Note

A Series Representation for the Titchmarsh-Weyl m-Function

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I. INTRODUCTION

We consider the second order linear differential equation
\[-y'' + qy = \lambda y\]  \quad on \quad [0, \infty) \tag{1.1}

where, among other conditions to be detailed later, \(q\) is a real-valued member of \(L[0, \infty)\) and \(\lambda\) is a complex parameter. It is known, see [7], that under these conditions (1.1) is limit point at infinity and regular at 0. A self adjoint boundary value problem is associated with (1.1) by imposing a boundary condition at the origin of the form
\[\cos \alpha y(0) + \sin \alpha y'(0) = 0 \quad (1.2)\]

for \(\alpha \in [0, \pi)\). A certain scaler-valued function known as the Titchmarsh-Weyl \(m\)-function is associated with (1.1) and (1.2) in the following way. Let \(\Theta_2\) and \(\phi_2\) denote the solutions of (1.1) which satisfy at 0 the conditions
\[
\phi_2(0) = -\sin \alpha \quad \phi_2'(0) = \cos \alpha \\
\Theta_2(0) = \cos \alpha \quad \Theta_2'(0) = -\sin \alpha.
\]

For each \(X > 0\) and \(\lambda\) with \(\text{Im}(\lambda) > 0\) we define \(C_2(X, \lambda)\) to be the circle which is the image of the real line under the map
\[
\xi \rightarrow M_2(X, \lambda, \xi) := \left\{ \Theta_2(X, \lambda) - \xi \Theta_2'(X, \lambda) \right\} \frac{\phi_2(X, \lambda) - \xi \phi_2'(X, \lambda)}{\phi_2(X, \lambda) - \xi \phi_2'(X, \lambda)}.
\]
It is known that as $X$ increases the circles $C_{\lambda}(X, \lambda)$ nest and, since (1.1) is limit point, they converge to a finite point as $X \to \infty$. We denote the limit point $m_{\lambda}(\lambda)$.

In this paper we consider in particular the cases $\alpha = 0$ and $\alpha = \pi/2$. The $m$-functions for different values of $\alpha$ are connected by the relation

$$m_{\lambda}(\lambda) = \frac{m_{\beta}(\lambda) \cos(\alpha - \beta) \sin(\alpha - \beta)}{m_{\beta}(\lambda) \sin(\alpha - \beta) + \cos(\alpha - \beta)}.$$ 

In particular we observe that $m_{0}(\lambda) = -1/m_{\pi/2}(\lambda)$.

In recent years there has been considerable interest in the form of $m(\lambda)$. In [3] the fact that $q \in L^{1}$ was exploited to give a convergent series representation for $m_{\pi/2}(\lambda)$ when $\lambda$ lay in a sector of the upper half plane. A similar approach was used in [6]. These were refined in [2] to show that the representation was valid for $\lambda$ outside a parabolic region containing the positive real axis. It was also shown that with the extra condition $tq(t) \in L^{1}[0, \infty)$ the representation held for all $\lambda$ with $|\lambda|$ sufficiently large.

In [5], $m_{\pi/2}(\lambda)$ was shown to be expressible as a quotient of two convergent series for all $\lambda$ with $|\lambda| > \lambda_{0}$ under the weaker hypothesis that $q \in L^{1}$. In this paper we return to the requirement that $(1 + t)q(t) \in L^{1}$ and, with a condition on the $L^{1}$ norm of $tq(t)$, derive a convergent series representation for $m_{0}(\lambda)$ which is valid for all $\lambda$ with $\operatorname{Im}\{\lambda\} \geq 0$ except for a compact interval on the negative real line.

2. The Results

Following the method of [3] we set

$$p_{1}(x, \lambda) := -\int_{x}^{\infty} e^{2\pi \lambda^{2}(t - x)} q(t) \ dt$$

$$p_{j+1}(x, \lambda) := \int_{x}^{\infty} p_{j}(t, \lambda)^{2} \ dt$$

for $j = 1, 2, \ldots$.

Theorem 1. If $\int_{0}^{\infty} (1 + t) |q(t)| \ dt < \infty$ and $\int_{0}^{\infty} t |q(t)| \ dt < \frac{1}{4} \log 5$ then $m_{0}(\lambda) = i\lambda^{1/2} + \sum_{n=1}^{\infty} p_{n}(0, \lambda)$ for all $\lambda$ with $\operatorname{Im}\{\lambda\} \geq 0$ except for a compact interval of the negative real line.
3. Proofs

There are a number of definitions of m-functions which are equivalent to that given in Section 1. We take as our starting point the definition of \( m_{n,2}(\lambda) \), or rather of \( C_{n,2}(X, \lambda) \), given by Atkinson in [1].

**DEFINITION.** For \( \text{Im}\{\lambda\} > 0 \) the Weyl discs, \( D_{n,2}(X, \lambda) \) made up of \( C_{n,2}(X, \lambda) \) together with its interior, consist of those \( m \in \mathbb{C} \) such that the solution of the Ricatti equation

\[
V' = -\lambda + q + V^2, \quad 0 \leq x \leq X
\]  

with \( V(0) = -1/m \) has \( \text{Im}\{V(X, \lambda)\} \geq 0 \).

We recall from Section 1 that (1.1) is in the limit point case at infinity and so the discs \( D_{n,2} \) nest and converge to a limit point, \( m_{n,2}(\lambda) \), as \( X \to \infty \).

In the sequel we write \( \lambda^{1/2} =: \alpha + i\beta \) and suppose in view of the requirement \( \text{Im}\{\lambda\} > 0 \) that \( \alpha, \beta > 0 \).

For arbitrary, but fixed, \( X \) we consider the solution, \( V_X \), of (3.1) with \( V_X(X, \lambda) = i\lambda^{1/2} \). It follows that \( -1/V_X(0, \lambda) \in D_{n,2}(X, \lambda) \).

For functions \( P_{n,X}(x, \lambda) \) to be chosen later we write \( V_X(x, \lambda) = i\lambda^{1/2} + \sum_{n=1}^\infty P_{n,X}(x, \lambda) \). We choose the functions \( P_{n,X} \) in an iterative fashion so that the function \( V_X \) is a solution of (3.1) which satisfies the condition at \( X \).

On substitution into (3.1) we see that the \( P_{n,X} \) must satisfy

\[
\sum_{n=1}^X P'_{n,X} = q - 2i\lambda^{1/2} \sum_{n=1}^X P_{n,X} - \sum_{n=1}^X P_{n,X} \sum_{m=1}^X P_{m,X}.
\]  

We choose \( P_{1,X} \) so that

\[
P'_{1,X} + 2i\lambda^{1/2} P_{1,X} = q \tag{3.3}
\]

and

\[
P_{1,X}(X, \lambda) = 0. \tag{3.4}
\]

In other words

\[
P_{1,X}(X, \lambda) = -\int_X^X e^{2i\lambda^{1/2}(t-x)}q(t) \, dt.
\]

We then choose \( P_{2,X} \) to satisfy

\[
P'_{2,X} + 2(i\lambda^{1/2} + P_{1,X}) P_{2,X} = -P_{1,X}^2
\]

\[
P_{2,X}(X, \lambda) = 0.
\]
Since $P_{1,X}$ has already been chosen this is a first order linear equation that we can solve to determine $P_{2,X}$. Proceeding inductively as in [3] we choose $P_{j+1,X}$ to satisfy

$$P_{j+1,X}' + 2\left(i\lambda^{1/2} + \sum_{n=1}^{j} P_{n,X}\right)P_{j+1,X} = -P_{j,X}^2$$

Again since $P_{j,X}$ has already been fixed we may regard (3.5) as a first order linear differential equation which we solve subject to (3.6) in order to determine $P_{j+1,X}$. This leads to the choice

$$P_{j+1,X}(x, \lambda) = \int_{x}^{X} e^{2\int_{t}^{X} i\lambda^{1/2} + \sum_{n=1}^{j} P_{n,X}(s, \lambda) ds} P_{j,X}(t, \lambda)^2 dt$$

for $j = 2, 3, 4, \ldots$.

We now show that the series of $P_{n,X}$ functions defined formally above is uniformly convergent for all $X > 0$, $0 \leq x \leq X$ for all $\alpha \geq 0$ and $\beta \geq 0$.

Recalling that $q \in L^1$ we define the decreasing function $a(x) := \int_{x}^{X} |q(t)| dt$.

**Lemma 1.** For any $X > 0$ and any $\lambda$ with $\beta \geq 0$

$$|P_{j,X}(x, \lambda)| \leq 2^{-j} \lambda a(x)$$

for all $x \in [0, X]$ and $j = 1, 2, 3, \ldots$.

**Proof.** We use induction on $j$. When $j = 1$ the result is trivial. Suppose the result is true for $j = k$. Then,

$$|P_{k+1,X}(x, \lambda)| \leq \int_{x}^{X} e^{2\sum_{n=1}^{k} \int_{t}^{X} P_{n,X}(s, \lambda) ds} |P_{k,X}(t, \lambda)|^2 dt$$

\begin{align*}
&\leq 2^{-2(k-1)} \int_{x}^{X} e^{2\sum_{n=1}^{k} \int_{t}^{X} a(S) ds} a(t)^2 dt \\
&\leq 2^{-2(k-1)} a(x) \int_{x}^{X} e^{4\int_{t}^{X} a(s) ds} a(t) dt \\
&\leq 2^{-2k} a(x)(e^{4\int_{0}^{X} a(s) ds} - 1) \\
&\leq 2^{-k} a(x)
\end{align*}

as asserted since

$$\int_{0}^{X} a(t) dt = \int_{t}^{X} \int_{t}^{X} |q(s)| ds dt = \int_{0}^{X} s |q(s)| ds < \frac{1}{3} \log 5$$
and, as $k > 2$ we have that

$$(e^4 I_k \alpha x) - 1) 2^{-k} \ll 1.$$ 

We note that the inequality of Lemma 1 holds for all $X$ and hence that the series $\sum_{n=1}^{X} P_{n\lambda}(X, \lambda)$ is uniformly convergent for $0 \leq x \leq X$ for all $X \leq \infty$.

It follows from (3.5) and Lemma 1 that

$$|P_{j+1, X}(x, \lambda)| \leq (2 |\lambda|^2 + 4a(x)) 2^{-j}a(x) + 2^{-2}a(x)^2$$

and hence that the differentiated series $\sum_{n=1}^{X} P'_{n\lambda}(X, \lambda)$ is uniformly convergent for all $\lambda$ with $\beta > 0, x \in [0, X]$ for all $X \leq \infty$.

Thus, the function $V_v(x, \lambda) = i\lambda^{1/2} + \sum_{n=1}^{X} P_{n\lambda}(x, \lambda)$ is the solution of (3.1) with $\text{Im}\{V_v(X, \lambda)\} = ia > 0$. According to the definition we then have

$$m_{\lambda}(\lambda) := \left( i\lambda^{1/2} + \sum_{n=1}^{X} p_{n\lambda}(0, \lambda) \right)^{-1} \in D_{\pi, 2}(X, \lambda).$$

Using the nesting property of the Weyl discs we also have $m_{\lambda}(\lambda) \in D_{\pi, 2}(X, \lambda)$ for all $X \geq X$.

Noting that the $p_{\lambda}(x, \lambda)$ functions of Section 2 are coincident with $P_{n\lambda}(x, \lambda)$ we thus have that

$$- \left( i\lambda^{1/2} + \sum_{n=1}^{X} p_{n\lambda}(0, \lambda) \right)^{-1} \in D_{\pi, 2}(X, \lambda)$$

for all $X > 0$ if $x > 0$ and $\beta > 0$. Since the $D_{\pi, 2}(X, \lambda)$ contract to a limit point as $X \to \infty$ it follows that

$$m_{\pi, 2}(\lambda) = - \left( i\lambda^{1/2} + \sum_{n=1}^{X} p_{n\lambda}(0, \lambda) \right)^{-1}$$

for $x > 0$ and $\beta > 0$. It further follows from (1.4) that

$$m_{\lambda}(\lambda) = i\lambda^{1/2} + \sum_{n=1}^{X} p_{n\lambda}(0, \lambda).$$ (3.8)

It remains only to justify (3.8) for $\text{Im}\{\lambda\} = 0$. It is clear from Lemma 1 that for $x, \beta > 0, p_{n\lambda}(0, \lambda)$ is continuous for $\text{Im}\{\lambda\} \geq 0$. The representation (3.8) thus admits a continuous extension to the real line. It is shown [4] that $m_{\lambda}(\lambda)$ has a continuous extension from the upper half plane to the positive real line. Thus (3.8) extends by continuity to $(0, \infty)$. 

REFERENCES