \|v\|_{X^{1/2,s,\alpha}}. \tag{3.4}$$

This type of estimate was systematically studied in [10], see also [5] for an elementary method. We will follow the idea in [10] to prove Proposition 3.1. Let  $Z$  be any abelian additive group with an invariant measure  $d\xi$ . In particular,  $Z = \mathbb{R}^2$  in this paper. For any  $k \geq 2$ , let  $\Gamma_k(Z)$  denote the hyperplane in  $\mathbb{R}^k$

$$\Gamma_k(Z) := \{(\xi_1, \dots, \xi_k) \in Z^k: \xi_1 + \dots + \xi_k = 0\}$$

endowed with the induced measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \dots, \xi_{k-1}, -\xi_1 - \dots - \xi_{k-1}) d\xi_1 \dots d\xi_{k-1}.$$

Note that this measure is symmetric with respect to permutation of the co-ordinates.

A function  $m : \Gamma_k(Z) \rightarrow \mathbb{C}$  is said to be a  $[k; Z]$ -multiplier, and we define the norm  $\|m\|_{[k; Z]}$  to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leq \|m\|_{[k; Z]} \prod_{j=1}^k \|f_j\|_{L^2} \tag{3.5}$$

holds for all test functions  $f_i$  on  $Z$ .

By duality and Plancherel's equality, it is easy to see that for (3.35), it suffices to prove

$$\left\| \frac{|\xi_3| \langle \xi_3 \rangle^s \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} (i(\tau_3 - \xi_3) + |\xi_3|^{2\alpha})^{-1/2+\delta}}{\langle i(\tau_2 - \xi_2) + |\xi_2|^{2\alpha} \rangle^{1/2} \langle i(\tau_1 - \xi_1) + |\xi_1|^{2\alpha} \rangle^{1/2}} \right\|_{[3; \mathbb{R}^2]} \lesssim 1. \tag{3.6}$$

By comparison principle (see [10]), it suffices to prove that

$$\begin{aligned} & \sum_{N_1, N_2, N_3} \sum_{L_1, L_2, L_3} \sum_H \frac{N_3 \langle N_3 \rangle^s \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle L_1 + N_1^{2\alpha} \rangle^{1/2} \langle L_2 + N_2^{2\alpha} \rangle^{1/2} \langle L_3 + N_3^{2\alpha} \rangle^{1/2-\delta}} \\ & \times \|\chi_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3; \mathbb{R}^2]} \lesssim 1, \end{aligned} \tag{3.7}$$

where  $N_i, L_i, H$  are dyadic,  $h(\xi) = \xi_1^3 + \xi_2^3 + \xi_3^3$  and

$$\chi_{N_1, N_2, N_3; H; L_1, L_2, L_3} = \chi_{|\xi_1| \sim N_1, |\xi_2| \sim N_2, |\xi_3| \sim N_3} \chi_{|h(\xi)| \sim H} \chi_{|\tau_1 - \xi_1^3| \sim L_1, |\tau_2 - \xi_2^3| \sim L_2, |\tau_3 - \xi_3^3| \sim L_3}. \tag{3.8}$$

The issues reduce to an estimate of

$$\|\chi_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3; \mathbb{R}^2]} \tag{3.9}$$

and dyadic summation. Since

$$\xi_1 + \xi_2 + \xi_3 = 0, \quad |h(\xi)| = |\xi_1^3 + \xi_2^3 + \xi_3^3| \sim N_1 N_2 N_3,$$

and

$$\tau_1 - \xi_1^3 + \tau_2 - \xi_2^3 + \tau_3 - \xi_3^3 + h(\xi) = 0,$$

then we have

$$\begin{aligned} N_{max} &\sim N_{med}, \\ L_{max} &\sim \max(L_{med}, H), \end{aligned} \tag{3.10}$$

where we define  $N_{max} \geq N_{med} \geq N_{min}$  to be the maximum, median, and minimum of  $N_1, N_2, N_3$  respectively. Similarly define  $L_{max} \geq L_{med} \geq L_{min}$ . It is known (see Section 4, [10]) that we may assume

$$N_{max} \gtrsim 1, \quad L_1, L_2, L_3 \gtrsim 1. \tag{3.11}$$

Therefore, from Schur’s test [10, Lemma 3.11] it suffices to prove that

$$\begin{aligned} &\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_1, L_2, L_3 \geq 1} \frac{N_3 \langle N_3 \rangle^s \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle L_1 + N_1^{2\alpha} \rangle^{1/2} \langle L_2 + N_2^{2\alpha} \rangle^{1/2} \langle L_3 + N_3^{2\alpha} \rangle^{1/2-\delta}} \\ &\times \|\chi_{N_1, N_2, N_3; L_{max}; L_1, L_2, L_3}\|_{[3; \mathbb{R}^2]} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \leq L_{max}} \frac{N_3 \langle N_3 \rangle^s \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle L_1 + N_1^{2\alpha} \rangle^{1/2} \langle L_2 + N_2^{2\alpha} \rangle^{1/2} \langle L_3 + N_3^{2\alpha} \rangle^{1/2-\delta}} \\ &\times \|\chi_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{[3; \mathbb{R}^2]} \end{aligned} \tag{3.13}$$

are both uniformly bounded for all  $N \gtrsim 1$ .

**Proposition 3.2.** (See [10, Proposition 6.1].) *Let dyadic numbers  $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$  obey (3.10), (3.11).*

(i) *If  $N_{max} \sim N_{min}$  and  $L_{max} \sim H$ , then we have*

$$(3.9) \lesssim L_{min}^{1/2} N_{max}^{-1/4} L_{med}^{1/4}. \tag{3.14}$$

(ii) *If  $N_2 \sim N_3 \gg N_1$  and  $H \sim L_1 \gtrsim L_2, L_3$ , then*

$$(3.9) \lesssim L_{min}^{1/2} N_{max}^{-1} \min\left(H, \frac{N_{max}}{N_{min}} L_{med}\right)^{1/2}. \tag{3.15}$$

*Similarly for permutations.*



(iii) In all other cases, we have

$$(3.9) \lesssim L_{\min}^{1/2} N_{\max}^{-1} \min(H, L_{\text{med}})^{1/2}. \tag{3.16}$$

In order to estimate the denominator in (3.12), (3.13), we will need the following proposition to reduce some cases.

**Proposition 3.3.** *Let  $k \in \mathbb{N}$ . Assume that  $a_1, a_2, \dots, a_k$  and  $b_1, b_2, \dots, b_k$  are non-negative numbers, and  $A_1 \leq A_2 \leq \dots \leq A_k, B_1 \leq B_2 \leq \dots \leq B_k$  are rearrange of  $\{a_i\}, \{b_i\}$  respectively. Then*

$$\prod_{i=1}^k (a_i + b_i) \geq \prod_{i=1}^k (A_i + B_i). \tag{3.17}$$

**Proof.** We apply an induction on  $k$ . The case  $k = 1$  is obviously. For  $k = 2$ , we have

$$\begin{aligned} (a_1 + b_1)(a_2 + b_2) &= a_1 a_2 + b_1 b_2 + a_1 b_2 + a_2 b_1 \\ &\geq A_1 A_2 + B_1 B_2 + A_1 B_2 + A_2 B_1 = (A_1 + B_1)(A_2 + B_2). \end{aligned}$$

We assume the lemma holds for all  $q \in \mathbb{N}, q \leq k - 1$ . Now we prove for  $k$ . If  $a_1 = A_1, b_1 = B_1$ , then we apply induction assumption for  $k - 1$  and get (3.17). Otherwise, we may assume  $a_1 = A_1, b_2 = B_1$ . By induction assumption for 2, then  $k - 1$ , we get

$$\begin{aligned} \prod_{i=1}^k (a_i + b_i) &= (a_1 + b_1)(a_2 + b_2) \prod_{i=3}^k (a_i + b_i) \\ &\geq (A_1 + B_1)(a_2 + b_1) \prod_{i=3}^k (a_i + b_i) \\ &\geq \prod_{i=1}^k (A_i + B_i), \end{aligned} \tag{3.18}$$

which completes the proof of the proposition.  $\square$

**Proof of Proposition 3.1.** We will prove the proposition using case-by-case analysis. We first bound (3.13). Since we have

$$N_3 \langle N_3 \rangle^s \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s} \lesssim N \langle N_{\min} \rangle^{-s} + N^{-2s} N_{\min} \langle N_{\min} \rangle^s \tag{3.19}$$

and from (iii) of Proposition 3.2, we obtain

$$\begin{aligned} (3.13) &\lesssim \sum_{N_{\max} \sim N_{\text{med}} \sim N} \sum_{L_i, L_{\max} \geq H} \frac{(N \langle N_{\min} \rangle^{-s} + N^{-2s} N_{\min} \langle N_{\min} \rangle^s) L_{\min}^{1/2} N_{\min}^{1/2}}{L_{\max}^{1/2-\delta} L_{\text{med}}^{1/2-\delta} L_{\min}^{1/2-\delta}} \\ &\lesssim \sum_{N_{\max} \sim N_{\text{med}} \sim N} \sum_{L_{\max} \geq H} (N \langle N_{\min} \rangle^{-s} + N^{-2s} N_{\min} \langle N_{\min} \rangle^s) L_{\max}^{-1+3\delta} N_{\min}^{1/2} \\ &\lesssim \sum_{N_{\min} \leq N^{-2}} (N + N^{-2s} N_{\min}) N_{\min}^{1/2} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{N^{-2} \leq N_{\min} \leq 1} (N + N^{-2s} N_{\min}) N^{-2+6\delta} N_{\min}^{-1/2+3\delta} \\
 & + \sum_{N_{\min} \geq 1} (N N_{\min}^{-s} + N^{-2s} N_{\min}^{1+s}) N^{-2+6\delta} N_{\min}^{-1/2+3\delta} \\
 & \lesssim 1,
 \end{aligned} \tag{3.20}$$

provided that  $-1 < s \leq 0$ .

We next bound (3.12), which is more complicated. We first assume that (3.14) applies. Then we have

$$\begin{aligned}
 (3.12) & \lesssim \sum_{N_{\max} \sim N_{\min} \sim N} \sum_{L_1, L_2, L_3 \geq 1} \frac{N^{3/4-s} L_{\min}^{1/2} L_{\text{med}}^{1/4} \langle L_{\min} + N^{2\alpha} \rangle^{-1/2+\delta}}{\langle L_{\max} + N^{2\alpha} \rangle^{1/2-\delta} \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \lesssim \sum_{N_{\max} \sim N_{\min} \sim N} \sum_{L_{\text{med}}} \frac{N^{3/4-s} L_{\text{med}}^{1/4+\delta}}{N^{3/2-3\delta} \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \lesssim N^{-\frac{3}{4}-\frac{\alpha}{2}-s+4\delta} \lesssim 1,
 \end{aligned} \tag{3.21}$$

provided that  $-\frac{3}{4} - \frac{\alpha}{2} < s \leq 0$ .

If (3.16) applies, from Proposition 3.3, we obtain

$$\begin{aligned}
 (3.12) & \lesssim \sum_{N_i} \sum_{L_i} \frac{(N \langle N_{\min} \rangle^{-s} + N^{-2s} N_{\min} \langle N_{\min} \rangle^s) L_{\min}^{1/2} N^{-1} L_{\text{med}}^{1/2}}{(L_{\max} + N^{2\alpha})^{1/2-\delta} \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta} \langle L_{\min} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \lesssim \sum_{N_i} \frac{(N \langle N_{\min} \rangle^{-s} + N^{-2s} N_{\min} \langle N_{\min} \rangle^s) N^{-1+4\alpha\delta}}{(N^2 N_{\min} + N^{2\alpha})^{1/2-3\delta}} \\
 & \lesssim \sum_{N_{\min} \leq N^{2\alpha-2}} \frac{(N + N^{-2s} N_{\min}) N^{-1+4\alpha\delta}}{N^{\alpha-6\delta}} \\
 & + \sum_{N^{2\alpha-2} \leq N_{\min} \leq 1} \frac{(N + N^{-2s} N_{\min}) N^{-1+4\alpha\delta}}{N^{1-6\delta} N_{\min}^{1/2-3\delta}} \\
 & + \sum_{N_{\min} \geq 1} \frac{(N N_{\min}^{-s} + N^{-2s} N_{\min}^{1+s}) N^{-1+4\alpha\delta}}{N^{1-6\delta} N_{\min}^{1/2-3\delta}} \\
 & \lesssim N^{-\alpha+10\delta} + N^{-2s-3+\alpha+6\delta} + N^{-2s-2+6\delta} + N^{-s-3/2+7\delta} \\
 & \lesssim 1,
 \end{aligned} \tag{3.22}$$

provided that  $-1 < s \leq 0$ .

If (3.15) applies, we have three cases:

$$N_2 \sim N_3 \gg N_1, \quad L_1 \gtrsim L_2, L_3, \tag{3.23}$$

$$N_1 \sim N_3 \gg N_2, \quad L_2 \gtrsim L_1, L_3, \tag{3.24}$$

$$N_1 \sim N_2 \gg N_3, \quad L_3 \gtrsim L_1, L_2. \tag{3.25}$$

If (3.23) holds, then we have

$$\begin{aligned}
 (3.12) & \lesssim \sum_{N_i} \sum_{L_i} \frac{N \langle N_{\min} \rangle^{-s} L_{\min}^{1/2} N^{-1} \min(H, \frac{N_{\max}}{N_{\min}} L_{\text{med}})^{1/2}}{N_{\min}^{1/2} N \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta} \langle L_{\min} + N^{2\alpha} \rangle^{1/2}} \\
 & \lesssim \sum_{N_i} \sum_{L_{\text{med}} \geq NN_{\min}^2} \frac{N \langle N_{\min} \rangle^{-s} \log(L_{\text{med}}) N^{-1} N_{\min}^{1/2} N}{N_{\min}^{1/2} N \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \quad + \sum_{N_i} \sum_{L_{\text{med}} \leq NN_{\min}^2} \frac{N \langle N_{\min} \rangle^{-s} \log(L_{\text{med}}) L_{\text{med}}^{1/2} N^{-1} N_{\min}^{-1/2} N^{1/2}}{N_{\min}^{1/2} N \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & = A_1 + A_2. \tag{3.26}
 \end{aligned}$$

We first bound  $A_1$ :

$$\begin{aligned}
 A_1 & \lesssim \sum_{N^{-2} \leq N_{\min} \leq 1} \sum_{L_{\text{med}} \geq NN_{\min}^2} \frac{L_{\text{med}}^\delta}{\langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \quad + \sum_{N_{\min} \geq 1} \sum_{L_{\text{med}} \geq NN_{\min}^2} \frac{L_{\text{med}}^\delta N_{\min}^{-s}}{\langle L_{\text{med}} \rangle^{1/2-\delta}} \\
 & \lesssim N^{-\alpha+7\delta} + \sum_{N_{\min} \geq 1} N_{\min}^{-s-1+4\delta} N^{-1/2+2\delta} \lesssim 1, \tag{3.27}
 \end{aligned}$$

provided  $-1 < s \leq 0$ .

For  $A_2$ , we have

$$\begin{aligned}
 A_2 & \lesssim \sum_{N^{-1/2} \leq N_{\min} \leq 1} \sum_{L_{\text{med}} \leq NN_{\min}^2} \frac{L_{\text{med}}^{\delta+1/2} N_{\min}^{-1} N^{-1/2}}{\langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} + \sum_{N_{\min} \geq 1} \sum_{L_{\text{med}} \leq NN_{\min}^2} \frac{L_{\text{med}}^{\delta+1/2} N_{\min}^{-1-s} N^{-1/2}}{\langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \lesssim \sum_{N^{-1/2} \leq N_{\min} \leq 1} N^{2\delta-1/2} N_{\min}^{4\delta-1} + \sum_{N_{\min} \geq 1} N_{\min}^{-1-s+4\delta} N^{-1/2+2\delta} \\
 & \lesssim 1, \tag{3.28}
 \end{aligned}$$

provided  $-1 < s \leq 0$ .

From symmetry, the case (3.23) is identical to the case (3.24). Now we assume that (3.25) holds, and we obtain

$$\begin{aligned}
 (3.12) & \lesssim \sum_{N_i} \sum_{L_i} \frac{N^{-2s} \langle N_{\min} \rangle^s N_{\min} L_{\min}^{1/2} N^{-1} \min(H, \frac{N_{\max}}{N_{\min}} L_{\text{med}})^{1/2}}{N_{\min}^{1/2-\delta} N^{1-2\delta} \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta} \langle L_{\min} + N^{2\alpha} \rangle^{1/2}} \\
 & \lesssim \sum_{N_i} \sum_{L_{\text{med}} \geq NN_{\min}^2} \frac{N^{-2s} \langle N_{\min} \rangle^s N_{\min} \log(L_{\text{med}}) N^{-1} N_{\min}^{1/2} N}{N_{\min}^{1/2-\delta} N^{1-2\delta} \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2-\delta}} \\
 & \quad + \sum_{N_i} \sum_{L_{\text{med}} \leq NN_{\min}^2} \frac{N^{-2s} \langle N_{\min} \rangle^s N_{\min} \log(L_{\text{med}}) L_{\text{med}}^{1/2} N^{-1} N_{\min}^{-1/2} N^{1/2}}{N_{\min}^{1/2-\delta} N^{1-2\delta} \langle L_{\text{med}} + N^{2\alpha} \rangle^{1/2}} \\
 & = B_1 + B_2. \tag{3.29}
 \end{aligned}$$

We first bound  $B_1$ :

$$\begin{aligned}
 B_1 &\lesssim \sum_{N^{-2} \leq N_{\min} \leq 1} \sum_{L_{\text{med}} \geq NN_{\min}^2} \frac{N^{-2s-1+2\delta} N_{\min}^{1+\delta} L_{\text{med}}^\delta}{(L_{\text{med}} + N^{2\alpha})^{1/2-\delta}} + \sum_{N_{\min} \geq 1} \sum_{L_{\text{med}} \geq NN_{\min}^2} \frac{N^{-2s-1+2\delta} N_{\min}^{1+\delta+s} L_{\text{med}}^\delta}{(L_{\text{med}} + N^{2\alpha})^{1/2-\delta}} \\
 &\lesssim \sum_{N^{-2} \leq N_{\min} \leq 1} \frac{N^{-2s-1+2\delta} N_{\min}^{1+\delta}}{(NN_{\min}^2 + N^{2\alpha})^{1/2-2\delta}} + \sum_{N_{\min} \geq 1} \frac{N^{-2s-1+2\delta} N_{\min}^{1+\delta+s}}{(NN_{\min}^2 + N^{2\alpha})^{1/2-2\delta}}. \tag{3.30}
 \end{aligned}$$

We discuss it in the following two cases. If  $1/2 \leq \alpha \leq 1$ , then

$$B_1 \lesssim N^{-2s-1-\alpha+6\delta} + \sum_{N_{\min} \geq N^{\alpha-1/2}} N^{-2s-3/2+4\delta} N_{\min}^{5\delta+s} + \sum_{1 \leq N_{\min} \leq N^{\alpha-1/2}} N^{-2s-1-\alpha+6\delta} N_{\min}^{1+\delta+s}, \tag{3.31}$$

provided that  $-\frac{3}{5-2\alpha} < s \leq 0$ . If  $0 < \alpha \leq 1/2$ , then

$$\begin{aligned}
 B_1 &\lesssim \sum_{N^{\alpha-1/2} \leq N_{\min} \leq 1} N^{-2s-3/2+4\delta} N_{\min}^{5\delta} + \sum_{N_{\min} \geq 1} N^{-2s-3/2+4\delta} N_{\min}^{5\delta+s} + \sum_{N^{-2} \leq N_{\min} \leq N^{\alpha-1/2}} N^{-2s-1-\alpha+6\delta} N_{\min}^{1+\delta} \\
 &\lesssim 1, \tag{3.32}
 \end{aligned}$$

provided that  $-3/4 < s \leq 0$ .

For  $B_2$ , we have

$$B_2 \lesssim \sum_{N^{-1/2} \leq N_{\min} \leq 1} \sum_{L_{\text{med}} \leq NN_{\min}^2} \frac{N^{-2s-3/2+2\delta} N_{\min}^\delta L_{\text{med}}^{1/2+\delta}}{(L_{\text{med}} + N^{2\alpha})^{1/2}} + \sum_{N_{\min} \geq 1} \sum_{L_{\text{med}} \leq NN_{\min}^2} \frac{N^{-2s-3/2+2\delta} N_{\min}^{\delta+s} L_{\text{med}}^{1/2+\delta}}{(L_{\text{med}} + N^{2\alpha})^{1/2}}$$

and get

$$B_2 \lesssim \sum_{N^{-1/2} \leq N_{\min} \leq 1} \frac{N^{-2s-1+3\delta} N_{\min}^{1+3\delta}}{(NN_{\min}^2 + N^{2\alpha})^{1/2}} + \sum_{N_{\min} \geq 1} \frac{N^{-2s-1+3\delta} N_{\min}^{1+s+3\delta}}{(NN_{\min}^2 + N^{2\alpha})^{1/2}}.$$

If  $1/2 \leq \alpha \leq 1$ , then

$$\begin{aligned}
 B_2 &\lesssim N^{-2s-1-\alpha+3\delta} + \sum_{N_{\min} \geq N^{\alpha-1/2}} N^{-2s-3/2+3\delta} N_{\min}^{s+3\delta} \\
 &\quad + \sum_{1 \leq N_{\min} \leq N^{\alpha-1/2}} N^{-2s-1-\alpha+3\delta} N_{\min}^{1+s+3\delta} \\
 &\lesssim 1, \tag{3.33}
 \end{aligned}$$

provided that  $-\frac{3}{5-2\alpha} < s \leq 0$ . If  $0 < \alpha \leq 1/2$ , then

$$\begin{aligned}
 B_2 &\lesssim \sum_{N^{-1/2} \leq N_{\min} \leq N^{\alpha-1/2}} N^{-2s-1-\alpha+3\delta} N_{\min}^{1+3\delta} \\
 &\quad + \sum_{N^{\alpha-1/2} \leq N_{\min} \leq 1} N^{-2s-3/2+3\delta} N_{\min}^{3\delta} + \sum_{N_{\min} \geq 1} N^{-2s-3/2+3\delta} N_{\min}^{s+3\delta} \\
 &\lesssim 1, \tag{3.34}
 \end{aligned}$$

provided that  $-3/4 < s \leq 0$ . Therefore, we complete the proof of Proposition 3.1.  $\square$

**Proposition 3.4.** *If  $s \leq s_\alpha$ , then for any  $0 < \delta \ll 1$ , there does not exist  $C > 0$  such that for any  $u, v \in S$ ,*

$$\|\partial_\chi(uv)\|_{\chi^{-1/2+\delta,s,\alpha}} \leq C \|u\|_{\chi^{1/2,s,\alpha}} \|v\|_{\chi^{1/2,s,\alpha}}. \tag{3.35}$$

**Proof.** From the proof of Proposition 3.1, we see that the restriction on  $s$  is caused by high–high interaction, and hence we construct the worst case. The idea is due to C. Kenig, G. Ponce and L. Vega [5]. In view of definition, (3.35) is equivalent to

$$\begin{aligned} & \left\| \frac{\xi(1+|\xi|)^s}{(1+|\xi|^{2\alpha}+|\tau-\xi^3|)^{1/2-\delta}} \int \frac{f(\xi_1, \tau_1)(1+|\xi_1|)^{-s} f(\xi-\xi_1, \tau-\tau_1)(1+|\xi-\xi_1|)^{-s} d\xi_1 d\tau_1}{(|\xi_1|^{2\alpha}+|\tau_1-\xi_1^3|)^{1/2} (|\xi-\xi_1|^{2\alpha}+|\tau-\tau_1-(\xi-\xi_1)^3|)^{1/2}} \right\|_{L^2_{\xi,\tau}} \\ & \lesssim \|f\|_{L^2_{\xi,\tau}}^2. \end{aligned} \tag{3.36}$$

If  $0 < \alpha \leq 1/2$ , fix  $N \gg 1$ , we set

$$f(\xi, \tau) = \chi_A(\xi, \tau) + \chi_{-A}(\xi, \tau),$$

where

$$A = \{(\xi, \tau) \in \mathbb{R}^2 \mid N \leq \xi \leq N+1, N \leq |\tau - \xi^3| \leq 2N\},$$

and

$$-A = \{(\xi, \tau) \in \mathbb{R}^2 \mid -(\xi, \tau) \in A\}.$$

Clearly,

$$\|f\|_{L^2_{\xi,\tau}} \sim N^{1/2}. \tag{3.37}$$

On the other hand,  $A$  contains a rectangle with  $(N, N^3 + N)$  as a vertex, with dimension  $N^{-1} \times N^2$  and longest side pointing in the  $(1, 3N^2)$  direction. Therefore,

$$|f * f(\xi, \tau)| \gtrsim N \chi_R(\xi, \tau), \tag{3.38}$$

where  $R$  is a rectangle centered at the origin of dimensions  $N^{-1} \times N^2$  and longest side pointing in the  $(1, 3N^2)$  direction. Taking the one-third rectangle away from origin, then we have  $|\xi| \sim 1$ , and therefore (3.36) implies that

$$N^{-1+2\delta} N^{-2s} N^{-1} N N^{-1/2} N \lesssim N, \tag{3.39}$$

which implies that  $s > -3/4$ .

If  $1/2 \leq \alpha \leq 1$ , then take

$$f(\xi, \tau) = \chi_B(\xi, \tau) + \chi_{-B}(\xi, \tau),$$

where

$$B = \{(\xi, \tau) \in \mathbb{R}^2 \mid N \leq \xi \leq N + N^{\alpha-1/2}, N^{2\alpha} \leq |\tau - \xi^3| \leq 2N^{2\alpha}\}, \tag{3.40}$$

and

$$-B = \{(\xi, \tau) \in \mathbb{R}^2 \mid -(\xi, \tau) \in B\}.$$

Clearly,

$$\|f\|_{L^2_{\xi, \tau}} \sim N^{\frac{3\alpha}{2} - \frac{1}{4}}. \tag{3.41}$$

On the other hand,  $B$  contains a rectangle with  $(N, N^3 + N^{2\alpha})$  as a vertex, with dimension  $N^{2\alpha-2} \times N^{\alpha+3/2}$  and longest side pointing in the  $(1, 3N^2)$  direction. Therefore,

$$|f * f(\xi, \tau)| \gtrsim N^{3\alpha-1/2} \chi_R(\xi, \tau), \tag{3.42}$$

where  $R$  is a rectangle centered at the origin of dimensions  $N^{2\alpha-2} \times N^{\alpha+3/2}$  and longest side pointing in the  $(1, 3N^2)$  direction. Taking the one-third rectangle away from origin, then we have  $|\xi| \sim N^{\alpha-1/2}$ , and therefore (3.36) implies that

$$N^{(\alpha-1/2)(1+s)} N^{(\alpha+3/2)(-1/2+\delta)} N^{-2s} N^{-2\alpha} N^{3\alpha-1/2} N^{\alpha-1} N^{\alpha/2+3/4} \lesssim N^{3\alpha-1/2}, \tag{3.43}$$

which implies that  $s > -3/(5 - 2\alpha)$ .  $\square$

**Remark 3.5.** The constant in Proposition 3.1 depends on  $\alpha$ , which is the main reason for gaining  $\delta$ -order derivative in time in the bilinear estimates. In proving global well-posedness we also need to exploit the smoothing effect of the dissipative term and then  $L^2$  conservation law. Therefore, the result of Theorem 1.1 is dependent of  $\epsilon$ .

#### 4. Uniform LWP for KdV–B equation

In this section we study the uniform local well-posedness for the KdV–Burgers equation. We will prove a time-localized version of Theorem 1.2 where  $T = T(\|\phi\|_{H^s})$  is small. In view of Remark 3.5, the space  $X^{b,s}$  we used in the last section is not proper in this situation. We will use the space  $F^s$ . Let us recall that (1.1) is invariant in the following scaling

$$u(x, t) \rightarrow \lambda^2 u(\lambda x, \lambda^3 t), \quad \phi(x) \rightarrow \lambda^2 \phi(\lambda x), \quad \epsilon \rightarrow \lambda^{3-2\alpha} \epsilon, \quad \forall 0 < \lambda \leq 1. \tag{4.1}$$

This invariance is very important in the proof of Theorem 1.2 and also crucial for the uniform global well-posedness in the next section. We first show that  $F^s(T) \hookrightarrow C([0, T], H^s)$  for  $s \in \mathbb{R}, T \in (0, 1]$  in the following proposition.

**Proposition 4.1.** *If  $s \in \mathbb{R}, T \in (0, 1]$ , and  $u \in F^s(T)$ , then*

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^s(T)}. \tag{4.2}$$

**Proof.** In view of definition, it suffices to show that for  $k \in \mathbb{Z}_+, t \in [0, 1]$ ,

$$\|\eta_k(\xi) \mathcal{F}_x u(t)\|_{L^2} \lesssim \|\eta_k(\xi) \mathcal{F} u\|_{X_k}. \tag{4.3}$$

From the fact

$$\eta_k(\xi) \mathcal{F}_x u(t) = \sum_{j \in \mathbb{Z}_+} \int_{\mathbb{R}} \eta_j(\tau - \xi^3) \eta_k(\xi) \mathcal{F}(u)(\tau) e^{it\tau} d\tau,$$

we easily see that (4.3) follows from the Minkowski’s inequality, Cauchy–Schwarz inequality and the definition of  $X_k$ .  $\square$

We prove an embedding property of the space  $N^s$  in the next proposition which can be viewed as a dual version of Proposition 4.1. This property is important in proving the limit behavior in Section 6.

**Proposition 4.2.** *If  $s \in \mathbb{R}$  and  $u \in L_t^2 H_x^s$ , then*

$$\|u\|_{N^s} \lesssim \|u\|_{L_t^2 H_x^s}. \tag{4.4}$$

**Proof.** We may assume  $s = 0$ . By definition it suffices to prove that for  $k \in \mathbb{Z}_+$ ,

$$\|(i + \tau - \xi^3)^{-1} \eta_k(\xi) \mathcal{F}(u)\|_{X_k} \lesssim \|\eta_k(\xi) \mathcal{F}(u)\|_{L^2}, \tag{4.5}$$

which immediately follows from the definition of  $X_k$ .  $\square$

As in the last section we will mainly work on the corresponding integral equation of Eq. (1.1). But for technical reason we will mainly work on the following integral equation

$$u(t) = \psi(t) [W_\epsilon^\alpha(t) \phi_1 - L(\partial_x(\psi^2 u^2))(x, t)], \tag{4.6}$$

where  $\psi$  is as in (3.3) and

$$L(f)(x, t) = W_0(t) \int_{\mathbb{R}^2} e^{ix\xi} \frac{e^{it\tau'} - e^{-\epsilon|t|\xi|^{2\alpha}}}{i\tau' + \epsilon|\xi|^{2\alpha}} \mathcal{F}(W_0(-t)f)(\xi, \tau') d\xi d\tau'. \tag{4.7}$$

One easily sees that

$$\chi_{\mathbb{R}_+}(t) \psi(t) L(f)(x, t) = \chi_{\mathbb{R}_+}(t) \psi(t) \int_0^t W_\epsilon^\alpha(t - \tau) f(\tau) d\tau. \tag{4.8}$$

Indeed, taking  $w = W_0(\cdot)f$ , the right-hand side of (4.8) can be rewritten as

$$\begin{aligned} & W_0(t) \left[ \chi_{\mathbb{R}_+}(t) \psi(t) \int_{\mathbb{R}^2} e^{ix\xi} e^{-\epsilon|t|\xi|^{2\alpha}} \widehat{w}(\xi, \tau') \int_0^t e^{i\tau\tau'} e^{e^{\epsilon|\xi|^{2\alpha}} \tau} d\tau d\xi d\tau' \right] \\ &= W_0(t) \left[ \chi_{\mathbb{R}_+}(t) \psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \frac{e^{it\tau'} - e^{-\epsilon|t|\xi|^{2\alpha}}}{i\tau' + \epsilon|\xi|^{2\alpha}} \widehat{w}(\xi, \tau') d\xi d\tau' \right]. \end{aligned}$$

Thus, if  $u$  solves (4.6) then  $u$  is a solution of (3.1) on  $[0, 1]$ . We first prove a uniform estimate for the free solution.

**Proposition 4.3.** *Let  $s \in \mathbb{R}$ . There exists  $C > 0$  such that for any  $0 \leq \epsilon \leq 1$*

$$\|\psi(t) W_\epsilon^\alpha(t) \phi\|_{F^s} \leq C \|\phi\|_{H^s}, \quad \forall \phi \in H^s(\mathbb{R}). \tag{4.9}$$

**Proof.** We only prove the case  $0 < \epsilon \leq 1$ . By definition of  $F^s$ , it suffices to prove that for  $k \in \mathbb{Z}_+$

$$\|\eta_k(\xi)\mathcal{F}(\psi(t)W_\epsilon^\alpha(t)\phi)\|_{X_k} \lesssim \|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}. \tag{4.10}$$

In view of the definition, if  $k = 0$ , then by Taylor’s expansion

$$\begin{aligned} & \|\eta_0(\xi)\mathcal{F}(\psi(t)W_\epsilon^\alpha(t)\phi)\|_{X_0} \\ & \lesssim \sum_{j=0}^\infty 2^{j/2} \left\| \eta_0(\xi)\widehat{\phi}(\xi)\mathcal{F}_t \left( \psi(t) \sum_{n \geq 0} \frac{(-1)^n \epsilon^n |\xi|^{2n\alpha}}{n!} |t|^n \right) (\tau)\eta_j(\tau) \right\|_{L_{\xi,\tau}^2} \\ & \lesssim \sum_{n \geq 0} \frac{4^n}{n!} \|\eta_0(\xi)\widehat{\phi}(\xi)\|_{L^2} \| |t|^n \psi(t) \|_{H^1} \lesssim \|\eta_0(\xi)\widehat{\phi}(\xi)\|_{L^2}, \end{aligned}$$

which is the estimate (4.10), as desired. We now consider the cases  $k \geq 1$ . We first observe that if  $|\xi| \sim 2^k$ , then for any  $j \geq 0$ ,

$$\|P_j(e^{-\epsilon|\xi|^{2\alpha}|t|})(t)\|_{L^2} \lesssim \|P_j(e^{-\epsilon 2^{2k\alpha}|t|})(t)\|_{L^2}, \tag{4.11}$$

which follows from Plancherel’s equality and the fact that

$$\mathcal{F}(e^{-|t|})(\tau) = C \frac{1}{1 + |\tau|^2}.$$

It follows from the definition that

$$\begin{aligned} \|\eta_k(\xi)\mathcal{F}(\psi(t)W_\epsilon^\alpha(t)\phi)\|_{X_k} & \lesssim \sum_{j=0}^\infty 2^{j/2} \|\eta_k(\xi)\widehat{\phi}(\xi)\eta_j(\tau)\mathcal{F}_t(\psi(t)e^{-\epsilon|t||\xi|^{2\alpha}})(\tau)\|_{L_{\xi,\tau}^2} \\ & \lesssim \sum_{j=0}^\infty 2^{j/2} \|\eta_k(\xi)\widehat{\phi}(\xi)P_j(\psi(t)e^{-\epsilon|t||\xi|^{2\alpha}})(t)\|_{L_{\xi,t}^2} \\ & \lesssim \sum_{j=0}^\infty 2^{j/2} \|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2} \sup_{|\xi| \sim 2^k} \|P_j(\psi(t)e^{-\epsilon|t||\xi|^{2\alpha}})(t)\|_{L_t^2}. \end{aligned}$$

It suffices to show that for any  $k \geq 1$ ,

$$\sum_{j=0}^\infty 2^{j/2} \sup_{|\xi| \sim 2^k} \|P_j(\psi(t)e^{-\epsilon|t||\xi|^{2\alpha}})(t)\|_{L_t^2} \lesssim 1. \tag{4.12}$$

We may assume  $j \geq 100$  in the summation. Using the para-product decomposition, we have

$$u_1 u_2 = \sum_{r=0}^\infty [(P_{r+1}u_1)(P_{\leq r+1}u_2) + (P_{\leq r}u_1)(P_{r+1}u_2)], \tag{4.13}$$

and

$$P_j(u_1 u_2) = P_j \left( \sum_{r \geq j-10} [(P_{r+1}u_1)(P_{\leq r+1}u_2) + (P_{\leq r}u_1)(P_{r+1}u_2)] \right) := P_j(I + II). \tag{4.14}$$



Now we take  $u_1 = \psi(t)$  and  $u_2 = e^{-\epsilon|t||\xi|^{2\alpha}}$ . It follows from Bernstein’s estimate, Hölder’s inequality and (4.11) that

$$\begin{aligned} \sum_{j \geq 100} 2^{j/2} \|P_j(II)\|_{L^\infty_{\xi} L^2_t} &\lesssim \sum_{j \geq 100} 2^{j/2} \sum_{r \geq j-10} \|P_{r+1}u_2\|_{L^\infty_{\xi} L^2_t} \|P_{\leq r+1}u_1\|_{L^\infty_{\xi,t}} \\ &\lesssim \sum_{j \geq 100} 2^{(j-r)/2} \sum_{r \geq j-10} 2^{r/2} \|P_{r+1}u_2\|_{L^\infty_{\xi} L^2_t} \\ &\lesssim \sum_r 2^{r/2} \|P_{r+1}(e^{-\epsilon|t|^{2k\alpha}})\|_{L^2_t} \lesssim 1, \end{aligned} \tag{4.15}$$

where we used the fact that  $\dot{B}^{1/2}_{2,1}$  has a scaling invariance and  $e^{-|t|} \in \dot{B}^{1/2}_{2,1}$ . The first term  $P_j(I)$  in (4.14) can be handled in an easier way. Therefore, we complete the proof of the proposition.  $\square$

From the proof we see that  $F^s$  norm has the same scale in time as  $B^{1/2}_{2,1}$  and  $e^{-\epsilon|t|}$ . If applying  $X^{1/2+s}$  norm, one cannot get a uniform estimate. Similarly for the inhomogeneous linear operator we get

**Proposition 4.4.** *Let  $s \in \mathbb{R}$ . There exists  $C > 0$  such that for all  $v \in S(\mathbb{R}^2)$  and  $0 \leq \epsilon \leq 1$ ,*

$$\|\psi(t)L(v)\|_{F^s} \leq C\|v\|_{N^s}. \tag{4.16}$$

**Proof.** The idea is essential due to Molinet and Ribaud [7]. See also Section 5 in [3]. We only prove the case  $0 < \epsilon \leq 1$ . In view of definition, it suffices to prove that if  $k \in \mathbb{Z}_+$ ,

$$\|\eta_k(\xi)\mathcal{F}(\psi(t)L(v))\|_{X_k} \lesssim \|(i + \tau - \xi^3)^{-1}\eta_k(\xi)\mathcal{F}(v)\|_{X_k}. \tag{4.17}$$

We set

$$w(\tau) = W_0(-\tau)v(\tau), \quad k_\xi(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau'} - e^{-\epsilon|t||\xi|^{2\alpha}}}{i\tau' + \epsilon|\xi|^{2\alpha}} \widehat{w}(\xi, \tau') d\tau'.$$

Therefore, by the definition, it suffices to prove that

$$\sum_{j=0} 2^{j/2} \|\eta_k(\xi)\eta_j(\tau)\mathcal{F}_t(k_\xi)(\tau)\|_{L^2_{\xi,\tau}} \lesssim \sum_{j=0} 2^{-j/2} \|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}}. \tag{4.18}$$

We first write

$$\begin{aligned} k_\xi(t) &= \psi(t) \int_{|\tau| \leq 1} \frac{e^{it\tau} - 1}{i\tau + \epsilon|\xi|^{2\alpha}} \widehat{w}(\xi, \tau) d\tau + \psi(t) \int_{|\tau| \leq 1} \frac{1 - e^{-\epsilon|t||\xi|^{2\alpha}}}{i\tau + \epsilon|\xi|^{2\alpha}} \widehat{w}(\xi, \tau) d\tau \\ &\quad + \psi(t) \int_{|\tau| \geq 1} \frac{e^{it\tau}}{i\tau + \epsilon|\xi|^{2\alpha}} \widehat{w}(\xi, \tau) d\tau - \psi(t) \int_{|\tau| \geq 1} \frac{e^{-\epsilon|t||\xi|^{2\alpha}}}{i\tau + \epsilon|\xi|^{2\alpha}} \widehat{w}(\xi, \tau) d\tau \\ &= I + II + III - IV. \end{aligned}$$

We now estimate the contributions of  $I-IV$ . First, we consider the contribution of  $IV$ :

$$\begin{aligned} \sum_{j=0} 2^{j/2} \|\eta_k(\xi) P_j(IV)(t)\|_{L^2_{\xi,t}} &\leq \sum_{j=0} 2^{j/2} \sup_{\xi \in I_k} \|\eta_k(\xi) P_j(\psi(t)e^{-\epsilon|t|\xi|^{2\alpha}})(t)\|_{L^2_t} \\ &\quad \times \int_{|\tau| \geq 1} \frac{\|\eta_k(\xi) \widehat{w}(\xi, \tau)\|_{L^2_{\xi}}}{|\tau|} d\tau \\ &\lesssim \sum_{j=0} 2^{-j/2} \|\eta_k(\xi) \eta_j(\tau) \widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}}, \end{aligned}$$

where we use Taylor expansion for  $k = 0$  and (4.12) for  $k \geq 1$ . Next, we consider the contribution of III. Setting  $g(\xi, \tau) = \frac{\widehat{w}(\xi, \tau)}{|i\tau + \epsilon|\xi|^{2\alpha}} \chi_{|\tau| \geq 1}$  we have

$$\begin{aligned} \sum_{j=0} 2^{j/2} \|\eta_k(\xi) P_j(III)(t)\|_{L^2_{\xi,t}} &\lesssim \sum_{j=0} 2^{j/2} \|\eta_k(\xi) \eta_j(\tau) \widehat{\psi} *_\tau g(\xi, \tau)\|_{L^2_{\xi,\tau}} \\ &\lesssim \sum_{j \geq 1} 2^{j/2} \left\| \frac{\eta_j(\tau') \|\eta_k(\xi) \widehat{w}(\xi, \tau')\|_{L^2_{\xi}}}{|i\tau'|} \chi_{|\tau'| \geq 1} \right\|_{L^2_{\tau'}} \\ &\lesssim \sum_{j=0} 2^{-j/2} \|\eta_k(\xi) \eta_j(\tau) \widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}}, \end{aligned}$$

where we used the fact that  $B_{2,1}^{1/2}$  is a multiplication algebra and that  $\mathcal{F}^{-1}(|\widehat{\psi}|) \in B_{2,1}^{1/2}$ . Thirdly, we consider the contribution of II. For  $\epsilon|\xi|^{2\alpha} \geq 1$ , as for IV, we get

$$\begin{aligned} \sum_{j=0} 2^{j/2} \|\eta_k(\xi) P_j(II)(t)\|_{L^2_{\xi,t}} &\lesssim \sum_{j=0} 2^{j/2} \sup_{\xi \in I_k} \|\eta_k(\xi) P_j(\psi(1 - e^{-\epsilon|t|\xi|^{2\alpha}}))(t)\|_{L^2_t} \int \frac{\|\widehat{w}(\xi, \tau)\|_{L^2_{\xi}}}{|\tau|} d\tau \\ &\lesssim \sum_{j=0} 2^{-j/2} \|\eta_k(\xi) \eta_j(\tau) \widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}}. \end{aligned}$$

For  $\epsilon|\xi|^{2\alpha} \leq 1$ , using Taylor's expansion, we have

$$\begin{aligned} \sum_{j=0} 2^{j/2} \|\eta_k(\xi) P_j(II)(t)\|_{L^2_{\xi,t}} &\lesssim \sum_{n \geq 1} \sum_{j=0} 2^{j/2} \left\| \eta_k(\xi) \int_{|\tau| \leq 1} \frac{\widehat{w}(\xi, \tau)}{i\tau + \epsilon|\xi|^{2\alpha}} d\tau P_j(|t|^n \psi(t)) \frac{\epsilon^n |\xi|^{2\alpha n}}{n!} \right\|_{L^2_{\xi,t}} \\ &\lesssim \left\| \int_{|\tau| \leq 1} \frac{\epsilon|\xi|^{2\alpha} |\eta_k(\xi) \widehat{w}(\xi, \tau)|}{|i\tau + \epsilon|\xi|^{2\alpha}|} d\tau \right\|_{L^2_{\xi}} \lesssim \sum_{j=0} 2^{-j/2} \|\eta_k(\xi) \eta_j(\tau) \widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}}, \end{aligned}$$

where in the last inequality we used the fact  $\| |t|^n \psi(t) \|_{B_{2,1}^{1/2}} \leq \| |t|^n \psi(t) \|_{H^1} \leq C2^n$ . Finally, we consider the contribution of I:

$$I = \psi(t) \int_{|\tau| \leq 1} \sum_{n \geq 1} \frac{(i\tau)^n}{n!(i\tau + \epsilon|\xi|^{2\alpha})} \widehat{w}(\tau) d\tau.$$

Thus, we get

$$\begin{aligned} & \sum_{j=0}^{\infty} 2^{j/2} \|\eta_k(\xi) P_j(I)(t)\|_{L^2_{\xi,t}} \\ & \lesssim \sum_{n \geq 1} \left\| \frac{t^n \psi(t)}{n!} \right\|_{B^{1/2}_{2,1}} \left\| \int_{|\tau| \leq 1} \frac{|\tau|}{|i\tau + \epsilon|\xi|^{2\alpha}} |\eta_k(\xi) \widehat{w}(\xi, \tau)| d\tau \right\|_{L^2_{\xi}} \\ & \lesssim \sum_{j=0}^{\infty} 2^{-j/2} \|\eta_k(\xi) \eta_j(\tau) \widehat{w}(\xi, \tau)\|_{L^2_{\xi,\tau}}. \end{aligned}$$

Therefore, we complete the proof of the proposition.  $\square$

In order to apply the standard fixed-point machinery, we next turn to a bilinear estimate in  $F^S$ . The proof is divided into several cases. We will use the estimate for the characterization multiplier in Proposition 3.2. The first case is *low*  $\times$  *high*  $\rightarrow$  *high* interaction.

**Proposition 4.5.** *If  $k \geq 10$ ,  $|k - k_2| \leq 5$ , then for any  $u \in F^S$ ,  $v \in F^S$*

$$\|(i + \tau - \xi^3)^{-1} \eta_k(\xi) i \xi \widehat{P}_0 u * \widehat{P}_{k_2} v\|_{X_k} \lesssim \|\widehat{P}_0 u\|_{X_0} \|\widehat{P}_{k_2} v\|_{X_{k_2}}. \tag{4.19}$$

**Proof.** For simplicity of notation we only prove the case that  $k = k_2$ , since the other cases can be handled in the same way. From definition of  $X_k$ , we get

$$\|(i + \tau - \xi^3)^{-1} \eta_k(\xi) i \xi \widehat{P}_0 u * \widehat{P}_k v\|_{X_k} \lesssim 2^k \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \|1_{D_{k,j}} u_{0,j_1} * v_{k,j_2}\|_2, \tag{4.20}$$

where

$$u_{0,j_1} = \eta_0(\xi) \eta_{j_1}(\tau - \xi^3) \widehat{u}, \quad v_{k,j_2} = \eta_k(\xi) \eta_{j_2}(\tau - \xi^3) \widehat{v}.$$

Thus, in view of definition it suffices to show that

$$\|1_{D_{k,j}} u_{0,j_1} * v_{k,j_2}\|_2 \lesssim 2^{-k} 2^{(j_1+j_2)/2} \|u_{0,j_1}\|_2 \|v_{k,j_2}\|_2. \tag{4.21}$$

By duality and  $\xi_1^3 + \xi_2^3 - (\xi_1 + \xi_2)^3 = -3\xi_1\xi_2(\xi_1 + \xi_2)$ , (4.21) is equivalent to

$$\begin{aligned} & \left| \iint u(\xi_1, \tau_1) v(\xi_2, \tau_2) g(\xi_1 + \xi_2, \tau_1 + \tau_2 - 3\xi_1\xi_2(\xi_1 + \xi_2)) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right| \\ & \lesssim 2^{-k} 2^{(j_1+j_2)/2} \|u\|_2 \|v\|_2 \|g\|_2 \end{aligned} \tag{4.22}$$

for any  $u, v, g \in L^2$  supported in  $I_0 \times I_{j_1}$ ,  $I_k \times I_{j_2}$ ,  $I_k \times I_j$  respectively. Therefore, it suffices to show that

$$\begin{aligned} & \int_{|\xi_1| \leq 2} \int_{|\xi_2| \sim 2^k} u(\xi_1) v(\xi_2) g(\xi_1 + \xi_2, -3\xi_1\xi_2(\xi_1 + \xi_2)) d\xi_1 d\xi_2 \\ & \lesssim 2^{-k} \|u\|_2 \|v\|_2 \|g\|_2 \end{aligned} \tag{4.23}$$

for any  $u, v, g \in L^2$  supported in  $I_0$ ,  $I_k$ ,  $I_k \times \widetilde{I}_{j_{max}}$  respectively where  $j_{max} = \max(j, j_1, j_2)$  and  $\widetilde{I}_{j_{max}} = \bigcup_{l=-3}^3 I_{j_{max}+l}$ .

Indeed, by changing the co-ordinates  $\mu_1 = \xi_1, \mu_2 = \xi_1 + \xi_2$ , the left side of (4.23) is bounded by

$$\int_{|\mu_1| \leq 2} \int_{|\mu_2| \sim 2^k} u(\mu_1)v(\mu_2 - \mu_1)g(\mu_2, -3\mu_1(\mu_2 - \mu_1)\mu_2) d\mu_1 d\mu_2. \tag{4.24}$$

Since in the integration area

$$\left| \frac{\partial}{\partial \mu_1} [-3\mu_1(\mu_2 - \mu_1)\mu_2] \right| \sim 2^{2k}, \tag{4.25}$$

then by Cauchy–Schwarz inequality we get

$$\begin{aligned} (4.24) &\lesssim \|u\|_2 \|v\|_2 \|g(\mu_2, -3\mu_1(\mu_2 - \mu_1)\mu_2)\|_{L^2_{|\mu_1| \leq 2, |\mu_2| \sim 2^k}} \\ &\lesssim 2^{-k} \|u\|_2 \|v\|_2 \|g\|_2, \end{aligned} \tag{4.26}$$

which completes the proof.  $\square$

**Proposition 4.6.** *If  $k \geq 10, |k - k_2| \leq 5$  and  $1 \leq k_1 \leq k - 9$ . Then for any  $u, v \in F^s$*

$$\|(i + \tau - \xi^3)^{-1} \eta_k(\xi) i \xi \widehat{P_{k_1} u} * \widehat{P_{k_2} v}\|_{X_k} \lesssim k^3 2^{-k/2} 2^{-k_1} \|\widehat{P_{k_1} u}\|_{X_{k_1}} \|\widehat{P_{k_2} v}\|_{X_{k_2}}. \tag{4.27}$$

**Proof.** We only prove the case  $k = k_2$ . From the definition, we get

$$\|(i + \tau - \xi^3)^{-1} \eta_k(\xi) i \xi \widehat{P_{k_1} u} * \widehat{P_k v}\|_{X_k} \lesssim 2^k \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \|1_{D_{k,j}} u_{k_1, j_1} * v_{k, j_2}\|_2, \tag{4.28}$$

where

$$u_{k_1, j_1} = \eta_{k_1}(\xi) \eta_{j_1}(\tau - \xi^3) \widehat{u}, \quad v_{k, j_2} = \eta_k(\xi) \eta_{j_2}(\tau - \xi^3) \widehat{v}.$$

By checking the support properties of the functions  $u_{k_1, j_1}, v_{k_2, j_2}$  and using the fact that  $|\xi_1^3 + \xi_2^3 - (\xi_1 + \xi_2)^3| \sim 2^{2k+k_1}$ , we get that  $1_{D_{k,j}} u_{k_1, j_1} * v_{k, j_2} \equiv 0$  unless  $j_{max} \geq 2k + k_1 - 10$ . Using (3.15), we get

$$\begin{aligned} &2^k \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \|1_{D_{k,j}} u_{k_1, j_1} * v_{k, j_2}\|_2 \\ &\lesssim 2^k \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} 2^{j_{min}/2} 2^{-k/2} 2^{-k_1/2} 2^{j_{med}/2} \|u_{k_1, j_1}\|_2 \|v_{k, j_2}\|_2 \\ &\lesssim 2^k \sum_{j_{max} \geq 2k+k_1-10} k^3 2^{-k/2} 2^{-k_1/2} 2^{-j_{max}/2} \|\widehat{P_{k_1} u}\|_{X_{k_1}} \|\widehat{P_k v}\|_{X_k} \\ &\lesssim k^3 2^{-k/2} 2^{-k_1} \|\widehat{P_{k_1} u}\|_{X_{k_1}} \|\widehat{P_k v}\|_{X_k}, \end{aligned} \tag{4.29}$$

which completes the proof of the proposition.  $\square$

The second case is *high  $\times$  high  $\rightarrow$  low*. This case is the worst and where the condition is imposed. This is easy to be seen, since  $s \leq 0$  and  $\|u\|_{F^s}, \|v\|_{F^s}$  are small for  $u, v$  with very high frequency.

**Proposition 4.7.** *If  $k \geq 10$ ,  $|k - k_2| \leq 5$ , then for any  $u, v \in F^5$*

$$\|(i + \tau - \xi^3)^{-1} \eta_0(\xi) i \xi \widehat{P_k u} * \widehat{P_{k_2} v}\|_{X_0} \lesssim k^3 2^{-3k/2} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_{k_2} v}\|_{X_{k_2}}. \tag{4.30}$$

**Proof.** As before we assume  $k = k_2$ . From the definition, we get

$$\|(i + \tau - \xi^3)^{-1} \eta_0(\xi) i \xi \widehat{P_k u} * \widehat{P_k v}\|_{X_0} \lesssim \sum_{k'=-\infty}^0 2^{k'} \sum_{j, j_1, j_2=0} 2^{-j/2} \|1_{D_{k',j}} u_{k, j_1} * v_{k, j_2}\|_2, \tag{4.31}$$

where

$$u_{k, j_1} = \eta_k(\xi) \eta_{j_1}(\tau - \xi^3) \widehat{u}, \quad v_{k, j_2} = \eta_k(\xi) \eta_{j_2}(\tau - \xi^3) \widehat{v}. \tag{4.32}$$

We may assume that  $k' \geq -10k$  and  $j, j_1, j_2 \leq 10k$ . Otherwise, from the following simple estimate which follows from Hölder's inequality and Young's inequality

$$\|1_{D_{k',j}} u_{k, j_1} * v_{k, j_2}\|_2 \lesssim 2^{j_{min}/2} 2^{k'/2} \|u_{k, j_1}\|_2 \|v_{k, j_2}\|_2$$

we immediately obtain (4.30). For the same reason as in the proof of last proposition, we see that  $j_{max} \geq 2k + k' - 10$ . Using (3.15), we get

$$\begin{aligned} & \| (i + \tau - \xi^3)^{-1} \eta_0(\xi) i \xi \widehat{P_k u} * \widehat{P_k v} \|_{X_0} \\ & \lesssim \sum_{k'=-10k}^0 2^{k'} \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \| 1_{D_{k',j}} u_{k, j_1} * v_{k, j_2} \|_2 \\ & \lesssim \sum_{k'=-10k}^0 \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} 2^{k'} 2^{j_{min}/2} 2^{-k/2} 2^{-k'/2} 2^{j_{med}/2} \| u_{k, j_1} \|_2 \| v_{k, j_2} \|_2 \\ & \lesssim \sum_{k'=-10k}^0 \sum_{j_{max} \geq 2k+k'} k^2 2^{-k/2} 2^{k'/2} 2^{-j_{max}/2} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_k v}\|_{X_k} \\ & \lesssim k^3 2^{-3k/2} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_k v}\|_{X_k}. \end{aligned} \tag{4.33}$$

Therefore, we complete the proof of the proposition.  $\square$

**Proposition 4.8.** *If  $k \geq 10$ ,  $|k - k_2| \leq 5$  and  $1 \leq k_1 \leq k - 9$ , then for any  $u, v \in F^5$*

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_1}(\xi) i \xi \widehat{P_k u} * \widehat{P_{k_2} v}\|_{X_{k_1}} \lesssim k^3 2^{-3k/2} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_{k_2} v}\|_{X_{k_2}}. \tag{4.34}$$

**Proof.** As before we assume  $k = k_2$ . From the definition of  $X_{k_1}$ , we get

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_1}(\xi) i \xi \widehat{P_k u} * \widehat{P_k v}\|_{X_{k_1}} \lesssim 2^{k_1} \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \|1_{D_{k_1,j}} u_{k, j_1} * v_{k, j_2}\|_2, \tag{4.35}$$

where  $u_{k, j_1}, v_{k, j_2}$  are as in (4.32). For the same reason as before we have  $j_{max} \geq 2k + k_1 - 10$  and we may assume  $j, j_1, j_2 \leq 10k$ . It follows from (3.15) that the right-hand side of (4.35) is bounded by

$$\begin{aligned} & \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} 2^{k_1} 2^{j_{\min}/2} 2^{-k/2} 2^{-k_1/2} 2^{j_{\text{med}}/2} \|u_{k, j_1}\|_2 \|v_{k, j_2}\|_2 \\ & \lesssim \sum_{j_{\max} \geq 2k+k_1} k^2 2^{-k/2} 2^{k_1/2} 2^{-j_{\max}/2} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_k v}\|_{X_k} \\ & \lesssim k^3 2^{-3k/2} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_k v}\|_{X_k}. \end{aligned}$$

Therefore we complete the proof of the proposition.  $\square$

**Proposition 4.9.** *If  $k \geq 10$ ,  $|k - k_2| \leq 5$  and  $k - 9 \leq k_1 \leq k + 10$ , then for any  $u, v \in F^S$*

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_1}(\xi) i \xi \widehat{P_k u} * \widehat{P_{k_2} v}\|_{X_{k_1}} \lesssim k^3 2^{-3k/4} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_{k_2} v}\|_{X_{k_2}}. \tag{4.36}$$

**Proof.** As before we assume  $k = k_2$ . From the definition of  $X_{k_1}$ , we get

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_1}(\xi) i \xi \widehat{P_k u} * \widehat{P_k v}\|_{X_{k_1}} \lesssim 2^{k_1} \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \|1_{D_{k_1, j}} u_{k, j_1} * v_{k, j_2}\|_2, \tag{4.37}$$

where  $u_{k, j_1}, v_{k, j_2}$  are as in (4.32). For the same reason as before we have  $j_{\max} \geq 2k + k_1 - 10$  and we may assume  $j, j_1, j_2 \leq 10k$ . It follows from (3.14) that the right-hand side of (4.39) is bounded by

$$\sum_{j, j_1, j_2 \geq 0} 2^{-j/2} 2^{k_1} 2^{j_{\min}/2} 2^{-k/4} 2^{j_{\text{med}}/4} \|u_{k, j_1}\|_2 \|v_{k, j_2}\|_2 \lesssim k^3 2^{-3k/4} \|\widehat{P_k u}\|_{X_k} \|\widehat{P_k v}\|_{X_k},$$

which completes the proof of the proposition.  $\square$

The final case is *low*  $\times$  *low*  $\rightarrow$  *low* interaction. Generally speaking, this case is always easy to handle in many situations.

**Proposition 4.10.** *If  $0 \leq k_1, k_2, k_3 \leq 100$ , then for any  $u, v \in F^S$*

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_1}(\xi) i \xi \widehat{P_{k_2} u} * \widehat{P_{k_3} v}\|_{X_{k_1}} \lesssim \|\widehat{P_{k_2} u}\|_{X_{k_2}} \|\widehat{P_{k_3} v}\|_{X_{k_3}}. \tag{4.38}$$

**Proof.** From the definition of  $X_{k_1}$ , we get that

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_1}(\xi) i \xi \widehat{P_{k_2} u} * \widehat{P_{k_3} v}\|_{X_{k_1}} \lesssim 2^{k_1} \sum_{j, j_1, j_2 \geq 0} 2^{-j/2} \|1_{D_{k_1, j}} u_{k_2, j_1} * v_{k_3, j_2}\|_2, \tag{4.39}$$

where  $u_{k_2, j_1}, v_{k_3, j_2}$  are as in (4.32). By checking the support properties of the function  $u_{k_2, j_1}, v_{k_3, j_2}$ , we get that  $1_{D_{k_1, j}} u_{k_2, j_1} * v_{k_3, j_2} \equiv 0$  unless  $|j_{\max} - j_{\text{med}}| \leq 10$  or  $j_{\max} \leq 1000$  where  $j_{\max}, j_{\text{med}}$  are the maximum and median of  $j, j_1, j_2$  respectively. It follows immediately from Young’s inequality that

$$\|1_{D_{k, j}} u_{k_1, j_1} * v_{k_2, j_2}\|_{L_{\xi, \tau}^2} \lesssim 2^{k_1} 2^{j_i} \|u_{k_1, j_1}\|_2 \|v_{k_2, j_2}\|_2, \quad i = 1, 2. \tag{4.40}$$

From definition and summing in  $j_i$ , we complete the proof of the proposition.  $\square$

With these propositions in hand, we are able to prove the bilinear estimate. The idea is to decompose the bilinear product using para-product, and then divide it into many cases according to the interactions. Finally we use discrete Young’s inequality.

**Proposition 4.11.** Fix any  $s \in (-3/4, 0]$ ,  $\forall s \leq \sigma \leq 0$ , there exists  $C > 0$  such that for any  $u, v \in F^\sigma$ ,

$$\|\partial_x(uv)\|_{N^\sigma} \leq C(\|u\|_{F^s}\|v\|_{F^\sigma} + \|v\|_{F^s}\|u\|_{F^\sigma}). \tag{4.41}$$

**Proof.** In view of definition, we get that

$$\|\partial_x(uv)\|_{N^\sigma}^2 = \sum_{k_3 \in \mathbb{Z}_+} 2^{2\sigma k_3} \|(i + \tau - \xi^3)^{-1} \eta_{k_3}(\xi) i \xi \widehat{u} * \widehat{v}\|_{X_{k_3}}^2. \tag{4.42}$$

We decompose  $\widehat{u}, \widehat{v}$  and get

$$\|(i + \tau - \xi^3)^{-1} \eta_{k_3}(\xi) i \xi \widehat{u} * \widehat{v}\|_{X_{k_3}} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}_+} \|(i + \tau - \xi^3)^{-1} \eta_{k_3}(\xi) i \xi \widehat{P_{k_1} u} * \widehat{P_{k_2} v}\|_{X_{k_3}}. \tag{4.43}$$

By checking the support properties we get that  $\eta_{k_3}(\xi) \widehat{P_{k_1} u} * \widehat{P_{k_2} v} \equiv 0$  unless  $|k_{max} - k_{med}| \leq 5$  where  $k_{max}, k_{med}$  are the maximum and median of  $k_1, k_2, k_3$  respectively. We may assume that  $k_1 \leq k_2$  from symmetry. By dividing the summation into *high*  $\times$  *high*, *high*  $\times$  *low* four parts, we get that the right-hand side of (4.43) is bounded by

$$\left( \sum_{j=1}^4 \sum_{k_1, k_2 \in A_j} \right) \|(i + \tau - \xi^3)^{-1} \eta_{k_3}(\xi) i \xi \widehat{P_{k_1} u} * \widehat{P_{k_2} v}\|_{X_{k_3}}, \tag{4.44}$$

where  $A_j, j = 1, 2, 3, 4$ , are defined by

$$\begin{aligned} A_1 &= \{k_2 \geq 10, |k_2 - k_3| \leq 5, k_1 \leq k_2 - 10\}; \\ A_2 &= \{k_2 \geq 10, |k_2 - k_3| \leq 5, k_2 - 9 \leq k_1 \leq k_2 + 10\}; \\ A_3 &= \{k_2 \geq 10, |k_2 - k_1| \leq 5, k_3 \leq k_1 - 10\}; \\ A_4 &= \{k_1, k_2, k_3 \leq 100\}. \end{aligned}$$

Therefore, (4.41) follows from Propositions 4.5–4.10, discrete Young’s inequality and the assumption that  $s > -3/4$ .  $\square$

We next show (1.1) is uniformly (on  $0 < \epsilon \leq 1$ ) locally well-posed in  $H^s, -3/4 < s \leq 0$ . The procedure is quite standard. See [5], for instance. By the scaling (4.1), we see that  $u$  solves (1.1) if and only if  $u_\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^3 t)$  solves

$$\partial_t u_\lambda + \partial_x^3 u_\lambda + \epsilon \lambda^{3-2\alpha} |\partial_x|^{2\alpha} u_\lambda + \partial_x(u_\lambda^2) = 0, \quad u_\lambda(0) = \lambda^2 \phi(\lambda \cdot). \tag{4.45}$$

Since  $-3/4 < s \leq 0$ ,

$$\|\lambda^2 \phi(\lambda x)\|_{H^s} = O(\lambda^{3/2+s} \|\phi\|_{H^s}) \quad \text{as } \lambda \rightarrow 0, \tag{4.46}$$

thus we can first restrict ourselves to considering (1.1) with data  $\phi$  satisfying

$$\|\phi\|_{H^s} = r \ll 1. \tag{4.47}$$

As in the last section, we will mainly work on the integral equation (4.6). We define the operator

$$\Phi_\phi(u) = \psi(t) W_\epsilon^\alpha(t) \phi - \psi(t) L(\partial_x(\psi^2 u^2)), \tag{4.48}$$

where  $L$  is defined by (4.7). We will prove that  $\Phi_\phi(\cdot)$  is a contraction mapping from

$$\mathcal{B} = \{w \in F^s : \|w\|_{F^s} \leq 2cr\} \tag{4.49}$$

into itself. From Propositions 4.2–4.4 we get if  $w \in \mathcal{B}$ , then

$$\begin{aligned} \|\Phi_\phi(w)\|_{F^s} &\leq c\|\phi\|_{H^s} + \|\partial_x(\psi(t)^2 w^2(\cdot, t))\|_{N^s} \\ &\leq cr + c\|w\|_{F^s}^2 \leq cr + c(2cr)^2 \leq 2cr, \end{aligned} \tag{4.50}$$

provided  $r$  satisfies  $4c^2r \leq 1/2$ . Similarly, for  $w, h \in \mathcal{B}$

$$\begin{aligned} \|\Phi_\phi(w) - \Phi_\phi(h)\|_{F^s} &\leq c\|L\partial_x(\psi^2(\tau)(u^2(\tau) - h^2(\tau)))\|_{F^s} \\ &\leq c\|w + h\|_{F^s}\|w - h\|_{F^s} \\ &\leq 4c^2r\|w - h\|_{F^s} \leq \frac{1}{2}\|w - h\|_{F^s}. \end{aligned} \tag{4.51}$$

Thus  $\Phi_\phi(\cdot)$  is a contraction. There exists a unique  $u \in \mathcal{B}$  such that

$$u = \psi(t)W_\epsilon^\alpha(t)\phi - \psi(t)L(\partial_x(\psi^2 u^2)). \tag{4.52}$$

Hence  $u$  solves the integral equation (3.1) in the time interval  $[0, 1]$ .

We prove now that  $u \in X^{1/2, s, \alpha}$ . Indeed, from the slightly modified argument as the proof for Propositions 2.1, 2.3 [7], we can show that

$$\begin{aligned} \|\psi(t)W_\epsilon^\alpha(t)\phi\|_{X^{1/2, s, \alpha}} &\lesssim \|\phi\|_{H^s}; \\ \|\psi(t)L(v)\|_{X^{1/2, s, \alpha}} &\lesssim \|v\|_{X^{-1/2, s, \alpha}} + \left(\int \langle \xi \rangle^{2s} \left(\int \frac{|\widehat{v}(\tau)|}{|\tau + \epsilon|\xi|^{2\alpha}} d\tau\right)^2 d\xi\right)^{1/2} \lesssim \|v\|_{N^s}, \end{aligned}$$

which then imply  $u \in X^{1/2, s, \alpha}$ , as desired. For general  $\phi \in H^s$ , by using the scaling (4.1) and the uniqueness in Theorem 1.1, we immediately obtain that Theorem 1.2 holds for a small  $T = T(\|\phi\|_{H^s}) > 0$ .

### 5. Uniform global well-posedness for KdV–B equation

In this section we will extend the uniform local solution obtained in the last section to a uniform global solution. The standard way is to use conservation law. Let  $u$  be a smooth solution of (1.1), multiply  $u$  and integrate, then we get

$$\frac{1}{2}\|u(t)\|_2^2 + \epsilon \int_0^t \|A^\alpha u(\tau)\|_2^2 d\tau = \frac{1}{2}\|\phi\|_2^2. \tag{5.1}$$

By a standard limit argument, (5.1) holds for  $L^2$ -strong solution. Thus if  $\phi \in L^2$ , then we get that (1.1) is uniformly globally well-posed.

For  $\phi \in H^s$  with  $-3/4 < s < 0$ , there is no such conservation law. We will follow the idea in [2] (I-method) to extend the solution. Let  $m : \mathbb{R}^k \rightarrow \mathbb{C}$  be a function. We say  $m$  is symmetric if



$m(\xi_1, \dots, \xi_k) = m(\sigma(\xi_1, \dots, \xi_k))$  for all  $\sigma \in S_k$ , the group of all permutations on  $k$  objects. The symmetrization of  $m$  is the function

$$[m]_{sym}(\xi_1, \xi_2, \dots, \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} m(\sigma(\xi_1, \xi_2, \dots, \xi_k)). \tag{5.2}$$

We define a  $k$ -linear functional associated to the multiplier  $m$  acting on  $k$  functions  $u_1, \dots, u_k$ ,

$$\Lambda_k(m; u_1, \dots, u_k) = \int_{\xi_1 + \dots + \xi_k = 0} m(\xi_1, \dots, \xi_k) \widehat{u}_1(\xi_1) \cdots \widehat{u}_k(\xi_k). \tag{5.3}$$

We will often apply  $\Lambda_k$  to  $k$  copies of the same function  $u$ .  $\Lambda_k(m; u, \dots, u)$  may simply be written  $\Lambda_k(m)$ . By the symmetry of the measure on hyperplane, we have  $\Lambda_k(m) = \Lambda_k([m]_{sym})$ .

The following statement may be directly verified by using the KdV–B equation (1.1). Compared to the KdV equation, the KdV–B equation has one more term caused by the dissipation.

**Proposition 5.1.** *Suppose  $u$  satisfies the KdV–B equation (1.1) and that  $m$  is a symmetric function. Then*

$$\frac{d}{dt} \Lambda_k(m) = \Lambda_k(mh_k) - \epsilon \Lambda_k(m\beta_{\alpha,k}) - i \frac{k}{2} \Lambda_{k+1}(m(\xi_1, \dots, \xi_{k-1}, \xi_k + \xi_{k+1})(\xi_k + \xi_{k+1})), \tag{5.4}$$

where

$$h_k = i(\xi_1^3 + \xi_2^3 + \dots + \xi_k^3), \quad \beta_{\alpha,k} = |\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + \dots + |\xi_k|^{2\alpha}.$$

We follow the I-method [2] to define a set of modified energies. Let  $m : \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary even  $\mathbb{R}$ -valued function and define the operator by

$$\widehat{I}f(\xi) = m(\xi) \widehat{f}(\xi). \tag{5.5}$$

We define the modified energy  $E_1^2(t)$  by

$$E_1^2(t) = \|Iu(t)\|_{L^2}^2. \tag{5.6}$$

By Plancherel and the fact that  $m$  and  $u$  are  $\mathbb{R}$ -valued, and  $m$  is even,

$$E_1^2(t) = \Lambda_2(m(\xi_1)m(\xi_2)).$$

Using (5.4), we have

$$\begin{aligned} \frac{d}{dt} E_1^2(t) &= \Lambda_2(m(\xi_1)m(\xi_2)h_2) - \epsilon \Lambda_2(m(\xi_1)m(\xi_2)\beta_{\alpha,2}) \\ &\quad - i \Lambda_3(m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)). \end{aligned} \tag{5.7}$$

The first term vanishes. The second term is non-positive, hence good. We symmetrize the third term to get

$$\frac{d}{dt} E_1^2(t) = -\epsilon \Lambda_2(m(\xi_1)m(\xi_2)\beta_{\alpha,2}) + \Lambda_3(-i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}). \tag{5.8}$$

Let us denote

$$M_3(\xi_1, \xi_2, \xi_3) = -i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}. \tag{5.9}$$

Form the new modified energy

$$E_1^3(t) = E_1^2(t) + \Lambda_3(\sigma_3),$$

where the symmetric function  $\sigma_3$  will be chosen momentarily to achieve a cancellation. Applying (5.4) gives

$$\begin{aligned} \frac{d}{dt} E_1^3(t) &= -\epsilon \Lambda_2(m(\xi_1)m(\xi_2)\beta_{\alpha,2}) + \Lambda_3(M_3) \\ &\quad + \Lambda_3(\sigma_3 h_3) - \epsilon \Lambda_3(\sigma_3 \beta_{\alpha,3}) - \frac{3}{2} i \Lambda_4(\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)). \end{aligned} \tag{5.10}$$

Compared to the KdV case [2], there is one more term to cancel, so we choose

$$\sigma_3 = -\frac{M_3}{h_3 - \epsilon \beta_{\alpha,3}} \tag{5.11}$$

to force the three  $\Lambda_3$  terms in (5.10) to cancel. Hence if we denote

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -i\frac{3}{2}[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}, \tag{5.12}$$

then

$$\frac{d}{dt} E_1^3(t) = -\epsilon \Lambda_2(m(\xi_1)m(\xi_2)\beta_{\alpha,2}) + \Lambda_4(M_4). \tag{5.13}$$

Similarly defining

$$E_1^4(t) = E_1^3(t) + \Lambda_4(\sigma_4)$$

with

$$\sigma_4 = -\frac{M_4}{h_4 - \epsilon \beta_{\alpha,4}}, \tag{5.14}$$

we obtain

$$\frac{d}{dt} E_1^4(t) = -\epsilon \Lambda_2(m(\xi_1)m(\xi_2)\beta_{\alpha,2}) + \Lambda_5(M_5), \tag{5.15}$$

where

$$M_5(\xi_1, \dots, \xi_5) = -2i[\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)]_{sym}. \tag{5.16}$$

Now we give pointwise bounds for the multipliers. We will only be interested in the value of the multiplier on the hyperplane  $\xi_1 + \xi_2 + \dots + \xi_k = 0$ . There is a flexibility of choosing the multiplier  $m$ . In application, we consider  $m(\xi)$  is smooth, monotone, and of the form

$$m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| > 2N. \end{cases} \tag{5.17}$$

It is easy to see that if  $m$  is of the form (5.17), then  $m^2$  satisfies

$$\begin{aligned} m^2(\xi) &\sim m^2(\xi') \quad \text{for } |\xi| \sim |\xi'|, \\ (m^2)'(\xi) &= O\left(\frac{m^2(\xi)}{|\xi|}\right), \\ (m^2)''(\xi) &= O\left(\frac{m^2(\xi)}{|\xi|^2}\right). \end{aligned} \tag{5.18}$$

We will need two mean value formulas which follow immediately from the fundamental theorem of calculus. If  $|\eta|, |\lambda| \ll |\xi|$ , then we have

$$|a(\xi + \eta) - a(\xi)| \lesssim |\eta| \sup_{|\xi'| \sim |\xi|} |a'(\xi')|, \tag{5.19}$$

and the double mean value formula that

$$|a(\xi + \eta + \lambda) - a(\xi + \eta) - a(\xi + \lambda) + a(\xi)| \lesssim |\eta||\lambda| \sup_{|\xi'| \sim |\xi|} |a''(\xi')|. \tag{5.20}$$

In order to use the formulas, we extend the surface supported multiplier  $\sigma_3$  to the whole space as in [6].

**Proposition 5.2.** *If  $m$  is of the form (5.17), then for each dyadic  $\lambda \leq \mu$  there is an extension of  $\sigma_3$  from the diagonal set*

$$\{(\xi_1, \xi_2, \xi_3) \in \Gamma_3(\mathbb{R}), |\xi_1| \sim \lambda, |\xi_2|, |\xi_3| \sim \mu\}$$

to the full dyadic set

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, |\xi_1| \sim \lambda, |\xi_2|, |\xi_3| \sim \mu\}$$

which satisfies

$$|\partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} \sigma_3(\xi_1, \xi_2, \xi_3)| \leq C m^2(\lambda) \mu^{-2} \lambda^{-\beta_1} \mu^{-\beta_2 - \beta_3}, \tag{5.21}$$

where  $C$  is independent of  $\epsilon$ .

**Proof.** Since on the hyperplane  $\xi_1 + \xi_2 + \xi_3 = 0$ ,

$$h_3 = i(\xi_1^3 + \xi_2^3 + \xi_3^3) = 3i\xi_1\xi_2\xi_3$$

is with a size about  $\lambda\mu^2$  and

$$M_3(\xi_1, \xi_2, \xi_3) = -i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{\text{sym}} = i(m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3),$$

if  $\lambda \sim \mu$ , we extend  $\sigma_3$  by setting

$$\sigma_3(\xi_1, \xi_2, \xi_3) = -\frac{i(m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3)}{3i\xi_1\xi_2\xi_3 - \epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})}, \tag{5.22}$$

and if  $\lambda \ll \mu$ , we extend  $\sigma_3$  by setting

$$\sigma_3(\xi_1, \xi_2, \xi_3) = -\frac{i(m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 - m^2(\xi_1 + \xi_2)(\xi_1 + \xi_2))}{3i\xi_1\xi_2\xi_3 - \epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})}. \tag{5.23}$$

From (5.19) and (5.18), we see that (5.21) holds.  $\square$

We define on the hyperplane  $\{(\xi_1, \xi_2, \xi_3) \in \Gamma_3(\mathbb{R}), |\xi_1| \approx \lambda, |\xi_2|, |\xi_3| \approx \mu\}$

$$\sigma_3^-(\xi_1, \xi_2, \xi_3) = -\frac{i(m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3)}{3i\xi_1\xi_2\xi_3 + \epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})}, \tag{5.24}$$

and extend it as for  $\sigma_3$ . Then (5.21) also holds for  $\sigma_3^-$ , and on the hyperplane  $\xi_1 + \xi_2 + \xi_3 = 0$  we get

$$|\sigma_3(\xi_1, \xi_2, \xi_3) - \sigma_3^-(\xi_1, \xi_2, \xi_3)| \lesssim \frac{\epsilon|\xi|_{\max}^{2\alpha}m^2(|\xi|_{\min})|\xi|_{\min}}{(\xi_1\xi_2\xi_3)^2 + \epsilon^2|\xi|_{\max}^{4\alpha}}, \tag{5.25}$$

where

$$|\xi|_{\max} = \max(|\xi_1|, |\xi_2|, |\xi_3|), \quad |\xi|_{\min} = \min(|\xi_1|, |\xi_2|, |\xi_3|).$$

Now we give the pointwise bounds for  $\sigma_4$  which is key to estimate the growth of  $E_I^4(t)$ . It has the same bound as in the KdV case.

**Proposition 5.3.** Assume  $m$  is of the form (5.17). In the region where  $|\xi_i| \sim N_i, |\xi_j + \xi_k| \sim N_{jk}$  for  $N_i, N_{jk}$  dyadic,

$$\frac{|M_4(\xi_1, \xi_2, \xi_3, \xi_4)|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{m^2(\min(N_i, N_{jk}))}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \tag{5.26}$$

**Proof.** From symmetry, we can assume that  $N_1 \geq N_2 \geq N_3 \geq N_4$ . Since  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ , then  $N_1 \sim N_2$ . We can also assume that  $N_1 \sim N_2 \gtrsim N$ , otherwise  $M_4$  vanishes, since  $m^2(\xi) = 1$  if  $|\xi| \leq N$ . If  $\max(N_{12}, N_{13}, N_{14}) \ll N_1$ , then  $\xi_3 \approx -\xi_1, \xi_4 \approx -\xi_1$ , which contradicts that  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ . Hence we get  $\max(N_{12}, N_{13}, N_{14}) \sim N_1$ . The right side of (5.26) may be reexpressed as

$$\frac{m^2(\min(N_i, N_{jk}))}{N_1^2(N + N_3)(N + N_4)}. \tag{5.27}$$

Since  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ , then  $h_4 = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)$ , and we can write that

$$\begin{aligned} CM_4(\xi_1, \xi_2, \xi_3, \xi_4) &= [\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{\text{sym}} \\ &= \sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4) + \sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4)(\xi_2 + \xi_4) \\ &\quad + \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3)(\xi_2 + \xi_3) + \sigma_3(\xi_2, \xi_3, \xi_1 + \xi_4)(\xi_1 + \xi_4) \\ &\quad + \sigma_3(\xi_2, \xi_4, \xi_1 + \xi_3)(\xi_1 + \xi_3) + \sigma_3(\xi_3, \xi_4, \xi_1 + \xi_2)(\xi_1 + \xi_2) \\ &= [\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4) - \sigma_3^-( -\xi_3, -\xi_4, \xi_3 + \xi_4)](\xi_3 + \xi_4) \\ &\quad + [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)](\xi_2 + \xi_4) \end{aligned}$$

$$\begin{aligned}
 &+ [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-( -\xi_2, -\xi_3, \xi_2 + \xi_3)](\xi_2 + \xi_3) \\
 &= I + II + III.
 \end{aligned}
 \tag{5.28}$$

The bound (5.26) will follow from case-by-case analysis.

**Case 1.**  $|N_4| \gtrsim \frac{N}{2}$ .

**Case 1a.**  $N_{12}, N_{13}, N_{14} \gtrsim N_1$ .

For this case, we just use (5.21), then we get

$$\frac{|M_4(\xi_1, \xi_2, \xi_3, \xi_4)|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|M_4(\xi_1, \xi_2, \xi_3, \xi_4)|}{|h_4|} \lesssim \frac{m^2(N_4)}{N_1 N_2 N_3 N_4},
 \tag{5.29}$$

which is acceptable.

**Case 1b.**  $N_{12} \ll N_1, N_{13} \gtrsim N_1, N_{14} \gtrsim N_1$ .

Contribution of I. We just use (5.21), then we get

$$\frac{|I|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|I|}{|h_4|} \lesssim \frac{m^2(\min(N_4, N_{12}))}{N_1 N_2 N_3 N_4},
 \tag{5.30}$$

which is acceptable.

Contribution of II. We first write

$$\begin{aligned}
 II &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)](\xi_2 + \xi_4) \\
 &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( \xi_1, \xi_3, \xi_2 + \xi_4)](\xi_2 + \xi_4) \\
 &\quad + [\sigma_3^-( \xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)](\xi_2 + \xi_4) \\
 &= II_1 + II_2.
 \end{aligned}
 \tag{5.31}$$

Then from (5.25) we get

$$\frac{II_1}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{II_1}{|\epsilon\beta_{\alpha,4}|} \lesssim \frac{m^2(N_4)}{N_1 N_1 N_1 N_3}.
 \tag{5.32}$$

We now consider  $II_2$ . If  $N_{12} \gtrsim N_3$ , then using (5.19) and (5.21), or else if  $N_{12} \ll N_3$ , then using (5.19) twice and (5.21), then

$$\frac{II_2}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{II_2}{h_4} \lesssim \frac{m^2(N_4)}{N_1 N_1 N_1 N_3}.
 \tag{5.33}$$

Contribution of III. This is identical to II.

**Case 1c.**  $N_{12} \ll N_1, N_{13} \ll N_1, N_{14} \gtrsim N_1$ .

Since  $N_{12} \ll N_1, N_{13} \ll N_1$ , then  $N_1 \sim N_2 \sim N_3 \sim N_4$ .

Contribution of I. We first write

$$\begin{aligned}
 I &= [\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4) - \sigma_3^-(\xi_1, \xi_2, \xi_3 + \xi_4)](\xi_3 + \xi_4) \\
 &\quad + [\sigma_3^-(\xi_1, \xi_2, \xi_3 + \xi_4) - \sigma_3^-(-\xi_3, \xi_2, \xi_3 + \xi_4)](\xi_3 + \xi_4) \\
 &\quad + [\sigma_3^-(-\xi_3, \xi_2, \xi_3 + \xi_4) - \sigma_3^-(-\xi_3, -\xi_4, \xi_3 + \xi_4)](\xi_3 + \xi_4) \\
 &= I_1 + I_2 + I_3.
 \end{aligned}
 \tag{5.34}$$

We use (5.25) for the first term and (5.21), (5.19) for the last two terms, then we get

$$\frac{I}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{I_1}{|\epsilon\beta_{\alpha,4}|} + \frac{I_2}{|h_4|} + \frac{I_3}{|h_4|} \lesssim \frac{m^2(N_{12})}{N_1^4}.
 \tag{5.35}$$

Contribution of II. This is identical to I.

Contribution of III. We first write

$$\begin{aligned}
 III &= [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-(-\xi_2, -\xi_3, \xi_2 + \xi_3)](\xi_2 + \xi_3) \\
 &= [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-(\xi_1, \xi_4, \xi_2 + \xi_3)](\xi_2 + \xi_3) \\
 &\quad + 1/2[\sigma_3^-(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-(-\xi_2, -\xi_3, \xi_2 + \xi_3) \\
 &\quad - \sigma_3^-(-\xi_3, -\xi_2, \xi_2 + \xi_3) + \sigma_3^-(\xi_4, \xi_1, \xi_2 + \xi_3)](\xi_2 + \xi_3) \\
 &= III_1 + III_2.
 \end{aligned}
 \tag{5.36}$$

We use (5.25) for the first term and (5.20) four times for the second term, then we get

$$\frac{III}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{III_1}{|\epsilon\beta_{\alpha,4}|} + \frac{III_2}{|h_4|} \lesssim \frac{m^2(N_1)}{N_1^4}.
 \tag{5.37}$$

**Case 1d.**  $N_{12} \ll N_1, N_{13} \gtrsim N_1, N_{14} \ll N_1$ .

This case is identical to Case 1c.

**Case 2.**  $N_4 \ll N/2$ .

In this case we have  $m^2(\min(N_i, N_{jk})) = 1$ , and  $N_{13} \sim |\xi_1 + \xi_3| = |\xi_2 + \xi_4| \sim N_1$ . We discuss this case in the following two subcases.

**Case 2a.**  $N_1/4 > N_{12} \gtrsim N/2$ .

Since  $N_4 \ll N/2$  and  $|\xi_3 + \xi_4| = |\xi_1 + \xi_2| \gtrsim N/2$ , then  $N_3 \gtrsim N/2$ . From  $|h_4| \sim N_{12}N_1^2$ , then we bound the six terms in (5.28) respectively, and get

$$\frac{|M_4|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|M_4|}{|h_4|} \lesssim \frac{1}{N_1^2 N_3 N},
 \tag{5.38}$$

which is acceptable.

**Case 2b.**  $N_{12} \ll N/2$ .

Since  $N_{12} = N_{34} \ll N/2$  and  $N_4 \ll N/2$ , then we must have  $N_3 \ll N/2$ , and  $N_{13} \sim N_{14} \sim N_1$ .

Contribution of I. Since  $N_3, N_4, N_{34} \ll N/2$ , then we have  $\sigma_3^-( -\xi_3, -\xi_4, \xi_3 + \xi_4) = 0$ . Thus it follows from (5.21) that

$$\frac{|I|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)|}{N_1^2} \lesssim \frac{1}{N_1^4}. \tag{5.39}$$

Contribution of II and III. We have two items of  $N_3, N_4, N_{12}$  in the denominator, which will cause a problem. Thus we cannot deal with II and III separately, but we need to exploit the cancellation between II and III. We rewrite

$$\begin{aligned} II + III &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)](\xi_2 + \xi_4) \\ &\quad + [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-( -\xi_2, -\xi_3, \xi_2 + \xi_3)](\xi_2 + \xi_3) \\ &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)]\xi_4 \\ &\quad + [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-( -\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_3 \\ &\quad + [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)] \\ &\quad + \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-( -\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\ &= J_1 + J_2 + J_3. \end{aligned} \tag{5.40}$$

We first consider  $J_1$ . From

$$\begin{aligned} \frac{|J_1|}{|h_4 - \epsilon\beta_{\alpha,4}|} &\leq \frac{|[\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)]\xi_4|}{|h_4|} \\ &\quad + \frac{|[\sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4)]\xi_4|}{|\epsilon\beta_{\alpha,4}|}, \end{aligned} \tag{5.41}$$

and (5.25) for the second term, (5.19) if  $N_{12} \ll N_3$  (in this case,  $N_3 \sim N_4$ ), and (5.21) if  $N_{12} \gtrsim N_3$  for the first term, then we get

$$\frac{|J_1|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{1}{N_1^4}. \tag{5.42}$$

The term  $J_2$  is identical to the term  $J_1$ . Now we consider  $J_3$ . We first assume that  $N_{12} \gtrsim N_3$ . Then by the symmetry of  $\sigma_3$ , we get

$$\begin{aligned} J_3 &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-( -\xi_2, -\xi_4, \xi_2 + \xi_4) \\ &\quad + \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3^-( -\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\ &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2 - \xi_3, \xi_3, \xi_2) \\ &\quad + \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2 - \xi_4, \xi_4, \xi_2)]\xi_2. \end{aligned} \tag{5.43}$$

From (5.19) and  $N_{12} \gtrsim N_3$ , we get

$$\frac{|J_3|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|J_3|}{|h_4|} \lesssim \frac{1}{N_1^4}. \tag{5.44}$$

If  $N_{12} \ll N_3$ , then  $N_3 \sim N_4$ . We first write

$$\begin{aligned}
 J_3 &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-(\xi_1, \xi_3, \xi_2 + \xi_4) \\
 &\quad + \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3) - \sigma_3^-(-\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\
 &\quad + [\sigma_3^-(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-(-\xi_2, -\xi_4, \xi_2 + \xi_4) \\
 &\quad + \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(\xi_1, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\
 &\quad + [\sigma_3^-(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-(-\xi_2, \xi_3, \xi_2 + \xi_4) \\
 &\quad + \sigma_3(\xi_1, -\xi_3, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\
 &= J_{31} + J_{32} + J_{33}.
 \end{aligned} \tag{5.45}$$

It follows from (5.19) that

$$\frac{|J_{33}|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|J_{33}|}{|h_4|} \lesssim \frac{1}{N_1^4}. \tag{5.46}$$

It remains to bound  $J_{31}$  and  $J_{32}$ . First we consider  $J_{31}$ . Since  $m^2(\xi_3) = 1$ , we rewrite  $J_{31}$  by

$$\begin{aligned}
 J_{31} &= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3^-(\xi_1, \xi_3, \xi_2 + \xi_4) \\
 &\quad + \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3) - \sigma_3^-(-\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\
 &= A(\xi_1, \xi_3, \xi_2 + \xi_4)(m^2(\xi_1)\xi_1 + \xi_3 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4))\xi_2 \\
 &\quad + A(-\xi_2, -\xi_3, \xi_2 + \xi_3)(-m^2(\xi_2)\xi_2 - \xi_3 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3))\xi_2 \\
 &= [A(\xi_1, \xi_3, \xi_2 + \xi_4) - A(-\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_3\xi_2 \\
 &\quad - [A(\xi_1, \xi_3, \xi_2 + \xi_4) - A(-\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\
 &\quad \times [-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)] \\
 &\quad + A(\xi_1, \xi_3, \xi_2 + \xi_4)\xi_2 \\
 &\quad \times [m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4) - m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)],
 \end{aligned} \tag{5.47}$$

where

$$A(\xi_1, \xi_2, \xi_3) = \frac{2\epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})}{|\xi_1\xi_2\xi_3|^2 + \epsilon^2(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})^2}.$$

It is easy to see that  $A(\xi_1, \xi_2, \xi_3)$  satisfies

$$|\partial_{\xi_i} A(\xi_1, \xi_2, \xi_3)| \lesssim \frac{|A(\xi_1, \xi_2, \xi_3)|}{|\xi_i|}, \quad i = 1, 2, 3. \tag{5.48}$$

For the first two terms in (5.47) we use (5.19) by writing

$$\begin{aligned}
 A(\xi_1, \xi_3, \xi_2 + \xi_4) - A(-\xi_2, -\xi_3, \xi_2 + \xi_3) &= A(\xi_1, \xi_3, \xi_2 + \xi_4) - A(-\xi_2, \xi_3, \xi_2 + \xi_4) \\
 &\quad + A(-\xi_2, \xi_3, \xi_2 + \xi_4) - A(-\xi_2, \xi_3, \xi_2 + \xi_3).
 \end{aligned}$$

For the third term, we note that



$$\begin{aligned}
 & m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4) - m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3) \\
 &= m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4) - m^2(\xi_2)\xi_2 \\
 &\quad - m^2(\xi_2 + \xi_3 + \xi_4)(\xi_2 + \xi_3 + \xi_4) + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3),
 \end{aligned} \tag{5.49}$$

thus we can apply (5.20). Therefore, we get

$$\frac{|J_{31}|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|J_{31}|}{|\epsilon\beta_{\alpha,4}|} \lesssim \frac{1}{N_1^4}. \tag{5.50}$$

Last we consider  $J_{32}$ . We denote

$$\begin{aligned}
 B(\xi_1, \xi_2, \xi_3) &= \frac{1}{i\xi_1\xi_2\xi_3 - \epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})} - \frac{1}{i\xi_1\xi_2\xi_3} \\
 &= \frac{\epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})}{[i\xi_1\xi_2\xi_3 - \epsilon(|\xi_1|^{2\alpha} + |\xi_2|^{2\alpha} + |\xi_3|^{2\alpha})]i\xi_1\xi_2\xi_3}.
 \end{aligned} \tag{5.51}$$

It is easy to see that  $B(\xi_1, \xi_2, \xi_3)$  satisfies

$$|\partial_{\xi_i} B(\xi_1, \xi_2, \xi_3)| \lesssim \frac{|B(\xi_1, \xi_2, \xi_3)|}{|\xi_i|}, \quad i = 1, 2, 3. \tag{5.52}$$

Let

$$\tilde{\sigma}_3(\xi_1, \xi_2, \xi_3) = \frac{M(\xi_1, \xi_2, \xi_3)}{i\xi_1\xi_2\xi_3}, \tag{5.53}$$

then we can rewrite  $J_{32}$  by

$$\begin{aligned}
 J_{32} &= [\sigma_3^-(-\xi_2, \xi_3, \xi_2 + \xi_4) - \sigma_3^-(-\xi_2, -\xi_4, \xi_2 + \xi_4) \\
 &\quad + \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(\xi_1, -\xi_3, \xi_2 + \xi_3)]\xi_2 \\
 &= B(-\xi_2, \xi_4, \xi_2 + \xi_4)[-m^2(-\xi_2)\xi_2 - \xi_4 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)]\xi_2 \\
 &\quad + B(\xi_1, \xi_4, \xi_2 + \xi_3)[m^2(\xi_1)\xi_1 + \xi_4 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)]\xi_2 \\
 &\quad - B(\xi_2, \xi_3, \xi_2 + \xi_4)[-m^2(-\xi_2)\xi_2 + \xi_3 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)]\xi_2 \\
 &\quad - B(\xi_1, -\xi_3, \xi_2 + \xi_3)[m^2(\xi_1)\xi_1 - \xi_3 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)]\xi_2 \\
 &\quad + [\tilde{\sigma}_3(-\xi_2, \xi_3, \xi_2 + \xi_4) - \tilde{\sigma}_3(\xi_1, -\xi_3, \xi_2 + \xi_3) \\
 &\quad - \tilde{\sigma}_3(-\xi_2, -\xi_4, \xi_2 + \xi_4) + \tilde{\sigma}_3(\xi_1, \xi_4, \xi_2 + \xi_3)]\xi_2.
 \end{aligned} \tag{5.54}$$

For the first four terms in (5.54), we can bound them by the same way as for  $J_{31}$ , using (5.52) and the symmetry of  $B$  that  $B(\xi_1, -\xi_2, \xi_3) = B(-\xi_1, \xi_2, \xi_3)$ . For the last term, it follows from (5.53) and  $m^2(\xi_3) = m^2(\xi_4) = 1$  that

$$\begin{aligned}
 J_L &= [\tilde{\sigma}_3(-\xi_2, \xi_3, \xi_2 + \xi_4) - \tilde{\sigma}_3(\xi_1, -\xi_3, \xi_2 + \xi_3) \\
 &\quad - \tilde{\sigma}_3(-\xi_2, -\xi_4, \xi_2 + \xi_4) + \tilde{\sigma}_3(\xi_1, \xi_4, \xi_2 + \xi_3)]\xi_2 \\
 &= \frac{-m^2(\xi_2)\xi_2 + \xi_3 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{-\xi_2\xi_3(\xi_2 + \xi_4)}\xi_2
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{-m^2(\xi_2)\xi_2 - \xi_4 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{\xi_2\xi_4(\xi_2 + \xi_4)}\xi_2 \\
 & + \frac{m^2(\xi_1)\xi_1 + \xi_4 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_1\xi_4(\xi_2 + \xi_3)}\xi_2 \\
 & - \frac{m^2(\xi_1)\xi_1 - \xi_3 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{-\xi_1\xi_3(\xi_2 + \xi_3)}\xi_2.
 \end{aligned} \tag{5.55}$$

Note that there is a cancellation. Therefore,

$$\begin{aligned}
 J_L = & - \frac{\xi_3 + \xi_4}{\xi_3\xi_4} \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{\xi_2(\xi_2 + \xi_4)}\xi_2 \\
 & + \frac{\xi_3 + \xi_4}{\xi_3\xi_4} \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_1(\xi_2 + \xi_3)}\xi_2.
 \end{aligned} \tag{5.56}$$

We rewrite (5.56) by

$$\begin{aligned}
 & - \frac{\xi_3 + \xi_4}{\xi_3\xi_4} \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4) + m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_2(\xi_2 + \xi_4)}\xi_2 \\
 & + \frac{\xi_3 + \xi_4}{\xi_3\xi_4} [m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)] \left[ \frac{1}{\xi_1(\xi_2 + \xi_3)} + \frac{1}{\xi_2(\xi_2 + \xi_4)} \right] \xi_2.
 \end{aligned}$$

Therefore, we use (5.20) for the first term, and (5.19) for the second term, and finally we conclude that

$$\frac{|J_L|}{|h_4 - \epsilon\beta_{\alpha,4}|} \lesssim \frac{|J_L|}{|h_4|} \lesssim \frac{1}{N_1^4}, \tag{5.57}$$

which completes the proof of the proposition.  $\square$

With the estimate of  $\sigma_4$ , we immediately get the estimate of  $M_5$ . We have the same bound as in the KdV case.

**Proposition 5.4.** *If  $m$  is of the form (5.17), then*

$$|M_5(\xi_1, \dots, \xi_5)| \lesssim \left[ \frac{m^2(N_{*45})N_{45}}{(N + N_1)(N + N_2)(N + N_3)(N + N_{45})} \right]_{sym}, \tag{5.58}$$

where

$$N_{*45} = \min(N_1, N_2, N_3, N_{45}, N_{12}, N_{13}, N_{23}).$$

So far we have showed that the multipliers  $M_i$ ,  $i = 3, 4, 5$ , have the same bounds as for the KdV equation. We list now some propositions.

**Proposition 5.5.** *Let  $w_i(x, t)$  be functions of space–time with Fourier support  $|\xi| \sim N_i$ ,  $N_i$  dyadic. Then*

$$\left| \int_0^\delta \int \prod_{i=1}^5 w_i(x, t) dx dt \right| \lesssim \prod_{j=1}^3 \|w_j\|_{F^{1/4}(\delta)} \|w_4\|_{F^{-3/4}(\delta)} \|w_5\|_{F^{-3/4}(\delta)}. \tag{5.59}$$

**Proof.** It follows from the same argument as for the proof of Lemma 5.1 in [2] with Proposition 2.1.  $\square$

**Proposition 5.6.** *If the associated multiplier  $m$  is of the form (5.17) with  $s = -3/4+$ , then*

$$\left| \int_0^\delta \Lambda_5(M_5; u_1, \dots, u_5) dt \right| \lesssim N^{-\beta} \prod_{i=1}^5 \|Iu_i\|_{F^0(\delta)}, \tag{5.60}$$

where  $\beta = 3 + \frac{3}{4}-$ .

**Proof.** This proposition can be proved by following the proof of Lemma 5.2 in [2] and using Proposition 5.5. We omit the details.  $\square$

**Proposition 5.7.** *Let  $I$  be defined with the multiplier  $m$  of the form (5.17) and  $s = -3/4$ . Then*

$$|E_7^4(t) - E_7^2(t)| \lesssim \|Iu(t)\|_{L^2}^3 + \|Iu(t)\|_{L^2}^4. \tag{5.61}$$

**Proof.** Since  $E_7^4(t) = E_7^2(t) + \Lambda_3(\sigma_3) + \Lambda_4(\sigma_4)$  and the bound for  $\sigma_3, \sigma_4$  are the same as in the KdV case, this proposition follows immediately from Lemma 6.1 in [2].  $\square$

We state a variant local well-posedness result which follows from slight argument in the last section. This is used to iterate the solution in the I-method.

**Proposition 5.8.** *If  $s > -3/4$ , then (1.1) is uniformly locally well-posed for data  $\phi$  satisfying  $I\phi \in L^2(\mathbb{R})$ . Moreover, the solution exists on a time interval  $[0, \delta]$  with lifetime*

$$\delta \sim \|I\phi\|_{L^2}^{-\alpha}, \quad \alpha > 0, \tag{5.62}$$

and the solution satisfies the estimate

$$\|Iu\|_{F^s(\delta)} \lesssim \|I\phi\|_{L^2}. \tag{5.63}$$

With these propositions and the scaling (4.1), we can show Theorem 1.2 by using the same argument in [2]. We omit the details.

### 6. Limit behavior

In this section we prove our third result. It is well known that (1.4) is completely integrable and has infinite conservation laws, and as a corollary one obtains that let  $v$  be a smooth solution to (1.4), for any  $k \in \mathbb{Z}_+$ ,

$$\sup_{t \in \mathbb{R}} \|v(t)\|_{H^k} \lesssim \|v_0\|_{H^k}. \tag{6.1}$$

There are less symmetries for (1.1). We can still expect that the  $H^k$  norm of the solution remains bounded for a finite time  $T > 0$ , since the dissipative term behaves well for  $t > 0$ . We already see that for  $k = 0$  from (5.1). Now we prove for  $k = 1$  which will suffice for our purpose. We do not pursue for  $k \geq 2$ .

Assume  $u$  is a smooth solution to (1.1). Let  $H[u] = \int_{\mathbb{R}} (u_x)^2 - \frac{2}{3}u^3 + u^2 dx$ , then by Eq. (1.1) and partial integration

$$\begin{aligned} \frac{d}{dt} H[u] &= \int_{\mathbb{R}} 2u_x \partial_x (u_t) - 2u^2 u_t + 2uu_t dx \\ &= \int_{\mathbb{R}} 2u_x (-u_{xxxx} - \epsilon |\partial_x|^{2\alpha} \partial_x u - (u^2)_{xx}) dx \\ &\quad + \int_{\mathbb{R}} 2u^2 (u_{xxx} + \epsilon |\partial_x|^{2\alpha} u + (u^2)_x) dx + \int_{\mathbb{R}} -2\epsilon (\Lambda^\alpha u)^2 dx \\ &= \int_{\mathbb{R}} -2\epsilon (\Lambda^{1+\alpha} u)^2 + 2\epsilon u^2 \Lambda^{2\alpha} u - 2\epsilon (\Lambda^\alpha u)^2 dx \\ &\leq -\epsilon \int_{\mathbb{R}} (\Lambda^{2\alpha} u)^2 + 2u^2 \Lambda^{2\alpha} u dx, \end{aligned}$$

where we denote  $\Lambda = |\partial_x|$ . Thus we have

$$\frac{d}{dt} H[u] + \frac{\epsilon}{2} \|\Lambda^{2\alpha} u\|_2^2 \lesssim \|u\|_4^4. \tag{6.2}$$

Using Gagliardo–Nirenberg inequality

$$\|u\|_3^3 \lesssim \|u\|_2^{5/2} \|u_x\|_2^{1/2}, \quad \|u\|_4^4 \lesssim \|u\|_2^3 \|u_x\|_2$$

and Cauchy–Schwarz inequality, we get

$$\sup_{[0,T]} \|u(t)\|_{H^1} + \epsilon^{1/2} \left( \int_0^T \|\Lambda^{2\alpha} u(\tau)\|_2^2 d\tau \right)^{1/2} \leq C(T, \|\phi\|_{H^1}), \quad \forall T > 0. \tag{6.3}$$

Assume  $u_\epsilon$  is an  $L^2$ -strong solution to (1.1) obtained in the last section and  $v$  is an  $L^2$ -strong solution to (1.4) in [2], with initial data  $\phi_1, \phi_2 \in L^2$  respectively. We still denote by  $u_\epsilon, v$  the extension of  $u_\epsilon, v$ . From the scaling (4.1), we may assume first that  $\|\phi_1\|_{L^2}, \|\phi_2\|_{L^2} \ll 1$ . Let  $w = u_\epsilon - v, \phi = \phi_1 - \phi_2$ , then  $w$  solves

$$\begin{cases} w_t + w_{xxx} + \epsilon |\partial_x|^{2\alpha} u_\epsilon + (w(v + u_\epsilon))_x = 0, & t \in \mathbb{R}_+, x \in \mathbb{R}, \\ v(0) = \phi. \end{cases} \tag{6.4}$$

We first view  $\epsilon |\partial_x|^{2\alpha} u_\epsilon$  as a perturbation to the difference equation of the KdV equation, and consider the integral equation of (6.4)

$$w(x, t) = W_0(t)\phi - \int_0^t W_0(t - \tau) [\epsilon |\partial_x|^{2\alpha} u_\epsilon + (w(v + u_\epsilon))_x] d\tau, \quad t \geq 0. \tag{6.5}$$

Then  $w$  solves the following integral equation on  $t \in [0, 1]$ ,

$$\begin{aligned}
 w(x, t) = \psi(t) & \left[ W_0(t)\phi - \int_0^t W_0(t - \tau) \chi_{\mathbb{R}_+}(\tau) \psi(\tau) \epsilon |\partial_x|^{2\alpha} u_\epsilon(\tau) d\tau \right. \\
 & \left. - \int_0^t W_0(t - \tau) \partial_x (\psi^2(\tau) w(v + u_\epsilon))(\tau) d\tau \right]. \tag{6.6}
 \end{aligned}$$

By Proposition 4.2 and Propositions 4.3, 4.4, 4.11, we get

$$\|w\|_{F^0} \lesssim \|\phi\|_{L^2} + \epsilon \|u_\epsilon\|_{L^2_{[0,2]} \dot{H}^{2\alpha}_x} + \|w\|_{F^0} (\|v\|_{F^0} + \|u_\epsilon\|_{F^0}). \tag{6.7}$$

Since from Theorem 1.2 we have

$$\|v\|_{F^0} \lesssim \|\phi_2\|_{L^2} \ll 1, \quad \|u_\epsilon\|_{F^0} \lesssim \|\phi_1\|_{L^2} \ll 1,$$

then we get that

$$\|w\|_{F^0} \lesssim \|\phi\|_{L^2} + \epsilon \|u_\epsilon\|_{L^2_{[0,2]} \dot{H}^{2\alpha}_x}. \tag{6.8}$$

From Proposition 4.1 and (6.3) we get

$$\|u_\epsilon - v\|_{C([0,1],L^2)} \lesssim \|\phi_1 - \phi_2\|_{L^2} + \epsilon^{1/2} C(\|\phi_1\|_{H^1}, \|\phi_2\|_{L^2}). \tag{6.9}$$

For general  $\phi_1, \phi_2 \in L^2$ , using the scaling (4.1), then we immediately get that there exists  $T = T(\|\phi_1\|_{L^2}, \|\phi_2\|_{L^2}) > 0$  such that

$$\|u_\epsilon - v\|_{C([0,T],L^2)} \lesssim \|\phi_1 - \phi_2\|_{L^2} + \epsilon^{1/2} C(T, \|\phi_1\|_{H^1}, \|\phi_2\|_{L^2}). \tag{6.10}$$

Therefore, (6.10) automatically holds for any  $T > 0$ , due to (5.1) and (6.3).

**Proof of Theorem 1.3.** For fixed  $T > 0$ , we need to prove that  $\forall \eta > 0$ , there exists  $\sigma > 0$  such that if  $0 < \epsilon < \sigma$  then

$$\|S_T^\epsilon(\varphi) - S_T(\varphi)\|_{C([0,T];H^s)} < \eta. \tag{6.11}$$

We denote  $\varphi_K = P_{\leq K} \varphi$ . Then we get

$$\begin{aligned}
 \|S_T^\epsilon(\varphi) - S_T(\varphi)\|_{C([0,T];H^s)} & \leq \|S_T^\epsilon(\varphi) - S_T^\epsilon(\varphi_K)\|_{C([0,T];H^s)} \\
 & \quad + \|S_T^\epsilon(\varphi_K) - S_T(\varphi_K)\|_{C([0,T];H^s)} \\
 & \quad + \|S_T(\varphi_K) - S_T(\varphi)\|_{C([0,T];H^s)}. \tag{6.12}
 \end{aligned}$$

From Theorem 1.2 and (6.10), we get

$$\|S_T^\epsilon(\varphi) - S_T(\varphi)\|_{C([0,T];H^s)} \lesssim \|\varphi_K - \varphi\|_{H^s} + \epsilon^{1/2} C(T, K, \|\varphi\|_{H^s}). \tag{6.13}$$

We first fix  $K$  large enough, then let  $\epsilon$  go to zero, therefore (6.11) holds.  $\square$

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