Centralizers of involutory automorphisms of groups of odd order

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Abstract

The following theorem is proved. Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$ such that $G = [G, \phi]$. Suppose that $C_G(\phi)$ has a nilpotent subgroup of index $n$. Then the index $[G' : F(G')]$ is bounded by a function depending only on $n$.

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1. Introduction

Let $G$ be a finite group admitting an automorphism $\phi$ of order two (such automorphisms are called involutory). It is well known that the structure of $C_G(\phi)$ has strong impact on that of $G$. One of the best illustrations for this is the well-known elementary result that if $C_G(\phi) = 1$, then $G$ is abelian. A result of Hartley and Meixner says that if $C_G(\phi)$ is of order $m$, then $G$ possesses a subgroup which is nilpotent of class at most two and has index bounded by a function depending on $m$ only [4]. Now assume that $G$ is a finite group of odd order. By the Feit–Thompson theorem [2] $G$ is soluble. Kovács and Wall showed that if $C_G(\phi)$ is abelian, then $G'$, the derived group of $G$, is nilpotent [6]. The situation where $C_G(\phi)$ is nilpotent was considered by Ward, who proved in [13] that in this case $G$ coincides with the third term of the upper Fitting series of $G$. As usual, we denote the Fitting subgroup of $G$ by $F(G)$ and define $F_{i+1}(G)$ inductively by
\( F_0(G) = 1 \) and \( F_{i+1}(G)/F_i(G) = F(G/F_i(G)) \) for \( i = 1, 2, \ldots \). Asar sharpened this by showing that if \( C_G(\phi) \) is nilpotent, then so are both \([G, \phi]'\) and \( G/[G, \phi] \) \([1]\). Recall that \([G, \phi]\) is by definition the subgroup of \( G \) generated by the elements of the form \( x^{-1}x^\phi \), where \( x \in G \). This subgroup is always normal in \( G \). It follows that if \( G = [G, \phi] \) and \( C_G(\phi) \) is nilpotent, then \( C_G(\phi) \leq F(G) \). In \([8]\) we gave another proof of Asar’s result. Some further results on involutory automorphisms of groups of odd order can be found in \([9–11]\). In \([8]\) we also gave an example (due to Hartley) showing that even if \( G = [G, \phi] \), \( F(CG(\phi)) \) need not be contained in \( F(G) \).

Let us recall here the famous result by J.G. Thompson that if \( \psi \) is an automorphism of prime order of a finite group \( G \) such that \((|G|, |\psi|) = 1\), then \( F(CG(\phi)) \leq F_4(G) \) and \( F(CG(\phi)) \leq F_3(G) \) in case \(|G|\) is odd \([12]\).

Despite the fact that in general \( F(CG(\phi)) \not\leq F(G) \), it seems some progress along those lines is possible. In particular, there are some indirect evidences that there exists a constant \( C \) such that if \( G = [G, \phi] \), then

\[
|C_G(\phi) \cap F(G)|^C \geq |F(CG(\phi))|.
\]

The goal of the present paper is to establish the following theorem.

**Theorem 1.1.** Let \( G \) be a finite group of odd order admitting an involutory automorphism \( \phi \) such that \( G = [G, \phi] \). Suppose that \( C_G(\phi) \) has a nilpotent subgroup of index \( n \). Then the index \([G': F(G')]\) is bounded by a function depending only on \( n \).

It is straightforward from Theorem 1.1 that the index of \( C_G(\phi) \cap F(G) \) in \( F(CG(\phi)) \) is bounded by a function of \( n \).

2. Preliminaries

Throughout the article we use the term “\([a, b, c \ldots]\)-bounded” to mean “bounded from above by some function depending only on the parameters \( a, b, c \ldots \)” If \( H \) is a group with an automorphism \( \phi \), we write

\( H_\phi \) for \( C_H(\phi) \) and \( H_{-\phi} \) for the set \( \{x \in H; x^\phi = x^{-1}\} \).

The first lemma is a collection of well-known facts about involutory automorphisms. In the sequel we will frequently use it without any reference.

**Lemma 2.1.** Let \( G \) be a finite group of odd order admitting an involutory automorphism \( \phi \). Then we have

1. \( G = G_\phi G_{-\phi} = G_{-\phi} G_\phi \), and the subgroup generated by \( G_{-\phi} \) is exactly \([G, \phi]\).
2. If \( N \) is any \( \phi \)-invariant normal subgroup of \( G \) we have \( (G/N)_\phi = G_\phi N/N \), and \( (G/N)_{-\phi} = \{gN; g \in G_{-\phi}\} \).
3. The normal closure of \( G_\phi \) contains \( G' \). If \( G_\phi \) is nilpotent of class \( c \), then \([G, \phi]\) contains \( \gamma_{c+1}(G) \).
4. \( G_\phi \) normalizes the set \( G_{-\phi} \).
In fact some of the statements of Lemma 2.1 are well known without assuming that the automorphism has order two. In particular we have the following lemma (see for example [3, 6.2.2, 6.2.4]).

**Lemma 2.2.** Let $A$ be a group of automorphisms of a finite group $G$ with $(|A|, |G|) = 1$.

1. If $N$ is any $A$-invariant normal subgroup of $G$, then $C_{G/N}(A) = C_G(A)N/N$.

The next lemma is due to Hartley [5, Lemma 2.6].

**Lemma 2.3.** Let $A$ be a group of automorphisms of a finite group $G$ with $(|A|, |G|) = 1$. Let $\{N_i; i \in I\}$ be a family of normal $A$-invariant subgroups of $G$ and $N = \prod_i N_i$. Then $C_N(A) = \prod_i C_{N_i}(A)$.

In what follows $G$ will always denote a finite group of odd order admitting an involutory automorphism $\phi$. The following elementary lemma is very well known so the proof is omitted.

**Lemma 2.4.** Suppose that $G = [G, \phi]$. Let $N$ be a $\phi$-invariant normal subgroup such that either $N \leq G - \phi$ or $N \leq G\phi$. Then $N \leq Z(G)$.

Every element $x \in G$ can be written uniquely as a product $x = x_\phi x_{-\phi}$, where $x_\phi \in G\phi$ and $x_{-\phi} \in G - \phi$. Given a subset $X \subseteq G$, we write $X_\phi$ and $X_{-\phi}$ for the sets $\{x_\phi; \text{ where } x \in X\}$ and $\{x_{-\phi}; \text{ where } x \in X\}$, respectively.

**Lemma 2.5.** Let $X, Y$ be two commuting subsets of $G$ and assume that $Y$ is $\phi$-invariant. Then both $X_\phi$ and $X_{-\phi}$ commute with $Y$.

**Proof.** Since $Y$ is $\phi$-invariant, we conclude that $\langle X, X_\phi \rangle \leq C_G(Y)$. Clearly, $\langle X, X_\phi \rangle = \langle X_\phi, X_{-\phi} \rangle$ so the lemma follows. \qed

**Lemma 2.6.** Let $x \in G - \phi$ and $a \in G\phi$ and suppose that $[x, a] \in G\phi$. Then $[x, a] = 1$.

**Proof.** We have $xax^{-1} = (xax^{-1})_\phi$, whence $xax^{-1} = x^{-1}ax$. Since $G$ has odd order, it follows that $x$ commutes with $a$. \qed

**Lemma 2.7.** Let $m$ be a positive integer such that $|G\phi| \leq m$. Then

1. $G$ has a normal $\phi$-invariant subgroup $H$ such that $H' \leq G\phi$ and the index $[G : H]$ is $m$-bounded.
2. If $G = [G, \phi]$, then the order of $G'$ is $m$-bounded.

**Proof.** Part 1 is Lemma 3.4 in [5]. Let us prove Part 2. We will use induction on $m$. Let $H$ be as in Part 1. If $H' \neq 1$, by induction the result holds for $G/H'$ and since $|G'| = |G'/H'||H'|$, there is nothing to prove. Suppose that $H$ is abelian and let $M = H \cap G\phi$, $N = (M^G)$. Since both the order of $M$ and the index of $C_G(M)$ in $G$ are $m$-bounded, we conclude that so is the order of $N$. 


Again considering $G/N$, in the case that $N \neq 1$ the result follows by induction. Therefore we can assume that $H \cap G_\phi = 1$. It follows that $H \leq G_\phi$ and so, by Lemma 2.4, $H \leq Z(G)$. Thus, the index $[G : Z(G)]$ is $m$-bounded and the lemma follows from the Schur theorem [7, p. 102].

Given an element $x \in G$ and a subset $L \subseteq G$, we denote by $\rho^\phi_x(L)$ the minimal $\phi$-invariant subgroup of $G$ containing $x^{-1}Lx$. Given several elements $x_1, \ldots, x_k \in G$, we define inductively

$$\rho^\phi_{x_1,\ldots,x_k}(L) = \rho^\phi_{x_k}(\rho^\phi_{x_1,\ldots,x_{k-1}}(L)).$$

Subgroups of this type will play an important rôle in the subsequent proofs.

**Lemma 2.8.** Let $L$ be a $\phi$-invariant normal abelian subgroup of $G$ and $x \in G_\phi$. Then $\rho^\phi_x(L - \phi)$ contains $L - \phi$.

**Proof.** Let us denote $x^{-1}L_\phi x$ by $K$. It is clear that $\rho^\phi_x(L - \phi) = \langle K - \phi, K_\phi \rangle$. Therefore it is sufficient to prove that $K_\phi = L - \phi$. If $|K_\phi| < |L - \phi|$, then there exist two distinct elements $l_1, l_2 \in L - \phi$ such that $(x^{-1}l_1x)_\phi = (x^{-1}l_2x)_\phi$. But then $x^{-1}l_1l_2^{-1}x \in L_\phi$. So $(x^{-1}l_1l_2^{-1}x)_\phi = x^{-1}l_1l_2^{-1}x$ and we obtain $x(l_1l_2^{-1}x) = x^{-1}l_1l_2^{-1}x$. Thus, $l_1l_2^{-1}$ is conjugate to its inverse. Since $G$ has odd order, we conclude that $l_1 = l_2$. We have shown that the inequality $|K_\phi| < |L_\phi|$ is impossible and so $K_\phi = L - \phi$. □

**Lemma 2.9.** Let $x \in G_\phi$ and $H \leq G_\phi$. Then $(H^x)_\phi$ has at least as many elements as $H$.

**Proof.** Suppose that $|(H^x)_\phi| < |H|$. Then there exist two distinct elements $h_1, h_2 \in H$ such that $(x^{-1}h_1x)_\phi = (x^{-1}h_2x)_\phi$. Write $x^{-1}h_1x = h_0g_1$ and $x^{-1}h_2x = h_0g_2$, where $h_0 = (x^{-1}h_1x)_\phi = (x^{-1}h_2x)_\phi$ and $g_1, g_2 \in G_\phi$. Then $x^{-1}h_2^{-1}h_1x = g_2^{-1}g_1$. We see that $xg_2^{-1}g_1x^{-1} \in G_\phi$ so $(xg_2^{-1}g_1x^{-1})_\phi = xg_2^{-1}g_1x^{-1}$. We obtain $xg_2^{-1}g_1x^{-1} = x^{-1}g_2g_1^{-1}x$ whence it follows that $g_2^{-1}g_1$ is conjugate in $G$ to its inverse. Since $G$ has odd order, $g_1 = g_2$ and $h_1 = h_2$, a contradiction. □

**Lemma 2.10.** Let $L$ be a $\phi$-invariant normal abelian subgroup of $G$ and $x \in G_\phi$. Suppose that $C_L(x) = 1$. Then $\rho^\phi_x(L - \phi) = L$.

**Proof.** As in Lemma 2.8 let us denote $x^{-1}L_\phi x$ by $K$. Since $\rho^\phi_x(L - \phi) = \langle K - \phi, K_\phi \rangle$, it is sufficient to prove that $K_\phi = L - \phi$ and $K_\phi = L_\phi$. The equality $K_\phi = L - \phi$ follows from Lemma 2.8 so it remains to show that $K_\phi = L_\phi$.

Suppose that $|K_\phi| < |L_\phi|$. Then either there exist two distinct elements $l_1, l_2 \in L_\phi$ such that $(x^{-1}l_1x)_\phi = (x^{-1}l_2x)_\phi$ or $|L_\phi| < |L_\phi|$. In the former case $x^{-1}l_1l_2^{-1}x \in L_\phi$. So $(x^{-1}l_1l_2^{-1}x)_\phi = (x^{-1}l_1l_2^{-1}x)_\phi$ and we obtain $x(l_1l_2^{-1}x) = x^{-1}l_1l_2^{-1}x$. It follows that $x^2$ commutes with $l_1l_2^{-1}$. Since $G$ has odd order, so does $x$. By the hypothesis $C_L(x) = 1$, a contradiction. If $|L_\phi| < |L_\phi|$, then $L_\phi \cap x^{-1}L_\phi x \neq 1$. Choose a non-trivial element $a \in L_\phi \cap x^{-1}L_\phi x$. We have $xax^{-1} \in L_\phi$ and so, by Lemma 2.6, it follows that $x$ commutes with $a$, a contradiction. □

**Lemma 2.11.** Let $L$ be a $\phi$-invariant subgroup of $G$ and $x \in G_\phi$. Suppose that the order of $\rho^\phi_x(L)$ is the same as that of $L$. Then $x$ normalizes $L$. 
Proof. We have \( \rho_x^\phi(L) = (x^{-1}Lx, (x^{-1}Lx)^\phi) \). Since this has the same order as \( L \), it follows that \( x^{-1}Lx = (x^{-1}Lx)^\phi \). Taking into account that \( x^\phi = x^{-1} \), we conclude that \( x^2 \) normalizes \( L \). Recall that \( x \) has odd order. It becomes clear that \( x \) normalizes \( L \). \( \square \)

3. The proof of the theorem

We start this section with a technical result that will be crucial in the proof of the theorem.

Lemma 3.1. Suppose that \( G = [G, \phi] \) and \( |C_G(\phi)| \leq m \). Let \( G(\phi) \) act faithfully and irreducibly on an abelian \( p \)-group \( V \), where \( p \) is an odd prime. Then there exist an \( m \)-bounded constant \( k \) and elements \( x_1, \ldots, x_k \in G_{-\phi} \) such that \( V = \rho_{x_1, \ldots, x_k}^\phi(V_{-\phi}) \).

Proof. Because \( G(\phi) \) acts irreducibly on \( V \), it is clear that any element of \( Z(G) \) is fixed-point-free, that is, \( C_V(g) = 1 \) for every \( g \in Z(G) \). Therefore if \( Z(G)_{-\phi} \neq 1 \), the result is immediate from Lemma 2.10. Assume that \( Z(G)_{-\phi} = 1 \). By Lemma 2.7 \( G \) contains a normal \( \phi \)-invariant subgroup \( H \), of \( m \)-bounded index, such that \( H' \leq C_G(\phi) \). Then, by Lemma 2.4, \( H' \leq Z(G) \).

Suppose that there exist non-commuting elements \( x, y \in H_{-\phi} \) and set \( h = [x, y] \). Obviously, \( H \) is a \( p' \)-group. This is because \( H \) acts faithfully on \( V \). Set \( U = [V, x] \) and \( W = [V, y] \). We notice that \( x \) normalizes any subgroup of \( V \) containing \( U \) and, likewise, \( y \) normalizes any subgroup of \( V \) containing \( W \). Furthermore, we observe that \( x \) and \( y \) are fixed-point-free on \( U \) and \( W \), respectively. By Lemma 2.10, we conclude that \( U = \rho_x^\phi(U_{-\phi}) \) and \( W = \rho_y^\phi(W_{-\phi}) \). Since \( x^\phi \) acts trivially on \( V/U^x \), it follows that \( x^\phi \) acts trivially on \( V/\rho_x^\phi(U) \). Also, we know that \( y \) acts trivially on \( V/\rho_y^\phi(V_{-\phi}) \). We remark that in view of Lemma 2.8 \( \rho_x^\phi(V_{-\phi}) \) contains both \( V_{-\phi} \) and \( U \). Hence \( \rho_x^\phi, y(V_{-\phi}) \) contains both \( \rho_y^\phi(V_{-\phi}) \) and \( U^x \). Therefore elements \( x^\phi \) and \( y \) act trivially on \( V/\rho_x^\phi, y(V_{-\phi}) \). It follows that \( h \) also acts trivially on \( V/\rho_x^\phi, y(V_{-\phi}) \). On the other hand, since \( h \) is central, \( h \) is fixed-point-free on \( V \) and, by Lemma 2.2, \( h \) is also fixed-point-free on \( V/\rho_x^\phi, y(V_{-\phi}) \). Thus, \( h \) is both fixed-point-free and trivial on \( V/\rho_x^\phi, y(V_{-\phi}) \). It follows that \( V = \rho_x^\phi, y(V_{-\phi}) \).

We will now assume that any two elements in \( H_{-\phi} \) commute. Then \( H_{-\phi} \) is a \( \phi \)-invariant abelian subgroup of \( m \)-bounded index. Let \( A \) be the intersection of all the conjugates of \( H_{-\phi} \).

Then \( A \) is normal and is contained in \( G_{-\phi} \). By Lemma 2.4 \( A \leq Z(G) \). Since \( Z(G)_{-\phi} = 1 \), it follows that \( A = 1 \) and \( G \) has \( m \)-bounded order, say \( k \). In that case \( V \) has order at most \( p^k \). Set \( V_0 = V_{-\phi} \). If \( V_0 = V \), the lemma is immediate. If \( V_0 \neq V \), by Lemma 2.11, there exists \( x_1 \in G_{-\phi} \) such that the order of \( \rho_x^\phi(V_0) \) is greater than that of \( V_0 \). Set \( V_1 = \rho_x^\phi(V_0) \). Again if \( V_1 \neq V \), by Lemma 2.11, there exists \( x_2 \in G_{-\phi} \) such that the dimension of \( V_2 = \rho_x^\phi(V_1) \) is greater than that of \( V_1 \). Continuing the argument we find a sequence of length at most \( k \) of not necessarily distinct elements \( x_1, \ldots, x_k \in G_{-\phi} \) such that \( V_k = \rho_x^\phi, y(V_{-\phi}) = V \). \( \square \)

In the proof of the theorem we will also require the following result obtained in [8].

Proposition 3.2. Suppose that \( G = [G, \phi] \). Let \( N \) be a normal \( \phi \)-invariant subgroup and suppose that \( N_{-\phi} \) has a normal Sylow \( p \)-subgroup \( P \). Then \( P \leq F(G) \). In particular, if \( N_{-\phi} \) is nilpotent, then \( N_{-\phi} \leq F(G) \).

We are now ready to embark on the proof of our main theorem. We will restate it in the following equivalent form.
Theorem 3.3. Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$ such that $G = [G, \phi]$. Denote the index $[G_\phi : F(G_\phi)]$ by $n$. Then the index $[G' : F(G')]$ is bounded by a function depending only on $n$.

Proof. Let $C = G_\phi \cap F_2(G)$ and $D = F(G_\phi) \cap F_2(G)$. We will use induction on $n$. Suppose that $N$ is a nilpotent $\phi$-invariant normal subgroup of $G$ such that $C_G(N)$ contains elements of $C - D$ and suppose that the theorem applies to $G/N$. Consider the natural action of $G/\phi$ on $N$. Some elements of $C - D$ lie in the kernel of the action and so in such a situation the induction works. Thus, the induction hypothesis will be that if $N$ is a nilpotent $\phi$-invariant normal subgroup of $G$ such that $G'N/N$ has a nilpotent subgroup of $n$-bounded index and $C_G(N)$ contains elements of $C - D$, then $G'$ has a subgroup $G_0$ of $n$-bounded index such that the product $NG_0$ is nilpotent. In view of Lemma 2.7 this is equivalent to saying that $G_\phi$ has a subgroup $H$ of $n$-bounded index such that the product $NH$ is nilpotent.

We know from the Thompson result [12] that the Fitting height $h(G)$ of $G$ is bounded in terms of $n$ alone so we will also use induction on $h(G)$. Let $F = F(G)$. By induction we assume that the theorem applies to $G/F$. Therefore the image of $G'$ in $G/F$ has a nilpotent subgroup of $n$-bounded index. It follows that the index $[G_\phi : C]$ is bounded by a function depending only on $n$ while the index $[C : D]$ is, of course, bounded by $n$. By [3, 6.1.6] we can assume that $F$ is abelian.

Define subgroups $T$, $S$, $R$ such that

(i) $T$ is the maximal $\phi$-invariant normal subgroup of $G$ with the property that $T_\phi \subseteq D$;
(ii) $T \subseteq S$ and $S/T = Z(G/T)$;
(iii) $RS/S$ is a minimal $\phi$-invariant normal subgroup of $G/S$.

The subgroup $T$ is determined uniquely. By Lemma 2.3 this is exactly the product of all $\phi$-invariant normal subgroups $N$ of $G$ such that $N_\phi \subseteq D$. It is clear that $F \subseteq T$. Moreover, by Proposition 3.2, $T_\phi \subseteq F$. Therefore $T/F \subseteq Z(G/F)$.

Let $\gamma_\infty(FD)$ denote the intersection of all terms of the lower central series of the subgroup $FD$. Since $(FD)_\phi$ is nilpotent, it follows that $\gamma_\infty(FD) \subseteq (FD)_\phi$. As $F$ is abelian, we have $(FD)_\phi = F_\phi$. In particular we deduce that $\gamma_\infty(FD) \subseteq F_\phi$. The inclusion $T/F \subseteq Z(G/F)$ implies that $T$ normalizes $\gamma_\infty(FD)$ and so, by Lemma 2.4, $T$ commutes with $\gamma_\infty(FD)$ because $T_\phi \subseteq F$. Let $M = C_F(T)$ and $E/M$ be the Fitting subgroup of $G/M$. Since $FD/M$ is nilpotent, it follows that $D \subseteq E$ and so $\phi$ has only boundedly many fixed points in $G/E$. Therefore, by Lemma 2.7, the derived group of $G/E$ has bounded order. We conclude that $G'$ contains a subgroup $G_1$ of bounded index such that $G_1/M$ is nilpotent.

Consider the action of $G$ on $M$. If $C_G(M)$ contains elements of $C - D$, then by induction on $n$ the derived group $G'$ contains a subgroup $G_2$ of bounded index such that the product $MG_2$ is nilpotent. It is clear that $G_1 \cap G_2$ is nilpotent and the theorem follows.

If $C_G(M)$ does not contain elements of $C - D$, then $C_G(M) = T$. Therefore, the quotient $\bar{G} = G/T$ faithfully acts on $M$. In what follows for any subset $X$ of $G$ we denote by $\bar{X}$ the image of $X$ in $\bar{G}$. The definition of $T$ ensures that every $\phi$-invariant normal subgroup of $\bar{G}$ contains images of some elements of $C - D$. Therefore the non-trivial elements of $\bar{S}$ are contained in $\bar{C} - \bar{D}$. We wish to show that $[\bar{R}, \bar{D}] = 1$. Clearly, $R \leq F_2(G)$. Hence $\bar{R} \leq Z_2(F(\bar{G}))$, the second term of the upper central series of $F(\bar{G})$. If $\bar{S} \neq Z(F(\bar{G}))$, then $\bar{R}$, being a minimal normal subgroup of $\bar{G}/\bar{S}$, must be contained in $Z(F(\bar{G}))$, whence $[\bar{R}, \bar{D}] = 1$. Suppose $\bar{S} = Z(F(\bar{G}))$. We observe that $(R/S)_\phi \neq 1$ because otherwise $\bar{R}$ would be contained in $Z(\bar{G})$. 

(Lemma 2.4) and we would have $R = S$ whence the result is immediate. On the one hand, we have $[\bar{R}_{-\phi}, \bar{D}] \leq Z(F(\bar{G})) = \bar{S}$. On the other hand, by Lemma 2.6, the inclusion $[\bar{R}_{-\phi}, \bar{D}] \leq \bar{G}_{\phi}$ implies $[\bar{R}_{-\phi}, \bar{D}] = 1$. Furthermore, $[\bar{R}_{\phi}, \bar{D}]$ is contained in $\bar{D}$ because $\bar{D}$ is normal in $\bar{G}_{\phi}$ and, on the other hand, $[\bar{R}_{\phi}, \bar{D}] \leq Z(F(\bar{G})) = \bar{S}$. Since $\bar{S} \cap \bar{D} = 1$, we conclude that $[\bar{R}_{\phi}, \bar{D}] = 1$. Combining this with the earlier established $[\bar{R}_{-\phi}, \bar{D}] = 1$, it follows that $[\bar{R}, \bar{D}] = 1$. So if $Q$ is the quotient $\text{RS}/S$, the automorphism $\phi$ has only boundedly many fixed points in the group $G/C_G(Q)$ acting on $Q$. Now we are in a position to use Lemma 3.1. It tells us that there exist boundedly many elements $x_1, \ldots, x_k \in G_{-\phi}$ such that

$$\bar{R} \leq \rho_{x_1, \ldots, x_k}^\phi (\bar{R}_{-\phi}) \bar{S}.$$ 

Put $\bar{U} = \rho_{x_1, \ldots, x_k}^\phi (\bar{R}_{-\phi})$. Suppose first that $\bar{R} = \bar{U}$.

Let $P$ be a Sylow $p$-subgroup of $D$. The group $\bar{R}_{-\phi}$ normalizes $M\bar{P}$ because $[\bar{R}_{-\phi}, \bar{D}] = 1$ and, consequently, it normalizes $K = \gamma_\infty(M\bar{P})$. Clearly, $K$ is a $p'$-group and so, by Lemma 2.2, $K = [O_{p'}(M), P]$. Since $D \leq F(G_{\phi})$, it follows that $[K_{\phi}, P] = 1$ and so $K = K_{-\phi}$. By Lemma 2.4 we conclude that $\bar{R}_{-\phi}$ centralizes $K$. Hence $(\bar{R}_{-\phi})^{x_1}$ centralizes $K_{x_1}$ while since $\bar{P}$ normalizes $M(\bar{R}_{-\phi})^{x_1}$ and because $[K_{x_1}, P] \leq M_{-\phi}$, it follows that $[K_{x_1}, P]$ centralizes both $\bar{R}_{-\phi}$ and $(\bar{R}_{-\phi})^{x_1}$. Taking into account that $[K_{x_1}, P]$ is $\phi$-invariant (because $[K_{x_1}, P] \leq M_{-\phi}$), by Lemma 2.5 we deduce that $[K_{x_1}, P]$ centralizes

$$\bar{R}_1 = [(\bar{R}_{-\phi})^{x_1}, (\bar{R}_{-\phi})^{x_1} \phi \rho_{x_1}^\phi (\bar{R}_{-\phi})].$$

Further, $[K_{x_1}, P]^{x_2}$ centralizes $\bar{R}_1^{x_2}$. As above, because $[[K_{x_1}, P]^{x_2}, P]$ is $\phi$-invariant, by Lemma 2.5 we deduce that $[[K_{x_1}, P]^{x_2}, P]$ centralizes

$$\bar{R}_2 = [\bar{R}_1^{x_2}, \bar{R}_1^{x_2} \phi] = \rho_{x_1, x_2}^\phi (\bar{R}_{-\phi}).$$

Eventually we obtain that

$$[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P]$$

centralizes $\bar{R} = \rho_{x_1, \ldots, x_k}^\phi (\bar{R}_{-\phi})$. Suppose that

$$[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P] = 1.$$

Then $[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P], P^{x_1-1}] = 1$. Let $P_1 = (P^{x_1-1})_\phi \cap P$. Since $[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P] \leq M_{-\phi}$, it follows that the centralizer of $[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]$ is $\phi$-invariant and by Lemma 2.5 we deduce that $[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P], P_1] = 1$. Clearly, also $[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k-1}, P_1, P_1] = 1$. (We just replaced the last $P$ by its subgroup $P_1$.) Since $K$ is a $p'$-group, it follows from Lemma 2.2 that

$$[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k-1}, P_1] = 1.$$

Now we obtain

$$[[[K_{x_1}, P]^{x_2}, P]^{x_3}, \ldots]^{x_{k-2}}, P], P_1^{x_{k-1}}] = 1.$$
Put \( P_2 = \langle (x_{k-1} P_1 x_{k-1}^{-1}) \phi \rangle \cap P_1 \). We deduce that

\[
[\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots]^{x_{k-2}}, P_2] = 1.
\]

We can define inductively \( P_{i+1} = \langle (x_{k-i} P_i x_{k-i}^{-1}) \phi \rangle \cap P_i \) and show that

\[
[\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_{k-i}}, P_i] = 1
\]

for \( i = 1, \ldots, k \). In the end we obtain \([K, P_k] = 1\). From this we derive that \([O_{p'}(F), P_k] = 1\) and consequently \( P_k \triangleleft F(G)\). Suppose \( P \) has index \( j \) in the Sylow \( p \)-subgroup of \( C \). By Lemma 2.9 the index of \( (P^{x_k}) \phi \) is at most \( j \), too. Therefore the index of \( P_1 \) is at most \( j^2 \). Further, the index of \( P_2 \) is at most \( j^k \). Continuing this argument we conclude that the index of \( P_k \) is at most \( j^{2^k} \).

Thus, the Sylow \( p \)-subgroup of \( FC/F \) has order at most \( j^{2^k} \). Moreover, by Proposition 3.2, any prime divisor of \( |FC/F| \) is a divisor of \( n \). Hence the order of \( FC/F \) is \( n \)-bounded. Taking into account that the index \( [G_\phi : C] \) is likewise \( n \)-bounded, it follows that \([G_\phi F/F] \) is \( n \)-bounded and the theorem is now immediate from Lemma 2.7. We have just proved the theorem under the additional assumption that \([\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P] = 1\). In general we have

\[
[\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P \subseteq M_1 = C_F(\bar{R}).
\]

Thus, there exists a subgroup \( D_1 \) of \( n \)-bounded index in \( D \) such that \( FD_1/M_1 \) is nilpotent. Consider the action of \( G(\phi) \) on \( M_1 \). Since \( R \) contains elements of \( C - D \), it follows that some elements of \( C - D \) lie in the kernel of the action. So by induction on \( n \) there is a subgroup \( D_2 \) of bounded index in \( D \) such that \([M_1, D_1] \) is nilpotent. Put \( D_0 = D_1 \cap D_2 \). Obviously \( FD_0 \) is a subnormal nilpotent subgroup so \( D_0 \triangleleft F \). Thus, in the case that \( \bar{R} = \bar{U} \) the theorem follows.

We will now assume that \( \bar{R} \neq \bar{U} \). Then \( \bar{U} \) is not normal in \( \bar{G} \). By Lemma 2.11 there exists \( y \in G_\phi \) such that \( \rho_{\bar{R}}^\phi(\bar{U}) \) has order greater than \( |\bar{U}| \). Since \( \bar{R} = \bar{U}_\phi \), it follows that \([\bar{G}_\phi \cap \rho_{\bar{U}}^\phi(\bar{U})] > |\bar{U}_\phi| \). Taking into account that \( \bar{R}_\phi \leq \bar{U}_\phi \bar{S} \) we conclude that either \( \bar{U}_\phi \) or \( \rho_{\bar{U}}^\phi(\bar{U}) \) contains a non-trivial element \( \tilde{s} \in \bar{S} \). Keeping notation introduced in the previous paragraph, we obtain that \( \tilde{s} \) commutes with

\[
[\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P \].
\]

Now put \( M_2 = C_F(\tilde{s}) \). Since \( \tilde{s} \in Z(\bar{G}) \), the subgroup \( M_2 \) is normal in \( G \). We will exploit the fact that \( M_2 \) contains

\[
[\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P \].
\]

If \([\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P \] = 1, the theorem can be proved precisely as we did in the previous paragraph when we had

\[
[\ldots[[K^{x_1}, P]^{x_2}, P]^{x_3}, \ldots, P]^{x_k}, P \] = 1.
\]

Thus, there exists a subgroup \( G_3 \) of bounded index in \( G' \) such that \( G_3 M_2/M_2 \) is nilpotent. Further, since \( \tilde{s} \) acts on \( M_2 \) trivially, the induction on \( n \) allows us to assume that \( G' \) contains a subgroup \( G_4 \) of bounded index such that the product \( M_2 G_4 \) is nilpotent. The subgroup \( G_3 \cap G_4 \) is nilpotent and has \( n \)-bounded index in \( G' \). The proof is now complete. \( \square \)
References