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# Generic Existence of a Nonempty Compact Set of Fixed Points

TOMÁS DOMÍNGUEZ BENAVIDES

*Facultad de Matemáticas, Universidad de Sevilla, Sevilla, Spain*

*Submitted by Ky Fan*

Let $X$ be a complete metric space, $\mathscr{M}$ a set of continuous mappings from $X$ into itself, endowed with a metric topology finer than the compact-open topology. Assuming that there exists a dense subset $\mathscr{N}$ contained in $\mathscr{M}$ such that for every mapping $T$ in $\mathscr{N}$ the set $\{x \in X: Tx = x\}$ is nonempty, it is proved that most mappings (in the Baire category sense) in $\mathscr{M}$ do have a nonempty compact set of fixed points. Some applications to $\alpha$-nonexpansive operators, semiaccretive operators and differential equations in Banach spaces are derived.

## INTRODUCTION

A property is said to be generic in a Baire space $E$ if it holds in a residual subset of $E$. In [7, 11] the generic existence of a fixed point for a metric space $\mathscr{M}$ of mappings on a complete metric space $X$ has been studied. In both papers it is assumed that there exists a dense subset $\mathscr{N}$ of $\mathscr{M}$ such that every mapping $T$ in $\mathscr{N}$ has a unique fixed point. Proofs in these papers heavily rest upon the uniqueness of the fixed point in $\mathscr{N}$. The main result in this paper is a generic existence theorem of fixed points (Theorem 1, Section 1) with the weaker assumption: For every mapping $T$ in $\mathscr{N}$ the set $\{x \in X: Tx = x\}$ is a nonempty set. We also prove that fixed points of $T$ continuously depends (in a weak sense, see, (Theorem 1(ii)) on the mapping $T$). In Section 2, we apply Theorem 1 to prove the generic existence (and weak continuous dependence) of fixed points for $\alpha$-nonexpansive mappings (i.e., mappings which satisfy $\alpha(T(A)) \leqslant \alpha(A)$ for every bounded set $A$, where $\alpha(\cdot)$ denotes the Kuratowski's noncompactness measure [9, p. 412]) and $\alpha$-nonexpansive mappings defined on cones. Generic problems concerning $\alpha$-nonexpansive mappings have been treated in [2, 5]. In Section 3, Theorem 1 is applied to prove that most of the semiaccretive mappings (i.e., compact perturbations of accretive mappings, see the definition in Section 3) satisfying a boundary condition do have a zero. In Section 4, using Theorem 1, we answer to an open question proposed in [11] about the



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generic existence of solution of a differential equation in a Banach space with Caratheodory's hypotheses. Further application are indicated in Sections 2 and 4.

## 1. Main Result

In the following $X$ will be a complete metric space, $\mathbb{N}$ the set of the positive integers and $\mathscr{M}$ a set of continuous mappings from $X$ into $X$ endowed with a metric finer than the compact open topology.

Theorem 1. *Assume that there exists a subset $\mathscr{N}$ of $\mathscr{M}$ satisfying:*

(a) *$\mathscr{N}$ is dense in $\mathscr{M}$.*

(b) *For every $f$ in $\mathscr{N}$ the set of the solutions of*

$$x = f(x) \tag{1}$$

*is nonempty.*

(c) *For every $f$ in $\mathscr{N}$ and every sequence $\{x_n\}$ such that $x_n$ is a solution of*

$$x = f_n(x), \tag{2}$$

*where $\{f_n\} \to f$ in $\mathscr{M}$, there exists a subsequence of $\{x_n\}$ which converges to a solution of* (1).

*Then, there exists a residual subset $\mathscr{M}'$ of $\mathscr{M}$ such that the following conditions are satisfied by every $f$ in $\mathscr{M}'$:*

(i) *$\{x \in X: x = f(x)\}$ is a nonempty compact set.*

(ii) *If $\{f_n\} \to f$ in $\mathscr{M}$ and $x_n$ is a solution of* (2), *there exists a subsequence of $\{x_n\}$ which converges to a solution of* (1).

*Proof.* Let $\{x_n\}$ be a sequence in $X$. Define

$$A(\{x_n\}) = \sup\{d(x_n, x_m): n, m \in \mathbb{N}\}$$

$$B(\{x_n\}) = \inf\ \{A(\{x'_n\}): \{x'_n\} \text{ is a subsequence of } \{x_n\}\}.$$

For every $f$ in $\mathscr{M}$ denote

$$V(f) = \sup\{B(\{x_n\}): \{f_n\} \to f \text{ in } \mathscr{M} \text{ and } x_n \text{ is a solution of } (2)\}.$$

*Claim* 1. Assume $V(f)=0$. If $\{f_n\}\to f$ in $\mathscr{M}$ and $x_n$ is a solutions of (2) there exists a Cauchy subsequence of $\{x_n\}$.

Since $B(\{x_n\})=0$ there exists a subsequence $\{x_n^1\}$ such $A(\{x_n^1\})<1$. By induction, if $\{x_n^m\}$ is a subsequence of $\{x_n\}$ satisfying $A(\{x_n^m\})<1/m$ we can construct a subsequence $\{x_n^{m+1}\}$ of $\{x_n^m\}$ such that $A(\{x_n^{m+1}\})<1/(m+1)$. Consider the diagonal sequence $\{x_n^n\}$. For every real number $\varepsilon>0$ let $N$ be an integer number greater than $1/\varepsilon$. Since $x_n^n$ is a term of $\{x_n^N\}$ for $n>N$ we obtain that $\{x_n^n\}$ is a Cauchy sequence.

*Claim* 2. For every $f$ in $\mathscr{N}$, $V(f)=0$.

Let $f$ be a mapping in $\mathscr{N}$, $\{f_n\}$ a sequence converging to $f$ in $\mathscr{M}$, and $x_n$ a solution of (2). By (c) there exists a Cauchy subsequence $\{x_n'\}$ of $\{x_n\}$. For every real number $\varepsilon>0$ we can choose a subsequence $\{x_n''\}$ of $\{x_n'\}$ such that $A(\{x_n''\})<\varepsilon$. Thus $B(\{x_n\})<\varepsilon$ that implies $V(f)=0$.

*Claim* 3. At every mapping $f$ in $V^{-1}(\{0\})$, $V$ is continuous.

Otherwise, for some $f$ in $\mathscr{M}$ with $V(f)=0$ we can find a real number $\eta>0$ and a sequence $\{f_n\}\to f$ in $\mathscr{M}$ such that $V(f_n)>\eta$ for every $n$ in $\mathbb{N}$. Then, for each $n$ there exists a sequence $\{f_{n,m}\}\to f_n$ in $\mathscr{M}$ and a sequence $\{x_{n,m}\}$ of solutions of $x=f_{n,m}(x)$ such that $B(\{x_{n,m}\})>\eta/2$. We can assume $d(f_{n,m},f_n)<1/n$ for every $m\geqslant n$. For each $n$ we can construct a subsequence $\{x_{n,\phi(m)}\}$ ($\phi$: $\mathbb{N}\to\mathbb{N}$ order preserving) satisfying

$$d(x_{n,\phi(m)},x_{n,\phi(m')})>\eta/4 \tag{3}$$

for every $m,m'\in\mathbb{N}$ $(m\neq m')$. In order to do that choose $\phi(1)=1$. Let $\phi$ be defined for $j=1,\ldots,k-1$, satisfying (3). There exists $m>\phi(k-1)$ such that $d(x_{n,m},x_{n,\phi(j)})>\eta/4$ $(j=1,\ldots,k-1)$ because otherwise we could choose for some $x_{n,\phi(j)}$ $(j\leqslant k-1)$ a subsequence of $\{x_{n,m}\}$ formed by all points whose distance to $x_{n,\phi(j)}$ is less or equal than $\eta/4$, contradicting $B(\{x_{n,m}\})>\eta/2$. Denote again $\{x_{n,m}\}$ the subsequence which satisfies (3). Hence we have

$$d(x_{n,m},x_{n,m'})>\eta/4 \qquad \text{for every} \quad n,m,m'\in\mathbb{N}, \quad m\neq m'. \tag{4}$$

By $A\leqslant B$ we shall mean that $A$ and $B$ are ordered subset of $\mathbb{N}$ and $A$ is contained in $B$, so that $\{(n,x_n): n\in A\}$ is a subsequence of $\{x_n\}$ if $A\leqslant\mathbb{N}$. Consider the sequence $\{(n,x_{n,n}): n\in\mathbb{N}\}$. Since $\{f_{n,n}\}\to f$ and $V(f)=0$ there exists $N_0\leqslant\mathbb{N}$ such that the sequence $\{(n,x_{n,n}): n\in N_0\}$ converges to a solution $x_0$ of (1). Consider the sequence $\{(n,x_{n,n+1}): n\in N_0\}$. We can construct a subsequence $\{(n,x_{n,n+1}): n\in N_1\}$ $(N_1\leqslant N_0)$ converging to a solution $x_1$ of (1). By induction we can construct sequences $\{n,x_{n,n+k}): n\in N_k\}$ $(N_k\leqslant N_{k-1})$ converging to solutions $x_k$ of (1). Since

$V(f)=0$ we choose by Claim 1 a Cauchy subsequence $\{(k, x_k): k \in K\}$ ($K \leqslant \mathbb{N}$) of $\{x_k\}$. Hence for $k, k' \in K$ ($k < k'$) large enough one has $d(x_k, x_{k'}) < \eta/8$. Since $\{(n, x_{n,n+k}): n \in N_{k'}\}$ and $\{(n, x_{n,n+k'}): n \in N_{k'}\}$ converge to $x_k$ and $x_{k'}$ respectively, we obtain for $n \in N_{k'}$ large enough $d(x_{n,n+k}, x_{n,n+k'}) < \eta/4$ contradicting (4).

*Claim* 4. The subset $V^{-1}(\{0\})$ is a residual subset of $\mathscr{M}$.

It suffices to note that $V^{-1}(\{0\}) = \bigcap_{n=1}^{\infty} V^{-1}([0, 1/n))$ and by Claims 2 and 3 $V^{-1}([0,1/n))$ contains a dense open subset of $\mathscr{M}$.

We can easily complete the proof. If $f$ is in $\mathscr{M}' = V^{-1}(\{0\})$, let $\{f_n\}$ be a sequence in $\mathscr{N}$ which converges to $f$. Let $x_n$ be a solution of (2). By Claim 1 there exists a Cauchy subsequence, which converges to a solution $x$ of (1). Continuous dependence follows in the same way. From (ii) every sequence in the set $\{x \in X: f(x) = x\}$ contains a converging subsequence. Thus this set is compact.

## 2. Applications to $\alpha$-Nonexpansive Mappings

Let $X$ be a Banach space and $C$ a bounded closed nonempty convex subset of $X$. If $\alpha(\cdot)$ denotes the Kuratowski's noncompactness measure and $T$ from $C$ into $C$ is a continuous mapping satisfying $\alpha(T(A)) < \alpha(A)$ for every set $A$ contained in $C$, it is known [10, p. 127] that the set $\{x \in X: Tx = x\}$ is a nonempty compact set. This result is not longer true if $T$ is $\alpha$-nonexpansive, i.e., $T$ satisfies $\alpha(T(A)) \leqslant \alpha(A)$, even if $X$ is uniformly convex [10, p. 126]. By applying Theorem 1 we prove the generic existence of fixed points of $\alpha$-nonexpansive mappings.

Theorem 2. *Let $\mathscr{M}$ be the set of all continuous mappings $T: C \to C$ which satisfy $\alpha(T(A)) \leqslant \alpha(A)$ for every subset $A$ of $C$, endowed with the topology of uniform convergence. Then, there exists a residual subset $\mathscr{M}'$ of $\mathscr{M}$ such that for every $T$ in $\mathscr{M}'$ the following conditions are satisfied*:

(i) *$\{x \in C: Tx = x\}$ is a nonempty compact set.*

(ii) *If $\{T_n\}$ converges to $T$ in $\mathscr{M}$ and $x_n$ is a fixed point of $T_n$ there exists a subsequence of $\{x_n\}$ which converges to some fixed point of $T$.*

*Proof.* Let $\mathscr{N}$ be the set of all $\alpha$-Lipschitz operators with constant less than 1. Since $(1 - 1/n)T \to T$ $\mathscr{N}$ is a dense subset of $\mathscr{M}$. By a Darbo's result [3] the set $\{x \in X: Tx = x\}$ is a nonempty compact set for every $T$ in $\mathscr{N}$. To prove condition (c) in Theorem 1 let $\{T_n\}$ be a sequence converging in $\mathscr{M}$ to $T \in \mathscr{N}$, $x_n$ a fixed point of $T_n$ and assume $\alpha(\{x_n: n \in \mathbb{N}\}) > 0$. Choose

$\varepsilon = (1 - k_T)/3\alpha(\{x_n : n \in \mathbb{N}\})$, where $k_T$ is te $\alpha$-Lipschitz constant of $T$. For $N$ large enough so that $d(T, T_n) < \varepsilon$ for $n \geqslant N$ we have

$$\begin{aligned}\alpha(\{x_n : n \in \mathbb{N}\}) = \alpha(\{x_n : n \geqslant N\}) &= \alpha(\{T_n x_n : n \geqslant N\}) \\ &\leqslant 2\varepsilon + \alpha(T(\{x_n : n \geqslant N\})) \\ &< 2\varepsilon + k_T \alpha(\{x_n : n \in \mathbb{N}\}) < \alpha(\{x_n : n \in \mathbb{N}\})\end{aligned}$$

Then $\{x_n : n \in \mathbb{N}\}$ is a relatively compact set.

*Remark.* Condition (i) in Theorem 2 has been proved in [2, 5] with a different technique. In [5] it is proved that the successive approximation are precompact and a similar result for upper semicontinuous $\alpha$-nonexpansive mappings which are compact convex valued. This result (replacing precompactness of the successive approximation by (ii)) can be derived from the result corresponding to Theorem 1 for multivalued mapping and the fixed point theorem of Fan [6]. Theorem 2 also holds if we consider the $\beta$-measure defined by $\beta(A) = \inf\{\varepsilon > 0$: $A$ can be covered by finitely many balls with diameter less than $\varepsilon\}$.

Theorem 1 can be also applied to quasibounded $\alpha$-nonexpansive mappings defined on a cone. We recall [10, p. 128] that a mapping $T$ defined from a cone $C$ in a Banach space $X$ into $X$ is said to be quasibounded if $T$ maps bounded sets into bounded sets and

$$q(T) = \lim_{\substack{|x| \to \infty \\ x \in C}} |x|^{-1} |Tx| < +\infty.$$

It is known [10, p. 129] that every $\alpha$-contractive quasibounded mapping $T: C \to C$ with $q(T) < 1$ has a fixed point.

THEOREM 3. *Let $\mathcal{M}$ be the space of all quasibounded continuous mappings $T: C \to C$ that satisfy: $(q_1)$ $\alpha(T(A)) \leqslant \alpha(A)$ for every bounded set $A$ contained in $C$; $(q_2)$ $q(T) \leqslant 1$, endowed with the uniform convergence on bounded sets topology. Then, there exists a residual subset $\mathcal{M}'$ of $\mathcal{M}$ such that for every $T$ in $\mathcal{M}'$ one has:*

(i) *$\{x \in X: Tx = x\}$ is a nonempty compact set.*

(ii) *If $\{T_n\} \to T$ in $\mathcal{M}$ and $x_n$ is a fixed point of $T_n$, there exists a subsequence of $\{x_n\}$ which converges to a fixed point of $T$.*

*Proof.* It runs as that of Theorem 2, $\mathcal{N}$ being the set of all quasibounded $\alpha$-Lipschitz mappings $T$ with Lipschitz constant $k_T < 1$ and $q(T) < 1$. To prove condition (c) in Theorem 1, let $T$ be a mapping in $\mathcal{N}$ and $\{T_n\}$ a sequence converging to $T$ in $\mathcal{M}$. If $q(T) < h < 1$ choose $R > 0$ such that

$|Tx| < h|x|$ for every $x$ in $C$, $|x| > R$. Denote $M = \sup\{|Tx|: |x| \leqslant R\}$, $r = \max\{R, M+1\}$, $C_r = \{x \in C: |x| \leqslant r\}$. Choose $n_0$ large enough such that $|T_n(x)| \leqslant r$ for every $n \geqslant n_0$ and $x$ in $C_r$. Then $T_n$ maps $C_r$ into $C_r$ for $n \geqslant n_0$. As in the proof of Theorem 2, we can obtain (ii).

## 3. Aplication to Semiaccretive Mappings

Let $X$ be a real Banach space with uniformly convex dual $X^*$, $D$ a bounded closed convex subset of $X$. Denote by $J: X \to X^*$ the (unique) duality mapping. If $T$ of $D$ into $X$ is a strongly accretive mapping, (i.e., there exists a constant $c > 0$ such that

$$\langle J(x-y), Tx - Ty \rangle \geqslant c\,|x-y|^2$$

for every $x, y$ in $D$) and $T$ satisfies $(I-T)(\partial D) \subset D$, it is known [12] that $T$ has a unique zero in $D$. De Blasi [6] proves that this result holds for most accretive mappings (i.e., mappings which satisfy

$$\langle J(x-y), Tx - Ty \rangle \geqslant 0$$

for every $x, y$ in $D$) in the sense of the Baire category.

Definition [1, p. 234, 236]. A mapping $T: D \to X$ is said to be strongly semiaccretive if there exists a continuous mapping $S$ of $D \times D$ into $X$ for which the following conditions are all satisfied:

$(s_1)$ $Tx = S(x, x)$ for every $x$ in $D$.

$(s_2)$ For every $x$ in $D$, the mapping $S_x: D \to X$ defined by $S_x\,y = S(x, y)$ is strongly accretive.

$(s_3)$ The mapping $x \to S_x$ is a continuous mapping from $D$ into a relatively compact set of the space of continuous mappings from $D$ into $X$, endowed with the topology of uniform convergence.

A mapping $T: D \to X$ is said to be semiaccretive if there exists a continuous mapping $S$ of $D \times D$ into $X$ for which $(s_1)$ and the following conditions are satisfied

$(s_2')$ For every $x$ in $D$, the mapping $S_x: D \to X$ is accretive.

$(s_3')$ The mapping $x \to S_x$ is a continuous mapping from the weak topology on $D$ to the topology of uniform convergence on the set of continuous mappings from $D$ into $X$.

We say that a mapping $T: D \to X$ is weakly semiaccretive if there exists a continuous mapping $S: D \times D \to X$ for which $(s_1)$, $(s_2')$, and $(s_3)$ are satisfied.

Since $X^*$ is uniformly convex, condition $(s_3')$ implies condition $(s_3)$. Thus, every semiaccretive mapping is a weakly semiaccretive mapping. If $T$ is strongly semiaccretive and satisfies $(I-T)(\partial D) \subset D$ it is known [1, Theorem 13.15] that there exists a point $x$ in $D$ such that $Tx = 0$. This result is also valid [1, Theorems 13.17] if $T$ is semiaccretive and $X$ is a uniformly convex space. The Theorem 4 shows that this result still remains valid for most semiaccretive and weakly semiaccretive mappings in arbitrary Banach spaces with uniformly convex dual.

THEOREM 4. *Let $X$ be a Banach space with uniformly convex dual, $D$ a closed bounded convex subset of $X$. Denote $\mathscr{M}$ (resp. $\mathscr{S}$) the set of semiaccretive mappings (resp. weakly semiaccretive mappings) from $D$ into $X$ satisfying $(I - T)(\partial D) \subset D$, endowed with the topology of uniform convergence. Then, there exists a residual subset $\mathscr{M}'$ of $\mathscr{M}$ (resp. $\mathscr{S}'$ of $\mathscr{S}$) such that every $T$ in $\mathscr{M}'$ (resp. $\mathscr{S}'$) has a zero in $D$.*

*Proof.* We may assume without loss of generality that 0 is an interior point of $D$, since translation of the independent variable for a mapping does not affect its accretiveness and hence its semiaccretiveness. Denote by $\mathscr{A}$ the set of the strongly semiaccretive mapping $T$ that satisfy for some constant $c_T > 0$ and for every $x, u, v$ in $D$ the inequality $\langle J(u-v); S_x u - S_x v\rangle \geqslant c_T |u-v|^2$, where $S$ is the mapping given for $T$ by the semiaccretiveness. For every $T$ in $\mathscr{M}$ or $\mathscr{S}$ the mapping $T_\varepsilon = (1-\varepsilon)T + \varepsilon I$ is in $\mathscr{A}$ (for $S_\varepsilon(x, y) = (1-\varepsilon) S(x, y) + \varepsilon y$) and $I - T_\varepsilon$ maps $\partial D$ into $D$. Thus $\mathscr{A} \cap \mathscr{M}$ (resp. $\mathscr{A} \cap \mathscr{S}$) is a dense subset of $\mathscr{M}$ (resp. $\mathscr{S}$). To prove Theorem 1(c) let $\{T_n\}$ be a sequence converging in $\mathscr{M}$ or $\mathscr{S}$ to a mapping $T$ in $\mathscr{A}$. Let $x_n$ be a zero of $T_n$. From $(s_3)$ there exists a subsequence of $\{S_{x_n}\}$, denoted again by $\{S_{x_n}\}$ which converges uniformly to a mapping $U: D \to X$ satisfying $\langle J(u-v), Uu - Uv\rangle \geqslant c_T |u-v|^2$ for every $u, v$ in $D$. Let $L$ be the diameter of $D$ and $\varepsilon > 0$ arbitrary. Choose $N$ large enough so that $d(T_n, T) < \varepsilon$, $d(S_{x_n}, U) < \varepsilon$ for $n > N$. For every $n, m > N$ one has

$$\begin{aligned}\langle J(x_n - x_m), Tx_n - Tx_m\rangle &= \langle J(x_n - x_m), Ux_n - Ux_m\rangle \\ &\quad + \langle J(x_n - x_m), T_m x_m - Tx_m\rangle \leqslant 2L\varepsilon.\end{aligned}$$

Furthermore

$$\begin{aligned}c_T |x_n - x_m|^2 &\leqslant \langle J(x_n - x_m), Ux_n - Ux_m\rangle \\ &= \langle J(x_n - x_m), S(x_n, x_n) - S(x_m, x_m)\rangle \\ &\quad + \langle J(x_n - x_m), Ux_n - S(x_n, x_n)\rangle \\ &\quad + \langle J(x_n - x_m), S(x_m, x_m) - Ux_m\rangle \leqslant 4L\varepsilon.\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Let $\Lambda$ be defined by $\Lambda(T) = I - T$. Applying Theorem 1 to $\Lambda(\mathscr{M})$ (resp. $\Lambda(\mathscr{S})$) the poof is complete.

THEOREM 5. *Let X be a Banach space with uniformly convex dual, D a closed bounded convex subset of X. Denote by $\mathscr{A}$ the set of all continuous mappings from D into X which satisfy the following conditions*

($c_1$) *$T = A + C$, where A is accretive and C is a compact continuous mapping from D into X.*

($c_2$) *$(I - A - C)$ maps $\partial D$ into D.*

*Then, there exists a residual subset $\mathscr{A}'$ of $\mathscr{A}$ such that every T in $\mathscr{A}'$ has a zero in D.*

*Proof.* It runs as that of Theorem 4.

## 4. APPLICATIONS TO DIFFERENTIAL EQUATIONS

Let $X$ be a Banach space, $I$ an interval in the set $\mathbb{R}$ of real numbers and $f$ a mapping from $I \times X$ into $X$. Consider the Cauchy problem

$$x' = f(t, x), \qquad x(t_0) = x_0. \tag{5}$$

If (5) has unique solution for $(t_0, x_0)$ running in a dense subset $A$ of $I \times X$ and $f$ satisfies Caratheodory hypotheses for $X = \mathbb{R}$, Cafiero [3] proves that uniqueness holds almost everywhere in $I \times \mathbb{R}$. Vidossich [11] extends this result (in the sense of Baire category) to an arbitrary Banach space $X$, assuming that for every $(t_0, x_0)$ in $I \times X$ the mapping

$$x(\cdot) \to x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{6}$$

is a compact mapping from $C(I, X)$ into itself, $C(I, X)$ being the set of all continuous mappings from $I$ into $X$ with the compact open topology. Since in infinite-dimensional spaces, Caratheodory hypotheses do not assure the existence of solution, we can study an existence problem analogous to Cafiero uniqueness problem. This question was raised in [11]. By using Theorem 1 we can easily prove

THEOREM 6. *Let $f: I \times X \to X$ be a mapping which satisfies*

($e_1$) *f is continuous with respect to the second variable,*

($e_2$) *f is medible with respect to the first variable and there exists $h \in L^1_{loc}(I, \mathbb{R})$ such that $|f(t, x)| \leqslant h(t)$ for every $(t, x)$ in $I \times X$.*

*Assume in addition*

($e_3$) *There is a solution of* (5) *for* $(t_0, x_0)$ *running in a dense subset* $A$ *of* $I \times X$.

($e_4$) *If* $(t_0, x_0)$ *is in* $A$, $\{(t_n, x_n)\} \to (t_0, x_0)$ *in* $I \times X$ *and* $\phi_n$ *is a solution of* $x' = f(t, x)$; $x(t_n) = x_n$, *then there exists a subsequence of* $\{\phi_n\}$ *which converges to a solution of* (5).

*Then, existence of solution is a generic property in* $I \times X$.

*Proof.* Define for $(t_0, x_0)$ in $I \times X$ a mapping from $C(I, X)$ into $C(I, X)$ as in (6). It is easy to prove that this mapping, denoted by $F(t_0, x_0)$, is continuous on $C(\mathrm{I}, X)$. We can topologize $\{F(t, x): (t, x) \in I \times X\}$ setting $d(F(t, x), F(s, y)) = \max\{|t - s|, |x - y|\}$. This topology is finer than the topology of uniform convergence. By applying Theorem 1 we can complete easily the proof.

*Remark.* The technique used in Theorem 6 can be equally well applied to prove similar results for integral equations or functional equations. For instance, consider the nonlinear integral equation of Volterra type

$$x(t) = g(t) + \int_0^t a(t, s) f(s, x(s))\, ds, \tag{7}$$

where $g: I \to X$ is a continuous mapping, $f$ is as in Theorem 6, and $a$ is a continuous mapping from $I \times I$ into the set of bounded linear operators on $X$. If there exists solution for (7) when $g$ runs in a dense subset of $C(I, X)$ we can prove the analogue of Theorem 6.

## References


1. F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *in* "Proc. Sympos. Pure Math.," Vol. 18 II, Amer. Math. Soc., Providence, R.I., 1976.
2. G. J. Butler, Almost all 1-set-contractions have a fixed point, *Proc. Amer. Math. Soc.* **74** (2) (1979), 353–357.
3. F. Cafiero, Sul fenomeno di Peano nelle equazioni differenziali ordinarie del primo ordine, *Rend. Accad. Sci. Fis. Mat. Napoli.* **17** (1950), 51–61.
4. G. Darbo, Punti uniti in transformazioni a codominio non compatto, *Rend. Sem. Mat. Univ. Padova* **24** (1955), 84–92.
5. F. S. De Blasi, Some generic properties in fixed point theory, *J. Math. Anal. Appl.* **71** (1979), 161–166.
6. F. S. De Blasi and J. Myjak, Generic properties of contraction semigroups and fixed points of nonexpansive operators, *Proc. Amer. Math. Soc.* **77** (3) (1979), 341–347.
7. F. S. De Blasi, M. Kwapisz, and J. Myjak, Generic properties of functional equations, *Nonlinear Anal.* **2** (1977), 239–249.

8. K. FAN, Fixed point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 121–126.
9. K. KURATOWSKI, "Topology," Vol. I, Academic Press, New York, 1966.
10. R. H. MARTIN, "Nonlinear Operators and Differential Equations in Banach Spaces," Wiley–Interscience, New York, 1976.
11. G. VIDOSSICH, Existence, uniqueness and approximation of fixed points as a generic property, *Bol. Soc. Brasil. Mat.* **5** (1974), 17–29.
12. G. VIDOSSICH, How to get zeros of nonlinear operators using the theory of ordinary differential equations, *in* "Atas. Semana Analise Funcional," Sao Paulo, 1973.