Advances in Applied Mathematics 48 (2012) 340–353

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Steiner symmetrization using a finite set of directions

Daniel A. Klain

Department of Mathematical Sciences, University of Massachusetts Lowell, Lowell, MA 01854, USA

A R T I C L E   I N F O

Article history:
Received 14 May 2011
Accepted 19 September 2011
Available online 19 October 2011

MSC:
52A20

Keywords:
Convex body
Steiner symmetrization

A B S T R A C T

Let $v_1, \ldots, v_m$ be a finite set of unit vectors in $\mathbb{R}^n$. Suppose that an infinite sequence of Steiner symmetrizations are applied to a compact convex set K in $\mathbb{R}^n$, where each of the symmetrizations is taken with respect to a direction from among the v_i. Then the resulting sequence of Steiner symmetrals always converges, and the limiting body is symmetric under reflection in any of the directions v_i that appear infinitely often in the sequence. In particular, an infinite periodic sequence of Steiner symmetrizations always converges, and the set functional determined by this infinite process is always idempotent.

1. Introduction

Denote n-dimensional Euclidean space by $\mathbb{R}^n$, and let $\mathscr{K}_n$ denote the set of all compact convex sets in $\mathbb{R}^n$. Let $K \in \mathscr{K}_n$, and let u be a unit vector. Viewing K as a family of line segments parallel to u, slide these segments along u so that each is symmetrically balanced around the hyperplane $u^\perp$. By Cavalieri's principle, the volume of K is unchanged by this rearrangement. The new set, called the *Steiner symmetrization* of K in the direction of u, will be denoted by $\mathrm{s}_u K$. It is not difficult to show that $\mathrm{s}_u K$ is also convex, and that $\mathrm{s}_u K \subseteq \mathrm{s}_u L$ whenever $K \subseteq L$. A little more work verifies the following intuitive assertion: if you iterate Steiner symmetrization of K through a suitable sequence of unit directions, the successive Steiner symmetrals of K will approach a Euclidean ball in the Hausdorff topology on compact (convex) subsets of $\mathbb{R}^n$. A detailed proof of this assertion can be found in any of [11, p. 98], [16, p. 172], or [31, p. 313], for example.

For well over a century Steiner symmetrization has played a fundamental role in answering questions about isoperimetry and related geometric inequalities [14,15,26,27]. Steiner symmetrization appears explicitly in the titles of numerous papers (see e.g. [2,3,5,6,8–10,12,13,18–20,22,23,25,30]) and plays a key role in recent work such as [7,17,21,28,29].

E-mail address: daniel_klain@uml.edu.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2011.09.004

In spite of the importance of Steiner symmetrization throughout geometric analysis, many elementary questions about this construction remain open, including some concerning the following issue: Given a convex body K, under what conditions on the sequence of directions u_i does the sequence of Steiner symmetrals

$$\mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} K \tag{1}$$

converge? And if the sequence converges, what symmetries are satisfied by the limiting body?

The sequence of bodies (1) is called a *Steiner process*. If the limit

$$\lim_{i \to \infty} \mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} K \tag{2}$$

exists, the resulting body $\tilde{K}$ is called the limit of that Steiner process. In [3] it is shown that not every Steiner process converges, even if the directions u_i are dense in the sphere.

This article addresses the case in which an infinite Steiner process of the form (1) uses only a finite set of directions, each repeated infinitely often, whether in a periodic fashion, according to some more complex arrangement, or even completely at random.

Let $v_1, \ldots, v_m$ be a finite set of unit vectors in $\mathbb{R}^n$. Suppose that an infinite sequence of Steiner symmetrizations is applied to a compact convex set K in $\mathbb{R}^n$, where each of the symmetrizations is taken with respect to a direction from among the v_i. The main result of this article is Theorem 5.1, which asserts that the resulting sequence of Steiner symmetrals *always* converges. The limiting body is symmetric under reflection in any of the directions v_i that appear infinitely often in the sequence. In particular, an infinite periodic sequence of Steiner symmetrizations always converges, and the set functional determined by this infinite process is always idempotent.

2. Background and basic properties of Steiner symmetrization

Given a compact convex set K and a unit vector u, we have $\mathrm{s}_u K = K$ (or respectively, up to translation) if and only if K is symmetric under reflection across the subspace $u^{\perp}$ (respectively, up to translation). In particular, $\mathrm{s}_u K = K$ will hold for *every* direction u (or even a dense set of directions) if and only if K is a Euclidean ball centered at the origin.

Let $h_K : \mathbb{R}^n \to \mathbb{R}$ denote the support function of a compact convex set K; that is,

$$h_K(v) = \max_{x \in K} x \cdot v.$$

The standard separation theorems of convex geometry imply that the support function h_K characterizes the body K; that is, $h_K = h_L$ if and only if $K = L$. If K_i is a sequence in $\mathscr{K}_n$, then $K_i \to K$ in the Hausdorff topology if and only if $h_{K_i} \to h_K$ uniformly when restricted to the unit sphere in $\mathbb{R}^n$.

Given compact convex subsets $K, L \subseteq \mathbb{R}^n$ and $a, b \geqslant 0$, denote

$$aK + bL = \{ax + by \mid x \in K \text{ and } y \in L\}.$$

An expression of this form is called a *Minkowski combination* or *Minkowski sum*. Since K and L are convex sets, the set $aK + bL$ is also convex. Convexity also implies that $aK + bK = (a + b)K$ for all $a, b \geqslant 0$, although this does not hold for general sets. Support functions satisfy the identity $h_{aK+bL} = ah_K + bh_L$. (See, for example, any of [4,24,31].)

The following is also easy to prove (see, for example, [16, p. 169] or [31, p. 310]).

Proposition 2.1.

$$\mathrm{s}_u(K + L) \supseteq \mathrm{s}_u K + \mathrm{s}_u L.$$

Denote by $V_n(K)$ the n-dimensional volume of a set $K \subseteq \mathbb{R}^n$. Given $K, L \in \mathscr{K}_n$ and $\varepsilon > 0$, the function $V_n(K + \varepsilon L)$ is a polynomial in ε, whose coefficients are given by *Steiner's formula* [4,24,31]. In particular, the following derivative is well defined:

$$nV_{n-1,1}(K,L) = \lim_{\varepsilon \to 0} \frac{V_n(K+\varepsilon L) - V_n(K)}{\varepsilon} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} V_n(K+\varepsilon L). \tag{3}$$

The expression $V_{n-1,1}(K,L)$ is an example of a *mixed volume* of K and L. Important special cases appear when either of K or L is a unit Euclidean ball B:

$$\begin{aligned} nV_{n-1,1}(K,B) &= \text{Surface Area of } K, \\ \frac{2}{\omega_n} V_{n-1,1}(B,L) &= \text{Mean Width of } L \end{aligned} \tag{4}$$

where ω_n denotes the n-volume of the Euclidean unit ball B. We will denote the mean width of L by $W(L)$.

It follows from Proposition 2.1 and the volume invariance of Steiner symmetrization that

$$V_n(K+\varepsilon L) = V_n\big(\mathrm{s}_u(K+\varepsilon L)\big) \geqslant V_n(\mathrm{s}_u K + \varepsilon \mathrm{s}_u L),$$

so that

$$\frac{V_n(K+\varepsilon L) - V_n(K)}{\varepsilon} \geqslant \frac{V_n(\mathrm{s}_u K + \varepsilon \mathrm{s}_u L) - V_n(\mathrm{s}_u K)}{\varepsilon},$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0^+$, we have

$$V_{n-1,1}(K,L) \geqslant V_{n-1,1}(\mathrm{s}_u K, \mathrm{s}_u L) \tag{5}$$

for all $K, L \in \mathscr{K}_n$ and all unit directions u.

For $r \geqslant 0$ denote by rB the closed Euclidean ball of radius r centered at the origin. Since $\mathrm{s}_u B = B$, it follows from (4) and (5) that the surface area does not increase under Steiner symmetrization. Similarly, the mean width satisfies $W(\mathrm{s}_u K) \leqslant W(K)$ for all u.

From monotonicity it is also clear that, if $r, R \in \mathbb{R}$ such that

$$rB \subseteq K \subseteq RB \tag{6}$$

then

$$rB \subseteq \mathrm{s}_u K \subseteq RB. \tag{7}$$

Let R_K denote the minimum radius of any Euclidean n-ball containing K, and let r_K denote the maximal radius of any Euclidean n-ball contained inside K. It follows that

$$R_{\mathrm{s}_u K} \leqslant R_K \quad \text{and} \quad r_K \leqslant r_{\mathrm{s}_u K}. \tag{8}$$

It can also be shown using elementary arguments that Steiner symmetrization does not increase the diameter of a set [31, p. 310].

The following lemma will be useful in Section 5.

Lemma 2.2. *Suppose that $\{K_i\}$ is a convergent sequence of compact convex sets whose limit K has non-empty interior. Then, for all $0 < \tau < 1$, there is an integer $N > 0$ such that*

$$(1-\tau)K \subseteq K_i \subseteq (1+\tau)K$$

for all $i \geqslant N$.

Proof. Since K has interior, it has positive inradius r. Without loss of generality (translating as needed) we may assume that $rB \subseteq K$. For $\tau \in (0,1)$, choose N so that

$$K_i \subseteq K + r\tau B \quad \text{and} \quad K \subseteq K_i + r\tau B$$

for $i \geqslant N$. In this case,

$$K_i \subseteq K + r\tau B \subseteq K + \tau K = (1+\tau)K$$

and

$$K \subseteq K_i + r\tau B \subseteq K_i + \tau K,$$

so that $(1-\tau)K \subseteq K_i$. $\square$

It follows from Lemma 2.2 and the monotonicity property (7) that Steiner symmetrization is continuous with respect to K and u provided that $K \in \mathscr{K}_n$ has non-empty interior. (See also [16, p. 171] or [31, p. 312].)

Note that the interior condition is needed to guarantee continuity: Steiner symmetrization is *not* continuous at lower-dimensional sets. For example, consider a sequence of distinct unit line segments K_i with endpoints at $\pm u_i$, where $u_i \to u$. While the line segments K_i approach the line segment with endpoints at $\pm u$, their symmetrizations $\mathrm{s}_u K_i$ form a sequence of projected line segments in $u^\perp$ whose lengths approach zero, so that $\mathrm{s}_u K_i \to o$, the origin. But $\mathrm{s}_u K = K \neq o$, since K is already symmetric under reflection across $u^\perp$. See also [16, p. 170].

Denote by $\mathscr{K}^n_{r,R}$ the set of compact convex sets in $\mathbb{R}^n$ satisfying (6). By the Blaschke selection theorem $\mathscr{K}^n_{r,R}$ is compact. Since $\mathbb{S}^n$ is also compact, the function

$$(K,u) \mapsto \mathrm{s}_u K$$

is uniformly continuous on $\mathscr{K}^n_{r,R} \times \mathbb{S}^{n-1}$.

Moreover, it follows from monotonicity that Steiner symmetrization does respect the limits of decreasing sequences of sets, even if the limit has empty interior. More specifically, recall that if

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots \tag{9}$$

then

$$\lim_{m\to\infty} K_m = \bigcap_{m=1}^{\infty} K_m. \tag{10}$$

This follows from the fact that a pointwise limit of support functions of compact convex sets is always a uniform limit as well [24, p. 54]. We then have the following special case where continuity holds for Steiner symmetrization of a *descending* sequence of convex bodies, even when the limiting body is lower-dimensional.

Proposition 2.3. *Suppose that $\{K_m\}$ is a sequence of compact convex sets in $\mathbb{R}^n$ such that* (9) *holds, and let*

$$K = \lim_{m\to\infty} K_m = \bigcap_{m=1}^{\infty} K_m.$$

If u is a unit vector in $\mathbb{R}^n$, then

$$\mathrm{s}_u K = \lim_{m\to\infty} \mathrm{s}_u K_m = \bigcap_{m=1}^{\infty} \mathrm{s}_u K_m.$$

Proof. Denote by $\pi_u L$ the orthogonal projection of a compact convex set L onto the subspace $u^\perp$, and note that $\pi_u \mathrm{s}_u L = \pi_u L$ for all $L \in \mathscr{K}_n$. It follows from the monotonicity of s_u applied to the sequence (9) that

$$\mathrm{s}_u K_1 \supseteq \mathrm{s}_u K_2 \supseteq \mathrm{s}_u K_3 \supseteq \cdots,$$

so that the limit

$$L = \lim_{m\to\infty} \mathrm{s}_u K_m = \bigcap_{m=1}^{\infty} \mathrm{s}_u K_m$$

exists. Moreover, since $K \subseteq K_m$ for all m, it follows that $\mathrm{s}_u K \subseteq \mathrm{s}_u K_m$ as well, so that $\mathrm{s}_u K \subseteq L$. Note also that both $\mathrm{s}_u K$ and L are symmetric under reflection across $u^\perp$.

From the continuity of orthogonal projection we also have

$$\pi_u \mathrm{s}_u K = \pi_u K = \lim_{m\to\infty} \pi_u K_m = \lim_{m\to\infty} \pi_u \mathrm{s}_u K_m = \pi_u \lim_{m\to\infty} \mathrm{s}_u K_m = \pi_u L,$$

so that $\mathrm{s}_u K$ and L have the same orthogonal projection into $u^\perp$.

Finally, for each $x \in \pi_u L$, the linear slice of L perpendicular to x has length given by the infimum over m of the length of the linear slice of $\mathrm{s}_u K_m$ over the point x. Since Steiner symmetrization translates these slices (preserving their lengths), this is the same as the infimum over m of the length of the linear slice of K_m over the point x, which gives the length of linear slice of $\mathrm{s}_u K$ perpendicular to x. Hence, $L = \mathrm{s}_u K$. □

3. The layering function

Define the *layering function* of $K \in \mathscr{K}_n$ by

$$\Omega(K) = \int_0^\infty V_n(K \cap rB) e^{-r^2}\, dr.$$

Evidently the function Ω is monotonic and continuous on $\mathscr{K}_n$. The layering function vanishes on sets with empty interior and is strictly positive on sets with non-empty interior.

The following crucial property of Steiner symmetrization will be used in the sections that follow.

Theorem 3.1. *Suppose that $K \in \mathscr{K}_n$, and let u be a unit vector. Then*

$$\Omega(\mathrm{s}_u K) \geqslant \Omega(K). \tag{11}$$

If K has non-empty interior, then equality holds in (11) *if and only if $\mathrm{s}_u K = K$.*

In the proof of Theorem 3.1 we will use the following elementary fact: If D is a ball centered at the origin, and if X is a line segment, parallel to the unit vector u, having one endpoint in the interior of D and the other endpoint outside D, then Steiner symmetrization will strictly increase the slice length; that is,

$$|s_u X \cap D| > |X \cap D|. \tag{12}$$

To see this, let ℓ denote the line through X. Our conditions on the endpoints of X imply that $|\ell \cap D| > |X \cap D|$. Meanwhile, s_u fixes D and slides X parallel to u until it is symmetric about $u^\perp$. If $|X| < |\ell \cap D|$, then $s_u X$ will lie wholly inside D, so that $|s_u X \cap D| = |X| > |X \cap D|$ and (12) follows. If $|X| \geqslant |\ell \cap D|$, then $s_u X$ will cover the slice $\ell \cap D$ completely, so that $|s_u X \cap D| = |\ell \cap D|$ and (12) follows once again.

Proof of Theorem 3.1. Let u be a unit vector. The monotonicity of s_u implies that

$$s_u(K \cap rB) \subseteq s_u K \cap s_u rB = s_u K \cap rB,$$

so that

$$V_n(s_u K \cap rB) \geqslant V_n\big(s_u(K \cap rB)\big) = V_n(K \cap rB),$$

whence $\Omega(s_u K) \geqslant \Omega(K)$.

Evidently equality holds if $s_u K = K$. For the converse, suppose that K has non-empty interior, and that $s_u K \neq K$. Let ψ denote the reflection of $\mathbb{R}^n$ across the subspace $u^\perp$. Since $\psi K \neq K$ and K has non-empty interior, there is a point $x \in int(K)$ such that $\psi x \notin K$. Let D denote the ball around the origin of radius $|x|$, and let ℓ denote the line through x and parallel to u. The slice $K \cap \ell$ meets the boundary of D at x on one side of $u^\perp$, has an endpoint $x + \varepsilon u$ outside D and another endpoint $x - \delta u$ in the interior of D, where $\varepsilon, \delta > 0$. It follows from (12) that

$$|s_u K \cap \ell \cap D| > |K \cap \ell \cap D|.$$

Moreover, this holds for parallel slices through points x' in an open neighborhood of x. After integration of parallel slice lengths to compute volumes, we obtain

$$V_n(s_u K \cap rB) > V_n(K \cap rB)$$

for values of r in an open neighborhood of $|x|$. It follows that $\Omega(s_u K) > \Omega(K)$. □

In [11, p. 90] Eggleston proves a result similar to Theorem 3.1 for the surface area function. If $S(K)$ denotes the surface area of a compact convex set K having non-empty interior, then $S(s_u K) \leqslant S(K)$, with equality if and only if K and $s_u K$ are translates. The layering function Ω is more appropriate for our purposes, because the equality case in Theorem 3.1 is more stringent (even translates are not allowed).

4. Steiner processes

Let $\alpha = \{u_1, u_2, \ldots\}$ be a sequence of unit vectors in $\mathbb{R}^n$. Given $K \in \mathscr{K}_n$, denote

$$K_i = s_{u_i} \cdots s_{u_i} K \tag{13}$$

for $i = 1, 2, \ldots$.

Proposition 4.1. *The sequence of bodies* (13) *is uniformly bounded and therefore always has a convergent subsequence.*

Proof. Since K is compact, there exists $\rho \geqslant 0$ such that $K \subseteq \rho B$. Since Steiner symmetrization is monotonic, we have

$$\mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} K \subseteq \mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} \rho B = \rho B$$

as well, so that sequence is bounded. The Blaschke selection theorem [4,24,31] then implies that (13) has a convergent subsequence. □

Note that the original sequence $\{K_i\}$ defined by (13) does not necessarily converge to a limit. If $L = \lim_i K_i$ exists, we write $L = \mathrm{s}_\alpha K$. If L is the limit of some convergent subsequence of $\{K_i\}$, we say that L is a *subsequential limit* of $\mathrm{s}_\alpha K$.

Since the layering function Ω is weakly increasing under Steiner symmetrization by Theorem 3.1 and is also continuous and bounded above, the following is immediate.

Proposition 4.2. *If L is a subsequential limit of $\mathrm{s}_\alpha K$, then*

$$\Omega(L) = \sup_i \Omega(K_i).$$

Proposition 4.3. *If $\mathrm{s}_\alpha M$ exists, and if L is a subsequential limit of $\mathrm{s}_\alpha K$, then*

$$V_{n-1,1}(L, \mathrm{s}_\alpha M) = \inf_i V_{n-1,1}(K_i, \mathrm{s}_\alpha M).$$

Proof. We are given that $L = \lim_j K_{i_j}$ for some subsequence $\{K_{i_j}\}$ of (13). The continuity of mixed volumes implies that the sequence

$$V_{n-1,1}(K_{i_j}, \mathrm{s}_{u_{i_j}} \cdots \mathrm{s}_{u_1} M) \tag{14}$$

converges to $V_{n-1,1}(L, \mathrm{s}_\alpha M)$. Since $V_{n-1,1}(K_i, \mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} M)$ is decreasing with respect to i by (5), the corresponding subsequence (14) is also decreasing, and the proposition follows. □

In particular, we have the following.

Proposition 4.4. *Suppose that $\mathrm{s}_\alpha M$ exists. If $\mathrm{s}_\alpha K$ has a subsequential limits L_1 and L_2, then*

$$V_{n-1,1}(L_1, \mathrm{s}_\alpha M) = V_{n-1,1}(L_2, \mathrm{s}_\alpha M).$$

Because Steiner symmetrization may be discontinuous on sequences of bodies converging to lower-dimensional limits, the next proposition is sometimes helpful.

Proposition 4.5. *Suppose that*

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

is a descending sequence of compact convex sets in $\mathbb{R}^n$, and denote

$$C = \bigcap_m C_m.$$

If $s_\alpha C_m$ converges for each C_m, then $s_\alpha C$ converges to the limit

$$s_\alpha C = \bigcap_m s_\alpha C_m.$$

Proof. Let L be a subsequential limit of $s_\alpha C$. For each m let $D_m = s_\alpha C_m$. Since $C \subseteq C_m$ for each m, the subsequential limit L of $s_\alpha C$ lies inside each D_m, so that

$$L \subseteq \bigcap_m D_m = D.$$

Meanwhile, since Steiner symmetrization does not increase mean width, the non-negative sequence of values $W(s_{u_j} \cdots s_{u_2} s_{u_1} C)$ is decreasing, so that

$$\lim_j W(s_{u_j} \cdots s_{u_2} s_{u_1} C) = \inf_j W(s_{u_j} \cdots s_{u_2} s_{u_1} C) = \mu$$

exists. Since W is continuous, we must have $W(L) = \mu$. It also follows from (10) that

$$W(D) = \inf_m W(D_m) = \inf_m \inf_j W(s_{u_j} \cdots s_{u_2} s_{u_1} C_m) = \inf_j \inf_m W(s_{u_j} \cdots s_{u_2} s_{u_1} C_m).$$

By Proposition 2.3,

$$s_{u_j} \cdots s_{u_2} s_{u_1} C_m \to s_{u_j} \cdots s_{u_2} s_{u_1} C,$$

so that

$$W(s_{u_j} \cdots s_{u_2} s_{u_1} C_m) \to W(s_{u_j} \cdots s_{u_2} s_{u_1} C).$$

Hence,

$$W(D) = \inf_j W(s_{u_j} \cdots s_{u_2} s_{u_1} C) = \mu.$$

Since $L \subseteq D$ and $W(L) = W(D) = \mu$, it follows that $L = D$.

We have shown that every subsequential limit of $s_\alpha C$ has the same limit D. If the full sequence $s_\alpha C$ does not converge, there is a subsequence γ of $s_\alpha C$ that stays some distance $\varepsilon > 0$ from D. Since the sequence $s_\alpha C$ is uniformly bounded, so is the subsequence γ. The Blaschke selection theorem [31, p. 97] implies that γ, and therefore $s_\alpha C$, has a convergent subsequence γ'. By the previous argument γ' has limit D, contradicting the construction of γ. It follows that the original sequence $s_\alpha C$ converges, and therefore must converge to the limit D. □

These results together lead to the following uniqueness theorem.

Theorem 4.6. *Suppose that $K \in \mathscr{K}_n$ has non-empty interior. If $s_\alpha L = L$ for all subsequential limits L of $s_\alpha K$ then $s_\alpha K$ converges.*

Proof. By the Blaschke selection theorem, every subsequence of $s_\alpha K$ has a sub-subsequence converging to a limit. Suppose that L_1 and L_2 are two such limits.

We are given that $s_\alpha L_j = L_j$ for each j. By Proposition 4.4 and the volume invariance of Steiner symmetrization,

$$V_{n-1,1}(L_1, L_2) = V_{n-1,1}(L_2, L_2) = V_n(L_2) = V_n(K) = V_n(L_1).$$

Since $V_n(K) > 0$, the same is true of all symmetrals of K. It follows from the equality conditions of the Minkowski inequality for mixed volumes (see, for example, [24,31]) that L_1 and L_2 are translates, so that $L_2 = L_1 + x$ for some $x \in \mathbb{R}^n$.

Since $s_\alpha L_j = L_j$ for each j, it follows that $s_\alpha x = x$, so that $x \in u_i^\perp$ for each $u_i \in \alpha$. If the sequence α contains a basis for $\mathbb{R}^n$, then $x = 0$, and $L_1 = L_2$.

If the sequence α spans a proper subspace ξ of $\mathbb{R}^n$, then $x \in \xi^\perp$. Since every symmetrizing direction u_i of α lies in ξ, the supporting plane of K normal to x also supports each symmetral K_i, so that $h_{K_i}(x) = h_K(x)$ for all i. After taking limits it follows that

$$h_{L_1}(x) = h_K(x) = h_{L_2}(x) = h_{L_1+x}(x) = h_{L_1}(x) + x \cdot x,$$

so that $x \cdot x = 0$ and $L_2 = L_1$ once again.

We have shown that every convergent subsequence of $s_\alpha K$ converges to L_1. If the full sequence $s_\alpha K$ does not converge, there is a subsequence γ of $s_\alpha K$ that stays some distance $\varepsilon > 0$ from L_1. Since the sequence $s_\alpha K$ is uniformly bounded, so is the subsequence γ. The Blaschke selection theorem [31, p. 97] implies that γ, and therefore $s_\alpha K$, has a convergent subsequence γ'. By the previous argument γ' has limit L_1, contradicting the construction of γ. It follows that the original sequence $s_\alpha K$ converges, and therefore must converge to the limit L_1. □

The condition that $s_\alpha L = L$ for every subsequential limit L is required for the proof of Theorem 4.6 and does not hold for Steiner processes in general. Indeed, even when a Steiner process *converges*, it may not be the case that the limit is invariant under s_α. In other words, the converse of Theorem 4.6 is false.

A simple counterexample to the converse is constructed as follows. Let u and v be distinct non-orthogonal unit vectors in $\mathbb{R}^2$, and let α denote the sequence $\{u, v, v, \ldots\}$, where v is repeated forever. If K is any compact convex set in $\mathbb{R}^2$, then $s_\alpha K = s_v s_u K$, since s_v is idempotent. But $s_v s_u K \neq s_v s_u s_v s_u K$ in general (for example, if K is any line segment of positive length), so that $s_\alpha s_\alpha K \neq s_\alpha K$.

5. Steiner processes using a finite set of directions

Suppose that $\alpha = \{u_1, u_2, \ldots\}$ is a sequence of unit vectors such that each u_i is chosen from a given **finite** list of permitted directions $\{v_1, \ldots, v_m\}$.

Theorem 5.1. *Let $K \in \mathscr{K}_n$. The sequence $s_\alpha K$ has a limit $L \in \mathscr{K}_n$. Moreover, L is symmetric under reflection in each of the directions v_i occurring infinitely often in the sequence.*

In other words, a Steiner process using a finite set of directions *always* converges.

Proof of Theorem 5.1. To begin, suppose that K has non-empty interior. Without loss of generality (passing to a suitable tail of the sequence), we may assume that each of the directions v_i occurs infinitely often. In view of Theorem 4.6 it is then sufficient to show that every subsequential limit of $s_\alpha K$ is invariant under s_{v_i} for each i.

Let L denote the limit of some convergent subsequence of $s_\alpha K$. Since the list of distinct vectors v_i is finite, some v_i occurs infinitely often as the final iterate in this subsequence. Without loss of generality, relabel the directions $\{v_i\}$ so that v_1 is this recurring final direction. Passing to the sub-subsequence $\{K_{i_j}\}$ where this occurs, we are left with a sequence of the form

$$\{K_{i_j}\} = \{s_{v_1} s_{u_{i_j - 1}} \cdots s_{u_1} K\}$$

where each $u_{i_j} = v_1$.

Since every K_{i_j} is an s_{v_1} symmetral, it is immediate that $L = \lim_j K_{i_j}$ is symmetric under reflection across $v_1^\perp$.

Note that each successor to K_{i_j} in the original sequence K_i has the form

$$K_{i_j+1} = s_{u_{i_j+1}} s_{v_1} s_{u_{i_j-1}} \cdots s_{u_1} K.$$

The direction u_{i_j+1} must attain one of the values v_i infinitely often. Since $s_{v_1} s_{v_1} = s_{v_1}$, we may (without loss of generality) suppose this new direction is v_2, and that $v_2 \neq v_1$. Let us pass further to the sub-subsequence where every $u_{i_j+1} = v_2$. It now follows that

$$s_{v_2} L = \lim_j s_{v_2} K_{i_j} = \lim_j K_{i_j+1}.$$

Suppose that $s_{v_2} L \neq L$. In this case the strict monotonicity of Ω yields

$$\Omega(s_{v_2} L) - \Omega(L) > \varepsilon > 0$$

for some $\varepsilon > 0$. By the continuity of Ω and the definition of L there is an integer $M > 0$ such that

$$\Omega(s_{v_2} K_{i_j}) - \Omega(K_{i_t}) > \frac{\varepsilon}{2} > 0$$

for all $j, t > M$. But the monotonicity of Ω implies that

$$\Omega(K_{i_t}) \geqslant \Omega(K_{i_j+1}) = \Omega(s_{v_2} K_{i_j})$$

when $i_t > i_j$, a contradiction. It follows that

$$s_{v_2} L = L.$$

More generally, suppose that $L = s_{v_1} L = \cdots = s_{v_k} L$, where L is the limit of the subsequence K_{i_j}. For each j, let Q_j be the first successor of K_{i_j} in the original sequence K_i whose final iterated Steiner symmetrization uses a direction v_t for $t > k$. Again some particular v_t must appear infinitely often as the final direction for the symmetrals Q_j. Without loss of generality, and passing to subsequences as needed, suppose this direction is always v_{k+1}. Let $\tilde{Q}_j$ denote the immediate predecessor of each Q_j in the original sequence K_i, so that $Q_j = s_{v_{k+1}} \tilde{Q}_j$.

Again, passing to subsequences as needed, we may assume (by omitting repetitions) that each Q_j corresponds to a distinct entry of the original sequence K_i, so that Q_t appears strictly later than Q_j in the original sequence whenever $t > j$.

Since the subsequence $K_{i_j} \to L$ and L has non-empty interior, Lemma 2.2 implies that, for any given $\tau \in (0, 1)$,

$$(1 - \tau)L \subseteq K_{i_j} \subseteq (1 + \tau)L$$

for sufficiently large i_j. Since each $\tilde{Q}_j$ is a finite iteration of Steiner symmetrals of K_{i_j} using only directions from the list $\{v_1, \ldots, v_k\}$, and because $L = s_{v_1} L = \cdots = s_{v_k} L$, it follows from the monotonicity of Steiner symmetrization that

$$(1 - \tau)L \subseteq \tilde{Q}_j \subseteq (1 + \tau)L$$

for sufficiently large j, so that $\tilde{Q}_j \to L$ as well. It then follows from the monotonicity of $s_{v_{k+1}}$ that

$$(1 - \tau)s_{v_{k+1}} L \subseteq Q_j \subseteq (1 + \tau)s_{v_{k+1}} L.$$

In other words, $Q_j \to s_{v_{k+1}} L$.

Suppose that $s_{v_{k+1}} L \neq L$. In this case the strict monotonicity of Ω yields

$$\Omega(s_{v_{k+1}} L) - \Omega(L) > \varepsilon > 0$$

for some $\varepsilon > 0$. Since $Q_j \to s_{v_{k+1}} L$ and $\tilde{Q}_j \to L$, the continuity of Ω implies that

$$\Omega(Q_j) - \Omega(\tilde{Q}_t) > \frac{\varepsilon}{2} > 0$$

for all $j, t > M$, provided M is sufficiently large. But the monotonicity of Ω over the original sequence K_i implies that

$$\Omega(\tilde{Q}_t) \geqslant \Omega(Q_j) = \Omega(s_{v_{k+1}} \tilde{Q}_j)$$

when $t > j$, a contradiction. It follows that

$$s_{v_{k+1}} L = L.$$

It now follows that L is symmetric under reflection in each of the directions v_i, so that $s_\alpha L = L$. In other words L is a fixed point for the process s_α. Since this argument applies to every subsequential limit L of $s_\alpha K$, it follows from Theorem 4.6 that these subsequential limits are identical, and that the original sequence K_i converges to L.

Finally, suppose that K has empty interior. For each integer $m > 0$, the parallel body $C_m = K + \frac{1}{m} B$ has interior, so the limit of $s_\alpha C_m$ exists, by the previous argument. Since each $C_m \supseteq C_{m+1}$, and

$$K = \bigcap_m C_m,$$

it follows from Proposition 4.5 that the limit of $s_\alpha K$ exists, and is given by

$$s_\alpha K = \bigcap_m s_\alpha C_m.$$

Since each $s_\alpha C_m$ is symmetric under reflection in each of the directions v_i, the limit $s_\alpha K$ is also symmetric under each of those reflections. $\square$

Recall that if $K \in \mathscr{K}_n$ and $u \in \mathbb{S}^{n-1}$, then $s_u s_u K = s_u K$. This is a trivial consequence of the fact that $s_u K$ is symmetric under reflection across $u^\perp$, so that any subsequent iteration of s_u makes no difference. On the other hand, given two non-identical and non-orthogonal directions u and v, it may easily happen that

$$s_u s_v K \neq s_u s_v s_u s_v K.$$

More generally, there is no reason to believe that a Steiner process s_α (whether finite or infinite) is idempotent. However, the previous theorem implies that certain families of Steiner processes are indeed idempotent.

Corollary 5.2. *Let $v_1, \ldots, v_m$ be unit directions in $\mathbb{R}^n$, and let α be a sequence of directions, each of whose entries is taken from among the v_i, and in which each of the v_i occurs infinitely often.*

The map $s_\alpha : \mathscr{K}_n \to \mathscr{K}_n$ given by $K \mapsto s_\alpha K$ is well defined and idempotent.

Note that *every* direction in α must repeat infinitely often in the sequence to guarantee idempotence.

Proof of Corollary 5.2. It is an immediate consequence of Theorem 5.1 that the map $K \mapsto \mathrm{s}_\alpha K$ is well defined. Since each $\mathrm{s}_\alpha K$ is symmetric under reflection across each subspace $v_i^\perp$, it follows that $\mathrm{s}_{v_i}\mathrm{s}_\alpha K = \mathrm{s}_\alpha K$ for each i, so that $\mathrm{s}_\alpha \mathrm{s}_\alpha K = \mathrm{s}_\alpha K$. $\square$

It follows from Theorem 5.1 that *periodic* Steiner processes always converge to bodies that are symmetric under the subgroup of $O(n)$ generated by reflections through a given repeated set of directions $\{v_1, \ldots, v_m\}$. More precisely, we have the following.

Corollary 5.3. *Let $v_1, \ldots, v_m$ be unit directions in $\mathbb{R}^n$, and let α be the periodic sequence of directions given by*

$$\alpha = \{\underbrace{v_1, \ldots, v_m}, \underbrace{v_1, \ldots, v_m}, \ldots\}. \tag{15}$$

Then the limit of $\mathrm{s}_\alpha K$ exists for every $K \in \mathcal{K}_n$, and this limit is symmetric under reflection across each subspace $v_i^\perp$, so that the Steiner process s_α is idempotent.

A basis for $\mathbb{R}^n$ is said to be *irrational* if the angles between any two vectors in the basis are irrational multiples of π. The set of reflections across the coordinate planes of an irrational basis generate a dense subgroup of $O(n)$. Consequently, if a compact convex set K is symmetric under reflections across all of the directions from an irrational basis, then K must be symmetric under *all* reflections through the origin, so that K must be a Euclidean ball, centered at the origin.

Applying the previous results to an irrational basis of directions leads to the following generalization of a periodic construction described in [11, p. 98].

Corollary 5.4. *Let $v_1, \ldots, v_m$ be a set of unit directions in $\mathbb{R}^n$ that contains an irrational basis for $\mathbb{R}^n$. Suppose that $\alpha = \{u_1, u_2, \ldots\}$ is a sequence of unit vectors such that each u_i is chosen from the list of permitted directions $\{v_1, \ldots, v_m\}$, and such that each element of the irrational basis appears infinitely often in the sequence α. Then the limit of $\mathrm{s}_\alpha K$ exists and is a Euclidean ball for every $K \in \mathcal{K}_n$.*

In particular, if a periodic sequence of the form (15) contains an irrational basis for $\mathbb{R}^n$, then $\mathrm{s}_\alpha K$ is a Euclidean ball for every $K \in \mathcal{K}_n$. For a generalization of this special case to arbitrary compact sets, see also [7].

6. Open questions

1. Rate of convergence

While Theorem 5.1 guarantees convergence of infinite Steiner processes using a finite set of distinct directions, there remain questions about the rate of convergence for different distributions of the permitted set of directions. For example, given three normal vectors u, v, w to the edges of an equilateral triangle in $\mathbb{R}^2$ and various choices of α such as

$$\alpha = \{\underbrace{u, v, w}, \underbrace{u, v, w}, \ldots\},$$

$$\alpha = \{\underbrace{u, v, w}, v, \underbrace{u, v, w, u, v, w}, v, \underbrace{u, v, w, u, v, w, u, v, w}, v, \ldots\},$$

$$\alpha = \{u, v, w, \underbrace{u, v, u, v}, w, \underbrace{u, v, u, v, u, v}, w, \ldots\},$$

how does the rate of convergence of $s_\alpha K$ vary? If instead α is determined by a sequence of random choices from the set $\{u, v, w\}$, how is the rate of convergence related to the probability distribution for the choices of directions?

2. More general classes of sets

For most theorems regarding Steiner processes on convex bodies it is natural to ask whether similar results hold when the initial convex body is replaced by a more general kind of set, such as an arbitrary compact set in $\mathbb{R}^n$ (see, for example, [6,7,28–30]). While the proof of Theorem 5.1 above makes use of certain constructions that rely on convexity (such as mixed volumes, and the equality condition for the Brunn–Minkowski inequality), one can still ask whether Theorem 5.1 can be generalized to Steiner processes on arbitrary compact sets in $\mathbb{R}^n$. In [7] Burchard and Fortier show that this is the case when the finite set of repeated directions contains an irrational basis (as in Corollary 5.4). What happens if instead the finite set of directions generates a finite subgroup of reflections?

3. Cases of non-convergence

There also remain many questions about the cases in which Steiner processes fail to converge. In [3] a convex body K and a sequence of directions u_i are described for which the sequence of Steiner symmetrals

$$K_i = s_{u_i} \cdots s_{u_1} K$$

fails to converge in the Hausdorff topology. (For more such examples, see also [7].) More recently [1] it has been shown that such examples converge in *shape*: there is a corresponding sequence of isometries ψ_i such that the sequence $\{\psi_i K_i\}$ converges. However, many related questions remain open. How does this limiting shape depend on the initial body K and the sequence α of symmetrizing directions? What happens if K is permitted to be an arbitrary (possibly non-convex) compact set?

References

[1] G. Bianchi, A. Burchard, P. Gronchi, A. Volčič, private communication.
[2] G. Bianchi, P. Gronchi, Steiner symmetrals and their distance from a ball, Israel J. Math. 135 (2003) 181–192.
[3] G. Bianchi, D. Klain, E. Lutwak, D. Yang, G. Zhang, A countable set of directions is sufficient for Steiner symmetrization, Adv. in Appl. Math. 47 (2011) 869–873.
[4] T. Bonnesen, W. Fenchel, Theory of Convex Bodies, BCS Associates, Moscow, Idaho, 1987.
[5] J. Bourgain, J. Lindenstrauss, V.D. Milman, Estimates related to Steiner symmetrizations, in: Geometric Aspects of Functional Analysis (1987–1988), in: Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 264–273.
[6] A. Burchard, Steiner symmetrization is continuous in $W^{1,p}$, Geom. Funct. Anal. 7 (1997) 823–860.
[7] A. Burchard, M. Fortier, Convergence of random polarizations, arXiv:1104.4103v1 [math.FA], 2011.
[8] A. Cianchi, M. Chlebík, N. Fusco, The perimeter inequality under Steiner symmetrization: cases of equality, Ann. of Math. (2) 162 (2005) 525–555.
[9] A. Cianchi, N. Fusco, Strict monotonicity of functionals under Steiner symmetrization, in: Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, in: Quad. Mat., vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 187–220.
[10] A. Cianchi, N. Fusco, Steiner symmetric extremals in Pólya–Szegö type inequalities, Adv. Math. 203 (2006) 673–728.
[11] H. Eggleston, Convexity, Cambridge University Press, New York, 1958.
[12] K.J. Falconer, A result on the Steiner symmetrization of a compact set, J. Lond. Math. Soc. (2) 14 (1976) 385–386.
[13] R.J. Gardner, Symmetrals and X-rays of planar convex bodies, Arch. Math. (Basel) 41 (1983) 183–189.
[14] R.J. Gardner, The Brunn–Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002) 355–405.
[15] R.J. Gardner, Geometric Tomography, second ed., Cambridge University Press, New York, 2006.
[16] P. Gruber, Convex and Discrete Geometry, Springer-Verlag, New York, 2007.
[17] C. Haberl, F. Schuster, General L_p affine isoperimetric inequalities, J. Differential Geom. 83 (2009) 1–26.
[18] B. Klartag, V. Milman, Isomorphic Steiner symmetrization, Invent. Math. 153 (2003) 463–485.
[19] B. Klartag, V. Milman, Rapid Steiner symmetrization of most of a convex body and the slicing problem, Combin. Probab. Comput. 14 (2005) 829–843.
[20] M. Longinetti, An isoperimetric inequality for convex polygons and convex sets with the same symmetrals, Geom. Dedicata 20 (1986) 27–41.

[21] E. Lutwak, D. Yang, G. Zhang, Orlicz projection bodies, Adv. Math. 223 (2010) 220–242.
[22] P. Mani-Levitska, Random Steiner symmetrizations, Studia Sci. Math. Hungar. 21 (1986) 373–378.
[23] A. McNabb, Partial Steiner symmetrization and some conduction problems, J. Math. Anal. Appl. 17 (1967) 221–227.
[24] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge University Press, New York, 1993.
[25] P.R. Scott, Planar rectangular sets and Steiner symmetrization, Elem. Math. 53 (1998) 36–39.
[26] J. Steiner, Einfacher Beweis der isoperimetrische Hauptsätze, J. Reine Angew. Math. 18 (1838) 281–296.
[27] G. Talenti, The standard isoperimetric theorem, in: P. Gruber, J.M. Wills (Eds.), Handbook of Convex Geometry, North-Holland, Amsterdam, 1993, pp. 73–124.
[28] J. Van Schaftingen, Universal approximation of symmetrizations by polarizations, Proc. Amer. Math. Soc. 134 (2005) 177–186.
[29] J. Van Schaftingen, Approximation of symmetrizations and symmetry of critical points, Topol. Methods Nonlinear Anal. 28 (2006) 61–85.
[30] A. Volčič, Random Steiner symmetrizations of sets and functions, preprint.
[31] R. Webster, Convexity, Oxford University Press, New York, 1994.