Consequences of Topological Stability

MIKE HURLEY

Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106

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A diffeomorphism $f$ of a compact manifold $M$ is said to be topologically stable if any $C^0$ small perturbation of $f$ is continuously semiconjugate to $f$. (The precise definitions are contained in the next section.) Nitecki [15] has shown that all known examples of structurally stable diffeomorphisms (the Axiom A and Strong Transversality Condition diffeomorphisms) are topologically stable. Related results, due to Walters and to Kato and Morimoto, can be found in [29, 8, 9]. The purpose of the present paper is to present a foundation for the study of a converse to Nitecki's result. More precisely, we are interested in the question of whether every topologically stable diffeomorphism is topologically conjugate to a structurally stable one. It is an easy exercise to show that topological stability is a conjugacy class invariant, so it is easy to construct topologically stable diffeomorphisms that are not structurally stable. Hence an affirmative answer to this question is the strongest type of converse that one could expect. Yano [31] has established this converse for diffeomorphisms of the circle, and the present author in a joint work with P. Fleming [5] has established it for flows on compact surfaces. An interesting example of a topologically stable diffeomorphism that is not structurally stable is discussed in [10].

The basic tool in the present paper is the shadowing theorem of P. Walters [30], which shows that a topologically stable diffeomorphism $f$ satisfies the "pseudo-orbit tracing property," that is, that every $a$-chain (approximate $f$-orbit) can be shadowed (approximated by an actual $f$-orbit). Using this fact, we are able to show that the chain recurrent sets of topologically stable diffeomorphisms share many features with those of the Axiom A, Strong Transversality Condition diffeomorphisms. We also use the $C^0$ density theorem of M. Shub [24, 25, 27] to establish further similarities between these two classes of diffeomorphisms. Using these techniques, we are able to answer two questions raised in [15] by showing that neither the Newhouse examples [13] nor the Abraham–Smale examples [1] are topologically stable.

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The basic results of this paper are:

**Theorem A.** Suppose $f$ is a topologically stable diffeomorphism on a compact manifold $M$.

1. The periodic points of $f$ are dense in the chain recurrent set of $f$.
2. $f'$ has only a finite number of chain components (basic sets), and each of these is the closure of a single $f$-orbit.
3. If $f$ can be $C^0$ approximated by Morse-Smale diffeomorphisms, then the chain recurrent set of $f$ is finite.
4. $f$ has a finite set of periodic points the union of whose stable sets is dense in $M$.
5. If $X$ is a chain component of $f$ and $k$ is the least period of any periodic point in $X$, then $X$ consists of no more than $k$ chain components of $f^k$, and $f^k$ is mixing on each.

**Theorem B.** Neither the Newhouse examples [13] nor the Abraham-Smale examples [1] are topologically stable.

One can also define topological stability for flows generated by vector fields. Here the appropriate $C^0$ topology is the $C^0$ metric on the vector fields, and the semiconjugacies are allowed to reparameterize the time coordinate, but must preserve the direction of the trajectories (if $f, g: \mathbb{R} \times M \to M$ are flows on $M$, then $g$ is semiconjugate to $f$ if $h \circ g(t, x) = f(\tau_x(t), h(x))$, where $h$ is a continuous map from $M$ onto itself, each $\tau_x$ is an orientation-preserving homeomorphism of the real line, and the map $(x, t) \to \tau_x(t)$ is continuous as a map from $M \times \mathbb{R}$ to $M$.)

**Theorem C.** Suppose $f$ is a topologically stable flow on a compact manifold $M$. Then statements analogous to (1)–(4) of Theorem A are true of $f$. More precisely, (1) and (2) hold as stated, and

3. If $f$ can be $C^0$ approximated by a Morse-Smale flow, then the chain recurrent set of $f$ consists of a finite number of orbits, each of which is either fixed or periodic.

4. There is a finite collection of fixed points and periodic orbits of $f$ the union of whose stable sets are dense in $M$.

**Definitions and Proof of Theorem A**

We consider functions acting on a smooth, compact Riemannian manifold $M$. Let Homeo($M$) denote the collection of all homeomorphisms of $M$ to itself topologized by the $C^0$ metric 

$$d_0(f, g) = \sup \{d(f(x), g(x)) | x \in M \},$$
where \( d(\,\cdot\,) \) is a metric on \( M \) that is compatible with the Riemannian structure. A \( C^1 \) diffeomorphism \( f \) on \( M \) is topologically stable (sometimes called lower semistable) if there is a \( C^0 \) neighborhood \( U \) of \( f \) in \( \text{Homeo}(M) \) with the following property: for each \( g \in U \) there is a continuous surjection \( h_g : M \to M \) satisfying

\[
\begin{align*}
(i) \quad & h_g \circ g = f \circ h_g, \\
(ii) \quad & h_g \to \text{identity as } g \to f(C^0).
\end{align*}
\]

The map \( h_g \) is called a semiconjugacy from \( g \) to \( f \). We will often suppress the dependence of the semiconjugacy on \( g \) and write \( h \) for \( h_g \).

An important dynamical property of topologically stable diffeomorphisms is that they have the shadowing property. This was established by Walters in [30]. Several definitions are necessary in order to describe this property.

An \( \alpha \)-chain for \( f \) is a sequence \( \{x_i\} \) in \( M \) with the property that \( d(x_i, f(x_{i-1})) < \alpha \) for each \( i \). An \( \alpha \)-chain is an approximate \( f \)-orbit; indeed, it is sometimes called an \( \alpha \)-pseudo-orbit. The chain recurrent set of \( f \), \( \text{CR}(f) \), is the set of \( x \) in \( M \) with the property that for each \( \alpha > 0 \) there is a periodic \( \alpha \)-chain containing \( x \) (the \( \alpha \)-chain \( \{x_i\} \) is periodic of period \( k \) if \( x_{i+k} = x_i \) for all \( i \)) [4]. A basic problem is to determine when a chain recurrent point is approximable by periodic points of \( f \), or more generally to determine if each \( \alpha \)-chain can be approximated by an actual orbit of \( f \). The \( f \)-orbit of \( x \), \( \{f^i(x)\} \), is said to \( \beta \)-shadow (or \( \beta \)-trace) \( \{x_i\} \) if \( d(f^i(x), x_i) < \beta \) for each \( i \). P. Walters has shown that topologically stable diffeomorphisms enjoy the property that any \( \alpha \)-chain for sufficiently small \( \alpha \) can be \( \beta \)-shadowed [30].

**Theorem (Walters).** A topologically stable diffeomorphism has the shadowing property: given \( \beta > 0 \), there is \( \alpha > 0 \) such that every \( \alpha \)-chain can be \( \beta \)-shadowed.

By combining the shadowing theorem with the fact [15, 11] that the periodic points of a topologically stable diffeomorphism are dense in its nonwandering set, one can easily obtain the following proposition. We include a direct proof because it is both simple and instructive (the details of our proof are essentially the same as arguments found in [15] and [11]). \( \text{Per}(f) \) denotes the set of periodic points of \( f \); that is, the set of points \( p \) such that \( f^k(p) = p \) for some integer \( k \geq 1 \).

**Proposition 1.** If \( f \) is topologically stable, then the closure of \( \text{Per}(f) \) is equal to \( \text{CR}(f) \).

**Proof.** If \( M \) is one-dimensional this is an easy exercise for the reader. Otherwise, if we have a periodic \( \alpha \)-chain \( \{x_i\} \), then we can \( C^0 \) perturb \( f \) to get a diffeomorphism \( g \) such that \( \{x_i\} \) is a periodic orbit for \( g \) [16, Lemma 13;
and 30). From the definitions it is easy to see that if $h$ is a semiconjugacy from $g$ to $f$, then $h(\text{Per}(g)) \subset \text{Per}(f)$. Since we are assuming that we can find a semiconjugacy close to the identity, we see that $f$ has a periodic point near $x_0$. Because the $C^0$ distance between $f$ and $g$ goes to 0 as $\alpha$ approaches 0, this argument shows that we can find a sequence of periodic points of $f$ that converges to $x_0$, provided that $x_0 \in \text{CR}(f)$.

Next we describe the structure of $\text{CR}(f)$. A chain component of $f$ is a maximal chain transitive subset of $\text{CR}(f)$; that is, an equivalence class under the following equivalence relation: $xEy$ if and only if for each $\alpha > 0$ there are $\alpha$-chains going from $x$ to $y$ and from $y$ to $x$ (we say the $\alpha$-chain $\{x_i\}$ goes from $x_m$ to $x_n$ if $n > m$). Note that each chain component of $f$ is closed, $f$-invariant, and can be realized as the closure of a single $\alpha$-chain for any $\alpha > 0$.

We list for later reference two easily verified results. Lemma 2 is a direct consequence of compactness, and Lemma 3 is true because our semiconjugacies are continuous surjections. Denote the orbit of $x$ under a homeomorphism $g$ ($\{g^i(x)\}_{1 \to i \to \infty}$) by $O(x; g)$ or by $O(x)$ if the identity of $g$ is apparent.

**Lemma 2.** Suppose $g$ is a homeomorphism on $M$ and $x \in M$. Then $O(x; g)$ meets either one or two chain components of $g$. $O(x; g)$ meets only one chain component of $g$ if and only if it is contained in that chain component, and if and only if $x$ is in $\text{CR}(g)$.

**Lemma 3.** If $h$ is a semiconjugacy from $g$ to $f$, $hg = fh$, and if $D$ is a closed, nonempty, $f$-invariant subset of $M$, then $h^{-1}(D)$ is closed, nonempty, and $g$-invariant.

Recall that a periodic point of a diffeomorphism $g$ is said to be hyperbolic if the spectrum of the tangent map $(Dg^k)_p$ is disjoint from the unit circle in the complex plane (here $k$ is the period of $p$, $g^k(p) = p$). If $p$ is hyperbolic, then $\{x \ g^n(x) \to p$ as $n \to \infty\}$ is an immersed submanifold of $M$, called the stable manifold of $p$ for $g$. Its dimension is equal to the number of eigenvalues (counting multiplicity) of $(Dg^k)_p$ that are inside the unit circle. The unstable manifold of $p$ for $g$ is the stable manifold of $p$ for $g^{-1}$. See [14] or [28] for more details.

**Definition.** $g$ is a Morse–Smale diffeomorphism if

(i) $\text{CR}(g)$ consists of a finite number of hyperbolic periodic points, and

(ii) whenever $p$ and $q$ are periodic for $g$, then the stable manifold of $p$ is transverse to the unstable manifold of $q$.

Let $\text{MS}$ denote the set of Morse–Smale diffeomorphisms on $M$. 

PROPOSITION 4. *If* $f$ *is topologically stable and is in the $C^0$ closure of MS, then $\text{CR}(f)$ is finite.*

*Proof.* Suppose $g$ is Morse–Smale and $h$ is a semiconjugacy, $hg = fh$. By Lemma 3, $h^{-1}$ of any periodic orbit of $f$ is closed, nonempty, and $g$-invariant. It follows that if $\{O(p_j; f)\}$ is a collection of disjoint periodic orbits of $f$, then $\{h^{-1}(O(p_j; f))\}$ is a collection of pairwise disjoint, closed, nonempty, $g$-invariant sets. However, $g$ is Morse–Smale, so Lemma 2 shows that any nonempty closed, $g$-invariant set must contain a periodic orbit of $g$. Since $g$ has only a finite number of periodic orbits, this shows that $f$ can have only finitely many periodic orbits as well. Now an application of Proposition 1 shows that $\text{CR}(f) = \text{Per}(f)$. $\blacksquare$

The next proposition combines the ideas of Proposition 4 with Shub’s $C^0$ density theorem [24, 27]. We say $f$ satisfies Axiom A if

(i) $\text{CR}(f)$ has a hyperbolic structure. That is, there is a continuous splitting of the tangent space over $\text{CR}(f)$, $TM_x = E^s(x) \oplus E^u(x)$, and constants $C > 0$, $\lambda > 1$ such that $\|Df^n|E^s\| > C\lambda^n$ and $\|Df^n|E^u\| < C/\lambda^n$ for all $n > 0$.

(ii) $\text{CR}(f) = \text{closure}(\text{Per}(f))$.

If $f$ is Axiom A and $x$ is in $\text{CR}(f)$, then the stable manifold of $x$, $\{y|d(f^n(y), f^n(x)) \to 0 \text{ as } n \to \infty\}$, and the unstable manifold of $x$, $\{y|d(f^n(y), f^n(x)) \to 0 \text{ as } n \to -\infty\}$, are immersed submanifolds of $M$ whose dimensions are equal to those of $E^s(x)$ and $E^u(x)$, respectively. An Axiom A diffeomorphism $f$ is said to satisfy the Strong Transversality Condition if

(iii) for any $x, y$ in $\text{CR}(f)$, the stable manifold of $x$ is transverse to the unstable manifold of $y$.

Finally, $f$ is called a Smale diffeomorphism if it satisfies (i)–(iii) and

(iv) $\text{CR}(f)$ is zero-dimensional.

We should note that what we have called Axiom A is a stronger condition than what is usually meant by Axiom A (our condition is equivalent to Axiom A plus the so-called no-cycles property). However, the only Axiom A diffeomorphism that we will deal with are Smale diffeomorphisms, and in the presence of condition (iii) our definition and the usual one are equivalent. See [6], [14], or [28] for more details.

Of immediate interest to us is the fact that the chain recurrent set of a Smale diffeomorphism is composed of a finite number of chain components (usually called basic sets in this context), each of which has a dense orbit [2, 14, 28]. We also rely on Shub’s density theorem, which says that the set of Smale diffeomorphisms on $M$ is $C^0$ dense in Homeo($M$) [24, 27].
PROPOSITION 5. If \( h \) is a continuous semiconjugacy from \( g_1 \) to \( g_2 \), then \( g_1 \) has at least as many chain components as \( g_2 \) (this number may be infinite). In particular, if \( f \) is topologically stable, then \( \text{CR}(f) \) is composed of a finite number of chain components.

Proof. We may assume that \( \text{CR}(g_1) \) is composed of a finite number of chain components, \( X_1, X_2, \ldots, X_k \), since otherwise the first assertion is trivially true. An elementary \( \epsilon - \delta \) argument shows that each \( h(X_i) \) is contained in a single chain component of \( g_2 \). Let \( Y_i \) denote the chain component of \( g_2 \) that contains \( h(X_i) \), and let \( Y \) denote the union of the \( Y_i \). If \( g_2 \) has a chain component \( Z \) disjoint from \( Y \), then \( h^{-1}(Z) \) is closed, nonempty, \( g_1 \)-invariant, and disjoint from \( \text{CR}(g_1) \). Since this is impossible, \( g_2 \) has at most \( k \) chain components.

The second assertion follows from the first by taking \( g_2 \) to be \( f \) and \( g_1 \) to be any Smale diffeomorphism semiconjugate to \( f \).

PROPOSITION 6. If \( f \) is topologically stable, then each chain component of \( f \) contains a dense orbit.

Proof. Choose \( \alpha \) small enough that any \( \alpha \)-chain for \( f \) can be \( \beta \)-shadowed, where \( \beta \) is less than half the distance between any two chain components of \( f \). From the definition, if \( X \) is a chain component of \( f \), then there is an \( \alpha \)-chain in \( X \) that is dense in \( X \). By shadowing, there is an \( f \)-orbit \( O(x; f) \) that is wholly contained in the \( \beta \)-ball about \( X \) and such that \( X \) is contained in the closed \( \beta \)-ball about \( O(x; f) \). By the choice of \( \beta \), the closure of \( O(x; f) \) is disjoint from every chain component of \( f \) except \( X \). By Lemma 2, this forces \( x \) to lie in \( X \). In this way we show that for each \( \beta > 0 \) there is a point \( x_\beta \) in \( X \) such that \( O(x_\beta; f) \) is \( \beta \)-dense in \( X \). From here it is an exercise in point set topology to show, first, that if \( U \) is an open subset of \( X \) (open in the relative topology) then the union of all the iterates \( f^n(U) \) is dense in \( X \) and, second, that \( X \) has a dense orbit.

If \( p \) is periodic for \( f \), with period \( k \), then the stable set of \( p \), \( W^s(p; f) \) and the unstable set of \( p \), \( W^u(p; f) \) are defined by

\[
W^s(p; f) = \left\{ x \mid \lim_{n \to -\infty} f^{-n}(x) = p \right\}, \quad W^u(p; f) = \left\{ x \mid \lim_{n \to +\infty} f^n(x) = p \right\}.
\]

(If \( p \) is hyperbolic for \( f \), then these are the stable and unstable manifolds defined earlier. If \( p \) is not hyperbolic then in general these sets will not be manifolds.)

PROPOSITION 7. If \( f \) is topologically stable, then there is a finite collection of periodic points of \( f \), \( p_1, \ldots, p_k \), such that the union of their stable sets is dense in \( M \).
Proof. Let $g$ be a Smale diffeomorphism and $h$ a semiconjugacy, $hg = fh$. If $X$ is an attracting basic set of $g$, then $X$ is a periodic sink. (An attracting basic set must contain the unstable manifold of each of its points; each of these submanifolds has the same dimension as the unstable subspace $E_u(x)$ of $TM_x$. As $X$ is zero-dimensional, this forces $E_u(x) = 0$ and $E'(x) = TM_x$ for all $x$ in $X$. It follows that each periodic orbit in $X$ is a sink; since $X$ contains periodic points and has a dense orbit, this shows that $X$ must be a single periodic attractor.) Moreover, $g$ satisfies the shadowing property [23, 12], so there is a finite collection of attracting periodic points of $g$, $q_1, \ldots, q_k$, the union of whose stable manifolds is a dense subset of $M$ [3, Section 5; 7, Theorem B]. Let $p_i = h(q_i)$ for each $i$; note that each $p_i$ is periodic under $f$, and that $W'(p_i; f)$ contains $h(W'(q_i; g))$. Thus $M = h(M) = h(\text{closure}(\cup_i W'(q_i; g))) = \text{closure}(h(\cup_i W'(q_i; g))) \subset \text{closure}(\cup_i W'(p_i; f))$.

Next we establish a few basic results necessary for the proof of the next proposition. The length of the finite $\alpha$-chain $\{x_i\}$, $0 \leq i \leq k$, is $k + 1$.

Lemma. (a) Given $\alpha > 0$, $n > 0$, there is a $\beta > 0$ such that any $\beta$-chain for $g$ of length $jn + 1$, $x_0, x_1, x_2, \ldots, x_{jn}$, defines an $\alpha$-chain for $g^n$ of length $j + 1$, $x_0, x_1, \ldots, x_{jn}$.

(b) $\text{CR}(g) = \text{CR}(g^n)$.

(c) If $n$ is a positive integer, $X$ a chain component of $g$, and $p$ a periodic point of $g$ in $X$ with period $k$, then $X$ consists of no more than $k$ chain components for $g^n$.

Proof. Part (a) is essentially the fact that $g$ is uniformly continuous; (b) follows directly from (a). To prove (c), we shall show that any $x$ in $X$ is in the same chain component of $g^n$ as at least one point on the orbit of $p$. Let $\alpha > 0$ be given, and choose $\beta > 0$ as in (a). Select a finite $\beta$-chain from $x$ to $p$. Extend this chain by adding iterates of $p$ to the right-hand end of the given chain until you obtain a finite chain of length $jn + 1$, for some integer $j$. The last point of this chain will be one of the points $g^i(p)$ where $i$ is between 0 and $k - 1$. Further extend this chain by adding some $\beta$-chain from $g^i(p)$ back to $x$ to the right-hand end of the previous chain. If necessary, concatenate this periodic $\beta$-chain with itself several times to obtain a finite $\beta$-chain for $g$, beginning and ending at $x$, of length $mn + 1$ ($m$ some positive integer), whose $jn + 1$st entry is on the orbit of $p$. By applying (a), we see that $x$ and some element of the orbit of $p$ belong to a single $\alpha$-chain for $g^n$. We can repeat this argument for a sequence of $\alpha$'s decreasing to 0; since the orbit of $p$ is finite, we conclude that $x$ and some element of the $g$-orbit of $p$ lie in the same chain component of $g^n$. □
DEFINITION. A homeomorphism $g: T \to T$ is called mixing if for any pair of nonempty open sets $U, V$ in $T$, there is a constant $N$, depending on $U$ and $V$, such that $g^n(U) \cap V$ is nonempty whenever $n$ is greater than $N$.

PROPOSITION 8. Suppose $f$ is topologically stable, $X$ is a chain component of $f$, and that $k$ is the least period of any periodic orbit in $X$. Then $X$ consists of no more than $k$ chain components of $f^k$, and $f^k$ is mixing on each.

Proof. The first assertion follows directly from the lemma. To see that $f^k$ is mixing on each of its chain components in $X$, note first that $f^k$ will have the shadowing property because $f$ does. Fix a chain component $Y$ of $f^k$ in $X$. Let $U$ and $V$ be nonempty and open in $Y$. Choose points $u$ in $U$ and $v$ in $V$ and a positive constant $\beta$ small enough that the $\beta$-balls in $Y$ about $u$ and $v$ are contained in $U$ and $V$, respectively. Now choose a small enough that any $\alpha$-chain for $f^k$ can be $\beta$-shadowed, and find an $\alpha$-chain for $f^k$ that begins at $u$, includes some fixed point $p$ of $f^k$, and ends at $v$. Let the length of this chain be denoted by $N$. Because $p$ is fixed by $f^k$, we can find an $\alpha$-chain for $f^k$ of this type for any $n$ larger than $N$. By shadowing there will be a point $x(n)$ such that $x(n)$ is within $\beta$ of $u$ and $f^k(x(n))$ is within $\beta$ of $v$. By arguing as in the proof of Proposition 6, we can conclude that $f^k(x(n))$ lies in $Y$ (this may force a smaller choice of $\beta$, but this cases no trouble). Thus $f^k(x(n))$ meets $V$ for any $n$ larger than $N$.

COROLLARY. (i) If $f$ is topologically stable and $X$ is a connected chain component of $f$, then the restriction of $f$ to $X$ is mixing.

(ii) If $f$ is topologically stable, $M$ is connected, and $\text{CR}(f)$ has interior, then $\text{CR}(f) = M$, and $f$ is mixing.

Proof. (i) Chain components are closed, so connectedness implies that $X$ is a chain component of $f^n$ for all $n$, so the proof of Proposition 8 shows that $f^k$ is mixing on $X$ for some $k$. Combining compactness with the definition of mixing, it is easy to see that for any nonempty open set $U$ in $X$ and any positive constant $\delta$, there is a constant $N$ such that $f^n(U)$ is $\delta$-dense in $X$ whenever $n$ is larger than $N$. (This involves the uniform continuity of $f$.) Since $\delta$ can be chosen arbitrarily small, this shows that $f$ is mixing on $X$.

(ii) Theorem B of [7] shows that if $f$ satisfies the shadowing property, then any chain component of $f$ that has interior must be both an attractor and a repeller of $f$. This forces the chain component to be both open and closed in $M$, so by connectedness it is all of $M$. Now (i) applies to show $f$ is mixing.
Proof of Theorem B

The following two results answer questions posed in [15]. Together with the remark following the Corollary they indicate that in general one cannot expect topological stability to be a generic property of Diff'(M). Here Diff'(M) stands for the collection of C' diffeomorphisms on M with the uniform C' topology (f and g are C' close if and only if they and their kth derivatives, 1 \leq k \leq r, are uniformly close over M). A property of C' diffeomorphisms is said to be generic (or C' generic) if it is possessed by all diffeomorphisms in a residual subset of Diff'(M). (A subset is residual if it can be realized as a countable intersection of open and dense sets.)

**Corollary to Proposition 5.** The Newhouse examples [13] are not topologically stable: consequently topological stability is not a generic property in Diff'(M), provided that both r and the dimension of M are at least 2.

**Proof.** These examples include a second category set of diffeomorphisms (for r and M as stated) each of which has an infinite number of attracting periodic points. It is easy to see that each attracting periodic orbit is a chain component. Now Proposition 5 applies.

**Remark.** Theorem 1 of [18] shows that topological stability is not a generic property of Homeo(M) with the C^0 topology (for essentially the same reason that the Newhouse examples are not topologically stable: C^0 generically there are infinitely many chain components).

**Proposition 9.** The Abraham–Smale examples are not topologically stable; consequently topological stability is not a generic property of Diff^1(S^2 \times T^2).

**Proof.** These examples are described in [1]. For us the essential features are that there is an open set N in Diff^1(S^2 \times T^2) and disjoint open neighborhoods U, V in M such that if g \in N, then

1. \{x \mid g^n(x) \in U for all n\} is a single hyperbolic fixed point q_g and in fact \{x \mid g^n(x) \in U for all n \geq 0\} is contained in W^s(q_g).

2. \{x \mid g^n(x) \in V for all n\} is a compact hyperbolic set A_g, and in fact \{x \mid g^{-n}(x) \in V for all n \geq 0\} is contained in the union of the unstable manifolds for g of all the points of A_g. Denote this union as W^u(A_g).

3. W^u(q_g) \cap W^u(A_g) is nonempty.

4. Given \gamma > 0, there is \delta > 0 such that if \delta, g are in N and the C^1 distance from f to g is less than \delta, then
(i) \( d(q_f, q_g) < \gamma \),
(ii) \( A_f \) is contained in the \( \gamma \)-ball about \( A_g \),
(iii) \( A_g \) is contained in the \( \gamma \)-ball about \( A_f \).

(5) If \( N_1 = \{ g \in N; W^u(q_g) \cap W^u(p; g) = \emptyset \) for each periodic point \( p \in A_g \}, \) and \( N_2 = N - N_1 \), then both \( N_1 \) and \( N_2 \) are dense in \( N \).

Claim 1. If \( f \in N_1 \), then \( f \) is not topologically stable.

Proof. Let \( \delta > 0 \) be small enough that the \( \delta \)-ball about \( q_f \) is contained in \( U \), and the \( \delta \)-neighborhood about \( A_f \) is contained in \( V \). Choose \( g \in N_1 \), \( C' \) close enough to \( f \) such that there is a semiconjugacy from \( g \) to \( f \), \( hg = fh \), with \( d_0(h, id_U) < \delta \). Now choose \( x \in W^s(q_g) \cap W^u(p; g) \) where \( p \in A_g \) is periodic for \( g \). It follows that \( h(x) \in W^s(h(q_g); f) \cap W^u(h(p); f) \). A semiconjugacy takes fixed points to fixed points and periodic points to periodic points, so \( h(q_g) \) is fixed by \( f \) and is within \( \delta \) of \( q_f \). By the way \( \delta \) was chosen and by (1), this forces \( h(q_g) = q_f \). Similarly \( h(p) \) is periodic and its orbit under \( f \) stays within \( \delta \) of \( O(p; f) \subset A_f \), so by (2) \( h(p) \in A_f \). Thus \( h(x) \in W^s(q_f) \cap W^u(h(p); f) \). Since \( h(p) \) is periodic and is in \( A_f \), this contradicts the assumption that \( f \in N_1 \).

Claim 2. If \( f \in N_2 \), then \( f \) is not topologically stable.

Proof. Suppose \( y \in W^s(q_f) \cap W^u(p; f) \) where \( p \) is a periodic point in \( A_f \). By Hartman's theorem (the local stability of hyperbolic periodic points—see [14]) there is a neighborhood \( W \) of \( O(p; f) \) and a constant \( \delta > 0 \) such that

(a) the \( \delta \)-ball about \( q_f \) is in \( U \),
(b) the \( \delta \)-ball about \( A_f \) is in \( V \),
(c) if the \( C^1 \) distance from \( f \) to \( g \) is less than \( \delta \), then \( \{ x \in O(x; g) \subset W \} \) is a single hyperbolic periodic orbit of \( g \), \( O(p_g) \), and in fact \( \{ x \in g^{-n}(x) \subset W \) for all \( n > 0 \} \subset W^u(p_g) \). Now choose \( g \in N_1 \) such that
(d) the \( C^1 \) distance from \( f \) to \( g \) is less than \( \delta \) and there is a semiconjugacy from \( g \) to \( f \), \( hg = fh \), with
(e) \( d_0(h, id_U) < \delta \).

Consider a point \( z \) in \( h^{-1}(y) \). By (e) and (a), \( g^n(z) \) is in \( U \) for all large \( n \), so by (1) \( z \) is in \( W^s(q_g) \). Similarly, by (e) and (c), \( g^{-n}(z) \) is in \( W \) for all large \( n \), so \( z \) is in \( W^u(p_g) \). Finally, by (e) and (b), \( p_g \) is in \( A_g \). This forces \( g \in N_2 \), which is the desired contradiction. \( \square \)
The proofs of the various parts of Theorem A can be adapted to apply to flows. For the most part this is straightforward; we indicate some of the more important changes and we list references.

The definition of the chain recurrent set and of the related notions can be found in [4] and [7]. A simplifying feature of flows is that the chain components of a flow are exactly the connected components of its chain recurrent set.

The proof of Proposition 1 still works if the dimension of $M$ is at least 3. Dim($M$) = 1 is trivial, but when dim($M$) = 2 a different argument is needed. The problem is that because the flow lines locally disconnect a surface, it may not be possible to make a perturbation of the given flow in such a way as to change a periodic pseudo-orbit into an actual periodic orbit. For the same reason, Walters' proof of the shadowing theorem does not apply to this case. For a more detailed discussion, and a proof of the shadowing theorem for flows on surfaces, consult [5]; see also the discussion on page 1044 of [16]. The proof of Proposition 1 for flows on surfaces is contained in Lemma 10, below. Before we give that result, we indicate the other changes that have to be made to have Propositions 1–7 apply to flows. For the most part these changes are minor.

The conclusion in Proposition 4 must read that the chain recurrent set consists of a finite number of orbits, each of which is fixed or periodic; the same change must be made in Proposition 7. In the proof of Proposition 5 we require the flow version of Shub's density theorem. This is found in [17] and [32]. The flow version of Nitecki's semistability theorem is given in [23] and in [9].

**Definition.** $x$ in $M$ is nonwandering for the flow $f$ if whenever $U$ is a neighborhood of $x$ and $T$ is a real number, then $f(t, U) \cap U \neq \emptyset$ for some $t \geq T$. We denote the collection of nonwandering points of $f$ by $\Omega(f)$. If $f$ is a flow, then Per($f$) is the collection of all fixed points and periodic orbits of $f$. As the reader can easily verify, Per($f$) $\subset$ $\Omega(f)$ $\subset$ CR($f$).

**Lemma 10.** Suppose $f$ is a topologically stable $C^1$ flow on the compact surface $M$. Then CR($f$) consists of a finite number of fixed points and periodic orbits.

**Proof.** Our argument is to show

\[ \text{CR}(f) = \Omega(f) = \text{closure(Per}(f)) = \text{Per}(f). \quad (\ast) \]

By [26, Chapter 3] and [16], the first equality in (\ast) is equivalent to showing that if $f_n$ is a sequence of $C^1$ flows converging to $f$ in the $C^0$
topology, and if $x_n$ is a sequence of points with $x_n \in \Omega(f_n)$ for each $n$ and $x_n$ converging to some point $x$, then $x \in \Omega(f)$. To see that this is so, first apply Pugh's closing lemma [20, 21] to obtain for each $n$ a $C^1$ flow $g_n$ which is $C^1 - (1/n)$-close to $f_n$ and which satisfies $x_n \in \text{Per}(g_n)$. As in the proof of Proposition 1, we can use the existence of semiconjugacies to show that $x$ lies in the closure of $\text{Per}(f)$; it follows that $x \in \Omega(f)$. A similar use of the closing lemma shows that $\Omega(f)$ is the closure of $\text{Per}(f)$; this gives the second equality in (2). To obtain the third equality, combine the argument of the proofs of Propositions 4 and 5 with the density of the Morse–Smale flows [19, 22] to show that $\text{Per}(f)$ consists of a finite number of fixed points and periodic orbits.

As a final remark, we note that the combination of Walters' shadowing theorem and the semistability theorem of Nitecki gives an easy proof that the Axiom A, Strong Transversality flows and diffeomorphisms have the shadowing property. See [23] and [12] for direct proofs.

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REFERENCES


