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Positive Solutions for a Quasilinear Elliptic Equation of Kirchhoff Type

C. O. ALVES
Department of Mathematics, Federal University of Campina Grande
58109-970 Campina Grande, PB, Brazil
coalves@dme.ufcg.edu.br

F. J. S. A. CORRÊA
Department of Mathematics, Federal University of Pará
66075-110 Belém, PA, Brazil
fjulio@ufpa.br

T. F. MA*
Department of Mathematics, State University of Maringá
87020-900 Maringá, PR, Brazil
tofuma@correios.net.br

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Abstract—This paper is concerned with the existence of positive solutions to the class of nonlocal boundary value problems of the type

$$-M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u), \quad \text{in } \Omega, \qquad u=0, \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, M is a positive function, and f has subcritical growth.

Keywords—Variational methods, Nonlocal problems, Kirchhoff equation.

1. INTRODUCTION

In this paper, we consider the existence of positive solutions to the class of boundary value problems of the type

$$-M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u), \quad \text{in } \Omega, \qquad u=0, \quad \text{on} \partial\Omega, \tag{1}$$

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*Author to whom all correspondence should be addressed.
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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R} \to \mathbb{R}$ are continuous functions.

This problem is related to the stationary analogue of the Kirchhoff equation,

$$u_{tt} - M\left(\int_\Omega |\nabla_x u|^2 \, dx\right) \Delta_x u = f(x,t), \tag{2}$$

where $M(s) = as + b$, $a, b > 0$. It was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. The early classical studies dedicated to Kirchhoff equations were given by Bernstein [2] and Pohozaev [3]. However, equation (2) received much attention only after the paper by Lions [4], where an abstract framework to the problem was proposed. Some interesting results can be found, for example, in [5–7].

On the other hand, nonlocal boundary value problems like problem (1) model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [8–12] for some related works.

We are concerned in finding conditions on M and f for which problem (1) possesses a positive solution by mean of variational methods. To our best knowledge, the only variational approach to the problem (1) was given in Ma and Rivera [13], where minimization arguments were used. Here, we study some cases involving strongly indefinite functionals. In a first attempt to study problem (1) variationally, by analogy to superlinear problems of the type $-\Delta u = f(x,u)$, we are conduced to assume M nonincreasing. However, from the original motivation of our problem (equation (2)), the function M should contain the class of the linear functions with positive slopes. Then, to overcome this problem, we use truncation arguments and uniform *a priori* estimates of Gidas and Spruck type, see [14]. In this context, our main result state that if M does not grow too fast in a suitable interval near zero, then, problem (1) has a positive solution, see Theorem 5.

This work is organized as follows. In Section 2, we show the existence of positive solutions for the equation, $-M(\int_\Omega |\nabla u|^2 \, dx)\Delta u = u^p$. In Section 3, we establish a variational setting to the problem (1) and present our main result.

2. HOMOGENEOUS PERTURBATIONS

We begin by considering the problem,

$$-M\left(\|u\|^2\right) \Delta u = f(x), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{3}$$

where $\|u\| = (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$ is the usual norm of $H_0^1(\Omega)$ and M satisfies

$$M(t) \geq m_0, \qquad \forall t \geq 0, \tag{4}$$

for some $m_0 > 0$. Then, motivated by some arguments in [10,11], dedicated to equations of the type,

$$a\left(\int_\Omega u \, dx\right) \Delta u = f(x), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

we might study (3) by comparing it to the problem,

$$-\Delta w = f(x), \quad \text{in } \Omega, \qquad w = 0, \quad \text{on } \partial\Omega. \tag{5}$$

In fact, let $w > 0$ be a solution of (5) and put $u = \gamma w \|w\|^{-1}$. Then, $u > 0$ and

$$-M\left(\|u\|^2\right) \Delta u = -M\left(\gamma^2\right) \frac{\gamma}{\|w\|} \Delta w = M\left(\gamma^2\right) \frac{\gamma}{\|w\|} f.$$

This shows that such u is a positive solution of (3) if and only if γ solves $M(\gamma^2)\gamma = \|w\|$. We can summarize these remarks in the following theorem.

THEOREM 1. *Assume that M satisfies (4) and $f \geq 0$ is a nonzero Hölder continuous function in $\bar{\Omega}$. Then, problem (3) has as many positive solutions as the equation,*

$$M(t)\, t^{1/2} = \|w\| \qquad \text{(with respect to } t\text{)}, \tag{6}$$

where $w > 0$ is the unique solution of (5).

Now, we apply the above arguments to study the existence of positive solutions of the problem,

$$-M\left(\|u\|^2\right) \Delta u = u^p, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{7}$$

where $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ is $N = 1, 2$. This is done by comparing problem (7) with the semilinear problem,

$$-\Delta w = w^p, \quad \text{in } \Omega, \qquad w = 0, \quad \text{on } \partial\Omega, \tag{8}$$

which is well known to possess positive solutions.

THEOREM 2. *Assume that M satisfies (4). Then, problem (7) has at least as many positive solutions as the equation,*

$$\frac{M(t)}{t^{(p-1)/2}} = \|w\|^{1-p} \qquad \text{(with respect to } t\text{)}, \tag{9}$$

where w is a positive solution of (8). In addition, if

$$\lim_{t \to +\infty} \frac{M(t)}{t^{(p-1)/2}} = 0, \tag{10}$$

then, problem (7) has at least one positive solution.

PROOF. Let t be a solution of (9). Then, writing $\gamma = t^{1/2}\|w\|^{-1}$, we see that γw satisfies

$$M\left(\|\gamma w\|^2\right) = M(t) = \gamma^{p-1}.$$

Therefore, $u = \gamma w > 0$ is a solution of (7) since

$$-M\left(\|u\|^2\right) \Delta u = -M\left(\|\gamma w\|^2\right) \gamma \Delta w = \gamma^p w^p = u^p.$$

To see the last statement, we note that since $M \geq m_0 > 0$, one has

$$\lim_{t \to 0^+} \frac{M(t)}{t^{(p-1)/2}} = +\infty.$$

Then, in view of (10), by continuity, equation (9) has a solution, for any positive solution w of (8). ∎

REMARKS.

(i) The above argument is based on the p-homogeneity properties of $f(x,u) = u^p$ and can be easily extended to x-dependent nonlinearity like $f(x,u) = c(x)u^p$. If homogeneity is dropped, variational methods can still be applied, as discussed in the next section.

(ii) We notice that problem (7) with $0 < p < 1$ was considered in [8]. There, the existence results were obtained by using the method of sub- and super-solutions and by assuming other hypotheses for M. ∎

3. VARIATIONAL METHODS

In this section, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the subcritical growth condition,

$$|f(x,s)| \le C(1+|s|^p), \qquad \forall x \in \Omega, \quad \forall s \in \mathbb{R}, \tag{11}$$

where $C > 0$, $1 < p < (N+2)/(N-2)$ if $N \ge 3$ and $1 < p < \infty$ if $N = 1, 2$. A function $u \in H_0^1(\Omega)$ is called weak solution of (1) if

$$M\left(\|u\|^2\right)\int_\Omega \nabla u \nabla \phi\, dx - \int_\Omega f(x,u)\,\phi\, dx = 0, \qquad \text{for all } \phi \in H_0^1(\Omega).$$

Of course, if f is locally Lipschitz in $\bar{\Omega} \times \mathbb{R}$, then, weak solutions are also classical solutions. In addition, we see that weak solutions of (1) are critical points of the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\hat{M}\left(\|u\|^2\right) - \int_\Omega F(x,u)\, dx,$$

where

$$\hat{M}(t) = \int_0^t M(s)\, ds \qquad \text{and} \qquad F(x,t) = \int_0^t f(x,s)\, ds.$$

Since M is continuous and f has subcritical growth, the above functional is of class C^1 in $H_0^1(\Omega)$, and as a matter of fact, by combining the growth of M and f, we can easily obtain existence results by minimization arguments. We emphasize that our concern is with superlinear perturbations.

In order to use critical point theory we first derive an result about the Palais-Smale compactness condition. One says that a sequence (u_n) is a Palais-Smale sequence for the functional I if

$$I(u_n) \text{ is bounded} \qquad \text{and} \qquad \|I'(u_n)\|_* \to 0. \tag{12}$$

If every Palais-Smale sequence of I has a strongly convergent subsequence, then, one says that I satisfies the Palais-Smale condition ((PS) for short). We have the following lemma.

LEMMA 1. *Assume that conditions (4) and (11) hold. Then, any bounded Palais-Smale sequence of I has a strongly convergent subsequence.*

PROOF. Let (u_n) be a bounded (PS) sequence of I. Passing to a subsequence if necessary, there exists $u \in H_0^1(\Omega)$, such that $u_n \rightharpoonup u$ weakly. From the subcritical growth of f and the Sobolev embedding, we see that

$$\int_\Omega f(x,u_n)(u_n - u)\, dx \to 0,$$

and since

$$I'(u_n)(u_n - u) \to 0,$$

we conclude that

$$M\left(\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla(u_n - u)\, dx \to 0. \tag{13}$$

Hence, noting that M can be dropped in (13), we infer that $u_n \to u$ strongly in $H_0^1(\Omega)$. ∎

Now, let us show a basic existence result as a motivation to our main result Theorem 5.

THEOREM 3. *Assume that $f \in C(\bar{\Omega} \times \mathbb{R})$ is a locally Lipschitz function satisfying (11). Assume in addition that*

$$f(x,t) = o(t) \qquad (\text{as } t \to 0) \tag{14}$$

and, for some $\mu > 2$ and $R > 0$,

$$0 < \mu F(x,t) \le f(x,t)\,t, \qquad \forall\, |t| > R. \tag{15}$$

Then, if M is a function satisfying (4) and

$$\hat{M}(t) \geq M(t)\, t, \qquad \forall\, t \geq 0, \tag{16}$$

problem (1) has a positive solution.

PROOF. From the strong maximum principle, positive solutions of (1) are nonzero critical points of $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\hat{M}\left(\|u\|^2\right) - \int_\Omega F\left(x, u^+\right) dx, \tag{17}$$

where $u^+ = \max\{u, 0\}$. Then, according to the mountain pass theorem [15], I has a nonzero critical point if:

(i) $I(0) = 0$,
(ii) there exist $\rho, r > 0$, such that $I(u) \geq \rho$ if $\|u\| = r$,
(iii) there exists e, such that $\|e\| > r$ and $I(e) \leq 0$,

and (PS) holds for I. It turns out that using (16), we can see that I satisfies all the assumptions of the mountain pass theorem as it was $M = 1$. In fact, let us check the (PS) condition. Suppose that (u_n) satisfies (12). Then, letting C to denote several positive constants and using (15) and (16), we get (n was dropped)

$$\begin{aligned} C + C\|u\| &\geq I(u) - \frac{1}{\mu} I'(u)\, u \\ &= \frac{1}{2}\hat{M}\left(\|u\|^2\right) - \frac{1}{\mu} M\left(\|u\|^2\right)\|u\|^2 + \int_\Omega \left(\frac{1}{\mu} f\left(x, u^+\right) u - F\left(x, u^+\right)\right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) M\left(\|u\|^2\right)\|u\|^2 + \int_{|u| \geq R} F\left(x, u^+\right) dx - C. \end{aligned}$$

Since the first two terms in the above line are positive, we conclude that $\|u_n\|$ is bounded. Therefore, I satisfies (PS) from Lemma 1. ∎

REMARK. As we have pointed out before, accordingly, to the original meaning of the M, in the Kirchhoff equation (2), it should be an increasing function. Then,

$$\hat{M}(u) < \int_0^u M(u)\, ds = M(u)\, u, \qquad \forall\, u > 0,$$

and therefore, condition (16) cannot be satisfied. ∎

In what follows, we consider the existence of positive solutions of (1) where M may be increasing. To this end, first, we suppose that M is bounded. More precisely, we assume that there exist $m_1 \geq m_0$ and $t_0 > 0$, such that

$$M(t) = m_1, \qquad \forall\, t \geq t_0. \tag{18}$$

THEOREM 4. *Assume that $f \in (\bar{\Omega} \times \mathbb{R})$ is a locally Lipschitz function satisfying (11), (14), and (15). Assume, in addition, that M is a function satisfying (4) and (18) with*

$$\frac{m_0}{2} - \frac{m_1}{\mu} > 0. \tag{19}$$

Then, problem (1) has a positive solution.

PROOF. We argue as in Theorem 3 to show the functional I defined in (17) has a nonzero critical point. From (4) and (18), we see that

$$\hat{M}(t) \geq m_0 t, \quad \forall\, t \geq 0, \qquad \text{and} \qquad \hat{M}(t) \leq m_1 t + m_2, \quad \forall\, t \geq t_0, \tag{20}$$

where $m_2 = |\int_0^{t_0} M(s)\,ds - m_1 t_0|$. Using standard arguments, we infer that I satisfies

$$I(u) \geq C\|u\|^2 - C\|u\|^{p+1}, \qquad \forall\, u \in H_0^1(\Omega),$$

where C denotes several positive constants. If $\phi \geq 0$ is a nonzero function, we get from (15) and (20)

$$I(t\phi) \leq t^2 \frac{m_1}{2}\|\phi\|^2 - t^{\mu} C\|\phi\|_{\mu}^{\mu} + C \qquad (t > 0 \text{ large}).$$

Thus, clearly, I satisfies the mountain pass geometry and therefore, from the mountain pass theorem, it has a positive critical value provided that (PS) holds. Let (u_n) be a (PS) sequence of I and assume by contradiction that $\|u_n\| \to +\infty$. Then, proceeding as in Theorem 3, we have from (18) and (15),

$$I(u_n) - \frac{1}{\mu} I'(u_n)\,u_n \geq \left(\frac{m_0}{2} - \frac{m_1}{\mu}\right)\|u_n\|^2 + \int_{|u_n| \geq R} F(x, u_n^+)\,dx - C.$$

Then, from assumption (19), we conclude that

$$\|u_n\|^2 \leq C + C\|u_n\|,$$

which contradicts $\|u_n\| \to \infty$. Therefore, (u_n) is bounded and (PS) follows from Lemma 1. ∎

Our goal is to extend Theorem 4 to a larger class of M, including the increasing linear functions. This is done with truncation arguments and *a priori* estimates of Gidas and Spruck [14] type. Accordingly, if

$$\lim_{t\to\infty} \frac{f(x,t)}{t^p} = h(x), \qquad \text{uniformly in } \bar{\Omega}, \tag{21}$$

for some continuous function $h > 0$, then, any classical positive solution of

$$-\Delta u = f(x,u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

satisfies $u(x) \leq C_*$, where C_* depends only on p, h, and Ω. In order to establish our hypotheses, we prove a lemma which shows the relation of the H^1 norm of the solutions of problem (1) with $M(\|u\|^2)$.

LEMMA 2. *Let $f \in C(\bar{\Omega} \times \mathbb{R})$ be, such that*

$$|f(x,u)| \leq C_0|u|^q + C_1|u|^p, \qquad \forall\, x \in \Omega, \quad \forall\, s \in \mathbb{R}, \tag{22}$$

where $C_0 \geq 0$, $C_1 > 0$, $0 < q \leq p$, $1 < p < (N+2)/(N-2)$ if $N \geq 3$ or $1 < p < \infty$ if $N = 1, 2$. Then, if f satisfy (21) and M satisfies (4), there exists $\theta > 0$, not depending on M, such that

$$\|u\|^2 \leq \max\left\{ M\left(\|u\|^2\right)^{(2-p+q)/(p-1)},\ M\left(\|u\|^2\right)^{2/(p-1)} \right\}\theta, \tag{23}$$

for any positive classical solution u of (1).

PROOF. Let u be a positive solution of (1). Then,

$$w = \frac{u}{M\left(\|u\|^2\right)^{1/(p-1)}}$$

is a positive solution of

$$-\Delta w = g(x,w), \quad \text{in } \Omega, \qquad w = 0, \quad \text{on } \partial\Omega,$$

where

$$g(x,s) = \frac{f\left(x, M\left(\|u\|^2\right)^{1/(p-1)} s\right)}{M\left(\|u\|^2\right)^{p/(p-1)}}.$$

Now, since

$$\lim_{s\to\infty} \frac{g(x,s)}{s^p} = \lim_{s\to\infty} \frac{f\left(x, M\left(\|u\|^2\right)^{1/(p-1)} s\right)}{\left(M\left(\|u\|^2\right)^{1/(p-1)} s\right)^p} = h(x)$$

independent of $M > 0$, from Gidas-Spruck estimates there exists $C_* > 0$, not depending on M, such that

$$\|w\|_\infty \le C_*.$$

Therefore,

$$\|u\|_\infty \le M\left(\|u\|^2\right)^{1/(p-1)} C_*$$

and consequently,

$$\begin{aligned}
\|u\|^2 &= M\left(\|u\|^2\right)^{-1} \int_\Omega f(x,u)\, u\, dx \\
&\le M\left(\|u\|^2\right)^{-1} \left(C_0 \|u\|_\infty^{q+1} + C_1 \|u\|_\infty^{p+1}\right) |\Omega| \\
&\le \max\left\{M\left(\|u\|^2\right)^{(2-p+q)/(p-1)},\ M\left(\|u\|^2\right)^{2/(p-1)}\right\} \left(C_0 C_*^{q+1} + C_1 C_*^{p+1}\right) |\Omega|.
\end{aligned}$$

Then, we take $\theta = (C_0 C_*^{q+1} + C_1 C_*^{p+1})|\Omega|$. ∎

Our main result is the following.

THEOREM 5. *Assume that $f \in C(\bar{\Omega} \times \mathbb{R})$ is a locally Lipschitz function satisfying (14), (15), and (21). Assume in addition that M is a continuous function satisfying (4) and there exists $k > 0$, such that*

$$M(k) < \frac{\mu m_0}{2} \tag{24}$$

and

$$\max\left\{M(k)^{(2-p+q)/(p-1)},\ M(k)^{2/(p-1)}\right\} \le \frac{k}{\theta}, \tag{25}$$

where p, q, and θ are given in Lemma 2. Then, problem (1) has a positive solution.

PROOF. First, we note that assumptions (14) and (21) imply that (22) holds, and therefore, θ is well defined. Let us define the truncated function,

$$M_k(t) = \begin{cases} M(t), & \text{if } t \le k, \\ M(k), & \text{if } t > k. \end{cases} \tag{26}$$

Then, assumption (24) implies that M_k satisfies (19) with $m_1 = M(k)$. In addition, noting that (21) implies in (11), we can apply Theorem 4 to obtain $u_k > 0$, solution of the truncated problem,

$$-M_k\left(\|u\|^2\right) \Delta u = f(x,u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

From Lemma 2, we know that

$$\|u_k\|^2 \le \max\left\{M\left(\|u_k\|^2\right)^{(2-p+q)/(p-1)},\ M\left(\|u_k\|^2\right)^{2/(p-1)}\right\} \theta.$$

This implies that if $\|u_k\|^2 > k$, then, we get

$$k < \max\left\{M(k)^{(2-p+q)/(p-1)},\ M(k)^{2/(p-1)}\right\}\theta,$$

which contradicts (25). Therefore, $\|u_k\|^2 \le k$, which shows that u_k is, in fact, a positive solution of the (nontruncated) problem (1). ∎

EXAMPLE. Let us suppose that f is a given function satisfying (15) and (21), with $q = p$ in (22). Once computed μ and θ, we fix $m_0 > 0$ and $\kappa > 0$, such that

$$m_0 < \frac{\kappa^{(p-1)/2}}{\theta} < \frac{\mu}{2} m_0,$$

and define M as the line $M(s) = ms + m_0$ with

$$m = \left(\frac{\kappa^{(p-1)/2}}{\theta} - m_0\right)\frac{1}{\kappa}.$$

It follows that M satisfies the conditions (24) and (25) with $k = \kappa$. ∎

We finish this paper with a multiplicity result by combining local minimization and mountain pass type arguments. The model problem is

$$-M\left(\|u\|^2\right)\Delta u = \lambda u^q + f(x,u), \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{27}$$

where $0 < q < 1$, $\lambda > 0$, and f is superlinear. We note that f_1, defined by

$$f_1(x,u) = \lambda u^q + f(x,u),$$

satisfies condition (22) if, for example, (14) and (21) hold.

THEOREM 6. *Assume that hypotheses of Theorem 5 hold. Then, given $0 < q < 1$ there exists $\lambda_* > 0$, such that problem (27) has at least two positive solutions, for any $\lambda \in (0, \lambda_*)$.*

PROOF. First, we truncate M as in (26). Then, positive solutions of the truncated problem are critical points of the functional $I : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$I(u) = \frac{1}{2}\hat{M}_k\left(\|u\|^2\right) - \frac{\lambda}{q+1}\int_\Omega \left(u^+\right)^{q+1} dx - \int_\Omega F\left(x, u^+\right) dx. \tag{28}$$

Working as in Theorem 4, we infer that

$$I(u) \ge C\|u\|^2 - \lambda C\|u\|^{q+1} - C\|u\|^{p+1}, \qquad \forall u \in H_0^1(\Omega),$$

where C denotes several positive constants. Then, since $q+1 < 2 < p+1$, there are $\rho, r, \lambda_* > 0$, such that

$$I(u) \ge \rho, \qquad \text{for all } \|u\| = r \quad \text{and} \quad \lambda \in (0, \lambda_*). \tag{29}$$

Noting that I is bounded in $X = \bar{B}_r(0)$, it follows from Ekeland variational principle [16] applied to the metric space $(X, \|\cdot\|)$, that there exists a minimizing sequence $(u_n) \subset X$, such that

$$I(u_n) \to I_\infty = \inf_{u \in X} I(u) \qquad \text{and} \qquad I'(u_n) \to 0. \tag{30}$$

It follows that (u_n) is a bounded (PS) sequence of I in X and from Lemma 1, there exists $u \in X$, such $u_n \to u$ strongly. Moreover, since $0 < q < 1$, we have that $I_\infty < 0$, and therefore, $u \in B_r(0)$. This shows that u is a local minimum of I in $H_0^1(\Omega)$. Hence, I has critical a point u_λ^1 with negative energy. Now, continuing the analysis as in Theorem 4, we infer that I has a critical point u_λ^2 with positive energy from the mountain pass theorem. Then, we see that the truncated version of problem (27) has at least two positive solutions. Using Lemma 2, we deduce that $\|u_\lambda^1\|^2 \le k$ and $\|u_\lambda^2\|^2 \le k$, so that u_λ^1 and u_λ^2 are solutions of the original problem (27). ∎

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