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On Relational Homomorphisms of Automata

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This paper investigates the concepts of relational homomorphisms and their closely associated concepts of generalized congruence relations on automata which are in general incomplete, nondeterministic, and infinite. The concept of generalized isomorphism, which is a natural extension of the isomorphism concept in dealing with nondeterministic automata, is also studied.

LIST OF SYMBOLS

X, Y, Z, S, T, A	sets
$\theta, \delta, \lambda, \phi, \psi, \tau, \sigma$	relations
$\circ$	composition
$\cup$	union
$\cap$	intersection
$\subseteq$	inclusion
$\times$	direct product
$\forall$	for all
$\ni$	there exist
$\in$	membership
$\rightarrow$	implication
$\wedge$	logical and

I. INTRODUCTION

The closely related concepts of homomorphism and the substitution property are extremely useful in the structural theory of complete and deterministic automata. However, not much has been done on the structural theory of incomplete and nondeterministic automata. One of the reasons for this is probably the lack of tools to adequately study such structures. And it is our conviction that "relational homomorphisms" investigated in this paper are part of the tools desired for such studies. Although many authors (Ginzburg and Yoeli, 1960; Keisler, 1960;

Lyndon, 1959; and Thatcher, 1965), have investigated the concept of relational homomorphism on algebraic systems, their definitions fail to have the important feature that the relation $\phi \circ \phi^{-1}$ defined by the homomorphism ϕ satisfies the substitution property when extended to partial on relational structures such as incomplete and nondeterministic automata, whereas our definition in this paper does preserve this property. And because of this, we are able to extend many of the results such as isomorphism theorems in algebra to the more general structure of incomplete, nondeterministic automata.

This paper investigates the concept of generalized (or relational) homomorphism as well as the more restricted concept of generalized congruence relation. It can be shown that a generalized functional homomorphism on deterministic automata coincides with the ordinary homomorphism. Also, generalized congruence relation adsorbs the concepts of partition with substitution property and set systems, and is associated with generalized homomorphism similar to the relationship between homomorphism and congruence relation in the complete and deterministic case. Furthermore, each of the concepts mentioned above have two versions depending on whether the automata under consideration are complete. And this difference reveals why certain properties (such as substitution property) which hold in the complete and deterministic automata can be extended to the general case. Finally, properties of relational isomorphism, which is a weaker form of isomorphism, between automata are also studied.

II. PRELIMINARY

In this section we give basic definitions and general background which are necessary for the understanding of the following theory.

Let X, Y, and Z be arbitrary sets, $X' \subseteq X$, $Y' \subseteq Y$, $\theta \subseteq X \times Y$ and $\sigma \subseteq Y \times Z$, define *composition* "$o$" by the following rules:

$$\theta \circ \sigma = \{(x, z) \mid (\exists y \in Y)[(x, y) \in \theta \wedge (y, z) \in \sigma]\};$$

$$X' \circ \theta = \{y \in Y \mid (\exists x \in X')[(x, y) \in \theta]\};$$

$$\theta \circ Y' = \{x \in X \mid (\exists y \in Y')[(x, y) \in \theta]\}.$$

We define $I_X = \{(x, x) \mid x \in X\}$ and $\theta^{-1} = \{(y, x) \mid (x, y) \in \theta\}$.

An *automaton* M is a triple (S, A, δ), where S is an arbitrary nonempty set (of states), A is a finite set (of input symbols), and $\delta \subseteq (S \times A) \times S$ is called the *transition relation* of M. Clearly, M can also be represented by the triple $(S, A, \{\delta_a\}_{a \in A})$ where $\delta_a = \{(s, t) \mid ((s, a), t) \in \delta\}$. We also extend δ_a to δ_x for x in the free monoid A^* generated by A by the rules:

(i) $\delta_e = I_S$, where e is the empty word of A^*;

(ii) If $x = a_1 a_2 \cdots a_n$, then $\delta_x = \delta_{a_1} \circ \delta_{a_2} \circ \cdots \circ \delta_{a_n}$.

We say an automaton M is *deterministic* if δ_a is single-valued for all a in A. Otherwise, M is *nondeterministic.* We say M is *complete* if

$$(\forall a \in A)(\forall s \in S)[s \circ \delta_a \neq \varnothing].$$

Otherwise, M is said to be *incomplete.*

In this paper, we will restrict ourselves to the discussion of only those automata which have the same input alphabet A.

Let $M = (S, A, \delta)$ and $N = (T, A, \lambda)$ be two automata, we define the product automaton, $M \times N$, of M and N to be the automaton $(S \times T, A, \tau)$, where $\forall a \in A$,

$$\tau_a = \{((s, t), (s', t')) \mid (s, s') \in \delta_a \land (t, t') \in \lambda_a\}.$$

We say N is a *subautomaton* of M if and only if $T \subseteq S$ and $(\forall a \in A) \cdot [\lambda_a = \delta_a \cap T^2]$. Clearly, every subset of S determines a unique subautomaton of M. We say N is a *semi-complete subautomaton* of M if furthermore $(\forall a \in A)(\forall t \in T)\ [t \circ \delta_a \neq \varnothing \rightarrow t \circ \lambda_a \neq \varnothing]$. We say N is an *M-complete* subautomaton of M if N is a subautomaton of N and $(\forall a \in A)(\forall t \in T)(\forall s \in S)[(t, s) \in \delta_a \rightarrow (t, s) \in \lambda_a]$. In case N is complete, then we say N is a *complete subautomaton* of M.

In the sequel, we assume that automata M and N will always be represented by the triples (S, A, δ) and (T, A, λ) respectively unless stated otherwise.

III. GENERALIZED CONGRUENCE RELATIONS

DEFINITION 1. A relation[1] θ on S is said to satisfy the (dual) *substitution property* on M if and only if $(\forall a \in A)[\delta_a^{-1} \circ \theta \circ \delta_a \subseteq \theta]((\forall a \in A) \cdot [\delta_a \circ \theta \circ \delta_a^{-1} \subseteq \theta])$.

DEFINITION 2. $\theta \subseteq S^2$ is called a (dual) *generalized congruence* relation[2] (GCR) on M if and only if θ is reflexive, symmetric and satisfies the (dual) substitution property on M. We say θ is a *generalized produc-*

[1] Only binary relations will be considered in this paper.

[2] If θ is an equivalence relation, then a GCR becomes a congruence relation in the ordinary algebraic sense.

tive congruence relation (GPCR) on M if furthermore $(\forall aA)[\theta \circ \delta_a \circ S \subseteq \delta_a \circ S]$.

We note that in case M is a complete automaton, then the two concepts defined above are identical.

The following closure theorem follows directly from properties of sets.

THEOREM 1. *The partly ordered set of* GCR (GPCR) *on* M, *ordered by set inclusion, forms a complete, distributive lattice which is a sublattice of the lattice of all reflexive and symmetric relations on* S.

DEFINITION 3. A family of nonempty distinct subsets $C = \{S_i\}_{i \in I}$, I is an index set, of S is called a *cover with substitution property*[3] (or SP *cover* for short) on M if, and only if,

(i) $\bigcup_{i \in I} S_i = S$, and

(ii) $(\forall a \in A)(\forall S_i \in C)(\exists S_j \in C)[S_i \circ \delta_a \subseteq S_j]$.

We note that every SP cover C on M determines a unique GCR θ_C on M by the rule:

$$\theta_C = \{(s, t) \in S^2 \mid (\exists D \in C)[\{s, t\} \subseteq D]\}.$$

On the other hand, every GCR θ on M defines a SP cover C_θ on M by the rule:

$$C_\theta = \{T \subseteq S \mid (\forall s \in S)[s \in T \rightleftarrows (\forall t \in T)[(s, t) \in \theta]]\}.$$

The correspondence between GCR and SP cover on an automaton M is, nevertheless, not one to one. Since there are in general more than GCR as shown by the SP covers which define the following example.

Example. Let $M = (S, A, \delta)$ be defined as follows:

$S = \{1, 2, 3, 4, 5, 6\}$,

$A = \{a\}$,

$\delta_a = \{(1, 2), (1, 5), (2, 4), (2, 5), (3, 2), (4, 3), (5, 4), (6, 6), (6, 3)\}$.

Let $C_1 = \{\{1, 2, 3\}, \{2, 4, 5\}, \{3, 4, 6\}\}$ and $C_2 = C_1 \cup \{2, 3, 4\}$.
Clearly, $C_1 \neq C_2$. However, $\theta_{C_1} = \theta_{C_2}$.

Notation: If θ is a relation on M, and N a subautomaton of M, then denote by $\theta_N(\theta_T)$ the relation θ restricted to N (the set T).

THEOREM 2. *A* GCR θ *on* N *is a* GPCR *if, and only if,* N *is a subautomaton of a complete automaton* M *so that* θ *can be extended to a* GCR Θ *on* M *such that* $(\forall t \in T)[t \circ \theta = t \circ \Theta]$.

[3] Hartmanis and Stearns (1966) call it a set system and Yoeli (1963) calls it an admissible partition.

Proof. Suppose N is a subautomaton of a complete automaton M, and θ is an extension of θ on M such that $(\forall t \in T)[t \circ \theta = t \circ \Theta]$.

$$s \in \theta \circ \lambda_a \circ T \rightarrow (\exists t \in T)(\exists \mu \in T)[(s, t) \in \theta \wedge (t, u) \in \lambda_a].$$

M is complete implies that there exists $s' \in S$ such that $(s, s') \in \delta_a$. Since Θ is a GCR, so $(s', \mu) \in \Theta$. $\mu \in T$ implies $\mu \circ \theta = \mu \circ \Theta$, and hence s' must be an element of T such that $(s, s') \in \lambda_a$. Thus, $(\forall a \in A) \cdot [\theta \circ \lambda_a \circ T \subseteq \lambda_a \circ T]$ and θ is a GRCR on N.

Conversely, suppose θ is a GPCR on N. Let $C_\theta = \{T_i\}_{i \in I}$ for some index set I. Construct a complete automaton M by the following rules:

(i) $S = T \cup \{\beta\}$, where $\beta \notin T$;

(ii) $(\forall a \in A)[\delta_a = \lambda_a \cup \{\beta, \beta\} \cup \{(s, \beta) \mid s \circ \lambda_a = \emptyset\}$

Since θ is a GPCR, we see that if $s \circ \lambda_a = \emptyset$, then $(\forall T_i, T_j \in C_\theta)$ $[s \in T_i \wedge T_i \cap T_j \neq \emptyset \rightarrow T_j \circ \lambda_a = \emptyset]$. Now, it is quite clear that M is a complete automaton which has N as its subautomaton. If we let $C = \{T_i\}_{i \in I} \cup \{\beta\}$, and $\Theta = \theta_C$, then C is a SP cover on M and Θ is a GCR on M such that $\Theta_N = \theta$. Furthermore, $(\forall t \in T)[t \circ \theta = t \circ \Theta]$, and the theorem is proved.

THEOREM 3. *To each relation θ on S, there corresponds a unique maximal* GCR (GPCR) $m(\theta)$ *and a unique minimal* GCR (GPCR) $M(\theta)$ *on M satisfying $M(\theta) \subseteq \theta \subseteq m(\theta)$. Furthermore, if F is a family of relations σ on S, then*

(i) $M(\bigcup_{\sigma \in F} \sigma) = \bigcup_{\sigma \in F} M(\sigma)$;

(ii) $m(\bigcup_{\sigma \in F} \sigma) = \bigcup_{\sigma \in F} m(\sigma)$;

(iii) $M(\bigcap_{\sigma \in F} \sigma) = \bigcap_{\sigma \in F} M(\sigma)$;

(iv) $m(\bigcap_{\sigma \in F} \sigma) = \bigcap_{\sigma \in F} m(\sigma)$.

Proof. Define $M(\theta) = \sup C$, where C is the family of all GCR (GPCR) ρ on M such that $\rho \subseteq \theta$. Similarly, define $m(\theta) = \inf C'$, where C' is the family of all GCR (GPCR) δ on M such that $\theta \subseteq \delta$. Clearly, $M(\theta) \subseteq \theta \subseteq m(\theta)$, and properties (i) through (iv) follow directly from Theorem 1.

It is also clear that the above theorem holds for dual generalized congruence relations as well. For a given relation θ on M, we will denote by $M_d(\theta)$ and $m_d(\theta)$ the unique maximal and minimal dual GCR on M such that $M_d(\theta) \subseteq \theta \subseteq m_d(\theta)$.

DEFINITION 4. Let θ be a relation on a set X. Define the *trace*, $T(\theta)$, of θ by the rule:

$$T(\theta) = \begin{cases} 1, & \text{if there exist an } x \text{ in } X \text{ such that } (x, x) \in \theta. \\ 0, & \text{if for all } x \text{ in } X,\ (x, x) \notin \theta. \end{cases}$$

LEMMA 1. *Let θ and ρ be relations on a set X then*

(i) $T(\theta) = T(\theta^{-1})$;

(ii) $T(\theta \circ \rho) = T(\rho \circ \theta)$;

(iii) $\theta \subseteq \rho \rightleftarrows T(\theta \circ \bar{\rho}^{-1}) = 0$, where $\bar{\rho} = X^2 - \rho$.

Proof. We will only prove (iii) since (i) and (ii) are quite trivial. Suppose $\theta \subseteq \rho$. If there exists $x \in X$ such that $(x, x) \in \theta \circ \bar{\rho}^{-1}$, then there exists $y \in X$ such that $(x, y) \in \theta$ and $(y, x) \in \bar{\rho}^{-1}$. Since $(y, x) \in \bar{\rho}^{-1} \rightleftarrows (x, y) \notin \rho$ and since $\theta \subseteq \rho$, we arrive at a contradiction. Thus, $(\forall x \in X)[(x, x) \notin \theta \circ \bar{\rho}^{-1}]$ which implies that $T(\theta \circ \bar{\rho}^{-1}) = 0$.

Conversely, if $T(\theta \circ \bar{\rho}^{-1}) = 0$ and we assume that $\theta \nsubseteq \rho$. Then there exist x and y in X such that $(x, y) \in \theta$ but $(x, y) \notin \rho$. I.e., $(x, y) \in \bar{\rho}$. But then $(x, x) \in \theta \circ \bar{\rho}^{-1}$ which implies that $T(\theta \circ \bar{\rho}^{-1}) = 1$, a contradiction. Thus, we must have $\theta \subseteq \rho$

LEMMA 2. *If θ is a relation on S, then θ satisfies the substitution property on M if, and only if, $\bar{\theta} = S^2 - \theta$ satisfies the dual substitution property on M.*

$$\begin{aligned} \textit{Proof.}\quad \forall a \in A,\ \delta_a^{-1} \circ \theta \circ \delta_a \subseteq \theta &\rightleftarrows T(\delta_a^{-1} \circ \theta \circ \delta_a \circ \bar{\theta}^{-1}) = 0 \\ &\rightleftarrows T(\delta_a \circ \bar{\theta}^{-1} \circ \delta_a^{-1} \circ \theta) = 0 \\ &\rightleftarrows \delta_a \circ \bar{\theta}^{-1} \circ \delta_a^{-1} \subseteq \bar{\theta}^{-1} \\ &\rightleftarrows \delta_a \circ \bar{\theta} \circ \delta_a^{-1} \subseteq \bar{\theta}. \end{aligned}$$

Notations. If θ and ρ are two relations on a set X, then we denote by ρ_θ the relation $\theta^{-1} \circ \rho \circ \theta$. For each automaton M, let $S(M)$ be the semi-group generated by $\{\delta_a\}_{a \in A} \cup I_S$ under composition. For each $\alpha \in S(M)$, let $l(\alpha)$ be the number of elements of $\{\delta_a\}_{a \in A}$ contained in the minimum representation of α, and $l(I_S) = 0$.

DEFINITION 5. If θ is any relation on M, we define the SP *closure* θ^* of θ on M by the rule:

$$\theta^* = \bigcup_{\alpha \in S(M)} \theta_\alpha .$$

LEMMA 3. *Let θ, ρ be relations on M, then*

(i) *θ^* is the smallest relation satisfying the substitution property which contains θ;*

(ii) $\theta \subseteq \rho \rightarrow \theta^* \subseteq \rho^*$;

(iii) $\theta^{**} = \theta^*$.

Proof. We will only prove condition (i) here since the proof of (ii) and (iii) are quite trivial. It is obvious that $\theta \subseteq \theta^*$. For all a in A, $\delta_a^{-1} \circ \theta^* \circ \delta_a = \bigcap_{\alpha \in S(M)} \delta_a^{-1} \circ \theta_\alpha \circ \delta_a = \bigcup_{\alpha \in S(M)} \delta_a^{-1} \circ \alpha^{-1} \circ \theta \circ \alpha \circ \delta_a \subseteq \theta^*$.

If γ is another relation satisfying the substitution property and containing θ, then $\theta_\alpha \subseteq \gamma$ for $l(\alpha) = 0$. Assume that $\theta_\alpha \subseteq \gamma$, for all $\alpha \in S(M)$ such that $l(\alpha) \leqq n$, and let $\beta \in S(M)$ such that $l(\beta) = n + 1$. Then $\beta = \alpha \circ \delta_a$ for some $a \in A$ and some $\alpha \in S(M)$ such that $l(\alpha) \leqq n$, and

$$\theta_\beta = \delta_a^{-1} \circ \alpha^{-1} \circ \theta \circ \alpha \circ \delta_a \subseteq \delta_a^{-1} \circ \gamma \circ \delta_a \subseteq \gamma.$$

Thus, $\alpha \in S(M)$, $\theta_\alpha \subseteq \gamma$ and hence $\theta^* \subseteq \gamma$. This shows that θ^* is the minimal relation satisfying the substitution property and containing θ.

The following theorem gives rules for computing the maximal and minimal GCR a given relation on M.

THEOREM 4. *If θ is a reflexive and symmetric relation on M, then*

(i) $m(\theta) = \theta^*$;

(ii) $M(\theta) = M_d{}^\wedge(\hat{\theta})$ *where* $\hat{\theta} = \bar{\theta} \cup \bar{\theta}^{-1} \cup I_S$.

Proof. (i) If ρ is any GCR on M such that $\theta \subseteq \rho$. Then by lemma 3 above, $\theta^* \subseteq \rho$. In particular, $\theta^* \subseteq m(\theta)$. However, θ^* is a GCR containing θ, hence, $m(\theta) \subseteq \theta^*$ and therefore, $m(\theta) = \theta^*$.

(ii) We first note that $m_d(\theta)$ can also be obtained in a similar fashion as in (i). $m_d(\hat{\theta})$ satisfies the dual substitution property implies that $m_d(\theta)$ satisfies the substitution property by lemma 2. Since $m_d{}^\wedge(\hat{\theta})$ is reflexive and symmetric by definition, $m_d{}^\wedge(\hat{\theta})$ is a GCR. $\hat{\theta} \subseteq m_d(\hat{\theta}) \rightarrow m_d{}^\wedge(\hat{\theta}) \subseteq \hat{\theta} = \theta$. Thus, $m_d{}^\wedge(\hat{\theta}) \subseteq M(\theta)$. Since $m_d(\theta) \subseteq \rho$ for all dual GCR ρ on M such that $\hat{\theta} \subseteq \rho$, and since $M(\theta) \subseteq \theta$ and $M(\theta) = M^\wedge(\theta)$, so $M(\theta) \subseteq m_d{}^\wedge(\hat{\theta})$. This implies that $m_d{}^\wedge(\hat{\theta}) = M(\theta)$, and the theorem is proved.

IV. GENERALIZED HOMOMORPHISM

In this section, the concept of generalized (rational) homomorphism between automata and their relationship with generalized congruence relation will be discussed.

DEFINITION 6. $\phi \subseteq S \times T$ is called a *generalized homorphism*[4] (GH)

[4] In the conventional automata theory, a function $\phi: S \rightarrow T$ is a *homomorphism* from M to N if and only if $(\forall a \in A)[\delta_a \circ \phi = \phi \circ \lambda_a]$. It is not hard to show that in case M and N are complete and deterministic, and ϕ is a function from S to T, then ϕ is a *GH* from M to N if and only if ϕ is a homomorphism from M to N.

Ginzburg and Yoeli (1965) defined $\phi \subseteq S \times T$ to be a *weak homomorphism* from M to N if and only if (*i*) $I_S \subseteq \phi \circ \phi^{-1}$ and (ii) $(\forall x \in A^*)[\phi^{-1} \circ \delta_x \subseteq \lambda_x \circ \phi^{-1}]$. It is easy to see that weak homomorphism implies *GH* and that the two definitions are equivalent in case N is also complete.

from M to N if and only if

$$(\forall a \in A)[\delta^{-1} \circ \phi \circ \lambda_a \subseteq \phi].$$

It is clear that $\phi \subseteq S^2$ is a GH from M to itself if and only if satisfies the substitution property on M. Thus, following Theorem 1, the family of all GH from M to N forms a complete, distributive lattice under set inclusion.

THEOREM 5. *If $\phi \subseteq S \times T$, then the following statements are equivalent.*

(i) *ϕ is a* GH *from M to N.*

(ii) *ϕ^{-1} is a* GH *from N to M.*

(iii) $(\forall a \in A)[\lambda_a \subseteq \overline{\phi^{-1} \circ \delta_a \circ \bar{\phi}}]$.

(iv) *ϕ determines an $M \times N$-complete subautomaton of $M \times N$.*

Proof. (ii) $\rightleftarrows$ (i) $\rightleftarrows$ (iii). $\forall a \in A$,

$$\begin{aligned}\lambda_a^{-1} \circ \phi^{-1} \circ \delta_a \subseteq \phi^{-1} &\rightleftarrows \delta_a^{-1} \circ \phi \circ \lambda_a \subseteq \phi \\ &\rightleftarrows T(\delta_a^{-1} \circ \phi \circ \lambda_a \circ \bar{\phi}^{-1}) = 0 \\ &\rightleftarrows T(\bar{\phi}^{-1} \circ \delta_a^{-1} \circ \phi \circ \lambda_a) = 0 \\ &\rightleftarrows \bar{\phi}^{-1} \circ \delta_a^{-1} \circ \phi \subseteq \bar{\lambda}_a^{-1} \\ &\rightleftarrows \phi^{-1} \circ \delta_a \circ \bar{\phi} \subseteq \bar{\lambda}_a \\ &\rightleftarrows \lambda_a \subseteq \overline{\phi^{-1} \circ \delta_a \circ \bar{\phi}}.\end{aligned}$$

(i) $\rightleftarrows$ (iv). Suppose that ϕ is a GH from M to N. Let (ϕ, A, τ_a') be the subautomaton of $M \times N = (S \times T, A, \tau_a)$ determined by ϕ. If $(s, t) \in \phi$, and if there exists $(s', t') \in \phi \subseteq S \times T$ such that $((s, t), (s', t')) \in \tau_a$, then since $\delta_a^{-1} \circ \phi \circ \lambda_a \subseteq \phi$, and $(s, s') \in \delta_a$, $(t, t') \in \tau_a$, we must have $(s', t') \in \phi$; i.e., $((s, t), (s', t')) \in \phi^2$. Since $\tau_a' = \phi^2 \cap \tau_a$, $((s, t), (s', t')) \in \tau_a'$ and (ϕ, A, τ_a') is an $M \times N$-complete subautomaton of $M \times N$.

Hedetniemi's definition of a *full homomorphism* (1966), Keisler's definition of a *strong homomorphism* (1960), and Thatcher's definition of *homomorphism* (1965) coincide in the structure of automata. Namely, $\phi \subseteq S \times T$ is a homomorphism (in their sense) from M to N if and only if

$$(\forall a \in A)[\phi^{-1} \circ \delta_a \circ \phi = \lambda_a].$$

Clearly, if $I_S \subseteq \phi \circ \phi^{-1}$, then this definition implies Ginzburg and Yoeli's definition of weak homomorphism. On the other hand, Lyndon's definition of homomorphism (1959) embodies only half of the above definition, namely, $(\forall a \in A)[\phi^{-1} \circ \delta_a \circ \phi \subseteq \lambda_a]$.

Conversely, if ϕ determines an $M \times N$-complete subautomaton, then $\forall a \in A$,

$$\begin{aligned}(s, t) \in \delta_a^{-1} \circ \phi \circ \lambda_a &\rightarrow (\exists (s', t') \in \phi)[(s', s) \in \delta_a \wedge (t', t) \in \lambda_a] \\ &\rightarrow ((s', t'), (s, t)) \in \tau_a \\ &\rightarrow ((s', t'), (s, t)) \in \phi^2, \text{ since } (s', t') \in \phi\end{aligned}$$

and ϕ determines an $M \times N$-complete subautomaton

$$\rightarrow \delta_a^{-1} \circ \phi \circ \lambda_a \subseteq \phi.$$

THEOREM 6. *If $\phi \subseteq S \times T$, then there is a maximal* GH ϕ_M *and a minimal* GH ϕ_m *from M to N such that $\phi_M \subseteq \phi \subseteq \phi_m$.*

Proof. Define ϕ_m by the following rules:

(i) $\phi(1) = \phi$

(ii) $\forall k \geqq 1$, define

$$\phi(k + 1) = \phi(k) \cup \{(s, t) \mid (\exists a \in A)(\exists (s', t') \in \phi(k)) \cdot [(s', s) \in \delta_a \wedge (t', t) \in \lambda_a].$$

(iii) $\phi_m = \bigcup_{k>1} \phi(k)$.

We now define ϕ_M by the following rules:

(i) $\phi[1] = \phi$

(ii) $\forall k \geqq 1$, define

$$\phi[k + 1] = \phi[k] \cap \{(s, t) \mid (\forall a \in A)(\forall (s', t') \in S \times T) \cdot [(s, s') \in \delta_a \wedge (t, t') \in \lambda_a \rightarrow (s', t') \in \phi[k]]\}.$$

(iii) $\phi_M = \bigcap_{k>1} \phi[k]$.

It is easy to show that ϕ_m and ϕ_M do satisfy the conditions of the theorem.

DEFINITION 7. Let C be a family of nonempty subsets of S. The *quotient automaton, M/C, of M modulo C* is defined to be the automaton (C, A, δ_a^C) such that

$$(\forall a \in A)(\forall C_i, C_j \in C)[(C_i, C_j) \in \delta_a^C \rightleftarrows \emptyset \neq C_i \circ \delta_a \subseteq C_j].$$

If θ is a GCR, then we define the *quotient automaton, M/θ, of M modulo θ* to be $M/C_\theta = (C_\theta, A, {}^\theta_a)$, where $\delta_a^\theta = \delta_a^{C_\theta}$.

THEOREM 7. *Every GCR θ on M determines a unique* GH $\phi(\theta)$ *from M to M/θ. Conversely, if ϕ is a* GH *from M to a complete automaton N such that $I_S \subseteq \phi \circ \phi^{-1}$, then ϕ determines a unique* GCR $\theta(\phi)$ *on M.*

Proof. Let θ be a GCR on M. Let $C_\theta = \{C_i\}_{i \in I}$ where I is an index set. Define $\phi(\theta)$ from M to M/θ by the rule:

$$(\forall s \in S)(\forall C_i \in C_\theta)[(s, C_i) \in \phi(\theta) \rightleftarrows s \in C_i].$$

$$(s, C_i) \in \delta_a^{-1} \circ \phi(\theta) \circ \delta^\theta$$

$$\rightleftarrows (\exists s' \in S)(\exists C_j \in C)[(s', s) \in \delta_a \wedge (s', C_j) \in \phi(\theta) \wedge (C_j, C_i) \in \delta_a^\theta]$$

$$\rightarrow s' \in C_j \wedge C_j \circ \delta_a \subseteq C_i \rightarrow s \in C_i, \text{ since } s \in s' \circ \delta_a$$

$$\rightarrow \delta_a^{-1} \circ \phi(\theta) \circ \delta_a^\theta \subseteq \phi(\theta).$$

Thus, $\phi(\theta)$ is a GH from M to M/θ.

Conversely, let ϕ be a GH from M to a complete automaton N, and $I_S \subseteq \phi \circ \phi$. We define $\theta(\phi) = \phi \circ \phi^{-1}$. It is quite clear that $\theta(\phi)$ is reflexive and symmetric. Furthermore,

$$\forall a \in A,\ \delta_a^{-1} \circ \theta(\phi) \circ \delta_a = \delta_a^{-1} \circ \phi \circ \phi^{-1} \circ \lambda_a$$

$$\subseteq \delta_a^{-1} \circ \phi \circ \lambda_a \circ \lambda_a^{-1} \circ \phi^{-1} \circ \delta_a \subseteq \phi \circ \phi^{-1} = \theta(\phi).$$

Hence, $\theta(\phi)$ is indeed a GCR on M, and the theorem is proved.

V. GENERALIZED PRODUCTIVE HOMOMORPHISM

In the theory of complete and deterministic automata, we say two automata are structurally equivalent if, and only if, they are isomorphic; i.e., one automaton can be obtained from the other by renaming the states. In this section, a stronger version of the GH is given which we will utilize to compare the structures of incomplete, nondeterministic automata.

Definition 8. $\phi \subseteq S \times T$ is called a *generalized productive* homomorphism (GPH) from M to N if and only if $\forall a \in A$,

(i) $\delta_a^{-1} \circ \phi \circ \lambda_a \subseteq \phi$;
(ii) $\phi^{-1} \circ \delta_a \circ S \subseteq \lambda_a \circ T$;
(iii) $\phi \circ \lambda_a \circ T \subseteq \delta_a \circ S$.

Clearly, in case M and N are complete, the two definitions of homomorphisms coincide. Conditions (ii) and (iii) in the above definition guarantee that M and N have the same structure, even if they are incomplete, in the sense that a state produces a next state if, and only if, its corresponding states also produce next states under the same input.

It is also clear that the three conditions in definition 8 are equivalent to (i), (ii′) $\phi \circ \lambda_a \subseteq \delta_a \circ \phi$, and (iii′) $\phi^{-1} \circ \delta_a \subseteq \lambda_a \circ \phi^{-1}$.

In the following, we shall give an example to demonstrate the difference between GH and GPH.

Let $M = (S, A, \delta)$ and $N = (T, A, \lambda)$ such that $S = \{a, b, c, d\}$. $T = \{1, 2\}$, $A = \{a\}$, and δ_a and λ_a are defined by the following matrices.

δ_a	a	b	c	d
a	0	0	1	0
b	0	0	1	1
c	1	0	0	0
d	0	0	0	0

λ_a	1	2
1	0	1
2	1	0

If we define the two relations ϕ and ψ on $S \times T$ by the matrices

ϕ	1	2
a	1	0
b	0	0
c	0	1
d	0	0

and

ψ	1	2
a	1	0
b	1	0
c	0	1
d	0	1

Then ϕ is a GPH and ψ is a GH but not a GPH as shown below.

$$\delta_a^{-1} \circ \psi \circ \lambda_a = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \subseteq \psi \rightarrow \psi \text{ is a GH.}$$

However,

$$\psi \circ \lambda_a = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \nsubseteq \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \delta_a \circ \psi \rightarrow \psi \text{ is not a GPH.}$$

On the other hand,

$$\delta_a^{-1} \circ \phi \circ \lambda_a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \phi \to \phi \text{ is a GH.}$$

$$\phi \circ \lambda_a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \delta_a \circ \sigma, \text{ and}$$

$$\phi^{-1} \circ \delta_a = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \lambda_a \circ \phi$$

Thus ϕ is a GPH.

THEOREM 8. *A relation θ on S is a* GPCR *on M if and only if there exists an automaton N and a* GPH *ϕ from M to N such that $I_S \subseteq \phi \circ \phi^{-1} = \theta$.*

Proof. Suppose that ϕ is a GPH from M to N such that $I_S \subseteq \phi \circ \phi^{-1}$. Let $\theta = \phi \circ \phi^{-1}$. Clearly, θ is reflexive and symmetric. Furthermore, $a \in A$, we have

(i) $\theta \circ \delta_a \circ S = \phi \circ \phi^{-1} \circ \delta_s \circ S \subseteq \phi \circ \lambda_a \circ T \subseteq \delta_a \circ S$;

(ii) $(s, s') \in \delta_a^{-1} \circ \theta \circ \delta_a \rightleftarrows (s, s') \in \delta_a^{-1} \circ \phi \circ \phi^{-1} \circ \delta_a$

$\to (\exists u, v \in S)(\exists t \in T)[(u, s) \in \delta_a \wedge (u, t) \in \phi \wedge (t, v) \in \phi^{-1} \wedge (v, s') \in \delta_a] \to (\exists w \in T)[(t, w) \in \lambda_a]$, since $\phi^{-1} \circ \delta_a \circ S \subseteq \lambda_a \circ T \to (s, s') \in \delta_a^{-1} \circ \phi \circ \lambda_a \circ \lambda_a^{-1} \circ \phi^{-1} \circ \delta_a \subseteq \phi \circ \phi^{-1} = \theta$

Thus, θ is a GPCR on M.

Conversely, assume that θ is a GPCR on M. Let $C_\theta = \{C_i\}_{i \in I}$, where I is an index set. Let $N = M/\theta = (C_\theta, A, \delta_a^\theta)$, and define $\phi \subseteq S \times C_\theta$ by the rule:

$$(\forall s \in S)(\forall C_i \in C)[(s, C_i) \in \phi \rightleftarrows s \in C_i].$$

By the proof of theorem 6 above, it is clear that

$$(\forall a \in A)[\delta_a^{-1} \circ \phi \circ \delta_a^\theta \subseteq \phi].$$

Furthermore, $\forall a \in A$, we have

(i) $C_i \in \phi^{-1} \circ \delta_a \circ S \rightleftarrows (\exists s \in S)[(s, C_i) \in \phi \wedge s \circ \delta_a \neq \varnothing] \to (\exists C_j$

$\in C)\cdot[\varnothing \neq C_i \circ \delta_a \subseteq C_j]$, since C_θ is a cover with substitution property on $M \rightarrow \phi^{-1} \circ \delta_a \circ S \subseteq \delta_a^{\theta} \circ C_\theta$;

(ii) $s \in \phi \circ \delta_a^{\theta} \circ C_\theta \rightleftarrows (\exists C_i \in C_\theta)[(s, C_i) \in \phi \wedge C_i \circ \delta_a^{\theta} \neq \varnothing] \rightarrow (\exists t \in C_i)[t \circ \delta_a \neq \varnothing] \rightarrow s \circ \delta_a \neq \varnothing$, since $(s, t) \in \theta$ and $\theta \circ \delta_a \circ S \subseteq \delta_a \circ S \rightarrow \phi \circ \delta_a^{\theta} \circ C_\theta \subseteq \delta_a \circ S$.

Since $(\forall s \in S)(\exists C_i \in C_\theta)[(s, C_i) \in \phi]$, therefore ϕ is a GPH from M to N such that $I_S \subseteq \phi \circ \phi^{-1} = \theta$.

DEFINITION 9. $\phi \subseteq S \times T$ is called a *generalized isomorphism*[5] between M and N if, and only if, ϕ is a GPH from M to N such that $I_S \subseteq \phi \circ \phi^{-1}$ and $I_T \subseteq \phi^{-1} \circ \phi$.

Notation: We will denote by "$M \sim N$" the fact that there exists a generalized isomorphism between M and N.

We note that "$\sim$" defines an equivalence relation on the family of all automata over the same input alphabet A. Since clearly

(i) $M \overset{I_S}{\sim} M$;

(ii) $M \overset{\phi}{\sim} N \rightleftarrows N \overset{\phi^{-1}}{\sim} M$.

(iii) $M \overset{\phi_1}{\sim} N$ and $N \overset{\phi_2}{\sim} P \rightarrow M \overset{\phi_1 \circ \phi_2}{\sim} P$.

Notation: If N is a subautomaton of M, and θ a relation on M, then denote by $N(\theta)$ the subautomaton of M determined by $T \circ \theta$; i.e., $N(\theta) = (T \circ \theta, A, \sigma_a)$, where $\sigma_a = \delta_a = \delta_a \cap (T \circ \theta)^2$.

THEOREM 9. *If N is a complete subautomaton of M, θ a* GPCR *on M, ρ a* GPCR *on N, then ρ_θ is a* GPCR *on $N(\theta)$ and $N(\theta)/\rho_\theta \sim N/\rho$.*

Proof. We first note that $N(\theta) = (T \circ \theta, A, \sigma_a)$ is also a complete subautomaton of M. Since $s \in T \circ \theta \rightarrow (\exists t \in T)[(t, s) \in \theta]$. N being complete then implies that $(\forall a \in A)[t \circ \lambda_a \neq \varnothing]$. Since $\theta \circ \delta_a \circ S \subseteq \delta_a \circ S$, and $(t, s) \in \theta$, therefore $s \circ \delta_a \neq \varnothing$. Furthermore, $(\forall s' \in s \circ \delta_a) \cdot (\forall t' \in t \circ \lambda_a)\ [(t', s') \in \theta \wedge t' \in T]$, and thus $s' \in T \circ \theta$ which implies that $(\forall a \in A)(\forall s \in T \circ \theta)[s \circ \sigma_a \neq \phi]$.

We will now show that $\rho_\theta = \theta^{-1} \circ \rho \circ \theta = \theta \circ \rho \circ \theta$ is a GPCR on $N(\theta)$. Clearly, ρ_θ is reflexive and symmetric. $\forall a \in A$, we have

(i) $\sigma_a^{-1} \circ \rho \circ \sigma_a \subseteq \sigma_a^{-1} \circ \theta \circ \lambda_a \circ \lambda_a^{-1} \circ \rho \circ \lambda_a \circ \lambda_a^{-1} \circ \theta \circ \sigma_a \subseteq \sigma_\theta$;

[5] If M and N are complete and deterministic, then $\phi: S \rightarrow T$ is an *isomorphism* between M and N if and only if ϕ is a homorphism from M to N such that ϕ is also one to one. It is obvious that ϕ is an isomorphism between M and N implies that ϕ is a generalized isomorphism between M and N.

(ii) $s \in \theta \circ \rho \circ \theta \circ \sigma_a \circ (T \circ \theta) \rightleftarrows (\exists t, t' \in T)(\exists s', s'' \in S)$ $[(s, t) \in \theta \wedge (t, t') \in \rho \wedge (t', s') \in \theta \wedge (s', s'') \in \sigma_a \wedge$ $s'' \in T \circ \theta] \rightarrow \theta \circ \rho \circ (\theta \circ \sigma_a \circ (T \circ \theta)) \subseteq \theta \circ \rho \circ \sigma_a \circ T \subseteq$ $\theta \circ \sigma_a \circ T$, since N is complete and $t' \in T \rightarrow \theta \circ \sigma_a \circ T \subseteq$ $\sigma_a \circ T \circ \theta$, since $s \in T \circ \theta$ and $N(\theta)$ is complete.

Thus, ρ_θ is a GPCR on $N(\theta)$.

Let $C_{\rho_\theta} = \{X_i\}_{i \in I}$ and $C = \{Y_j\}_{j \in J}$, where I and J are index sets, then define $\phi \subseteq C_{\rho_\theta} \times C_\rho$ by the rule:

$$(X_i, Y_j) \in \phi \rightleftarrows (X_i \cap Y_j) \neq \varnothing.$$

Since $T \subseteq T \circ \theta$, so for each $X_i \in C_{\rho_\theta}$, there exists an $Y_j \in C_\rho$ such that $X_i \cap Y_j \neq \varnothing$ and vice versa. Therefore, $I_{C_{\rho_\theta}} \subseteq \phi \circ \phi^{-1}$ and $I_{C_\rho} \subseteq \phi^{-1} \circ \phi$. Let $N(\theta)/\rho_\theta = (C_{\rho_\theta}, A, \hat{\sigma}_a)$ and $N/\rho = (C, A, \hat{\lambda}_a)$. Then

$(\forall X_i \in C_{\rho_\theta})(\forall Y_j \in C_\rho)[(X_i, Y_j) \in \hat{\sigma}_a^{-1} \circ \phi \circ \hat{\lambda}_a] \rightleftarrows (\exists X_k \in C_{\rho_\theta})$ $(\exists Y_l \in C_\rho)\ [(X_k, X_i) \in \hat{\sigma}_a \wedge (X_k, Y_l) \in \phi \wedge (Y_l, Y_j) \in \hat{\lambda}_a] \rightarrow$ $X_k \cap Y_l \neq \varnothing \rightarrow X_i \cap Y_j \neq \varnothing$, since N is complete, $\rightarrow \hat{\sigma}_a^{-1} \circ \phi \circ \hat{\lambda}_a \subseteq$ ϕ. Also,

$(\forall Y_i \in C_\rho)[Y_i \in \phi^{-1} \circ \hat{\sigma}_a \circ C_{\rho_\theta}] \rightleftarrows (\exists X_j, X_k \in C_{\rho_\theta})[(X_j, Y_i) \in \phi \wedge$ $(X_j, X_k) \in \sigma_a] \rightarrow (\exists t \in T)\ [t \in X_j \cap T_i] \rightarrow X_k \cap T \neq \varnothing$, since N is complete $\rightarrow (\exists Y_l \in C_\rho)[X_k \cap Y_l \neq \varnothing]$, since C_ρ is a cover with substitution property on $T \rightarrow (\forall a \in A)[\phi^{-1} \circ \hat{\sigma}_a \circ C_{\rho_\theta} \subseteq \hat{\lambda}_a \circ C_\rho]$.

Similarly, we can prove that $(\forall a \in A)[\phi \circ \hat{\lambda}_a \circ C_\rho \subseteq \hat{\sigma}_a \circ C_{\rho_\theta}]$. Thus, ϕ is indeed a generalized isomorphism and hence $N(\theta)/\rho_\theta \sim N/\rho$.

THEOREM 10. *If ϕ is a* GPH *from M to N such that $I_S \subseteq \phi \circ \phi^{-1}$, then $M/\theta(\phi) \sim M\phi$. Where $M\phi = (S \circ \phi, A, \hat{\lambda}_a)$ is the subautomaton of N determined by $S \circ \phi$, and $\theta(\phi) = \phi \circ \phi^{-1}$.*

Proof. $\theta(\phi) = \phi \circ \phi^{-1}$ is a GPCR on M by Theorem 8. By the same theorem, there exists a GPH ψ such that $M \overset{\psi}{\sim} M/\theta(\phi)$. Since clearly, $M \overset{\phi}{\sim} M\phi$, so $M/\theta(\phi) \overset{\psi^{-1}\circ\phi}{\sim} M\phi$.

THEOREM 11. *If N is a semi-complete subautomaton of M, and θ a* GPCR *on M, then $N/\theta_N \sim N(\theta)/\theta$.*

Proof. θ determines a GPH ϕ from M to M/θ by Theorem 8. Let $\psi = \phi \mid N$, the restriction of ϕ to N. Since N is semi-complete subautomaton of M, we see that ψ is a GPH from N to $N(\theta)/\theta$ such that $(\forall t \in T)$ $[t \circ \psi = \{C_i \in C_\theta \mid t \in C_i\}]$. Furthermore, $I_T \subseteq \psi \circ \psi^{-1}$ and $I_{T\circ\theta/\theta} \subseteq \psi^{-1} \circ \psi$ and $\psi \circ \psi^{-1} = \theta \cap T^2 = \theta_N$, where $T \circ \theta/\theta$ is the state set of $N(\theta)/\theta$. Thus, by Theorem 10, we must have $N/\theta_N \overset{\psi}{\sim} N(\theta)/\theta$.

THEOREM 12. *If M is a complete automaton, θ and σ are GCR on M such that $\sigma \subseteq \theta$ then $(M/\sigma)/\theta_{\phi(\sigma)} \sim M/\theta$. Where $\phi(\sigma)$ is the* GH *from M to M/σ determined by σ.*

Proof. Let $C\sigma = \{X_i\}_{i \in I}$ and $C\theta = \{Y_j\}_{j \in J}$, where I and J are index sets. Sefine $\phi \subseteq C_\sigma \times C_\theta$ by the rule:

$$(\forall X_i \in C_\sigma)(\forall Y_j \in C_\theta)[(X_i, Y_j) \in \phi \rightleftarrows X_i \cap Y_j \neq \varnothing].$$

Let $M/\sigma = (C_\sigma, A, \delta_a')$ and $M/\delta = (C_\theta, A, \delta_a'')$. Then

$(X_i, Y_j) \in \delta_a'^{-1} \circ \phi \circ \delta_a''$

$\rightleftarrows (\exists X_i' \in C_\sigma)(\exists Y_j' \in C_\theta)[(X_i', X_i) \in \delta_a' \wedge (X_i', Y_j') \in \phi$

$\wedge (Y_j', Y_j) \in \delta_a'']$

$\rightarrow X_i' \cap Y_j' \neq \varnothing$

$\rightarrow (\forall X_i'' \in X_i' \circ \delta_a')(\forall Y_j'' \in Y_j' \circ \delta_a'')[X_i'' \cap Y_j'' \neq \varnothing]$

since M being complete implies that both M/σ and M/θ are complete

$\rightarrow X_i \cap Y_j \neq \varnothing$

$\rightarrow (\forall a \in A)[\delta_a'^{-1} \circ \phi \circ \delta_a'' \subseteq \phi]$.

Now, G/σ and G/θ are complete implies that $\forall a \in A$, $\phi^{-1} \circ \delta_a' \circ C_\sigma \subseteq \delta_a'' \circ C_\theta$ and $\phi \circ \delta_a'' \circ C_\theta \subseteq \delta_a' \circ C_\sigma$. Thus, ϕ is a GPH from G/σ to G/δ. By Theorem 10, we have $(M/\sigma)/\phi \circ \phi \overset{-1}{\sim} M/\theta$. We must now show that $\phi \circ \phi^{-1} = \theta_{\phi(\sigma)} = \phi(\sigma)^{-1} \circ \theta \circ \phi(\sigma)$.

$(\forall X_i, X_j \in C_\sigma)[(X_i, X_j) \in \phi \circ \phi^{-1}]$

$$\rightleftarrows (\exists Y_k \in C_\theta)[X_i \cap Y_k \neq \varnothing \wedge X_j \cap Y_k \neq \varnothing$$

$$\rightleftarrows (\exists x \in X_i)(\exists y \in X_j)[(x, y) \in \theta]$$

$$\rightleftarrows (X_i, X_j) \in \phi(\sigma)^{-1} \circ \theta \circ \phi(\sigma)$$

$$\rightarrow \theta_{\phi(\sigma)}' = \phi \circ \phi^{-1}$$

Therefore, $(M/\sigma)/\theta_{\phi(\sigma)} \sim M/\theta$.

VI. CONCLUDING REMARKS

In this paper a general approach to compare the structures of incomplete, nondeterministic automata has been developed via the concept of relational homomorphism. The concept of structural equivalence, we believe, is an important one in the sense that certain essential structures of two systems are preserved without demanding them to be the same. If we consider automata with output, then it can be shown that structural equivalence (with an additional condition on output) lies between the concepts of isomorphism and behavioral equivalence. Furthermore, in many cases when behavioral equivalence is demanded, structural equivalence is there also.

It appears that the results obtained in this paper may be used to discuss the structural properties of graphs or formal grammars.

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