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Note

Building blocks for the variety of absolute retracts

Richard C. Brewster[a], Gary MacGillivray[b]

[a]*Department of Mathematics and Statistics, Thompson Rivers University, PO BOX 3010, 900 McGill Road, Kamloops, BC, Canada V2C 5N3*
[b]*Department of Mathematics and Statistics, University of Victoria, PO BOX 3045 STN CSC, Victoria, BC, Canada V8W 3P4*

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Abstract

Given a graph H with a labelled subgraph G, a *retraction* of H to G is a homomorphism $r : H \to G$ such that $r(x) = x$ for all vertices x in G. We call G a *retract* of H. While deciding the existence of a retraction to a fixed graph G is NP-complete in general, necessary and sufficient conditions have been provided for certain classes of graphs in terms of *holes*, see for example Hell and Rival.

For any integer $k \geqslant 2$ we describe a collection of graphs that generate the variety $\mathscr{A}\mathscr{R}_k$ of graphs G with the property that G is a retract of H whenever G is a subgraph of H and no hole in G of size at most k is filled by a vertex of H. We also prove that $\mathscr{A}\mathscr{R}_k \subset \mathrm{NUF}_{k+1}$, where NUF_{k+1} is the variety of graphs that admit a near unanimity function of arity $k + 1$.

Keywords: Absolute retract; Variety; Near unanimity function

1. Introduction

We begin by introducing the terminology of products, retracts, and varieties, before describing the results of the paper. A graph is called *reflexive* if there is a loop at every vertex. For brevity, we use the term *graph* to mean finite, simple, reflexive graph.

Let G and H be graphs. A *homomorphism of H to G*, is a function $h : V(H) \to V(G)$ such that $h(x)h(y) \in E(G)$ whenever $xy \in E(H)$. The existence of such an edge-preserving mapping of the vertices of H to the vertices of G is denoted by $H \to G$, or $h : H \to G$ when the name of the function is important.

If G is a subgraph of H, then a *retraction of H to G* is a homomorphism $r : H \to G$ such that $r(x) = x$ for all $x \in V(G)$. If there exists a retraction of H to G, then G is called a *retract* of H.

Let $G_1, G_2, \ldots, G_n$ be graphs. The *strong product of* $G_1, G_2, \ldots, G_n$ is the graph $\prod_{i=1}^{n} G_i$ with vertices $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, and an edge joining $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ if and only if $x_i y_i \in E(G_i)$ for $i = 1, 2, \ldots, n$. We denote by G^k the strong product of k copies of G. Observe that since we are studying reflexive graphs, the product $K_2 \times K_2$ is a (reflexive) copy of K_4.

A set X of graphs is a *variety* if every retract of a graph in X is also in X, and the strong product of any finite number of graphs in X is also in X. Given a set S of graphs, the *variety generated by* S is the smallest variety that contains S.

E-mail address: gmacgill@uvic.ca (G. MacGillivray).

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Let G be a fixed graph. The *retraction problem for* G, denoted RET_G, is the problem of deciding if a given graph H, with G as a labelled subgraph, admits a retraction to G. It follows from results in [5] that RET_G is polynomial time solvable for each chordal graph G, and from results in [3] that RET_G is NP-complete for each graph G that admits a retraction to a cycle of length at least four. A dichotomy theorem that, assuming $\mathrm{P} \neq \mathrm{NP}$, characterizes the graphs G for which the problem is polynomial time solvable and states that the problem is NP-complete for all other graphs G, is unknown. Feder and Vardi [7] have established an equivalence between constraint satisfaction problems and retraction problems for (reflexive) graphs. Their conjecture that, assuming $\mathrm{P} \neq \mathrm{NP}$, each constraint satisfaction problem is either NP-complete or polynomial time solvable is thus equivalent to the assertion that such a dichotomy theorem for RET_G exists. We remark that dichotomy theorems for variants of the homomorphism problem do exist for several large classes, see [4,6,9].

Given the NP-completeness results above, it is not surprising that no theorem giving necessary and sufficient conditions for a graph G to be a retract of a graph H of which it is a labelled subgraph is known. However, RET_G is tractable for certain classes of graphs. Many authors have identified such classes by describing an obvious necessary condition, say $\mathscr{C}$, and then studying those graphs G for which the condition is also sufficient. The collection of all such G is the set of *absolute retracts with respect to* $\mathscr{C}$. See [12] for an excellent survey.

One necessary condition for G to be a retract of H is that G is an *isometric* subgraph of H, meaning that for all vertices x and y of G the distance between x and y in G equals the distance between x and y in H (i.e. $d_G(x, y) = d_H(x, y)$). Absolute retracts with respect to isometry have been characterized in a number of different ways [10,11,13,14] (and others, see [12]). One such characterization, due to Nowakowski and Rival [13], states that a graph G is a retract of any graph for which it is an isometric subgraph if and only if G is in the variety generated by the finite paths.

A generalization of the condition "G is an isometric subgraph of H" involving k-subsets of vertices of G, $k \geqslant 2$, was first studied in [11]. Let G be a graph. For an integer $k \geqslant 2$, a *k-hole* in G is a pair (L, f), where L is a k-subset of $V(G)$ and $f : L \to \mathbb{Z}^+$, such that the following two conditions are satisfied:

(i) no vertex $x \in V(G)$ satisfies $d(x, \ell) \leqslant f(\ell)$ for all $\ell \in L$, and
(ii) for any proper subset $L' \subset L$, there is a vertex y such that $d(y, \ell) \leqslant f(\ell)$ for all $\ell \in L'$.

Suppose G is a subgraph of H. A k-hole (L, f) in G is *filled* in H if some vertex $x \in V(H)$ satisfies $d_H(x, \ell) \leqslant f(\ell)$ for all $\ell \in L$. The vertex x is said to *fill* the hole.

As an example, consider a six cycle with vertices 0, 1, 2, 3, 4, 5 (in the natural order). The set $L = \{0, 2, 4\}$ and the function $f(0) = f(2) = f(4) = 1$ is a hole since no vertex in the cycle is simultaneously at distance at most one from each of 0, 2, and 4, and there is a vertex at distance at most one from any pair of vertices in L. Let H be the graph obtained by adding a single vertex v to the cycle, and joining v to each vertex in L. Observe that the cycle is an isometric subgraph of H; however, v fills the hole (L, f) and consequently H does not admit a retraction to the cycle.

We define $\mathscr{AR}_k$ to be the set of all finite graphs G with the property that for any graph H, G is a retract of H whenever G is a subgraph of H, and for $2 \leqslant i \leqslant k$, no vertex of H fills an i-hole in G. Observe that $\mathscr{AR}_2 \subseteq \mathscr{AR}_3 \subseteq \cdots$. We define $\mathscr{AR}$, the set of *absolute retracts*, to be $\bigcup_{k=2}^{\infty} \mathscr{AR}_k$. We remark that the inclusion $\mathscr{AR}_k \subseteq \mathscr{AR}_{k+1}$ is in fact strict, i.e. $\mathscr{AR}_k \neq \mathscr{AR}_{k+1}$, for $k = 2, 3, \ldots$ (see [12]). It transpires that, for each $k \geqslant 2$, the set $\mathscr{AR}_k$ is a variety (the proof of this fact is below), as is $\mathscr{AR}$. A characterization of $\mathscr{AR}$, attributed to an unpublished manuscript of Winkler, appears in [12].

Note that statement "no vertex in H fills a 2-hole in G" is equivalent to the statement that G is an isometric subgraph of H. Also, note that a graph which belongs to $\mathscr{AR}_k$ cannot have an ℓ-hole for $\ell > k$ as any filled hole is an obstruction to the existence of a retraction.

Hell and Rival [10] described a set Y of graphs such that $\mathscr{AR}_3$ is the variety of generated by Y. For the variety $\mathscr{AR}_k$, it might be possible to construct a set of generators by generalizing the so-called Y-graphs from [10] that generate $\mathscr{AR}_3$. However, this appears to quickly become unwieldy.

In Section 2 we present our main result: for each $k \geqslant 2$ we describe a collection S_k of graphs such that $\mathscr{AR}_k$ is the variety generated by $\bigcup_{n=2}^{k} S_n$. The collection $S_2 \cup S_3$ that generates $\mathscr{AR}_3$ admits a simpler description than the Y-graphs. (Our original motivation for this work was to find a simpler description of the Y-graphs.) A number of the ideas used in this paper can be seen to follow naturally from ideas in [10].

Feder and Vardi [7] proved that if there exists k such that the graph G admits a near unanimity function of arity k (defined in Section 3), then the problem RET_G is polynomial time solvable. In Section 3, we show that $\mathscr{AR}_k \subseteq \mathrm{NUF}_{k+1}$, where NUF_t is the variety of graphs that admit a near unanimity function of arity t.

2. The variety $\mathscr{AR}_k$

The following theorem is proved in [10] for $k = 2$ and $k = 3$, but the argument works for all integers $k \geqslant 2$.

Theorem 2.1 (*Hell and Rival [10]*). *For any integer $k \geqslant 2$, $\mathscr{AR}_k$ is a variety.*

Corollary 2.2 (*Hell and Rival [10]*). *The set $\mathscr{AR}$ of absolute retracts is a variety.*

The following concepts are from [10], but we include them here for completeness. Suppose $h : G \to H$ is a homomorphism. A k-hole (L, f) in G is *separated in H* if no vertex $x \in V(H)$ satisfies $d_H(x, h(\ell)) \leqslant f(\ell)$ for all $\ell \in L$. Such a homomorphism h is called a *separating map to H* for (L, f). If G is a subgraph of H, the inclusion map is a homomorphism of G to H. Moreover, it is a separating map to H for (L, f) if and only if (L, f) is not filled in H.

Let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ be elements of $\mathbb{R}^k$. We say that x *dominates* y if $x_i \geqslant y_i$ for $i = 1, 2, \ldots, k$.

Let G be a graph and let L be a k-subset of $V(G)$. The *distance labelling of G with respect to L* is the assignment to each vertex x of G, the k-tuple $(d(x, \ell_1), d(x, \ell_2), \ldots, d(x, \ell_k))$. Let $\mathscr{S}(G, L)$ be the set of all such k-tuples that arise in the distance labelling of G with respect to L, and let $M(G, L)$ be the largest integer that occurs in any component of a k-tuple in $\mathscr{S}(G, L)$, i.e. $M(G, L) = \max_{x,\ell_i} d(x, \ell_i)$.

A *distance matrix of size k* is a $k \times k$ symmetric matrix D with non-negative integer entries, such that $d_{ii} = 0$ for $i = 1, 2, \ldots, k$ and $d_{ac} \leqslant d_{ab} + d_{bc}$ for all $a, b, c \in \{1, 2, \ldots, k\}$.

Let D be a distance matrix of size k, and $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be a k-set. The graph $K = K(D, L)$ is the subdivision of the complete graph with vertex set L obtained by replacing each edge $\ell_i\ell_j$ by a path of length d_{ij}, $1 \leqslant i < j \leqslant k$. By construction, $d_{ij} = d_K(\ell_i, \ell_j)$.

Let D be a distance matrix and L a k-set. Consider the distance labelling of $K = K(D, L)$ with respect to L. Suppose $i < j$ and the path $\ell_i = v_0, v_1, v_2, \ldots, v_{d_{ij}} = \ell_j$ replaced the edge $\ell_i\ell_j$ in the construction of K. It is not hard to see that the k-tuple assigned to v_p is $(m_1, m_2, \ldots, m_{i-1}, p, m_{i+1}, m_{i+2}, \ldots, m_{j-1}, d_{ij} - p, m_{j+1}, m_{j+2}, \ldots, m_k)$, where for $r = 1, 2, \ldots, k$, we have $m_r = \min\{p + d_{ir}, d_{ij} - p + d_{jr}\}$.

Lemma 2.3. *Suppose G is a graph and (L, f) is a k-hole in G, where $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$. Further suppose D is the distance matrix whose entries are defined by $d_{ij} = d_G(\ell_i, \ell_j)$. Then the k-tuple $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$ does not dominate any element of $\mathscr{S}(K, L)$, where $K = K(D, L)$.*

Proof. For each pair of subscripts i and j with $1 \leqslant i < j \leqslant k$, let P_{ij} be a shortest $\ell_i - \ell_j$ path in G. Let F be the subgraph of G induced by $\bigcup_{1 \leqslant i < j \leqslant k} P_{ij}$.Consider the distance labelling of F with respect to L. The label assigned to the vertex at distance p from ℓ_i along P_{ij} is clearly dominated by

$$(m_1, m_2, \ldots, m_{i-1}, p, m_{i+1}, m_{i+2}, \ldots, m_{j-1}, d_{ij} - p, m_{j+1}, m_{j+2}, \ldots, m_k),$$

where $m_r = \min\{p + d_{ir}, d_{ij} - p + d_{jr}\}$. The latter is an element of $\mathscr{S}(K, L)$. By letting i, j and p vary, it follows that every element of $\mathscr{S}(K, L)$ dominates a label that occurs in the distance labelling of F, i.e. each element of $\mathscr{S}(F, L)$ is dominated by an element of $\mathscr{S}(K, L)$.

Now consider the distance labelling of G with respect to L. The label assigned to a vertex of F in this labelling of G is dominated by the label it receives in the distance labelling of F with respect to L. That is, each element of $\mathscr{S}(F, L)$ dominates an element of $\mathscr{S}(G, L)$. Therefore, each element of $\mathscr{S}(K, L)$ dominates an element of $\mathscr{S}(G, L)$. Since (L, f) is a k-hole in G, the k-tuple $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$ does not dominate any element of $\mathscr{S}(G, L)$. The result now follows. □

Let D be a distance matrix of size k. Let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be a k-set and $f : L \to \mathbb{N}$ be such that $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$ does not dominate any element of $\mathscr{S}(K, L)$, where $K = K(D, L)$. A k-tuple $(x_1, x_2, \ldots, x_k)$ of non-negative integers is called *admissible with respect to D* if $x_i + x_j \geqslant d_{ij}$, $1 \leqslant i, j \leqslant k$. (We think of an admissible k-tuple as a label that could be received by a vertex in the distance labelling of some graph containing L, where the distances between vertices in L are given by the entries in D.) A *k-separator* is a graph $R = R(D, L, f)$ defined as follows. The vertices of R are the admissible k-tuples dominated by $(M(K, L), M(K, L), \ldots, M(K, L))$ and not

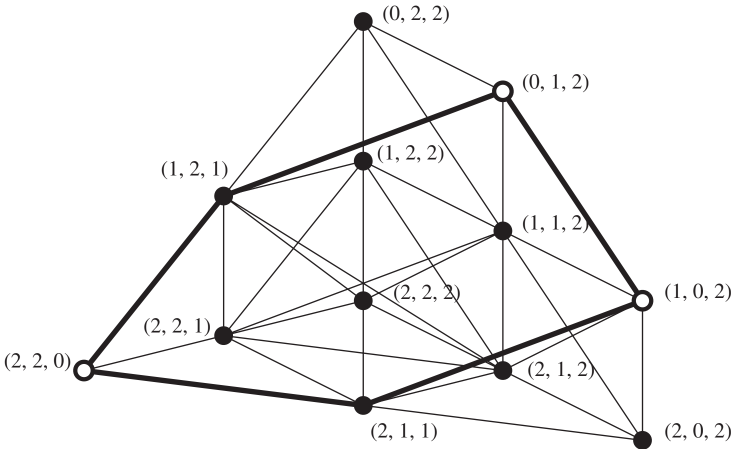

Fig. 1. An example of a 3-separator.

dominated by $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$. Two vertices $(x_1, x_2, \ldots, x_k)$, $(y_1, y_2, \ldots, y_k) \in V(R)$ are adjacent if and only if $|x_i - y_i| \leqslant 1$, $1 \leqslant i \leqslant k$.

Let D and L be as above. Observe that ith row of D is precisely the label received by ℓ_i in the distance labelling of $K = K(D, L)$ with respect to L. In particular, the label is admissible. However, by construction of $R(D, L, f)$, such a label is a vertex of $R(D, L, f)$. In other words, the rows of D correspond to a subset of vertices in $R(D, L, f)$. We let L_D denote this subset of vertices. Thus there is a natural correspondence between the elements of L and the elements of L_D, which we denote $\varphi : L \to L_D$.

An example of a 3-separator is given in Fig. 1. The distance matrix is

$$D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

and the function f is defined by $f(\ell_1) = f(\ell_2) = f(\ell_3) = 1$. The vertices $\varphi(\ell_1)$, $\varphi(\ell_2)$, and $\varphi(\ell_3)$ are shown in white, and the graph $K(D, L)$ is drawn in bold.

Lemma 2.4. *Suppose $x = (x_1, x_2, \ldots, x_k)$ is a vertex in $R = R(D, L, f)$ Then $d_R(x, \varphi(\ell_i)) \geqslant x_i$ for all $\ell_i \in L$.*

Proof. The ith coordinate of x is x_i, whereas, the ith coordinate of $\varphi(\ell_i)$ is 0. The result follows from the definition of adjacency in R. □

The following assertion, with $k = 3$ and 3-separator replaced by Y-graph, appears in the proof of Theorem 2 in [10].

Lemma 2.5. *Suppose (L, f) is a k-hole in G. Then there exists a k-separator R for which there is a separating map from G to R for (L, f).*

Proof. Let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$, and let D be the $k \times k$ distance matrix in which $d_{ij} = d_G(\ell_i, \ell_j)$. Let $R = R(D, L, f)$, and $K = K(D, L)$.

Consider the distance labelling of G with respect to L. Every label that arises is admissible, and none is dominated by $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$. Define a function $h : V(G) \to V(R)$ as follows. For each vertex $x \in V(G)$ with label $(x_1, x_2, \ldots, x_k)$, map x to $(M(K, L), M(K, L), \ldots, M(K, L))$ if $(x_1, x_2, \ldots, x_k)$ dominates $(M(K, L), M(K, L), \ldots, M(K, L))$, and otherwise map x to vertex $(x_1, x_2, \ldots, x_k)$ of R. It follows from the definitions of the distance labelling of G and adjacency in R that h is a homomorphism.

We remark that $h|_L$ is in fact the correspondence $\varphi : L \to L_D$ described above. Hence, for $i = 1, 2, \ldots, k$, the k-tuple $h(\ell_i) \in L_D$. Moreover, by Lemma 2.4 above there cannot be a vertex $y \in V(R)$ such that $d_R(y, h(\ell_i)) \leqslant f(\ell_i)$.

(Suppose to the contrary such a y does exist. Then $y_i \leqslant d_R(y, h(\ell_i)) \leqslant f(\ell_i)$, and y is not an admissible k-tuple, contrary to the assumption $y \in V(R)$.) It follows that h is a separating map to R for (L, f). □

We have just proved that given a particular hole in G, there is a homomorphism from G to R which separates the hole. The following result from [10] allows us construct a homomorphism (to a constructed target) which simultaneously separates all of the holes in G.

Lemma 2.6 (*Hell and Rival [10]*). *Let G be a fixed graph. Let N be a set of integers each of which is at least two, with $2 \in N$. If for each $n \in N$ and each n-hole (L, f) of G there is a graph $H(L, f)$ and a separating map to $H(L, f)$ for (L, f), then G is (isomorphic to) a subgraph of $P = \prod\{H(L, f) : (L, f)$ is an n-hole and $n \in N\}$ such that all n-holes, $n \in N$ of G are separated in P.*

Lemma 2.7. *Let $k \geqslant 2$ be an integer. Every k-separator $R(D, L, f)$ belongs to $\mathscr{AR}_k$.*

Proof. Suppose $R = R(D, L, f)$ is a subgraph of H, and no k-hole in R is filled in H. Consider the distance labelling of H with respect to L_D. Every label that arises is admissible with respect to D, and none is dominated by $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$. Define a function $h : V(H) \to V(R)$ as follows. For each vertex $x \in V(H)$ with label $(x_1, x_2, \ldots, x_k)$, map x to $(M(R, L_D), M(R, L_D), \ldots, M(R, L_D))$ if $(x_1, x_2, \ldots, x_k)$ dominates the k-tuple $(M(R, L_D), M(R, L_D), \ldots, M(R, L_D))$, and otherwise map x to the vertex $(x_1, x_2, \ldots, x_k)$ of R. It follows from the definitions of distance labelling of H and adjacency in R that h is a homomorphism. The definition of R further implies that h maps each vertex of R to itself.Therefore, h is a retraction. □

Theorem 2.8. *A graph belongs to $\mathscr{AR}_k$ if and only if it is in the variety of n-separators, $2 \leqslant n \leqslant k$.*

Proof. By Lemma 2.7, every n-separator belongs to $\mathscr{AR}_n$. Since $\mathscr{AR}_2 \subseteq \mathscr{AR}_3 \subseteq \cdots \subseteq \mathscr{AR}_k$, every n separator with $2 \leqslant n \leqslant k$ also belongs to $\mathscr{AR}_k$. Hence $\mathscr{AR}_k$ contains the variety of n-separators, $2 \leqslant n \leqslant k$.

On the other hand, Lemmas 2.5 and 2.6 together imply that the variety of n-separators, $2 \leqslant n \leqslant k$, contains $\mathscr{AR}_k$. To see this, suppose $G \in \mathscr{AR}_k$. By Lemma 2.5, there exists an n-separator R and a separating map $G \to R$ for each n-hole, $2 \leqslant n \leqslant k$. By Lemma 2.6, G is a subgraph of P, the product of all the separators. Moreover, since no n-hole, $2 \leqslant n \leqslant k$, of G is filled in P and $G \in \mathscr{AR}_k$, we have G is a retract of P. Thus G is in the variety of n-separators. □

3. Near unanimity functions

Let G be a graph. A *near unanimity function of arity k* is a homomorphism $g : G^k \to G$ which is *nearly unanimous*: $g(x_1, x_2, \ldots, x_k) = x$ whenever at least $k-1$ of $x_1, x_2, \ldots, x_k$ equal x.

We denote by NUF_k the set of graphs that admit a near unanimity function of arity k. The set NUF_k is a variety, see [2]. Connections between near unanimity functions and holes are studied in [2,7,12]. In particular, no graph with a k-hole can belong to NUF_k. (Also, see Theorem 3.2 below.)

Let $x_1, x_2, \ldots, x_t$ be integers. We denote by $\sigma_2(x_1, x_2, \ldots, x_t)$ the integer x_{i_2}, where $x_{i_1} \geqslant x_{i_2} \geqslant \cdots \geqslant x_{i_t}$.

For clarity we note in the following lemma that X is simply a set of k-tuples, and g is a function.

Lemma 3.1. *Let $X \subseteq \mathbb{N}^k$. Suppose $g : X^{k+1} \to X$ is defined by*

$$\begin{aligned} & g((x_{11}, x_{12}, \ldots, x_{1k}), (x_{21}, x_{22}, \ldots, x_{2k}), \ldots, (x_{(k+1)1}, x_{(k+1)2}, \ldots, x_{(k+1)k})) \\ & \quad = (\sigma_2(x_{11}, x_{21}, \ldots, x_{(k+1)1}), \sigma_2(x_{12}, x_{22}, \ldots, x_{(k+1)2}), \ldots, \sigma_2(x_{1k}, x_{2k}, \ldots, x_{(k+1)k})). \end{aligned}$$

Then, for any collection of $k+1$ elements, $v_1, v_2, \ldots, v_{k+1} \in X$, there exists v_i, $1 \leqslant i \leqslant k+1$, such that $g(v_1, v_2, \ldots, v_{k+1})$ dominates v_i.

Proof. The proof is by induction on k. The statement is clear when $k = 2$. Suppose it is true when $k = t$, for some $t \geqslant 2$. Consider the situation for $t+1$. Without loss of generality, $\sigma_2(x_{11}, x_{21}, \ldots, x_{(t+1)1}) = x_{21}$ and $x_{11} \geqslant x_{21}$. Let $\tilde{X}$

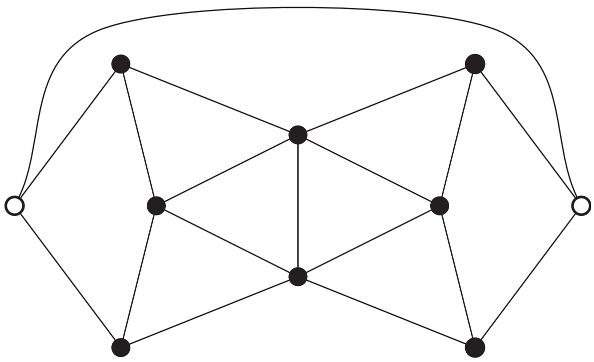

Fig. 2. The graph on the black vertices is in NUF_5, but not in $\mathscr{AR}_k$ for any k.

be the projection of X onto its last $(t-1)$ coordinates. That is, $\tilde{X} = \{(y_2, y_3, \ldots, y_t) | (y_1, y_2, y_3, \ldots, y_t) \in X\}$. Let $g' : \tilde{X}^t \to \tilde{X}$ be defined as in the statement of the lemma. Then, by the induction hypothesis,

$$\begin{aligned} &g'((x_{22}, x_{23} \ldots, x_{2t}), (x_{32}, x_{33} \ldots, x_{3t}), \ldots, (x_{(t+1)2}, x_{(t+1)3}, \ldots, x_{(t+1)t})) \\ &\quad = (\sigma_2(x_{22}, x_{32}, \ldots, x_{(t+1)2}), \sigma_2(x_{23}, x_{33}, \ldots, x_{(t+1)3}) \ldots, \sigma_2(x_{2t}, x_{3t}, \ldots, x_{(t+1)t})) \end{aligned}$$

dominates one of the $(t-1)$-tuples from $(x_{22}, x_{23} \ldots, x_{2t}), (x_{32}, x_{33} \ldots, x_{3t}), \ldots, (x_{(t+1)2}, x_{(t+1)3}, \ldots, x_{(t+1)t})$. Recall $x_{11} \geqslant x_{21} = \sigma_2(x_{11}, x_{21}, \ldots, x_{(t+1)1})$ and observe $\sigma_2(x_{1i}, x_{2i}, \ldots, x_{(t+1)i}) \geqslant \sigma_2(x_{2i}, x_{3i}, \ldots, x_{(t+1)i})$ for $i = 2, 3, \ldots, t$. Thus, the t-tuple, in X, $(\sigma_2(x_{11}, x_{21}, \ldots, x_{(t+1)1}), \sigma_2(x_{12}, x_{22}, \ldots, x_{(t+1)2}), \ldots, \sigma_2(x_{1t}, x_{2t}, \ldots, x_{(t+1)t}))$ dominates one of the following t-tuples: $(x_{21}, x_{22}, \ldots, x_{2t}), (x_{31}, x_{32}, \ldots, x_{3t}), \ldots (x_{(t+1)1}, x_{(t+1)2}, \ldots, x_{(t+1)t})$. The result now follows by induction. $\square$

The following result has been independently proved by Loten [12].

Theorem 3.2. *For all* $k \geqslant 2$, $\mathrm{NUF}_{k+1} \supseteq \mathscr{AR}_k$.

Proof. It is enough to show that every k-separator belongs to NUF_{k+1}.

Let D be a distance matrix of size k. Let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be a k-set and $f : L \to \mathbb{N}$ be such that $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$ does not dominate any vertex of $\mathscr{S}(K, L)$, where $K = K(D, L)$. Let $R = R(D, L, f)$. Define g on $V(R^{k+1})$ by

$$\begin{aligned} &g((x_{11}, x_{12}, \ldots, x_{1k}), (x_{21}, x_{22}, \ldots, x_{2k}), \ldots, (x_{(k+1)1}, x_{(k+1)2}, \ldots, x_{(k+1)k})) \\ &\quad = (\sigma_2(x_{11}, x_{21}, \ldots, x_{(k+1)1}), \sigma_2(x_{12}, x_{22}, \ldots, x_{(k+1)2}), \ldots, \sigma_2(x_{1(k+1)}, x_{2(k+1)}, \ldots, x_{k(k+1)})) = r. \end{aligned}$$

It follows from the definition that g is nearly unanimous. We will show that g is a homomorphism of R^{k+1} to R. By definition of σ_2 and $V(R)$, the k-tuple r is dominated by $(M(K, L), M(K, L), \ldots, M(K, L))$, where $K = K(D, L)$. By Lemma 3.1, r dominates a vertex of R. Since the vertices of R are admissible k-tuples not dominated by $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$, this implies that r is admissible and not dominated by $(f(\ell_1), f(\ell_2), \ldots, f(\ell_k))$. Therefore, r is a vertex of R. It remains to argue that g preserves edges. This is implied by the definition of $E(R)$ and the following observation. Suppose $|x_i - y_i| \leqslant 1$ for $i = 1, 2, \ldots, k+1$, then

$$|\sigma_2(x_1, x_2, \ldots, x_{k+1}) - \sigma_2(y_1, y_2, \ldots, y_{k+1})| \leqslant 1.$$

This completes the proof. $\square$

It is known that $\mathscr{AR}_2 = \mathrm{NUF}_3$, see [1], but it is unknown whether or not $\mathscr{AR}_3 = \mathrm{NUF}_4$. For $k \geqslant 4$, $\mathscr{AR}_k \subset \mathrm{NUF}_{k+1}$. An example, independently found by several people, and published in [12] is shown in Fig. 2. Let G be the graph induced by the black vertices in the figure, and let H be entire graph. Then H does not fill any hole in G, but H does not retract to G. Hence $G \notin \mathscr{AR}_k$ for any k. Yet G is a chordal graph with *leafage* 4 and by a result in [2] G does belong to NUF_5.

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