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# $P$ versus $NP$ and geometry

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ABSTRACT

In this primarily expository article, I describe geometric approaches to variants of $P$ versus $NP$, present several results that illustrate the role of group actions in complexity theory, and make a first step towards geometric definitions of complexity classes. My goal is to help bring geometry and complexity theory closer together.



## 1. Introduction

The purpose of this article is to explain some of the beautiful problems in geometry that arise in the study of **P** versus **NP** and to motivate geometers to work on them. The article is divided into three parts: (i) *holographic algorithms*, where surprising reductions in complexity are related to the geometry of complex Hermitian symmetric spaces, in particular the variety of pure spinors, (ii) comparing the complexity of computing the permanent and determinant polynomials, where local differential geometry, geometric invariant theory and representation theory all play a role, and (iii) first steps towards describing geometric (i.e., coordinate free) definitions of algebraic complexity classes. While these parts are formally unrelated, there are common themes arising in each case, most importantly, (possibly hidden) group actions.

Roughly speaking, a *problem* in complexity theory is a class of expressions to be evaluated (e.g. count the number of four colorings of a planar graph). An *instance* of a problem is a particular member of the class (e.g. count the number of four colorings of the complete graph with four vertices). **P** is

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the class of problems that admit an algorithm that solves any instance of it in a number of steps that depends polynomially on the size of the input data. One says that such problems "admit a polynomial time solution". **NP** is the class of problems where a proposed solution to an instance can be positively checked in polynomial time. The famous Cook's hypothesis is $\mathbf{P} \neq \mathbf{NP}$.

I will be concerned with two types of evaluations in this article, here is the first: For each $n$, let $V_n$ be a complex vector space and assume $\dim(V_n)$ grows exponentially with $n$. It is known that the pairing

$$\begin{aligned} V_n \times V_n^* &\to \mathbb{C} \\ (v, \alpha) &\mapsto \langle \alpha, v \rangle \end{aligned}$$

of the vector space with its dual requires on the order of $\dim(V_n)$ arithmetic operations to perform. However if $V_n$ has additional structure and $\alpha, v$ are in "special position" with respect to this structure, the pairing can be evaluated faster. A trivial example would be if $V_n$ were equipped with a basis and $v$ was restricted to be a linear combination of only the first few basis vectors. I will be concerned with more subtle examples such as the following: let $V_n = \Lambda^k \mathbb{C}^n$, then inside $V_n$ are the decomposable vectors (the cone over the Grassmannian $G(k, \mathbb{C}^n)$). If $\alpha, v$ are decomposable, Eq. (3.1.1) shows that the pairing $\langle \alpha, v \rangle$ can be evaluated in polynomial time in $n$. From a geometric perspective, this is one of the key ingredients to L. Valiant's *holographic algorithms* discussed in Sections 2 and 4. For $n$ large, the codimension of the Grassmannian is huge, so it would seem highly unlikely that any interesting problem could have $\alpha, v$ so special. However small Grassmannians are of small codimension. This leads to the second key ingredient to holographic algorithms. On the geometric side, if $[v_1] \in G(k_1, W_1)$ and $[v_2] \in G(k_2, W_2)$, then $[v_1 \otimes v_2] \in G(k_1k_2, W_1 \otimes W_2)$. Thus if our vectors can be thought of as being built out of vectors in smaller spaces, there is a much better chance of the vectors lying in the Grassmannian. Due to the nature of problems in complexity theory, this is exactly what occurs. The third key ingredient is that there is some flexibility in how the small vector spaces are equipped with the additional structure, and I show (Theorem 4.2.2) that even for **NP**-complete problems there is sufficient flexibility to allow everything to work up to this point. The difficulty occurs when one tries to glue together the small vector spaces compatibly for both $V_n$ and $V_n^*$, although even here, the "only" problem that can occur is one of signs, see Section 4.3.

The second type of evaluation I will be concerned with is that of sequences of (homogeneous) polynomials, $p_n \in S^{d(n)}\mathbb{C}^{v(n)}$, where the degree $d(n)$ and the number of variables $v(n)$ are required to grow at most polynomially with $n$. A generic such sequence is known to require an exponential (in $n$) number of arithmetic operations to evaluate and we are interested in characterizing the sequences where the evaluation can be done quickly. Again there are sequences such as $p_n = x_1^{d(n)} + \cdots + x_{v(n)}^{d(n)}$ where it is trivial to see that there is a polynomial time evaluation, but there are other, more subtle examples, such as $\det_n \in S^n\mathbb{C}^{n^2}$ where the fast evaluation occurs thanks to a group action (Gaussian elimination, see Section 7.1).

From a geometer's perspective, it is more interesting to look at the zero sets of the polynomials, to get sequences of hypersurfaces in projective spaces. Similar to the situation above regarding signs, if one changes the signs in the expression of the determinant, e.g., to all plus signs to obtain the permanent, one arrives at a **VNP**-hard sequence, where **VNP** is Valiant's algebraic analogue of **NP**, see Section 6 for a definition.

**Problem:** Determine geometric properties of sequences of hypersurfaces such that their defining equations admit polynomial time evaluations.

A very tentative step towards resolving this problem is taken in Section 9. A second problem is:

**Problem:** Determine geometric properties of sequences of hypersurfaces such that their defining equations are in the class **VNP**.

A first observation is that if a polynomial is easy to evaluate, then any specialization of it is also easy to evaluate, or in other words the polynomial associated to any linear section of its zero set is also easy to evaluate. This leads to Valiant's conjecture (Valiant, 1979a) that the permanent sequence $(\mathrm{perm}_m)$ cannot be realized as a linear projection of the determinant sequence $(\det_{n(m)})$ unless $n$ grows faster than any polynomial (Conjecture 7.3.3). The best results on this conjecture so far are due to Mignon and Ressayre (2004) who use local differential geometry. While the local differential geometry of the $\det_n$-hypersurface is essentially understood (see Theorem 7.4.1), a major difficulty in continuing their program is to distinguish the local differential geometry of the $\mathrm{perm}_m$-hypersurface from that of a generic hypersurface. Furthermore, the determinant hypersurface is so special it may be difficult to isolate exactly which of its properties are the key to it having a fast evaluation. Suggestions for overcoming this second difficulty are given in Section 8.3.

From the geometric point of view, a significant esthetic improvement towards approaching Valiant's conjecture is the *Geometric complexity theory* (GCT) program proposed by K. Mulmuley and M. Sohoni in Mulmuley (2001) and Mulmuley (2008). Instead of regarding the determinant itself, one considers its $GL_{n^2}$-orbit closure in $\mathbb{P}(S^n\mathbb{C}^{n^2})$ and similarly for the permanent. The problem becomes one to compare two algebraic varieties that are invariant under a group action. In Section 8.2 I briefly review the program, summarizing from Buergisser et al. (preprint). Even with the GCT program, one still begins with the determinant and permanent, and it might be useful to consider other sequences as well, as discussed in Section 8.3.

The examples up to this point indicate that sequences in **VP** that are not in $\mathbf{VP}_e$, the sequences of polynomials having "small" expressions, see Section 6 for a precise definition, (and analogously for **P**) should have some kind of symmetry, but that symmetry could be hidden. It would be very useful to be able to formalize the notion of "hidden symmetry" in this context. Similarly, it would be useful to have coordinate free definitions of complexity classes.

**Overview.** In Section 2, I describe how to convert a counting problem to a vector space pairing. In Section 3, I describe how the "big cell" in the Grassmannian (resp. the spinor variety) admits an interpretation as the set of vectors of minors (resp. sub-Pfaffians) in preparation for Section 4, where I review the reformulation of holographic algorithms of Landsberg et al. (preprint) and point out a consequence that all problems in **NP** are "nearly" holographic (Theorem 4.2.2). In Section 5 the results of Section 3 are generalized to all cominuscule varieties. In Section 6 I review the definitions of Valiant's complexity classes in preparation for Sections 7–9. Section 7 discusses Valiant's conjecture regarding the permanent as a projection of the determinant. There are two new results (Theorems 7.4.1 and 7.4.2) on the local differential geometry of the hypersurface $\{\det_n = 0\}$ relevant for complexity. The *Geometric Complexity Theory* program of Mulmuley and Sohoni is very briefly reviewed in Section 8. In Section 9 a coordinate free definition of the class $\overline{\mathbf{VP}_e}$ is given, where *joins* and *multiplicative joins* play a role, the latter perhaps being defined here for the first time, and a first step is taken towards a geometric definition of **VP**, using the idea of possibly hidden symmetries. Other than as noted above, the various sections can be read independently.

The results presented in this paper are preliminary — the main purpose of the paper is to indicate some of the deep and beautiful connections between the **P** versus **NP** problem and geometry. For connections with other areas of mathematics, see, e.g. Wigderson (2007).

I use the summation convention that repeated indices appearing up and down are to be summed over their range.

## 2. Holographic algorithms I: counting problems as vector space pairings $A^* \times A \to \mathbb{C}$

For simplicity of exposition, I restrict to the complexity problem of counting the number of solutions to equations $c_s$ over $\mathbb{F}_2$ with variables $x_i$. This problem is called *#SAT* in the complexity literature. (In complexity theory one usually deals with Boolean variables and clauses, which is essentially equivalent to equations over $\mathbb{F}_2$ but some care must be taken in the translation.)

It came as a shock to the complexity community when Valiant (2006) showed that a certain restricted counting problem (affectionately called "#Pl-Rtw-Mon-3CNF" in the complexity literature), where counting the number of solutions mod 2 is already #**P** complete, had the property that counting the number of solutions mod 7 could be done in polynomial time. Cai (2006) recognized Valiant's method could be formed in terms of pairings of tensors in dual spaces, and the discussion below follows his formulation. See Section 4.4 below for more on the history and further references.

To convert a counting problem to a vector space pairing, proceed as follows:

**Step 1.** To an instance of a problem construct a bipartite graph $\Gamma = (V_x, V_c, E)$ that encodes the problem. Here $V_x$, $V_c$ are the two sets of vertices and $E$ is the set of edges. $V_x$ corresponds to the set of variables, $V_c$ to the set of equations, and there is an edge $e_{is}$ joining the vertex of the variable $x_i$ to the vertex of the equation $c_s$ iff $x_i$ appears in $c_s$.

**Step 2.** Construct "local" tensors that encode the information at each vertex. To do this first associate to each edge $e_{is}$ a vector space $A_{is} = \mathbb{C}^2$ with basis $a_{is|0}, a_{is|1}$ and dual basis $\alpha_{is|0}, \alpha_{is|1}$ of $A_{is}^*$. Next, to each variable $x_i$ associate the vector space

$$A_i := \bigotimes_{\{s|e_{is}\in E\}} A_{is}$$

and the tensor

$$g_i := \otimes_{\{s|e_{is}\in E\}} a_{is|0} + \otimes_{\{s|e_{is}\in E\}} a_{is|1} \in A_i \tag{2.0.1}$$

which will encode that $x_i$ should be consistently assigned either 0 or 1 each time it appears. Now to each equation $c_s$ we associate a tensor in $A_s^* := \otimes_{\{i|e_{is}\in E\}} A_{is}^*$ that encodes that $c_s$ is satisfied. For example, say $x_i, x_j, x_k$ appear in $c_s$ and that

$$c_s(x_i, x_j, x_k) = x_i x_j + x_i x_k + x_j x_k + x_i + x_j + x_k + 1$$

which is satisfied over $\mathbb{F}_2$ as long as the variables $x_i, x_j, x_k$ are not all 0 or all 1. (This equation is called *3NAE* in the computer science literature.) More generally, say $c_s$ has $x_{i_1}, \ldots, x_{i_{d_s}}$ appearing and $c_s$ is $d_s$*NAE*, then one associates the tensor

$$r_s := \sum_{(\epsilon_1,\ldots,\epsilon_{d_s})\neq(0,\ldots,0),(1,\ldots,1)} \alpha_{i_1,s|\epsilon_1} \otimes \cdots \otimes \alpha_{i_{d_s},s|\epsilon_{d_s}}. \tag{2.0.2}$$

**Step 3.** Tensor all the local tensors from $V_x$ (resp. $V_c$) together to get two tensors in dual vector spaces with the property that their pairing counts the number of solutions. That is, consider $G := \otimes_i g_i$ and $R := \otimes_s r_s$ respectively elements of the vector spaces $A := \otimes_e A_e$ and $A^* := \otimes_e A_e^*$. Then, the pairing $\langle G, R\rangle$ counts the number of solutions.

**Remark 2.0.1.** Up until now I could have just taken each $A_{is} = \mathbb{Z}_2$. The reason for complex numbers was to allow a larger group action. This group action will destroy the local structure but leave the global structure unchanged. Valiant's inspiration for doing this was quantum mechanics, where particles are replaced by wave functions.

So far we have replaced our original counting problem with the problem of computing a pairing $A \times A^* \to \mathbb{C}$ where the dimension of $A$ is exponential in the size of the input data. If we had arbitrary vectors, then there is no way to perform this pairing in a number of steps that is polynomial in the size of the original data. We saw that if one is lucky, the pairing can be computed quickly. In the next section I describe the geometry underlying "getting lucky" and in the following section discussion how to make local changes of bases that simultaneously put each $g_i$ and $r_s$ into spinor varieties.

## 3. Detour: Grassmannians and spinor varieties

Mathematicians are used to viewing the Grassmannian as the variety parametrizing linear subspaces of a vector space, and the spinor variety as parametrizing isotropic subspaces. However in statistics, the "big cell" inside arises as the space parametrizing the set of minors of matrices (resp. Pfaffians of skew-symmetric matrices). We show how these second descriptions lead to the fast algorithms mentioned in the introduction.

### 3.1. The Grassmannian as a variety parametrizing minors of matrices

Let $W$ be a vector space and let $G(k, W)$ denote the Grassmannian of $k$-planes through the origin in $W$. Assume WLOG that $k \leq \dim W - k$. The Plücker embedding $G(k, W) \subset \mathbb{P}(\Lambda^k W)$ is obtained by, given a $k$-plane $E$, taking a basis $e_1, \ldots, e_k$ of $E$ and sending $E$ to the point $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(\Lambda^k W)$. The cone over the Grassmannian, $\hat{G}(k, W) \subset \Lambda^k W$ is thus the set of $v \in \Lambda^k W$, such that there exist $w_1, \ldots, w_k \in W$ with $v = w_1 \wedge \cdots \wedge w_k$.

The Grassmannian $G(k, W)$ admits a local parametrization as follows: Write $W = E \oplus F$ where $\dim E = k$. Let $1 \leq i, j \leq k$, $1 \leq s, t \leq n - k$, fix bases $e_1, \ldots, e_k$ of $E$ with dual basis $e^1, \ldots, e^k$ of $E^*$, and $f_1, \ldots, f_{n-k}$ of $F$ with dual basis $f^1, \ldots, f^{n-k}$. Say $E = [v_0]$, $v_0 \in \hat{G}(k, W)$ and we want to locally parametrize $G(k, W)$ around $[v_0]$. Choose our basis such that $e_j = w_j$ in the description of $v$ above. Let $x^s_j$ be linear coordinates on $E^* \otimes F \simeq T_{[v_0]}G(k, W)$. The local parametrization about $E = [v(0)]$ is

$$[v(x^s_i)] = [(e_1 + x^s_1 e_s) \wedge \cdots \wedge (e_k + x^s_k e_s)].$$

In what follows I will also need to work with $G(k, W^*)$, In our dual bases, a local parametrization about $E^* = \langle e^1, \ldots, e^k \rangle = [\alpha(0)]$ is

$$[\alpha(y^s_j)] = [(e^1 + y^1_s e^s) \wedge \cdots \wedge (e^k + y^k_s e^s)].$$

I next explain how to interpret the open subset of $G(k, W)$ described above as the vector of minors for $E^* \otimes F$.

For vector spaces $E, F$, $\Lambda^k(E \oplus F)$ has the following decomposition as a $GL(E) \times GL(F)$ module:

$$\Lambda^k(E \oplus F) = (\Lambda^k E \otimes \Lambda^0 F) \oplus (\Lambda^{k-1} E \otimes \Lambda^1 F) \oplus (\Lambda^{k-2} E \otimes \Lambda^2 F) \\ \oplus \cdots \oplus (\Lambda^1 E \otimes \Lambda^{k-1} F) \oplus (\Lambda^0 E \otimes \Lambda^k F).$$

Assume we have a volume form on $E$ so we may identify $\Lambda^s E \simeq \Lambda^{k-s} E^*$. We have the $SL(E) \times GL(F)$ decomposition:

$$\Lambda^k(E \oplus F) = (\Lambda^0 E^* \otimes \Lambda^0 F) \oplus (\Lambda^1 E^* \otimes \Lambda^1 F) \oplus (\Lambda^2 E^* \otimes \Lambda^2 F) \\ \oplus \cdots \oplus (\Lambda^{k-1} E^* \otimes \Lambda^{k-1} F) \oplus (\Lambda^k E^* \otimes \Lambda^k F).$$

Recall that $\Lambda^s E^* \otimes \Lambda^s F \subset S^s(E^* \otimes F)$ has the geometric interpretation as the space of $s \times s$ minors on $E \otimes F^*$, i.e., with any choices of bases, write an element $f$ of $E \otimes F^*$ as a matrix, then a basis of $\Lambda^s E^* \otimes \Lambda^s F$ evaluated on $f$ will give the set of $s \times s$ minors of $f$.

To see these minors explicitly, note that the bases of $E^*, F$ induce bases of the exterior powers. Expanding out $v$ above in such bases, (recall that the summation convention is in use)

$$\begin{aligned} v(x^s_i) &= e_1 \wedge \cdots \wedge e_k \\ &\quad + x^s_i e_1 \wedge \cdots \wedge e_{i-1} \wedge e_s \wedge e_{i+1} \wedge \cdots \wedge e_k \\ &\quad + (x^s_i x^t_j - x^s_j s^t_i) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_s \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_t \wedge e_{j+1} \cdots \wedge e_k \\ &\quad + \cdots \end{aligned}$$

i.e., writing $v$ as a row vector in the induced basis:

$$v = (1, x^s_i, x^s_i x^t_j - x^t_i x^s_j, \ldots) = (1, \Delta_{i,s}(x), \ldots, \Delta_{I,S}(x), \ldots, )$$

where we use the notation $I = (i_1, \ldots, i_p)$ $S = (s_1, \ldots, s_p)$ and $\Delta_{I,S}(x)$ denotes the corresponding $p \times p$ minor of $x$. Similarly $\alpha = (1, y^j_s, y^j_s y^i_t - y^i_s y^j_t, \ldots, )$.

Fix bases so $x, y$ are $k \times (n-k)$ matrices. I claim

$$\langle \alpha, v \rangle = \det(I_E + {}^t x y) \tag{3.1.1}$$

because the characteristic polynomial of a product of a $k \times \ell$ matrix ${}^t x$ with an $\ell \times k$ matrix $y$ is:

$$charpoly({}^t x y)(t) = \det(Id_E + t\,{}^t x y) = \sum_{I,S} \Delta_{I,S}(x) \Delta_{S,I}(y) t^{|I|}. \tag{3.1.2}$$

While (3.1.2) is no doubt classical, I include a proof as I did not find one in the literature.

For a linear map $f : A \to A$, recall the induced linear maps $f^{\wedge k} : \Lambda^k A \to \Lambda^k A$, where, if one chooses a basis of $A$ and represents $f$ by a matrix, then the entries of the matrix representing $f^{\wedge k}$ in the induced basis on $\Lambda^k A$ will be the $k \times k$ minors of the matrix of $f$. In particular, if $\dim A = \mathbf{a}$, then, $f^{\wedge \mathbf{a}}$ is multiplication by a scalar which is $\det(f)$.

Recall the decomposition:

$$End(E^* \oplus F) = (E^* \oplus F) \otimes (E^* \oplus F)^* = (E^* \otimes F) \oplus (E^* \otimes E) \oplus (F \otimes F^*) \oplus (F^* \otimes E).$$

To each $x \in E^* \otimes F$, $y \in E \otimes F^*$, associate the element

$$-x + Id_E + Id_F + y \in End(E^* \oplus F). \tag{3.1.3}$$

Note that

$$\det \begin{pmatrix} I_E & -{}^t x \\ y & I_F \end{pmatrix} = \det(I_E + {}^t xy).$$

Consider

$$(-x + Id_E + Id_F + y)^{\wedge n} = (Id_E)^{\wedge k} \wedge (Id_F)^{\wedge(n-k)} + (Id_E)^{\wedge k-1} \wedge (Id_F)^{\wedge(n-k-1)} \wedge (-x) \wedge y$$
$$+ (Id_E)^{\wedge(k-2)} \wedge (Id_F)^{\wedge(n-k-2)} \wedge (-x)^{\wedge 2} \wedge y^{\wedge 2} + \cdots + (Id_F)^{\wedge(n-2k)} \wedge (-x)^{\wedge k} \wedge y^{\wedge k}.$$

Let

$$e^1 \wedge \cdots \wedge e^k \wedge f_1 \wedge \cdots \wedge f_{n-k} \in \Lambda^n(E^* \otimes F)$$

be a volume form. All that remains to check is that when we re-order our terms that the signs work out correctly, which is left to the reader.

### *3.2. Spinor varieties*

For the interpretation of spinor varieties as maximal isotropic subspaces on a quadric, see any of Chevalley (1997), Harvey (1990) and Landsberg (2003). Here I simply define the spinor variety as the Zariski closure of the set of vectors of sub-Pfaffians of a skew-symmetric matrix with variables as entries. See Landsberg and Manivel (2002) for the connection with the classical definition.

For $x \in \Lambda^2 \mathbb{C}^{2n}$, the Pfaffian $\mathrm{Pf}(x) \in \mathbb{C}$ is defined by $x^{\wedge n} = \mathrm{Pf}(x) n! \Omega$, where $\Omega \in \Lambda^{2n}\mathbb{C}^{2n}$ is a volume form – it is a square root of $\det(x)$.

Let $E$ be an $n$-dimensional vector space equipped with a volume form. Define $(\hat{\mathbb{S}}_+)^0$ to be the image of the map

$$\Lambda^2 E \to \Lambda^{even} E =: \mathcal{S}_+$$
$$x \mapsto v = (1, x^i_j, \ldots, \mathrm{Pf}_I(x), \ldots, ) =: \mathrm{sPf}(x)$$

as $|I|$ varies over the even numbers from 0 to $\llcorner \frac{n}{2} \lrcorner$. The space of sub-Pfaffians of size $2p$ is parametrized by $\Lambda^{2p} E$. If $n$ is even, $\mathcal{S}_+$ is self-dual, and if $n$ is odd, its dual is $\mathcal{S}_- := \Lambda^{odd} E$ because $E$ is equipped with a volume form, so $\Lambda^{2p} E^* = \Lambda^{n-2p} E$.

Recall the decomposition

$$\Lambda^2(E \oplus E^*) = \Lambda^2 E \oplus E \otimes E^* \oplus \Lambda^2 E^*.$$

Consider $x + Id_E + y \in \Lambda^2(E \oplus E^*)$. Observe that

$$(x + Id_E + y)^{\wedge n} = \sum_{j=0}^{n} (Id_E)^{\wedge(n-j)} \wedge x^{\wedge j} \wedge y^{\wedge j} \in \Lambda^{2n}(E \oplus E^*).$$

Let $\Omega = e_1 \wedge e^1 \wedge e_2 \wedge e^2 \wedge \cdots \wedge e_n \wedge e^n \in \Lambda^{2n}(E \oplus E^*)$ be a volume form. The coefficient of the $j$-th term is the sum

$$\sum_{|I|=2j} \mathrm{sgn}(I)\, \mathrm{Pf}_I(x)\, \mathrm{Pf}_I(y)$$

where for an even set $I \subseteq [n]$, define $\sigma(I) = \sum_{i \in I} i$, and define $\operatorname{sgn}(I) = (-1)^{\sigma(I)+|I|/2}$. Put more invariantly, the $j$-th term is the pairing

$$\langle y^{\wedge j}, x^{\wedge j} \rangle.$$

For a matrix $z$ define a matrix $\tilde{z}$ by setting $\tilde{z}^i_j = (-1)^{i+j+1} z^i_j$. Let $z$ be an $n \times n$ skew-symmetric matrix. Then for every even $I \subseteq [n]$,

$$\operatorname{Pf}_I(\tilde{z}) = \operatorname{sgn}(I) \operatorname{Pf}_I(z).$$

For $|I| = 2p, p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$,

$$\operatorname{Pf}_I(\tilde{z}) = (-1)^{i_1+i_2+1} \cdots (-1)^{i_{2p-1}+i_{2p}+1} \operatorname{Pf}_I(z) = \operatorname{sgn}(I) \operatorname{Pf}_I(z).$$

Thus:

**Theorem 3.2.1** (Landsberg et al., preprint). *Let $z, y$ be skew-symmetric $n \times n$ matrices. Then*

$$\langle \operatorname{sPf}(z), \operatorname{sPf}^{\vee}(y) \rangle = \operatorname{Pf}(\tilde{z} + y).$$

*In particular, when n is even, the pairing $\mathcal{S}_+ \times \mathcal{S}_+ \to \mathbb{C}$ restricted to $(\hat{\mathbb{S}}_+)^0 \times (\hat{\mathbb{S}}_+)^0 \to \mathbb{C}$ can be computed in polynomial time. When n is odd, the pairing $\mathcal{S}_+ \times \mathcal{S}_- \to \mathbb{C}$ restricted to $(\hat{\mathbb{S}}_+)^0 \times (\hat{\mathbb{S}}_-)^0 \to \mathbb{C}$ can be computed in polynomial time.*

The first few spinor varieties are classical varieties in disguise (corresponding to coincidences of Lie groups in the first two cases and triality in the third):

$$\mathbb{S}_2 = \mathbb{P}^2 \subset \mathbb{P}^2$$
$$\mathbb{S}_3 = \mathbb{P}^3 \subset \mathbb{P}^3$$
$$\mathbb{S}_4 = Q^6 \subset \mathbb{P}^7.$$

In particular, although the codimension grows very quickly, it is small in these cases. The next case $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is not isomorphic to any classical homogeneous variety.

## 4. Holographic algorithms II: computing the vector space pairing in polynomial time

### 4.1. The $SL_2\mathbb{C}$ action

To try to move both $G, R$ to special position so that the pairing can be evaluated quickly, identify all the $A_e$ with a single $\mathbb{C}^2$, and allow $SL_2\mathbb{C}$ to act. This action is very cheap, and of course if we have it act simultaneously on $A$ and $A^*$, the pairing $\langle G, R \rangle$ will be unchanged. This step cannot always be carried out, otherwise Valiant would have proved $\mathbf{P} = \mathbf{NP}$.

To illustrate, we now restrict to $\#3SAT - NAE$, which is still **NP**-hard.

The tensor $g_i$ corresponding to a variable vertex $x_i$ is (2.0.1). The tensor corresponding to a NAE clause $r_s$ is (2.0.2) and $d_s = 3$ for all $s$. Let

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

be the basis change, the same in each $A_e$, sending $a_{e|0} \mapsto a_{e|0} + a_{e|1}$ and $a_{e|1} \mapsto a_{e|0} - a_{e|1}$ which induces the basis change $\alpha_{e|0} \mapsto \frac{1}{2}(\alpha_{e|0} + \alpha_{e|1})$ and $\alpha_{e|1} \mapsto \frac{1}{2}(\alpha_{e|0} - \alpha_{e|1})$ in $A^*_e$. Applying $T$, gives

$$T(a_{i,s_{i_1}|0} \otimes \cdots \otimes a_{i,s_{i_{d_i}}|0} + a_{i,s_{i_1}|1} \otimes \cdots \otimes a_{i,s_{i_{d_i}}|1}) = 2 \sum_{\{(\epsilon_1,\ldots,\epsilon_{d_i}) | \sum \epsilon_\ell = 0 \ (\mathrm{mod}\ 2)\}} a_{i,s_{i_1}|\epsilon_1} \otimes \cdots \otimes a_{i,s_{i_{d_i}}|\epsilon_{d_i}}.$$

Similarly

$$T\left(\sum_{(\epsilon_1,\epsilon_2,\epsilon_3)\neq(0,0,0),(1,1,1)} \alpha_{i_1,s|\epsilon_1}\otimes\alpha_{i_2,s|\epsilon_2}\otimes\alpha_{i_3,s|\epsilon_3}\right)$$
$$= 6\alpha_{i_1,s|0}\otimes\alpha_{i_2,s|0}\otimes\alpha_{i_3,s|0} - 2(\alpha_{i_1,s|0}\otimes\alpha_{i_2,s|1}\otimes\alpha_{i_3,s|1} + \alpha_{i_1,s|1}\otimes\alpha_{i_2,s|0}\otimes\alpha_{i_3,s|1} + \alpha_{i_1,s|1}\otimes\alpha_{i_2,s|1}\otimes\alpha_{i_3,s|0}).$$

After this change of basis $g_i \in \mathbb{S}_{\#\{s|e_{is}\in E\}}$ and $r_s \in \mathbb{S}_4$ for all $i, s$!

### 4.2. **NP**, *in fact* #**P** *is pre-holographic*

**Definition 4.2.1.** Let $P$ be a counting problem. We will say that $P$ is *pre-holographic* if it admits a formulation such that the vectors $g_i, r_s$ are all simultaneously representable as vectors of sub-Pfaffians.

The following was proved (although not stated) in Landsberg et al. (preprint):

**Theorem 4.2.2.** *Any problem in* **NP**, *in fact in* #**P**, *is pre-holographic.*

**Proof.** To prove the theorem it suffices to exhibit one #**P** complete problem that is pre-holographic. Counting the number of solutions to $\#3SAT - NAE$ is one such. $\square$

### 4.3. *What goes wrong*

While for $\#3SAT-NAE$ it is always possible to give $V$ and $V^*$ structures of the spin representations $\mathcal{S}_+$ and $\mathcal{S}_+^*$, so that $[G] \in \mathbb{P}V$ and $[R] \in \mathbb{P}V^*$ both lie in spinor varieties, these structures *may not be compatible*! What goes wrong is that the ordering of pairs of indices $(i, s)$ that is good for $V$ may not be good for $V^*$. The "only" thing that can go wrong are the signs of the sub-Pfaffians, see Landsberg et al. (preprint) for details.

In Landsberg et al. (preprint) we determine sufficient conditions for there to be a good ordering of indices and show that if the bipartite graph $\Gamma$ was planar, then these sufficient conditions hold.

### 4.4. *History*

In Valiant's original formulation of holographic algorithms (see Valiant, 2001, 2002a,b, 2004, 2005, 2008), the step of forming $\Gamma$ is the same, but then Valiant replaced the vertices of $\Gamma$ with weighted graph fragments to get a new weighted graph $\Gamma'$ in such a way that the number of (weighted) perfect matchings of $\Gamma'$ equals the answer to the counting problem. Then, if $\Gamma'$ is planar, one can appeal to the famous FKT algorithm (Kasteleyn, 1967; Temperley and Fisher, 1961) to compute the number of weighted perfect matchings in polynomial time. Valiant also found certain algebraic identities that were necessary conditions for the existence of such graph fragments.

Cai (Cai and Choudhary, 2006; Cai, 2006, 2007; Cai and Lu, 2007; Cai, 2007a,b, 2008) recognized that Valiant's procedure could be reformulated as a pairing of tensors as in steps two and three, and that the condition on the vertices was that the local tensors $g_i\, r_s$ could, possibly after a change of basis, be realized as a vector of sub-Pfaffians. In Cai's formulation one still appeals to the existence of $\Gamma'$ and the FKT algorithm in the last step.

## 5. Exponential pairings in polynomial time

In this section we show that the same phenomenon that we observed above for Grassmannians and spinor varieties holds for all *cominuscule* varieties, the homogeneous varieties that can be given the structure of a compact Hermitian symmetric space.

### 5.1. Cominuscule varieties

**Theorem 5.1.1.** *Let $V$ be a vector space of dimension $\binom{n}{k}$, $2^{n-1}$, $\binom{2n}{k}-\binom{2n}{k-2}$, $p^n$, or $\binom{n+p-1}{n}$. In each case there are explicit systems of degree two polynomial equations on $V$, $V^*$, such that if $\alpha \in V^*$ and $v \in V$ satisfy these equations, the pairing $\langle \alpha, v\rangle$, which naïvely requires $O(\dim V)$ arithmetic operations, can be computed in $O(n^4)$ operations.*

Theorem 5.1.1 is an immediate consequence of:

**Theorem 5.1.2.** *Let $V = V(n)$ be a cominuscule $G = G(n)$-module with $G/P \subset \mathbb{P}V$ the closed orbit and $G/P' \subset \mathbb{P}V^*$ the corresponding closed orbit in the dual space. Here $n$ is the rank of $G$. Then the pairing $V \times V^* \to \mathbb{C}$ restricted to $\hat{G}/P \times \hat{G}/P'$ can be computed in $O(n^4)$ arithmetic operations without divisions.*

The non-trivial cases are (where for notational convenience we use the rank of $G$ plus one in the $A_{n-1} = SL_n$-case):

| $V$ | $\dim V$ | $G$ | $G/P$ | $\mathfrak{g}/\mathfrak{p}$ |
|---|---|---|---|---|
| $\Lambda^k W$ | $\binom{n}{k}$ | $SL(W) = SL_n$ | $G(k, W)$ | $E^* \otimes F$ |
| $\mathcal{S}_+$ | $2^{n-1}$ | $D_n = Spin_{2n}$ | $\mathbb{S}_+$ | $\Lambda^2 E$ |
| $\Lambda^{\langle n\rangle} W$ | $\binom{2n}{n}-\binom{2n}{n-2}$ | $Sp(2n,\mathbb{C}) = Sp(W,\omega)$ | $G_{Lag}(n, 2n)$ | $S^2 E$ |
| $E_1 \otimes \cdots \otimes E_n$ | $p^n$ | $SL(E_1) \times \cdots \times SL(E_n)$ | $Seg(\mathbb{P}E_1 \times \cdots \times \mathbb{P}E_n)$ | $\oplus_j E_j'$ |
| $S^n E$ | $\binom{n+p-1}{n}$ | $SL(E)$ | $v_n(\mathbb{P}E)$ | $E' \circ \ell^{n-1}$ |

Explanations of $V$: $W$ is a vector space of dimension $n$ in the first case, $2n$ in the third, $\mathcal{S}_+$ is the (positive) half-spin representation of $Spin_{2n}$, $\Lambda^{\langle n\rangle} W = \Lambda^n W/(\Lambda^{n-2} W \wedge \omega)$ where $\omega \in \Lambda^2 W$ is a symplectic form. $E$, $E_j$ are vector spaces of dimension $p$ in the last two cases.

Explanations of $G/P$: $G(k, W)$ denotes the Grassmannian of $k$-planes in its Plucker embedding, $\mathbb{S}_+$ the "pure spinors" or *spinor variety*, $G_{Lag}(n, 2n)$ denotes the Lagrangian Grassmannian of $n$-planes isotropic for the symplectic form $\omega \in \Lambda^2\mathbb{C}^{2n}$, $Seg(\mathbb{P}E_1 \times \cdots \times \mathbb{P}E_n)$ denotes the Segre product, the projectivization of the set of decomposable tensors in $E_1 \otimes \cdots \otimes E_n$ and $v_n(\mathbb{P}E)$ denotes the Veronese variety of the projectivization of homogeneous polynomials of degree $n$ on $E^*$ that are $n$-th powers of a linear form.

Explanations of $\mathfrak{g}/\mathfrak{p}$: $\mathfrak{g}, \mathfrak{p}$ are the Lie algebras of $G, P$. Let $G_0$ denote the Levi factor of $P$. $G_0$ is respectively $S(GL(E) \times GL(F))$, $GL(E)$, $GL(E)$, $GL(E_1') \times \cdots \times GL(E_n')$, $GL(E')$. As a $G_0$-module, $\mathfrak{g}/\mathfrak{p}$ is the tangent space to $G/P$ at the point of $G/P$ corresponding to $Id \in G$. I have written $F = W/E$. Fix vectors $e_j \in E_j$, $e \in E$ and let $\ell_j, \ell$ respectively denote the lines they span, then $E_j' = \ell_1 \otimes \cdots \otimes \ell_{j-1} \otimes E_j/\ell_j \otimes \ell_{j+1} \otimes \cdots \otimes \ell_n$ and $E' = E/\ell$.

In each case $\mathfrak{g}/\mathfrak{p}$ is a space of endomorphisms and $V$ as a $G_0$-module is the sum of the spaces of all minors (of all sizes) of $\mathfrak{g}/\mathfrak{p}$, except in the spinor case, where one takes all sub-Pfaffians. $\oplus_j S_{2^j} E = \oplus_j S_{2\cdots 2} E$ denotes the irreducible $GL(E)$-submodule of $\Lambda^j E \otimes \Lambda^j E$ giving minors on $S^2 E \subset E \otimes E$.

It remains to prove the cases of the Lagrangian Grassmannian, the Segre and the Veronese.

### 5.2. Lagrangian Grassmannian case

The Lagrangian Grassmannian $G_{Lag}(n, 2n) \in \mathbb{P}\Lambda^{\langle n\rangle} W$ is a linear section of $G(n, 2n) \subset \mathbb{P}\Lambda^n W$. Here $\Lambda^{\langle n\rangle} W = \Lambda^n W/(\Lambda^{n-2} W \wedge \omega) = W_{\omega_n}^{Sp(2n,W)}$ and the quotient may be viewed as the complement to $\Lambda^{n-2} W \wedge \omega \subset \Lambda^n W$ to obtain the linear section.

The interpretation of an open subset (the "big cell") of $G_{Lag}(n, W)$ is as the set of vectors of (non-redundant) minors of symmetric matrices. The symplectic form enables the identification of $W/E \simeq E^*$ and the linear subspace of

$$E^* \otimes E^* = \Lambda^2 E^* \oplus S^2 E^*$$

corresponding to the tangent space is just $S^2 E^*$. See Landsberg (2003) for details.

The subspace of $\Lambda^j E^* \otimes \Lambda^j E^*$ giving rise to a non-redundant set of minors corresponds to the submodule $S_{2j}E^* \subset \Lambda^j E^* \otimes \Lambda^j E^*$.

For the Lagrangian Grassmannian case it suffices in (3.1.3) to take

$$-x + Id_E + y \in S^2(E \oplus E^*) = S^2 E \oplus E \otimes E^* \otimes S^2 E^*.$$

### 5.3. Segre and Veronese cases

The Segre is parametrized by a map $\phi$

$$(x^j_s) \mapsto (a^1_0 + x^j_1 a^1_j) \otimes (a^2_0 + x^j_2 a^2_j) \otimes \cdots \otimes (a^n_0 + x^j_n a^n_j) = (1, x^j_{s_1}, x^j_s x^k_{s_2}, \cdots x^1_{s_1} \cdots x^p_{s_p}),$$

where in each term $s_1 < \cdots < s_q$. Let $\phi^\vee$ denote the map to the dual Segre.

If $\alpha = \phi(x)$, $v = \phi^\vee(y)$ then

$$\langle \alpha, v \rangle = \sum_{I,S} x^I_S y^I_S$$

where $I = (i_1, \ldots, i_q)$, $i_1 \leq \cdots \leq i_q$, $1 \leq q \leq p$, and $S = (s_1, \ldots, s_r)$, $s_1 < \cdots < s_r$, $1 \leq r \leq n$. Here:

$$\langle \alpha, v \rangle = \det \begin{pmatrix} I_n & \begin{matrix}-x^1_1 & & \\ & \ddots & \\ & & -x^1_n\end{matrix} & \begin{matrix}-x^2_1 & & \\ & \ddots & \\ & & -x^2_n\end{matrix} & \cdots & \begin{matrix}-x^p_1 & & \\ & \ddots & \\ & & -x^p_n\end{matrix} \\ \begin{matrix}y^1_1 & & \\ & \ddots & \\ & & y^1_n\end{matrix} & I_n & & & \\ \begin{matrix}y^2_1 & & \\ & \ddots & \\ & & y^2_n\end{matrix} & & I_n & & \\ \vdots & & & \ddots & \\ \begin{matrix}y^p_1 & & \\ & \ddots & \\ & & y^p_n\end{matrix} & & & & I_n \end{pmatrix}.$$

The Veronese is parametrized by $(x^j) \mapsto (a_0 + x^j a_j)^p$ and the same matrix as above works replacing $x^j_s$ with $x^j$ for all $s$ and similarly for $y$.

## 6. Definitions of VP, VNP and VP$_e$

In the discussion above, the problem presented was far removed from geometry, and it was only after significant work that geometric objects appeared. Valiant (1979a) has proposed algebraic analogs of the complexity classes **P** and **NP** in terms of sequences of polynomials. Such classes should be closer to geometry, however, the properties of the resulting sequences of hypersurfaces relevant for geometry have yet to be determined. In this expository section, I briefly review the relevant definitions.

### 6.1. $\mathbf{VP}_e$

An elementary measure of the complexity of a (homogeneous) polynomial $p$ is as follows: given an expression for $p$, count the total number of additions plus multiplications present in the expression, and then take the minimum over all possible expressions.

**Example 6.1.1.**

$$p_n(x,y) = x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + y^n.$$

This expression for $p_n$ involves $n(n+1)$ multiplications and $n$ additions, but one can also write

$$p_n(x,y) = (x+y)^n$$

which requires $n$ multiplications and one addition to evaluate.

**Definition 6.1.2.** An *arithmetic circuit* $C$ is a finite, acyclic, directed graph with vertices of in degree 0 or 2 and exactly one vertex of out degree 0. In degree 0, inputs are labelled by elements of $\mathbb{C} \cup \{x_1, \ldots, x_n\}$ and in degree 2, vertices are called *computation gates* and labelled with $+$ or $*$. The *size* of $C$ is the number of vertices. From a circuit $C$, one can construct a polynomial $p_C$ in the variables $x_1, \ldots, x_n$.

If $C$ is a tree (i.e., all out degrees are at most one), then the size of $C$ equals the number of $+$'s and $*$'s used in the formula constructed from $C$.

**Definition 6.1.3.** For $f \in S^d\mathbb{C}^m$, the *expression size* $E(f)$ is the smallest size of a tree circuit that computes $f$. Define the class $\mathbf{VP}_e$ to be the set of sequences $(p_n)$ such that there exists a sequence $(C_n)$ of tree circuits, with the size of $C_n$ bounded by a polynomial in $n$, such that $C_n$ computes $p_n$.

It turns out that expression size is too naïve a measurement of complexity, as consider Example 6.1.1, we could first compute $z = x + y$, then $w = z^2$, then $w^2$ etc. until the exponent is close to $n$, for a significant savings in computation when $n$ is large.

### 6.2. $\mathbf{VP}$, $\mathbf{VP}_{ws}$ *and closures*

Circuits more general than trees allow one to use the results of previous calculation and gives rise to the class $\mathbf{VP}$:

**Definition 6.2.1.** The class $\mathbf{VP}$ is the set of sequences $(p_n)$ of polynomials of degree $d(n)$ in $v(n)$ variables where $d(n)$, $v(n)$ are bounded by polynomials in $n$ and such that there exists a sequence of circuits $(C_n)$ of polynomially bounded size such that $C_n$ computes $p_n$.

A polynomial $p(y_1, \ldots, y_m)$ is a *projection* of $q(x_1, \ldots, x_n)$ if we can set $x_i = a_i^s y_s + c_i$ for constants $a_i^s$, $c_i$ to obtain $p(y_1, \ldots, y_m) = q(a_1^s y_s + c_1, \ldots, a_n^s y_s + c_n)$. Geometrically, if we homogenize the polynomials by adding variables $y_0, x_0$, we can study the zero sets in projective space. Then $p$ is a projection of $q$ iff $\mathrm{Zeros}(p) \subset \mathbb{CP}^m$ is a linear section of $\mathrm{Zeros}(q) \subset \mathbb{CP}^n$. This is because if we consider a projection map $V \to V/W$, then $(V/W)^* \simeq W^\perp \subset V^*$.

**Definition 6.2.2.** A sequence $(p_n)$ is *hard* for a complexity class $\mathbf{C}$ defined by sequences of polynomials, if for all sequences $(q_m)$ in $\mathbf{C}$, $q_m$ can be realized as a projection of $p_{n(m)}$ where the function $n(m)$ is bounded by a polynomial in $m$. A sequence $(p_n)$ is *complete* for $\mathbf{C}$ if it is hard for $\mathbf{C}$ and if $(p_n) \in \mathbf{C}$.

A famous example of a sequence in $\mathbf{VP}$ is $\det_n \in S^n\mathbb{C}^{n^2}$, despite its apparently huge expression size. While it is known that $(\det_n) \in \mathbf{VP}$, it is not known whether or not it is $\mathbf{VP}$-complete. On the other hand, it is known that $(\det_n)$ is $\mathbf{VP}_e$-hard, although it is not known whether or not $(\det_n) \in \mathbf{VP}_e$. When complexity theorists and mathematicians are confronted with such a situation, what else do they do other than make another definition?

**Definition 6.2.3.** The class $\mathbf{VP}_{ws}$ is the set of sequences $(p_n)$ where $\deg(p_n)$ is bounded by a polynomial and such that there exists a sequence of circuits $(C_n)$ of polynomially bounded size such that $C_n$ represents $p_n$, and such that at any multiplication vertex, the component of the circuit of one of the two edges coming in is disconnected from the rest of the circuit by removing the multiplication vertex.

In Malod and Portier (2008) they show $(\det_n)$ is $\mathbf{VP}_{ws}$-complete, so Conjecture 7.3.3 may be rephrased as conjecturing $\mathbf{VP}_{ws} \neq \mathbf{VNP}$.

**Remark 6.2.4.** It is considered a major open question to determine whether or not $(\det_n) \in \mathbf{VP}_e$.

**Definition 6.2.5.** Given a complexity class $\mathbf{C}$ defined in terms of sequences of polynomials, we define a sequence $(p_n)$ to be in $\overline{\mathbf{C}}$ if there exists a curve of sequences $q_{n,t}$, such that for each fixed $t_0 \neq 0$, $(q_{n,t_0}) \in \mathbf{C}$ and for all $n$, $\lim_{t\to 0} q_{n,t} = p_n$.

### 6.3. VNP

The class **VNP** essentially consists of polynomials whose coefficients can be determined in polynomial time. Consider a sequence $h = (h_n) \in \mathbb{C}[x_1, \dots, x_n]_{\leq n}$ of (not necessarily homogeneous) polynomials of the form

$$h_n = \sum_{e\in\{0,1\}^n} g_n(e) x_1^{e_1} \cdots x_n^{e_n} \tag{6.3.1}$$

where $(g_n) \in \mathbf{VP}$. Define **VNP** to be the set of all sequences that are projections of sequences of the form $h$. For equivalent definitions, see e.g., Bürgisser et al. (1997, Section 21.2).

**Proposition 6.3.1** (*Valiant, 1979a*)**.** $(\mathrm{perm}_n) \in \mathbf{VNP}$, *in fact is* **VNP***-complete.*

**Conjecture 6.3.2** (*Valiant, 1979b, Valiant's hypothesis*)**.** $\mathbf{VP} \neq \mathbf{VNP}$.

It is known that $\mathbf{P} \neq \mathbf{NP}$ would imply $\mathbf{VP} \neq \mathbf{VNP}$ over finite fields.

## 7. Projecting the determinant to the permanent

### 7.1. *Complexity of* $(\det_n)$

For a vector space $V$, let $S^dV$ denote the space of homogeneous polynomials of degree $d$ on the dual space $V^*$. Let $E, F = \mathbb{C}^n$, and let $E \otimes F$ denote the space of linear maps $E^* \to F$. The polynomial $\det_n \in \Lambda^n E \otimes \Lambda^n F \subset S^n(E \otimes F)$ is the unique up to scale (nonzero) element of the one-dimensional vector space $\Lambda^n E \otimes \Lambda^n F$. $\det_n$ is invariant under the action of $SL(E)\times SL(F)$, as $\det(axb) = \det(a)\det(x)\det(b)$. Fix bases in $E, F$, so we may identify $E \otimes F$ with the space of $n \times n$ matrices and $SL(E)$ as the subgroup of all $n \times n$ matrices with determinant one. If $x \in E^* \otimes F^*$ is expressed as a matrix, letting $\mathfrak{S}_n$ denote the permutation group on $n$ elements, then

$$\det_n(x) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma) x^1_{\sigma(1)}, \dots, x^n_{\sigma(n)}.$$

In the naïve computation of $\det_n$ with this formula, one uses $(n-1)(n!)$ multiplications and $n! - 1$ additions. Nevertheless, one has the essentially classical:

**Proposition 7.1.1.** $(\det_n) \in \mathbf{VP}$. *More precisely,* $\det_n$ *can be evaluated by performing* $\mathcal{O}(n^4)$ *arithmetic operations.*

Fixing bases of $E, F$ and identifying $E^* \otimes F^*$ with the space of $n \times n$ matrices, there are subspaces of $E^* \otimes F^*$ on which det can be evaluated by performing $n$ arithmetic operations, for example the upper-triangular matrices which we will denote by $\mathfrak{b}$.

$\det_n$ is invariant under the action of the subgroup $U \subset SL(E)$ of all upper-triangular matrices with 1's on the diagonal as well as the group $\mathcal{W}$ of permutation matrices in $SL(E)$.

Proposition 7.1.1 essentially follows from:

**Proposition 7.1.2** (*Gaussian Elimination*). *Notations as above, given $x \in E^* \otimes F^*$, there exists $g$ in the group generated by $U$ and $\mathcal{W}$ such that $g \cdot x \in \mathfrak{b}$. Such a $g$ can be computed by performing a number of arithmetic operations that is polynomial in $n = \dim E$.*

**Proof of Proposition 7.1.1.** For sufficiently generic matrices the algorithm is clear and just using $U$ is sufficient. For an algorithm that works for arbitrary matrices, see, e.g., Blum et al. (1998), Malod and Portier (2008). □

*7.2. The permanent*

Define the permanent $\mathrm{perm}_n \in S^n(E \otimes F)$ to be the unique up to scale element of $S^nE \otimes S^nF \subset S^n(E \otimes F)$ invariant under the action of the diagonal matrices and permutation matrices acting on both the left and the right (i.e. the normalizers of the tori in $SL(E) \times SL(F)$). If $x \in E^* \otimes F^*$ is expressed as a matrix, then

$$\mathrm{perm}_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^1_{\sigma(1)}, \ldots, x^n_{\sigma(n)}.$$

*7.3. The permanent as a projection of the determinant*

**Theorem 7.3.1** (Valiant, 1982). *Every $f \in \mathbb{C}[x_1, \ldots, x_n]$ of expression size (see Section 6.1.3) $u$ is both a projection of $\det_{u+3}$ and $\mathrm{perm}_{u+3}$.*

In particular, any polynomial is the projection of some determinant.

**Example 7.3.2.** Let $f(x) = x_1x_2x_3 + x_4x_5x_6$, then

$$f(x) = \det \begin{pmatrix} 0 & x_1 & 0 & x_4 & 0 \\ 0 & 1 & x_2 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_5 \\ x_6 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Conjecture 7.3.3** (Valiant, 1979a). *Let $dc(\mathrm{perm}_m)$ be the smallest integer $n$ such that $\mathrm{perm}_m$ can be realized as a projection of $\det_n$. Then $dc(\mathrm{perm}_m)$ grows faster than any polynomial in $m$.*

*7.4. Differential invariants of $\det_n$*

This subsection discusses preliminary results of work with D. The and L. Manivel.

Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be varieties such that there is a linear space $L \simeq \mathbb{P}^m \subset \mathbb{P}^n$ such that $Y = X \cap L$.

Say $y \in Y = X \cap L$. Then the differential invariants of $X$ at $y$ will project to the differential invariants of $Y$ at $y$. A definition of differential invariants adequate for this discussion (assuming $X$, $Y$ are hypersurfaces) is as follows: choose local coordinates $(x^1, \ldots, x^{n+1})$ for $\mathbb{P}^n$ at $x = (0, \ldots, 0) \in X$ such that $T_xX = \langle \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \rangle$ and expand out a Taylor series for $X$:

$$x^{n+1} = r^2_{i,j}x^ix^j + r^3_{i,j,k}x^ix^jx^k + \cdots.$$

The zero set of $(r^2_{ij}dx^i \circ dx^j, \ldots, r^k_{i_1,\ldots,i_k}dx^{i_1} \circ \cdots \circ dx^{i_k})$ in $\mathbb{P}T_xX$ is independent of choices. I will refer to the polynomials $F_{\ell,x}(X)$ although they are not well defined individually. For more details see, e.g. Ivey and Landsberg (2003, Chap. 3).

One says that $X$ can approximate $Y$ to $k$-th order at $x \in X$ mapping to $y \in Y$ if one can project the differential invariants to order $k$ of $X$ at $x$ to those of $Y$ at $y$.

In Mignon and Ressayre (2004) it was shown that the determinant can approximate *any* polynomial to second order if $n \geq \frac{m^2}{2}$ and that $\mathrm{perm}_m$ is generic to order two, giving the lower bound $dc(\mathrm{perm}_m) \geq \frac{m^2}{2}$. The previous lower bound was $dc(\mathrm{perm}_m) \geq \sqrt{2}m$ due to Cai (1990) building on work of von zur Gathen (1987).

One can ask what happens at higher orders.

If $X \subset \mathbb{P}V$ is a quasi-homogeneous variety, i.e., a group $G$ acts linearly on $V$ and $X = \overline{G \cdot [v]}$ for some $[v] \in \mathbb{P}V$, then $T_{[v]}X$ is a $\mathfrak{g}([v])$-module, where $\mathfrak{g}([v])$ denotes the Lie algebra of the stabilizer of $[v]$ in $G$.

Let $e^1, \ldots, e^n$ be a basis of $E^*$ and $f^1, \ldots, f^n$ a basis of $F^*$, let $v = e^1 \otimes f^1 + \cdots + e^{n-1} \otimes f^{n-1}$, so $[v] \in \mathrm{Zeros}(\det_n)$ and $\mathrm{Zeros}(\det_n) = \overline{SL(E) \times SL(F) \cdot [v]}$.

Write $E' = v(F) \subset E^*$, $F' = v(E) \subset F^*$ and set $\ell_E = E^*/E'$, $\ell_F = F^*/F'$. Then, using $v$ to identify $F' \simeq (E')^*$, one obtains $T_{[v]}\mathrm{Zeros}(\det_n) = \ell_E \otimes F' \oplus (F')^* \otimes F' \oplus (F')^* \otimes \ell_F$ as a $\mathfrak{g}([v])$-*module*. Write an element of $T_{[v]}\mathrm{Zeros}(\det_n)$ as a triple $(x, A, y)$. In matrices,

$$v = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \qquad T_{[v]} \sim \begin{pmatrix} A & y \\ x & 0. \end{pmatrix}$$

Taking the $\mathfrak{g}([v])$-module structure into account, it is straightforward to show:

**Theorem 7.4.1.** *Let $X = \mathrm{Zeros}(\det_n) \subset \mathbb{P}^{n^2-1} = \mathbb{P}(E \otimes F)$, let $v = e_1 \otimes f_1 + \cdots + e_{n-1} \otimes f_{n-1} \in X$. With the notations above, there exist bases in which the differential invariants of $X$ at $[v]$ are the polynomials*

$$\begin{aligned} F_{2,[v]}(X) &= xy \\ F_{3,x}(X) &= xAy \\ &\vdots \\ F_{k,x}(X) &= xA^{k-2}y. \end{aligned}$$

Since the permanent hypersurface is not quasi-homogeneous, its differential invariants are more difficult to calculate. It is even difficult to write down a general point in a nice way (that depends on $m$, keeping in mind that we are not concerned with individual hypersurfaces, but sequences of hypersurfaces). For example, the point on the permanent hypersurface chosen in Mignon and Ressayre (2004) is not general as there is a finite group that preserves it. To get lower bounds it is sufficient to work with any point of the permanent hypersurface, but one will not know if the obtained bounds are sharp. To arrive at $dc(\mathrm{perm}_m)$ being an exponential function of $m$, one might expect to improve the exponent by one at each order of differentiation. The following theorem shows that this does not happen at order three.

The Mignon–Ressayre result implies that any hypersurface in $2n - 2$ variables defined by a homogeneous polynomial can be approximated to order two at any point by an affine linear projection of $\{\det_n = 0\} \subset \mathbb{C}^{n^2}$.

**Theorem 7.4.2.** *Any hypersurface in $n - 1$ variables can be approximated to order three at any point by an affine linear projection of $\{\det_n = 0\} \subset \mathbb{C}^{n^2}$.*

*In particular, $\{\mathrm{perm}_m = 0\} \subset \mathbb{C}^{m^2}$ can be approximated to order three at a general point by an affine linear projection of $\{\det_{m^2+1} = 0\} \subset \mathbb{C}^{(m^2+1)^2}$.*

**Proof.** The rank of $F_2$ for the determinant is $2(n - 1)$, whereas the rank of $F_2$ for the permanent, and of a general hypersurface in $q$ variables at a general point, is $q - 2$. so one would need to project to eliminate $(n - 1)^2$ variables to agree to order two.

Thus it is first necessary to perform a projection so that the matrix $A$, which has independent variables as entries becomes linear in the entries of $x, y$, write $A = A(x, y)$. The projected pair $F_2, F_3$ is still not generic because it has two linear spaces of dimension $n - 1$ in its zero set. This can be fixed by setting $y = L(x)$ for $L : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ a linear isomorphism. At this point one has $F_2 = L(x)x$, $F_3 = L(x)A(x, L(x))x$. Take $L$ to be the identity map, so the cubic is of the form $\sum_{i,j} x_i A_{ij}(x) x_j$ where the $A_{ij}(x)$ are arbitrary. This is an arbitrary cubic. $\square$

## 8. Geometric complexity theory approach to $\overline{\mathbf{VP}_{ws}}$ versus VNP

### 8.1. The Mulmuley–Sohoni program

In a series of papers (Mulmuley, 2001, 2008; Mulmuley and Sohoni, preprint-a; Mulmuley, preprint-c; Mulmuley and Narayaran, 2007; Mulmuley, 2007, preprint-b, preprint-a), K. Mulmuley and M. Sohoni outline an approach to prove $\overline{\mathbf{VP}_{ws}} \neq \mathbf{VNP}$. Let $\ell$ be a linear coordinate on $\mathbb{C}$, and take any linear inclusion $\mathbb{C} \oplus \mathbb{C}^{m^2} \subset \mathbb{C}^{n^2}$ to have $\ell^{n-m}\mathrm{perm}_m$ be a homogeneous degree $n$ polynomial on $\mathbb{C}^{n^2}$. Mulmuley and Sohoni observe that $\overline{\mathbf{VP}_{ws}} \neq \mathbf{VNP}$ is equivalent to the following assertion: Let $\overline{dc}(\mathrm{perm}_m)$ denote that the smallest value of $n$ such that $[\ell^{n-m}\mathrm{perm}_m] \in \overline{GL_{n^2} \cdot [\det_n]}$. Then $\overline{\mathbf{VP}_{ws}} \neq \mathbf{VNP}$ is equivalent to the statement $\overline{dc}(\mathrm{perm}_m)$ grows faster than any polynomial:

**Conjecture 8.1.3** (*Mulmuley, 2001*). *$\overline{dc}(\mathrm{perm}_m)$ grows faster than any polynomial in m.*

**Remark 8.1.4.** Recently in Landsberg (2010) it was shown that $\overline{dc}(\mathrm{perm}_m) \geq \frac{m^2}{2}$ and that there exist sequences $(p_m)$ with $\overline{dc}(p_m) < dc(p_m)$.

### 8.2. Description of the program to prove Conjecture 8.1.3 outlined in Mulmuley (2008)

For a complex projective variety $X \subset \mathbb{P}V$, let $I(X) \subset Sym(V^*)$ be the ideal of polynomials vanishing on $X$. Let $\mathbb{C}[X] = Sym(V^*)/I(X)$ denote the homogeneous coordinate ring. For complex projective varieties $X, Y \subset \mathbb{P}^N = \mathbb{P}V$, one has $X \subset Y$ iff $\mathbb{C}[Y]$ surjects onto $\mathbb{C}[X]$ (by restriction of functions). Mulmuley and Sohoni set out to prove:

**Conjecture 8.2.1** (*Mulmuley, 2001*). *Let $u(m)$ be a polynomial. There is a sequence of irreducible modules $M_m$ for $GL_{u(m)^2}$ such that $M_m$ appears in $\mathbb{C}[\overline{GL_{u(m)^2} \cdot [\ell^{u(m)-m}\mathrm{perm}_m]}]$ but not in $\mathbb{C}[\overline{GL_{u(m)^2} \cdot [\det_{u(m)}]}]$.*

In an attempt to find such a sequence of modules, Mulmuley and Sohoni consider $SL_{n^2} \cdot \det_n$ and $SL_{m^2} \cdot \mathrm{perm}_m$ because on the one hand their coordinate rings can be determined in principle using representation theory, and on the other hand they are closed affine varieties. They observe that any $SL_{n^2}$-module appearing in $\mathbb{C}[SL_{n^2} \cdot \det_n]$ must also appear in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]_k$ for some $k$. Regarding the permanent, for $n > m$, $SL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m$ is not closed, so they develop machinery to transport information about $\mathbb{C}[SL_{m^2} \cdot \mathrm{perm}_m]$ to $\mathbb{C}[\overline{GL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m}]$, including a notion of *partial stability*.

Mathematical aspects of this program are discussed in Buergisser et al. (preprint). The representation-theoretic information Mulmuley and Sohoni propose to exploit is studied in detail. In particular Buergisser et al. (preprint, Thm 5.7.1) is a precise description of conditions on *Kronecker coefficients* that are equivalent to Conjecture 8.2.1. In addition, suggestions are made for further geometric information that one could take into account that might imply a more tractable problem in representation theory.

The price of using $SL_{n^2}$ instead of $GL_{n^2}$ is that one loses the grading of the coordinate rings. On the other hand, in order to use $GL_{n^2}$, one must solve, or at least partially solve, an *extension problem*, which to even begin work on, means that one must determine the codimension one components of the boundaries in the orbit closures.

**Remark 8.2.2.** Recently in Bürgisser et al. (2009) evidence was given that the vanishing of Kronecker coefficients that would be necessary for Conjecture 8.2.1 is unlikely to occur.

### 8.3. Beyond determinant and permanent

Instead of considering $\det_n$, one could take a sufficiently generic $g_n \in GL_{n^2}$ and consider $p_n := \det_n + g_n \cdot \det_n$. Then the subgroup $G(p_n)$ of $GL_{n^2}$ preserving $p_n$ will be the same as that for a generic polynomial, although the sequence $(p_n)$ is still $\mathbf{VP}_{ws}$-complete. Thus just looking at the orbit, there would be *fewer* modules appearing in $\mathbb{C}[SL_{n^2} \cdot [p_n]]$ than in $\mathbb{C}[SL_{n^2} \cdot [\mathrm{perm}_n]]$. In particular the orbit closure is larger than that of the permanent. More generally, let $r(n)$ be a polynomial and take a sequence of points in $p_n \in \sigma_{r(n)}(\overline{GL_{n^2}\cdot[\det_n]})$, the $r$-th secant variety of $\overline{GL_{n^2}\cdot[\det_n]}$. One could study the differential invariants of these varieties to see how they project to the permanent as in Section 7 and consider GCT program using the varieties $\overline{GL_{n^2}\cdot p_n}$.

More examples of sequences of polynomials are given by the *immanants* defined by Littlewood in Littlewood (2006). Immanants generalize the determinant and permanent. Given a partition $\pi = (p_1, \dots, p_r)$ of $n$, and a vector space $V$ of dimension at least $r$, let $S_\pi V$ denote the corresponding irreducible $GL(V)$-module. $IM_\pi \in S^n\mathbb{C}^{n^2}$ may be defined as follows: consider $\mathbb{C}^{n^2} = E\otimes F$, where $E, F = \mathbb{C}^n$. Then $S^n(E\otimes F) = \oplus_\pi S_\pi E\otimes S_\pi F$ as a $GL(E)\times GL(F)$ module. Let $D^E \subset SL(E)$, $D^F \subset SL(F)$ denote the tori, i.e., the groups of diagonal matrices with determinant one. Let $\mathfrak{S}_n^E, \mathfrak{S}_n^F$ denote the groups of permutation matrices acting on the left and right, and let $\Delta(\mathfrak{S}_n) \subset \mathfrak{S}_n^E\times\mathfrak{S}_n^F$ denote the diagonal embedding. Then $IM_\pi \in S_\pi E\otimes S_\pi F$ is the unique (up to scale) element acted on trivially by $(D^E\times D^F)\ltimes\Delta(\mathfrak{S}_n)$.

In Ye (preprint), building on work in Duffner (1994) and Coelho (1996), it is shown that for all non-self-dual $\pi \neq (1^n), (n)$, that $G(IM_\pi) = ((D^E\times D^F)\ltimes\Delta(\mathfrak{S}_n))\ltimes\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by sending a matrix to its transpose.

Consider $IM_{(n-1,1)}$ and $IM_{(2,1^{n-1})}$. The first is **VNP**-complete and the second is in **VP**, see Bürgisser (2000), so one could attempt to apply the GCT program to them. By Ye (preprint) $G(IM_{(n-1,1)}) = G(IM_{(2,1^{n-1})})$ so $\mathbb{C}[SL_{n^2}\cdot[IM_{(n-1,1)}]] = \mathbb{C}[SL_{n^2}\cdot[IM_{(2,1^{n-1})}]]$. Without examining the boundaries of $\overline{GL_{n^2}\cdot[IM_{(n-1,1)}]}$ and $\overline{GL_{n^2}\cdot[IM_{(2,1^{n-1})}]}$ there is no way to distinguish them.

Such investigations will be the subject of future work.

## 9. Towards geometric definitions of complexity classes

As mentioned several times, symmetry, sometimes in hidden form, appears to play a central role in characterizing sequences in **VP** that are apparently not in $\mathbf{VP}_e$. To make a geometric study of complexity, it would be desirable to have coordinate free definitions. In this section I give a coordinate free and geometric definition of the class $\overline{\mathbf{VP}}_e$. I then give a coordinate free and geometric definition of a class $\mathbf{VP}_{hs}$ which is intended as a first attempt to geometrize the class **VP**. Unfortunately at this writing I have no idea for a proposed purely geometric definition of **VNP**. (S. Basu and M. Shub, in separate personal communications, have proposed that **VNP** should somehow be viewed as a bundle over **VP**, but I have been unable to make this precise.)

### 9.1. Joins and multiplicative joins

The *join* of projective varieties $X_1, \dots, X_r \subset \mathbb{P}V$, $J(X_1, \dots, X_r) \subset \mathbb{P}V$, is the Zariski closure of the points of the form $[p_1 + \cdots + p_r]$ with $[p_j] \in X_j$. The expected dimension of $J(X_1, \dots, X_r)$ is $\min(\sum \dim X_j + r - 1, \dim\mathbb{P}V)$. Let $\hat{T}_{[p]}X \subset V$ denote the affine tangent space of $X$ at $[p] \in X$. Terracini's lemma says that if $([p_1], \dots, [p_r]) \in X_1\times\cdots\times X_r$ is a general point, then

$$\hat{T}_{[p_1+\cdots+p_r]}J(X_1, \dots, X_r) = \hat{T}_{[p_1]}X_1 + \cdots + \hat{T}_{[p_r]}X_r.$$

One can similarly define joins in affine space. The expressions are the same without the brackets.

**Definition 9.1.1.** Let $X \subset \mathbb{P}S^aV$, $Y \subset \mathbb{P}S^bV$ be varieties. Define the *multiplicative join* of $X$ and $Y$, $MJ(X, Y)$, by

$$MJ(X, Y) := \{[pq] \mid [p] \in X,\ [q] \in Y\} \subset \mathbb{P}S^{a+b}V.$$

For varieties $X_j \subset \mathbb{P}S^{d_j}V$, define $MJ(X_1, \dots, X_r) \subset S^dV$ similarly (or inductively as $MJ(X, Y, Z) = MJ(X, MJ(Y, Z))$). In the special case $X_j = \mathbb{P}V \subset \mathbb{P}S^1V$, $MJ(\mathbb{P}V, \dots, \mathbb{P}V)$ is the Chow variety of polynomials that decompose into a product of linear factors.

Similarly, let $A_{d,v}$ denote the space of all polynomials of degree at most $d$ in $v$ variables. For affine varieties $X \subset A_{d_1,v}$, $Y \subset A_{d_2,v}$, $MJ(X, Y) \subset A_{d_1+d_2,v}$ is defined in the same way without brackets.

**Proposition 9.1.2.** *Let $X_j \subset \mathbb{P}S^{d_j}V$ be varieties and let $([p_1], \ldots, [p_r]) \in X_1 \times \cdots \times X_r$ be a general point. Then*

$$\hat{T}_{[p_1 \circ \cdots \circ p_r]} MJ(X_1, \ldots, X_r) = \hat{T}_{[p_1]} X_1 \circ p_2 \circ \cdots \circ p_r + \cdots + p_1 \circ \cdots \circ p_{r-1} \circ \hat{T}_{[p_r]} X_r.$$

*In particular, the expected dimension of $MJ(X_1, \ldots, X_r)$ is $\min(\dim X_1 + \cdots + \dim X_r, \dim \mathbb{P}S^d V)$.*

**Proof.** Let $p_j(t)$ be a curve in $X_j$ with $p_j(0) = p_j$. Differentiate the expression $p_1(t) \circ \cdots \circ p_r(t)$ at $t = 0$ to get the result. □

**Question 9.1.3.** What are the degenerate multiplicative joins, i.e., those that fail to be of the expected dimension?

### *9.2. A geometric characterization of $\overline{\mathbf{VP}}_e$*

Recall that the expression size $E(p)$ of a polynomial $p \in A_{d,v}$ is given by the number of internal nodes of the smallest tree circuit computing $p$. Define $\overline{E}(p)$ to be the smallest integer such that there is a curve $p_t$ with $\lim_{t \to 0} p_t = p_0$ and such that $E(p_t) = \overline{E}(p)$ for $t \neq 0$. By definition, a sequence $(p_n) \in A_{d(n),v(n)}$ is in $\mathbf{VP}_e$ (resp. $\overline{\mathbf{VP}}_e$) if there exists a polynomial $r(n)$ such that $E(p_n) \leq r(n)$ (resp. $\overline{E}(p_n) \leq r(n)$).

To a tree circuit $\Gamma$ associated to a polynomial $p$, associate an algebraic variety as follows: first form an new tree circuit $\Gamma'$ by collapsing all pairs of input nodes that are joined by a $+$ to a single input node, and repeat as many times as necessary until no pairs of input nodes are joined by a $+$. (I take this first step to eliminate the choice of coordinates involved in making the circuit.) Associate to each input node a copy of $\mathbb{P}V$.

Thus on $\Gamma'$, if any two input nodes are joined, they are joined by a $*$-node. Now perform a step by step procedure to eliminate all $*$-nodes joining pairs of input nodes. Take a $*$-node joined to two input nodes, and form a subtree containing all other $*$-nodes joined to it and an input node. Say there are $j_1 - 1$ such. Record the variety $MJ_{j_1} := MJ(\mathbb{P}V, \ldots, \mathbb{P}V)$ of $j_1$ copies of $\mathbb{P}V$. Collapse the subtree to a single input node and associate $MJ_{j_1}$ to this input node. Now start again, say we arrive at $j_2 - 1$ nodes in the subtree and record the variety $MJ_{j_2} = MJ(\mathbb{P}V, \ldots, \mathbb{P}V)$ of $j_2$ copies of $\mathbb{P}V$. Continue until we have recorded $p$ varieties of multiplicative joins of $V$ of various sizes.

We arrive at a new graph $\Gamma''$ all of whose $p$ input nodes have varieties $MJ_{j_i}$ associated to them and when input nodes are paired together by an internal node, the node is a $+$-node. Now perform a step by step procedure to eliminate all $+$'s joining pairs of input nodes. Take the first $+$, say that the variety $MJ_{j_{i_1}}$ is one of the input nodes and form a subtree consisting of all other $+$'s joined to it. Say there are $k - 1$ such. Record the variety $J(MJ_{j_{i_1}}, \ldots, MJ_{j_{i_k}})$. Collapse the subtree to a single input node and associate $J(MJ_{j_{i_1}}, \ldots, MJ_{j_{i_k}})$ to this input node. Continue until we have varieties of joins of multiplicative joins of various sizes as our new input nodes with all pairings of input nodes $*$-nodes.

Now continue as we did with $\Gamma'$, taking multiplicative joins (of the joins of multiplicative joins) until the further collapsed graph has all pairings of input nodes $+$'s, then go back to taking joins etc.

This process terminates after a number of steps fewer than the number of nodes of $\Gamma$, and one arrives at a variety $\Sigma_\Gamma$ of successive joins and multiplicative joins. By construction $p \in \Sigma_\Gamma$.

Note that for each such variety, there are many $\Gamma$ that are associated to it, but each has, up to the initial $v$ times the number of initial input nodes, the same expression size.

Let $\Sigma_R^{d,v}$ denote the union of all the varieties obtainable from a graph of at most $R$ internal nodes computing an element of $A_{d,v}$. There is a finite number of such, so $\Sigma_R^{d,v}$ is an algebraic variety. The above discussion implies

**Theorem 9.2.1.** *Let $p_n \in A_{d(n),v(n)}$ be a sequence with $d$, $v$ polynomials. Then $(p_n) \in \overline{\mathbf{VP}}_e$ iff there exists a polynomial $R(n)$ and $p_n \in \Sigma_{R(n)}^{d(n),v(n)}$. In other words the complexity class $\overline{\mathbf{VP}}_e$ is characterized by a sequence of algebraic varieties.*

**Remark 9.2.2.** One has to use the class $\overline{\mathbf{VP}}_e$ instead of $\mathbf{VP}_e$ because when taking joins one must include limits. It is not necessary to include limits when taking multiplicative joins.

**Corollary 9.2.3.** *A sequence $(p_n) \in A_{d(n),v(n)}$ is in $\overline{\mathbf{VP}}_e$ if either $d$ or $v$ is constant. A generic sequence in $A_{d(n),v(n)}$ is not in $\overline{\mathbf{VP}}_e$ if both $d, v$ grow at least linearly with respect to $n$.*

**Proof.** $\dim \Sigma_R^{d,v} \leq (v+1)(R+1)$. $\square$

### *9.3. Towards a geometric understanding of* **VP**

Recall that the determinant has the property that for each $n$ there is a subspace $\mathfrak{b}_n \subset \mathbb{C}^{n^2}$, such that $\det_n|_{\mathfrak{b}_n} \in \mathbf{VP}_e$ and moreover $G(\det_n) \cdot \mathfrak{b}_n = \mathbb{C}^{n^2}$. This perspective motivates the following definitions.

Define $\mathbf{VP}^{prim}$ to be the set of sequences $p_n \in A_{d(n),v(n)}$, where for each $n$, there exists a linear subspace $\Sigma_n \subset \mathbb{C}^{v(n)}$, such that the sequence $(p_n)|_{\Sigma_n}$ lies in $\mathbf{VP}_e$, and letting $G(n)$ denote the subgroup of $GL_{v(n)}$ preserving $(p_n)$, ask moreover that $G(n) \cdot \Sigma_n = \mathbb{C}^{v(n)}$. Clearly $\mathbf{VP}^{prim} \subset \mathbf{VP}$ as the action of $G(n)$ is cheap. $\mathbf{VP}^{prim}$ is modeled on $(\det_n)$ where $\Sigma_n$ is the upper-triangular matrices. Define $\mathbf{VP}_{hs}$ to be set of sequences $(p_n)$ such that there exists another sequence $(r_n)$ with $(r_n) \in \mathbf{VP}_e$, a polynomial $q(n)$, and sequences $(p_{n,j})$, $j = 1, \ldots, q(n)$ such that $(p_{n,j}) \in \mathbf{VP}^{prim}$ and $p(n) = r_n(p_{n,1}, \ldots, p_{n,q(n)})$. Then $\mathbf{VP}_{hs} \subseteq \mathbf{VP}$.

**Question 9.3.1.** What is the gap, if any, between **VP** and $\mathbf{VP}_{hs}$?

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