Triple positive solutions of boundary value problems for $p$-Laplacian dynamic equations on time scales

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Abstract

A new triple fixed-point theorem is applied to investigate the existence of at least three positive solutions of boundary value problems for $p$-Laplacian dynamic equations on time scales.

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1. Introduction

Let $T$ be a closed nonempty subset of $R$, and let $T$ have the subspace topology inherited from the Euclidean topology on $R$. In some of the current literature, $T$ is called a time scale (or measure chain). For notation, we shall use the convention that, for each interval $J$ of $R$,

$$J_T = J \cap T.$$ 

The theory of dynamical systems on time scales is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case; see [1,3,4,8,9,14–16] and the references therein. In
this paper, we are concerned with the existence of positive solutions of the $p$-Laplacian dynamic equation on a time scale
\[ \left[ \phi_p(u^\Delta(t)) \right]^\nu + g(t)f(u(t)) = 0, \quad t \in [0,T], \]

satisfying the boundary conditions
\[ u(0) - B_0(u^\Delta(0)) = 0, \quad u^\Delta(T) = 0, \] (2)
or
\[ u^\Delta(0) = 0, \quad u(T) + B_1(u^\Delta(T)) = 0, \] (3)

where $\phi_p(s)$ is $p$-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and

\begin{itemize}
  \item [(H1)] $f : R \to R^+$ is continuous ($R^+$ denotes the nonnegative reals);
  \item [(H2)] $g : T \to [0, +\infty)$ is left dense continuous (i.e., $g \in \text{C}\text{ld}(T, [0, +\infty))$) and does not vanish identically on any closed subinterval of $[0,T]_T$, where $\text{C}\text{ld}(T, [0, +\infty))$ denotes the set of all left dense continuous functions from $T$ to $[0, +\infty)$;
  \item [(H3)] $B_0(v)$ and $B_1(v)$ are both continuous odd functions defined on $R$ and satisfy that there exist $A,B > 0$ such that
  \[ Bv \leq Bj(v) \leq Av, \quad \text{for all } v \geq 0, \quad j = 0, 1. \]
\end{itemize}

We remark that by a solution $u$ of (1) and (2) (respectively (1) and (3)), we mean $u : T \to R$ which is delta differentiable, $u^\Delta$ and $(|u^\Delta|^{p-2}u^\Delta)^\nu$ are both continuous on $T^k \cap T_k$, and $u$ satisfies (1) and (2) (respectively (1) and (3)). If $u^\Delta(t) \leq 0$ on $[0,T] \cap T_k$, then we say $u$ is concave on $[0,T]_T$.

$p$-Laplacian problems with boundary conditions for ordinary differential equations and finite difference equations have been studied extensively; see [2,6,7,10,13,17–23] and references therein. However, there are not much concerning the $p$-Laplacian problems on time scales, see [3,14]. In this paper, by using a new triple fixed-point theorem due to Ren et al. [20] in a cone, we prove that there exist at least three positive solutions of (1) and (2) (respectively (1) and (3)). If $u^\Delta(t) \leq 0$ on $[0,T]_T$, then we state the triple fixed-point theorem. In Sections 3 and 4, by defining an appropriate Banach space and cones, we impose the growth conditions on $f$ which allow us to apply the triple fixed-point theorem in obtaining existence of at least three positive solutions of (1) and (2) (respectively (1) and (3)).

For convenience, we list the following well-known definitions which can be found in [1,4,8,9,15,16].

For $t < \sup T$ and $r > \inf T$, the forward jump operator $\sigma$ and the backward jump operator $\rho$ are well defined, respectively, by
\[ \sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T, \quad \rho(r) = \sup\{\tau \in T \mid \tau < r\} \in T, \]

for all $t,r \in T$. If $\sigma(t) > t$, $t$ is said to be right scattered, and if $\rho(r) < r$, $r$ is said to be left scattered. If $\sigma(t) = t$, $t$ is said to be right dense, and if $\rho(r) = r$, $r$ is said to be left dense. If $T$ has a right scattered minimum $m$, define $T_k = T - \{m\}$; otherwise set $T_k = T$. If $T$ has a left scattered maximum $M$, define $T^k = T - \{M\}$; otherwise set $T^k = T$. 


For \( x : T \to R \) and \( t \in T^k \), the delta derivative of \( x(t) \) is defined to be the number \( x/\Delta(t) \) \( \text{(provided it exists)} \) with the property that for any \( \varepsilon > 0 \) there is a neighborhood \( U \subset T \) of \( t \) such that
\[
\left| \left[ x(\sigma(t)) - x(s) \right] - x/\Delta(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s|,
\]
for all \( s \in U \). For \( x : T \to R \) and \( t \in T^k \), the nabla derivative of \( x(t) \) is defined to be the number \( x/\nabla(t) \) \( \text{(provided it exists)} \) with the property that for any \( \varepsilon > 0 \) there is a neighborhood \( V \subset T \) of \( t \) such that
\[
\left| \left[ x(\rho(t)) - x(s) \right] - x/\nabla(t)[\rho(t) - s] \right| < \varepsilon |\rho(t) - s|,
\]
for all \( s \in V \).

If \( T = R \), then \( x/\Delta(t) = x'(t) \). If \( T = Z \), then \( x/\Delta(t) = x(t + 1) - x(t) \) is the forward difference operator while \( x/\nabla(t) = x(t) - x(t - 1) \) is the backward difference operator.

A function \( F : T^k \to R \) is called a delta-antiderivative of \( f : T \to R \) provided \( F/\Delta(t) = f(t) \) holds for all \( t \in T^k \). In this case we define the delta integral of \( f \) by
\[
\int_a^t f(s) \Delta s = F(t) - F(a),
\]
for all \( a, t \in T \). A function \( \Phi : T_k \to R \) is called a nabla-antiderivative of \( f : T \to R \) provided \( \Phi/\nabla(t) = f(t) \) holds for all \( t \in T_k \). In this case we define the nabla integral by
\[
\int_a^t f(s) \nabla s = \Phi(t) - \Phi(a),
\]
for all \( a, t \in T \).

Throughout this paper, we assume \( T \) is a closed subset of \( R \) with \( 0 \in T_k, T \in T^k \).

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces, and we then state the triple fixed-point theorem for a cone preserving operator. The following definitions can be found in the book by Deimling [11] as well as in the book by Guo and Lakshmikantham [12].

**Definition 2.1.** Let \( E \) be a real Banach space. A nonempty, closed, convex set \( P \subset E \) is called a cone, if it satisfies the following two conditions:

(i) \( u \in P, \lambda \geq 0 \) implies \( \lambda u \in P \); and
(ii) \( u \in P, -u \in P \) implies \( u = 0 \).

Every cone \( P \subset E \) induces an ordering in \( E \) given by
\[
u \leq u \quad \text{if and only if} \quad v - u \in P.
\]

**Definition 2.2.** Given a cone \( P \) in a real Banach space \( E \), a functional \( \psi : P \to R \) is said to be increasing on \( P \), provided \( \psi(x) \leq \psi(y) \), for all \( x, y \in P \) with \( x \leq y \).
Definition 2.3. Given a nonnegative continuous functional $\gamma$ on a cone $P$ of a real Banach space $E$, we define for each $d > 0$ the set

$$P(\gamma, d) = \{ x \in P \mid \gamma(x) < d \}.$$ 

The following fixed-point theorem due to Ren et al. [20] (which is motivated by Avery and Henderson’s double fixed-point theorem [5]) will play an important role in the proof of our results. The origin in $X$ is denoted by $\theta$.

Theorem 2.1. Let $P$ be a cone in a real Banach space $E$. Let $\alpha, \beta$ and $\gamma$ be increasing, nonnegative, continuous functionals on $P$, such that for some $c > 0$ and $M > 0$,

$$\gamma(x) \leq \beta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq M \gamma(x),$$

for all $x \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers $a$ and $b$ with $a < b < c$ and

$$F: \overline{P(\gamma, c)} \to P$$

is a completely continuous operator such that:

\begin{enumerate}
  \item[(B1)] $\gamma(Fx) < c$, for all $x \in \partial P(\gamma, c)$;
  \item[(B2)] $\beta(Fx) > b$, for all $x \in \partial P(\beta, b)$;
  \item[(B3)] $P(\alpha, a) \neq \emptyset$, and $\alpha(Fx) < a$, for all $x \in \partial P(\alpha, a)$.
\end{enumerate}

Then $F$ has at least three fixed points $x_1, x_2$ and $x_3$ belonging to $\overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \quad \text{with} \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$ 

Remark 2.1. If the restriction $F\theta \neq \theta$ is imposed in Theorem 2.1, then there is the slightly stronger conclusion as following:

$F$ has at least three fixed points $x_1, x_2$ and $x_3$ belonging to $\overline{P(\gamma, c)}$ such that

$$0 < \alpha(x_1) < a < \alpha(x_2), \quad \text{with} \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$ 

3. Solutions of (1) and (2) in a cone

In this section, by defining an appropriate Banach space and cones, we impose the growth conditions on $f$ which allow us to apply the triple fixed-point theorem in establishing the existence of at least three positive solutions of (1) and (2). We note that, from the nonnegativity of $g$, $f$, a solution of (1) and (2) is nonnegative and concave on $[0, T]_T$.

Let the Banach space $E = C_{id}([0, T]_T, \mathbb{R})$ with norm $\| u \| = \sup_{t \in [0, T]_T} |u(t)|$, and define the cone, $P \subset E$, by

$$P = \{ u \in E \mid u \text{ is concave and nonnegative valued on } [0, T]_T, \text{ and } u^A(T) = 0 \}.$$ 

Lemma 3.1. [14] If $u \in P$, then

$$u(t) \geq \frac{t}{T} \| u \|, \quad t \in [0, T]_T,$$

where $\| u \| = \sup_{t \in [0, T]_T} |u(t)|$. 
\[ \eta = \min \left\{ t \in T \mid t \geq \frac{T}{2} \right\}, \]

and fix \( l \in T \) such that
\[ 0 < \eta < l < T, \]

and define the increasing, nonnegative, continuous functionals \( \gamma \), \( \beta \), and \( \alpha \) on \( P \), by
\[ \gamma(u) = \max_{t \in [0, \eta]} u(t) = u(\eta), \quad \beta(u) = \min_{t \in [\eta, l]} u(t) = u(\eta), \]

and
\[ \alpha(u) = \max_{t \in [0, l]} u(t) = u(l). \]

We see that, for each \( u \in P \),
\[ \gamma(u) = \beta(u) \leq \alpha(u). \]

In addition, for each \( u \in P \), Lemma 3.1 implies
\[ \|u\| \leq \frac{T}{\eta} \gamma(u), \quad \text{for all } u \in P. \]

For notational convenience, we denote \( m, M \) and \( \lambda_l \), by
\[ m = (B + \eta)\phi_q \left( \int_0^T g(r) \nabla r \right), \quad M = (A + \eta)\phi_q \left( \int_0^T g(r) \nabla r \right), \]
\[ \lambda_l = (A + l)\phi_q \left( \int_0^T g(r) \nabla r \right). \]

We note that \( u(t) \) is a solution of (1) and (2), if and only if
\[ u(t) = B_0 \left( \phi_q \left( \int_0^T g(r) f(u(r)) \nabla r \right) \right) + \int_0^t \phi_q \left( \int_s^T g(r) f(u(r)) \nabla r \right) \Delta s, \quad t \in [0, T]. \]

**Theorem 3.1.** Assume that conditions \((H_1), (H_2)\) and \((H_3)\) are satisfied. Let
\[ 0 < a < \frac{l}{T} b < b < \frac{lm}{TM} c, \]

and suppose that \( f \) satisfies the following conditions:

\( (C_1) \) \( f(w) < \phi_p \left( \frac{T}{M} \right), \) for \( w \in [0, \frac{T}{\eta} c] \);
\( (C_2) \) \( f(w) > \phi_p \left( \frac{b}{M} \right), \) for \( w \in [b, \frac{T}{\eta} b] \);
\( (C_3) \) \( f(w) < \phi_p \left( \frac{a}{M} \right), \) for \( w \in [0, \frac{T}{T} a] \).

Then, there exist at least three positive solutions \( u_1, u_2, u_3 \) of (1) and (2) such that
\[ 0 \leq \alpha(u_1) < a < \alpha(u_2), \quad \text{with } \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c. \]
Proof. Define a completely continuous integral operator $F: P \to E$ by

$$(Fu)(t) = B_0 \left( \phi_q \left( \int_0^T g(r) f\left( u(r) \right) \nabla r \right) \right) + \int_0^t \phi_q \left( \int_s^T g(r) f\left( u(r) \right) \nabla r \right) \Delta s, \quad u \in P,$$

for $t \in [0, T]_T$, we will seek fixed points of $F$ in the cone $P$. We note that from (4), if $u \in P$, then $(Fu)(t) \geq 0$ for $t \in [0, T]_T$. We now prove that the conditions of Theorem 2.1 hold with respect to $F$.

Let $u \in \overline{P(\gamma, c)}$, then $(Fu)(t) \geq 0$ for $t \in [0, T]_T$. It follows from (4) that

$$\frac{(Fu)}{\Delta_1}(t) = \phi_q \left( \int_t^T g(r) f\left( u(r) \right) \nabla r \right), \quad u \in P, \quad t \in [0, T]_T,$$

we see that $(Fu)/\Delta_1(t)$ is continuous and nonincreasing on $[0, T]_T$, and using Theorem 8.39 in [8] we obtain that $(Fu)/\Delta_1(t) \leq 0$ for $t \in [0, T]_T \cap T_\gamma$. Moreover, $(Fu)/\Delta_1(T) = 0$. This implies $Fu \in P$, and so $F: \overline{P(\gamma, c)} \to P$.

Now, if $u \in \partial P(\gamma, c)$, then $\gamma(u) = \max_{t \in [\eta, T]} u(t) = u(\eta) = c$. Recalling that $\|u\| \leq \frac{T}{\eta} \gamma(u) = \frac{T}{\eta} c$, we have

$$0 \leq u(t) \leq \frac{T}{\eta} c, \quad \text{for all } t \in [0, T]_T.$$

As a consequence of $(C_1)$,

$$f\left( u(s) \right) < \phi_p \left( \frac{c}{M} \right), \quad \text{for } s \in [0, T]_T.$$

Since $Fu \in P$, we get

$$\gamma(Fu) = (Fu)(\eta) = B_0 \left( \phi_q \left( \int_0^T g(r) f\left( u(r) \right) \nabla r \right) \right) + \int_0^\eta \phi_q \left( \int_s^T g(r) f\left( u(r) \right) \nabla r \right) \Delta s \\
\leq A \phi_q \left( \int_0^T g(r) f\left( u(r) \right) \nabla r \right) + \int_0^\eta \phi_q \left( \int_s^T g(r) f\left( u(r) \right) \nabla r \right) \Delta s \\
< (A + \eta) \phi_q \left( \int_0^T g(r) \nabla r \right) \frac{c}{M} = c.$$

Then, condition $(B_1)$ of Theorem 2.1 holds.

Let $u \in \partial P(\beta, b)$. Then $\beta(u) = \min_{t \in [\eta, T]} u(t) = u(\eta) = b$. This implies $u(t) \geq b$, $t \in [\eta, T]_T$, and since $u \in P$, we have $b \leq u(t) \leq \|u\| = u(T)$, for $t \in [\eta, T]_T$. Note that, $\|u\| \leq \frac{T}{\eta} \gamma(u) = \frac{T}{\eta} \beta(u) = \frac{T}{\eta} b$, for all $u \in P$. So,

$$b \leq u(t) \leq \frac{T}{\eta} b, \quad \text{for } t \in [\eta, T]_T.$$
From (C2), we have \( f(u(s)) > \frac{b}{m} \) for \( s \in [\eta, T] \), and so

\[
\beta(Fu) = (Fu)(\eta) = B_0 \left( \phi_q \left( \int_0^T g(r) f(u(r)) \, dr \right) \right) + \int_0^{\eta} \phi_q \left( \int_0^s g(r) f(u(r)) \, dr \right) \, ds
\]

\[
> (B + \eta) \phi_q \left( \int_0^\eta g(r) \, dr \right) = B.
\]

Then, condition (B2) of Theorem 2.1 holds.

We note that \( u(t) = \frac{a}{2} \), \( t \in [0, T] \), is a member of \( P(\alpha, a) \) and \( \alpha(u) = \frac{a}{2} < a \). So \( P(\alpha, a) \neq \emptyset \).

Now, let \( u \in \partial P(\alpha, a) \). Then \( \alpha(u) = \max_{t \in [0, l]} u(t) = u(l) = a \). This means that

\[
0 \leq u(t) \leq a, \quad t \in [0, l].
\]

Note that, \( \|u\| \leq \frac{T}{l} \gamma(u) \leq \frac{T}{l} \alpha(u) = \frac{T}{l} a \), for all \( u \in P \). So,

\[
0 \leq u(t) \leq \frac{T}{l} a, \quad t \in [0, T].
\]

From (C3), we get \( f(u(s)) < \frac{a}{\lambda l} \) for \( s \in [0, T] \), and so

\[
\alpha(Fu) = (Fu)(l) = B_0 \left( \phi_q \left( \int_0^T g(r) f(u(r)) \, dr \right) \right) + \int_0^{l} \phi_q \left( \int_0^s g(r) f(u(r)) \, dr \right) \, ds
\]

\[
\leq A \phi_q \left( \int_0^T g(r) f(u(r)) \, dr \right) + \int_0^{l} \phi_q \left( \int_0^s g(r) f(u(r)) \, dr \right) \, ds
\]

\[
< (A + l) \phi_q \left( \int_0^T g(r) \, dr \right) \frac{a}{\lambda l} = a.
\]

Then, condition (B3) of Theorem 2.1 holds.

Therefore, Theorem 2.1 implies that \( F \) has at least three fixed points which are positive solutions \( u_1, u_2 \) and \( u_3 \), belonging to \( P(\gamma, c) \), of (1) and (2) such that

\[
0 \leq \alpha(u_1) < a < \alpha(u_2), \quad \text{with} \quad \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c.
\]

The proof of Theorem 3.1 is complete. \( \square \)

### 4. Solutions of (1) and (3) in a cone

In this section, we use the triple fixed-point theorem to establish the existence of at least three positive solutions of (1) and (3).

Consider the Banach space \( E = C_{id}([0, T]_T, R) \) with norm \( \|u\| = \sup_{t \in [0, T]} |u(t)| \), and define the cone, \( P_1 \subseteq E \), by

\[
P_1 = \{ u \in E \mid u \text{ is concave and nonnegative valued on } [0, T]_T, \text{ and } u^\Delta(0) = 0 \}.
\]
Lemma 4.1. [14] If \( u \in P_1 \), then
\[
\begin{align*}
u(t) &\geq \frac{T-t}{T} \| u \|, \quad \text{for } t \in [0, T]_T,
\end{align*}
\]
where \( \| u \| = \sup_{t \in [0, T]_T} |u(t)| \).

Let \( h = \max \left\{ t \in T \mid 0 \leq t \leq \frac{T}{2} \right\} \),
and fix \( \tau \in T \) such that
\[
0 < \tau < h,
\]
and define the increasing, nonnegative and continuous functionals \( \gamma, \beta \) and \( \alpha \) on \( P \), by
\[
\gamma(u) = \max_{t \in [h, T]_T} u(t) = u(h), \quad \beta(u) = \min_{t \in [\tau, h]_T} u(t) = u(h),
\]
and
\[
\alpha(u) = \max_{t \in [\tau, T]_T} u(t) = u(\tau).
\]
It is easy to see that, for each \( u \in P \),
\[
\gamma(u) = \beta(u) \leq \alpha(u).
\]
In addition, for each \( u \in P \), Lemma 4.1 implies \( \gamma(u) = u(h) \geq \frac{T-h}{T} \| u \| \). Thus,
\[
\| u \| \leq \frac{T}{T-h} \gamma(u), \quad \text{for all } u \in P.
\]

Set
\[
m_1 = (B + T - \tau) \phi_q \left( \int_0^\tau g(r) \nabla r \right), \quad M_1 = (A + T - h) \phi_q \left( \int_0^T g(r) \nabla r \right),
\]
\[
\lambda_T = (A + T - \tau) \phi_q \left( \int_0^T g(r) \nabla r \right).
\]
We note that \( u(t) \) is a solution of (1) and (3), if and only if
\[
u(t) = B_1 \left( \phi_q \left( \int_0^T g(r) f(u(r)) \nabla r \right) \right) + \int_{t_0}^T \phi_q \left( \int_{t_0}^s g(r) f(u(r)) \nabla r \right) \Delta s, \quad t \in [0, T]_T.
\]

Theorem 4.1. Assume that conditions \((H_1)\), \((H_2)\) and \((H_3)\) are satisfied. Let
\[
0 < a < \frac{T - \tau}{T} b < b < \frac{(T - \tau)m_1}{TM_1} c,
\]
and suppose that \( f \) satisfies the following conditions:
\[
(D_1) \quad f(w) < \phi_p \left( \frac{\tau}{M_1} \right), \quad \text{for } w \in [0, \frac{T}{T-h} c];
\]
(D_2) \ f(w) > \varphi_P\left(\frac{b}{m_1}\right), \text{ for } w \in [b, \frac{T}{1-h}b];

(D_3) \ f(w) < \varphi_P\left(\frac{a}{m_1}\right), \text{ for } w \in [0, \frac{T}{1-h}a].

Then, there exist at least three positive solutions \( u_1, u_2, u_3 \) of (1) and (3) such that

\[ 0 \leq \alpha(u_1) < a < \alpha(u_2), \quad \text{with} \quad \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c. \]

**Proof.** Define a completely continuous integral operator \( F_1 : P_1 \to E \) by

\[
(F_1u)(t) = B_1\left(\phi_q\left(\int_0^T g(r)f(u(r))\nabla r\right)\right) + \int_t^T \phi_q\left(\int_0^s g(r)f(u(r))\nabla r\right)\Delta s, \quad u \in P,
\]

for \( t \in [0, T]_T \), the fixed point of \( F_1 \) in the cone \( P_1 \) is the solution of (1) and (3). In analogy to the proof of Theorem 3.1, we arrive at the conclusion. \( \Box \)

**Remark 4.1.** If adding in Theorem 3.1 (or Theorem 4.1) the restriction \( g(t)f(u) \neq 0 \) for \( t \in [0, T]_T \), then \( F \theta \neq \theta \) (or \( F_1 \theta \neq \theta \)). So by Remark 2.1, (1) and (2) (or (1) and (3)) has at least three positive solutions \( u_1, u_2 \) and \( u_3 \), belonging to \( P(\gamma, c) \) (or \( P_1(\gamma, c) \)), of (1) and (2) (or (1) and (3)) such that

\[ 0 < \alpha(u_1) < a < \alpha(u_2), \quad \text{with} \quad \beta(u_2) < b < \beta(u_3), \quad \gamma(u_3) < c. \]

**References**