ELSEVIER
Discrete Mathematics 199 (1999) 221–225
DISCRETE MATHEMATICS

Note

On perfect neighborhood sets in graphs

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Received 14 August 1997; revised 30 May 1998; accepted 8 June 1998

Abstract

Let $G=(V,E)$ be a graph and let $S\subseteq V$. The set S is a dominating set of G is every vertex of $V-S$ is adjacent to a vertex of S. A vertex v of G is called S-perfect if $|N[v]\cap S|=1$ where $N[v]$ denotes the closed neighborhood of v. The set S is defined to be a perfect neighborhood set of G if every vertex of G is S-perfect or adjacent with an S-perfect vertex. We prove that for all graphs G, $\Theta(G)=\Gamma(G)$ where $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G and where $\Theta(G)$ is the maximum cardinality among all perfect neighborhood sets of G. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

Let $G=(V,E)$ be a graph with vertex set V and edge set E, and let v be a vertex in V. The open neighbourhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is $N[v]=\{v\}\cup N(v)$. A set $S\subseteq V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. Equivalently, S is a dominating set of G if for every vertex v in V, $|N[v]\cap S|\geqslant 1$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G, while the *upper domination number* of G, denoted by $\Gamma(G)$, is the maximum cardinality of a minimal dominating

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[1] Research supported in part by the National Science Foundation under Grant CCR-9408167
[2] Research supported in part by the University of Natal and the South African Foundation for Research Development.

PII: S0012-365X(98)00306-9

set in G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [1] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes et al. [5,6].

In this paper we introduce the concept of perfect neighborhood sets in graphs. Let S be a subset of vertices of G. A vertex v of G is called *S-perfect* if $|N[v] \cap S| = 1$. The set S is defined to be a *perfect neighborhood* set of G if every vertex of G is S-perfect or adjacent to an S-perfect vertex. Equivalently, S is a perfect neighborhood set of G if for every $u \in V$, there exists a $v \in N[u]$ such that $|N[v] \cap S| = 1$. The *lower (upper) perfect neighborhood number* $\theta(G)$ (respectively, $\Theta(G)$) of G is defined to be the minimum (respectively, maximum) cardinality among all perfect neighborhood sets of G. We will refer to a perfect neighborhood set of cardinality $\theta(G)$ (respectively, $\Theta(G)$) as a θ-set (respectively, Θ-set) of G.

2. The parameter $\theta(G)$

In this section, we relate θ with other graph parameters. A set S of vertices of $G = (V, E)$ is a 2-dominating set of G if every vertex of $V - S$ is within distance 2 from some vertex of S. The minimum cardinality among all 2-dominating sets of G is called the 2-domination number and is denoted by $\gamma_2(G)$. Since every perfect neighborhood set of G is also a 2-dominating set of G, $\gamma_2(G) \leqslant \theta(G)$ for every graph G. Strict inequality may hold. For example, consider the bipartite graph G_n constructed as follows. Take a complete bipartite graph $K_{2,n}$ with partite sets $U = \{u, v\}$ and W. Attach $n+2$ vertex disjoint paths of length 2 to each of u and v, and attach a path of length 1 to each vertex of W. Then the resulting bipartite graph G_n satisfies $\theta(G_n) = n + 2$ and $\gamma_2(G_n) = 2$. Hence for every positive integer n, there exists a bipartite graph G_n with $\theta(G_n) - \gamma_2(G_n) = n$.

A *packing* of a graph is a set of vertices whose elements are pairwise at distance at least 3 apart in G. The *lower packing number* of G, denoted $\rho_L(G)$, is the minimum cardinality of a maximal packing of G. Since every maximal packing S of G is a perfect neighborhood set of G, $\theta(G) \leqslant \rho_L(G)$ for every graph G. Strict inequality may hold. For example, let T_n be the tree obtained from the disjoint union $2K_{1,n+1}$ of two stars each of order $n + 2$ by subdividing every edge exactly once and then joining the two vertices of degree $n+1$ with an edge. Then T_n satisfies $\theta(T_n) = 2$ and $\rho_L(T_n) = n+2$. Hence for every positive integer n, there exists a tree T_n with $\rho_L(T_n) - \theta(T_n) = n$.

Closely related to the concept of perfect neighborhood sets are irredundant sets. For a graph $G = (V, E)$, a subset S of vertices of G is defined to be *irredundant* if every vertex of S is S-perfect or adjacent with an S-perfect vertex. (Recall that S is a perfect neighborhood set of G if *every* vertex of V is S-perfect or adjacent with an S-perfect vertex.) The *irredundance number* of G, denoted by $ir(G)$, is the minimum cardinality taken over all maximal irredundant sets of vertices of G. We close this section with the following conjecture.[3]

Conjecture 1. For all graphs G $\theta(G) \leqslant ir(G)$.

3. Main result

In this section, we prove:

Theorem 1. *For all graphs G, $\Theta(G) = \Gamma(G)$.*

For each vertex v in a minimal dominating set D of a graph G, we let $PN(v, D)$, or simply $PN(v)$ if the set D is clear from the context, denote the set of all vertices that are adjacent with v but with no other vertex of D. We begin with the following lemma.

Lemma 1. *For any minimal dominating set D of a graph $G = (V, E)$, there exists a perfect neighborhood set of G of cardinality $|D|$.*

Proof. Let D be a minimal dominating set of G, and let D_1 consists of all isolated vertices in the subgraph $\langle D \rangle$ induced by D. Let $D_2 = D - D_1$. For each $v \in D_2$, $PN(v, D) \neq \emptyset$. For each $v \in D_2$, let $g(v) \in PN(v, D)$ and let $T = \{g(v) \mid v \in D_2\}$. Then $T \subseteq V - D$. Let $S = D_1 \cup T$. Then $|N[v] \cap S| = 1$ for every vertex of D, so all vertices of D are S-perfect. However, since D is a dominating set, every vertex in $V - D$ is therefore adjacent with an S-perfect vertex. Hence S is a perfect neighborhood set of cardinality $|D|$. □

An immediate corollary of Lemma 1 now follows.

Corollary 1. *For every graph G, $\theta(G) \leqslant \gamma(G)$ and $\Gamma(G) \leqslant \Theta(G)$.*

Lemma 2. *For every graph $G = (V, E)$, $\Theta(G) \leqslant \Gamma(G)$.*

Proof. Let S be a Θ-set of G. We show that G contains a minimal dominating set of cardinality at least $|S|$. Let $S_1 = \{v \in S \mid v \text{ has an } S\text{-perfect neighbor in } V - S\}$, and let $S_2 = S - S_1$. We show firstly that each vertex v of S_2 is isolated in $\langle S \rangle$. If this is not the case, then there is a $v \in S_2$ that is adjacent with some other vertex of S.

[3] This conjecture has attracted considerable interest since it was announced. In [2], the conjecture is proven true for all trees. In [3], the conjecture is shown to be true if G is claw-free or if G has a maximal irredundant set S of minimum cardinality for which the subgraph induced by S has at most six non-isolated vertices. Recently, Favaron [4] announced at the *16th British Combinatorial Conference* held in London in July 1997 that they have a counterexample to this conjecture. Their construction contains over two million vertices.

Thus $|N[v] \cap S| \geqslant 2$, so v is not S-perfect. But since S is a perfect neighborhood set of G, v must then have an S-perfect neighbor in $V - S$ and therefore v belongs to S_1, a contradiction. Hence each vertex of S_2 is isolated in $\langle S \rangle$. Thus each vertex of S_2 is S-perfect.

Let $T = N(S) \cap (V - S)$, and let $W = V - (S \cup T)$. Since $N[w] \cap S = \emptyset$ for each $w \in W$, we know that no vertex of W is S-perfect. Now let $T_1 = \{t \in T \mid t$ is an S-perfect neighbor of some vertex in $S\}$. Thus each vertex of T_1 is S-perfect and is adjacent with a unique vertex of S_1 and with no vertex of S_2. Let $T_2 = \{t \in T \mid t$ is adjacent with some vertex of $T_1 \cup S_2\}$, and let $T_3 = T - (T_1 \cup T_2)$.

Each vertex of $T - T_1$ is adjacent with at least two vertices of S, so no vertex of $T - T_1$ is S-perfect. In particular, no vertex of T_3 is S-perfect. Thus each vertex of T_3 must be adjacent with an S-perfect vertex. Since no vertex of T_3 is adjacent with any vertex of $S_2 \cup T_1$, and since no vertex of $T_2 \cup W$ is S-perfect, each vertex of T_3 must have an S-perfect neighbor in S. Among all subsets of S-perfect vertices of S that dominate all the vertices of T_3, let S_1' be one of minimum cardinality. Thus, each vertex of S_1' uniquely dominates at least one vertex of T_3; that is, for each $v \in S_1'$, there exists a vertex $t \in T_3$ such that t is adjacent with v but with no other vertex of S_1'. Since each vertex of S_1' is S-perfect, we know that each vertex of S_1' is isolated in $\langle S \rangle$. Furthermore since no vertex of T_3 is adjacent with any vertex of S_2, we know that $S_1' \subseteq S_1$. Let $S_1'' = S_1 - S_1'$.

We show next that $D = S_1' \cup S_2 \cup T_1$ is a dominating set of G. By definition, each vertex of S_1'' is adjacent with some vertex of T_1 and each vertex of T_2 is adjacent with some vertex of $S_2 \cup T_1$. We also know that the set S_1' dominates T_3. Since no vertex of W is S-perfect, each vertex of W must have an S-perfect neighbor from the set T. However, the only S-perfect vertices of T belong to the set T_1. Hence D is a dominating set of G. Thus there must exist a subset D^* of D that is a minimal dominating set of G.

It remains for us to show that $|D^*| \geqslant |S|$. Since each vertex of S_2 is isolated in $\langle D \rangle$, $S_2 \subseteq D^*$. For each $v \in S_1'$, we know there exists a vertex $t \in T_3$ such that t is adjacent with v but with no other vertex of D. Thus each vertex of S_1' uniquely dominates some vertex of T_3, so $S_1' \subseteq D^*$. Finally, $|D^* \cap T_1| \geqslant |S_1''|$ since each vertex of T_1 is adjacent with at most one vertex of S_1'' while no vertex of $S_1' \cup S_2$ is adjacent with any vertex of S_1''. Hence $|D^*| \geqslant |S_1'| + |S_1''| + |S_2| = |S|$. Thus D^* is a minimal dominating set of G of cardinality at least $|S|$. Consequently, $\Theta(G) = |S| \leqslant |D^*| \leqslant \Gamma(G)$. $\square$

Theorem 1 now follows immediately from Corollary 1 and Lemma 2.

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