Plumbing constructions of connected divides and the Milnor fibers of plane curve singularities

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ABSTRACT

A divide is the image of a generic, relative immersion of a finite number of copies of the unit interval or the unit circle into the unit disk. N. A'Campo defined for each connected divide a link in $S^3$ and proved that the link is fibered. In the present paper we show that the fiber surface of the fibration of a connected divide can be obtained from a disk by a successive plumbing of a finite number of positive Hopf bands. In particular, this gives us a geometric understanding of plumbing constructions of the Milnor fibers of isolated, complex plane curve singularities in terms of certain replacements of the curves of their real morsifications.

1. INTRODUCTION

A divide is the image of a generic, relative immersion of a finite number of copies of the unit interval or the unit circle into the unit disk. ‘Generic and relative’ mean the following: Let $D$ denote the unit disk and $\partial D$ its boundary. Then,

- there are neither triple points nor self-tangent points;
- both endpoints of each immersed interval lie on $\partial D$, and endpoints of all immersed intervals are distinct;
- the intersection of each immersed interval with $\partial D$ is transverse;
- immersed circles do not intersect $\partial D$.

Figure 1 is a typical example of a divide.

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Each divide determines a link in $S^3$ as follows: Let $(x_1, x_2) \in \mathbb{R}^2$ be coordinates of $\mathbb{R}^2$. The unit disk $D$ is given by $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$. Let $T(\mathbb{R}^2)$ denote the tangent bundle to $\mathbb{R}^2$ and $ST(\mathbb{R}^2)$ the unit sphere in $T(\mathbb{R}^2)$, that is,

$$ST(\mathbb{R}^2) := \{(x_1, x_2, u_1, u_2) \in T(\mathbb{R}^2) \mid x_1^2 + x_2^2 + u_1^2 + u_2^2 = 1\}.$$

The \textit{link} of a divide $P$ is the set in the 3-sphere $ST(\mathbb{R}^2)$ given by

$$L(P) := \{(x_1, x_2, u_1, u_2) \in ST(\mathbb{R}^2) \mid (x_1, x_2) \in P, (u_1, u_2) \in T_{(x_1, x_2)}(P)\},$$

where $T_{(x_1, x_2)}(P)$ is the set of tangent vectors to $P$ at $(x_1, x_2)$. By the generic condition of divides, $L(P)$ constitutes a regular link in $ST(\mathbb{R}^2)$.

The divide was introduced by N. A'Campo in [AC3,AC4]. The original idea of divides is a generalization of real morsified curves of isolated, complex plane curve singularities, and in consequence the links of divides succeed several properties of the links of plane curve singularities. The following fibration is a basic property of the link of a connected divide: \textit{If a divide} $P$ \textit{is connected then there is a fibration map} $\pi_{P, \eta} : ST(\mathbb{R}^2) \setminus L(P) \to S^1$, \textit{and moreover, the monodromy of the fibration is represented by a product of right-handed Dehn twists} (Theorem 3.2). The definition of right-handed Dehn twists is presented in Section 2. Further studies of the links of divides are developed in [C-P,Hi1,G-I1,G-I2,Kaw,G1,G2,Hi2,I2,Go-Hi-Yd].

In knot theory, a fibered link in $S^3$ has been investigated by using plumbings (cf. [S,Ha,Ga,Me-Mo,Go-Hi-Ym]). The definition of plumbings is presented also in Section 2. If a surface $A$ is a plumbing of another surface $B$ and a Hopf band, we say the surface $A$ is obtained from $B$ by plumbing a Hopf band, or the surface $B$ is obtained from $A$ by deplumbing a Hopf band. Recently, E. Giroux[Gi] proved, as a corollary of a study of contact structures of 3-manifolds, that any fiber surface in $S^3$ can be obtained from a disk by a combination of plumbings and deplumbings of Hopf bands, which answers a question of J. Harer stated in [Ha]. So, investigating plumbing constructions of fiber surfaces is significant of characterizing fibered links.

A fiber surface is called a \textit{positive Hopf plumbing} if it can be obtained from a disk by a successive plumbing of a finite number of positive Hopf bands (cf. [Me-Mo]).
M. Hirasawa constructed an algorithm for describing the Seifert fiber surface of the fibration of a connected divide together with its link diagram [Hi1]. Furthermore, in subsequent work [Hi2], he investigated how to find a plumbing construction from the described Seifert fiber surface and, as a consequence, proved that the fiber surface can be obtained from a disk by a combination of plumbings and deplumbings of positive Hopf bands (so-called a *stable positive Hopf plumbing*, cf. [Me-Mo]).

In the present paper we show that the fiber surface of the fibration of a connected divide can be obtained without using deplumbing operations.

**Theorem 1.1.** *The fiber surface of the fibration \( \pi_{P, \eta} : ST(\mathbb{R}^2) \setminus L(P) \rightarrow S^1 \), for a connected divide \( P \), is a positive Hopf plumbing.*

It has been known that for any isolated, complex plane curve singularity there is a connected divide \( P \) such that the fibration \( \pi_{P, \eta} : ST(\mathbb{R}^2) \setminus L(P) \rightarrow S^1 \) is topologically equivalent to the Milnor fibration of the singularity. This fact follows from the existence theorem of real morsifications of isolated, complex plane curve singularities, due to A'Campo [AC1, AC2] and S.M. Gusein-Zade [GZ1, GZ2, GZ3], and the equivalence theorem of Milnor fibrations and the fibrations of connected divides stated in [AC3].

The next is an immediate corollary of Theorem 1.1 and the above fact:

**Corollary 1.2.** *The fiber surface of the Milnor fibration of an isolated, complex plane curve singularity is a positive Hopf plumbing.*

Though the assertion is already known since the link of an isolated, complex plane curve singularity has a closed positive braid presentation, our plumbing construction is very special because it is closely related to the curve of a real morsification of the singularity. In fact, from the proof of Theorem 1.1 we will understand that there exists a one-to-one correspondence between the positive Hopf bands of the plumbing construction of the Milnor fiber and the quadratic singularities of the real morsified function. All plumbings will be done in local balls in \( S^3 \), so the correspondence for each quadratic singularity will be seen as a local phenomenon.

This paper will be organized as follows: In Section 2, we give definitions of Dehn twists, Hopf bands and plumbings, and in Section 3 we define a Morse function associated with a connected divide and state the fibration theorem of A'Campo for connected divides. In Section 4, we introduce U- and X-subdisk replacements for connected divides and prove that any other subdisk replacement can be realized by a combination of them. Section 5 is devoted to the proof of Theorem 1.1. In Section 6 we explain the plumbing construction of the Milnor fiber of an isolated, complex plane curve singularity by using the curve of its real morsification.

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2. DEFINITIONS OF DEHN TWISTS, HOPF BANDS AND PLUMBINGS

In this section we give the definitions of Dehn twists, Hopf bands and plumbings. For their fundamental properties, see for instance [K].

**Definition 2.1.** A **Dehn twist** on a simple closed curve $C$ in an oriented, smooth 2-dimensional manifold $S$ is a diffeomorphism obtained by cutting $S$ along $C$, twisting $2\pi$ and re-gluing. A Dehn twist is **right-handed** if the twist brings a man, standing on $S$ in front of $C$, to the right. If the twisting is performed in the opposite direction, it is called **left-handed**.

**Definition 2.2.** A **Hopf band** is an unknotted annulus, embedded in $S^3$, with one full twist. If the full twist is in the clockwise direction, the Hopf band is called **positive** and otherwise called **negative**.

Positive and negative Hopf bands are described in Figure 2. In either case, the boundary of the band is a Hopf link $L$. So there is a fibration $S^3 \setminus L \to S^1$ over $S^1$ and the band is a fiber surface of this fibration. The fibration (in particular, its monodromy) depends on whether the Hopf band is positive or negative. In the former case, the fibration is nothing but the Milnor fibration[Mi] of a quadratic singularity in $\mathbb{C}^2$ and the monodromy is represented by the right-handed Dehn twist along the core curve of the Hopf band. In the latter case, the monodromy of the fibration is represented by the left-handed Dehn twist along the core curve.

![positive Hopf band](image1)

**positive Hopf band**

![negative Hopf band](image2)

**negative Hopf band**

Fig. 2. Positive and negative Hopf bands.

**Definition 2.3.** A compact, oriented surface $R$ embedded in $S^3$ is a **plumbing** of two compact, oriented surfaces $R_1$ and $R_2$ in $S^3$ if they satisfy the following:

1. $R = R_1 \cup R_2$ such that
   1.1 $R_1 \cap R_2$ is a square disk with edges $a_1, b_1, a_2, b_2$ enumerated in this order,
   1.2 $a_i$ is contained in $\partial R_1$ and is a proper arc in $R_2$ for all $i$, and
   1.3 $b_i$ is contained in $\partial R_2$ and is a proper arc in $R_1$ for all $i$.
2. There exists 3-balls $B_1$ and $B_2$ in $S^3$ such that
   2.1 $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2 = S^2$.
Figure 3 is a plumbing of a surface and a positive Hopf band.

The plumbing operation has the following significant property.

Theorem 2.4 ([Ga]). Let $R$ be a plumbing of compact, oriented surfaces $R_1$ and $R_2$. Then $R$ is a fiber surface if and only if both $R_1$ and $R_2$ are fiber surfaces. Moreover, if $R$ is a fiber surface then the monodromy of the fibration of $R$ is represented by a product of the monodromies of $R_1$ and $R_2$.

We note that, in [Ga], the assertion is proved for not a plumbing but a Murasugi-sum, which is a generalized version of plumbings.

3. A MORSF FUNCTION ASSOCIATED WITH A CONNECTED DIVIDE

A divide is said to be connected if the set of immersed curves of the divide is connected. In this section, following [AC3] and [AC4] (cf. [I3]), we define a Morse function associated with a connected divide and state the fibration theorem for the links of connected divides.

Let $P$ be a divide in $D$. A region of $P$ is a connected component of $D \setminus P$. If a region of $P$ is bounded by only $P$ then it is called an inside region, and otherwise it is called an outside region. For each outside region, its intersection with $\partial D$ is called the outside boundary.

A Morse function $f : \mathbb{R}^2 \to \mathbb{R}$ is a function which has only quadratic singularities. A maximum (respectively saddle and minimum) of $f$ is a quadratic singularity with Morse index $-2$ (respectively 0 and 2).

Definition 3.1. Let $P$ be a connected divide in $D$. A Morse function $f_P$ associated with $P$ is a Morse function $f_P : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following:

(1) $0 \in \mathbb{R}$ is either a regular value or a critical value of only saddle singula-
rities of \( f_P \) so that the immersed curve \( X_0 := \{ x \in \mathbb{R}^2 \mid f_P(x) = 0 \} \) coincides with \( P \) in \( D \);

(2) each inside region of \( X_0 \) contains one maximum or minimum of \( f_P \);

(3) each double point of \( X_0 \) corresponds to a saddle of \( f_P \), and \( f_P \) is locally given by \( (X + Y)(X - Y) \) with local coordinates \( (X, Y) \) centered at the double point;

(4) there are no singularities of \( f_P \) in \( D \) other than those in (2) and (3);

(5) if the outside boundary of an outside region of \( X_0 \) is \( \partial D \) then the outside boundary is a level set of \( f_P \);

(6) if an outside region of \( X_0 \) is not in case (5) then there is just one point, in its outside boundary, at which a level set of \( f_P \) intersects \( \partial D \) tangentially.

Let \( P \) be a connected divide and \( f_P \) a Morse function associated with \( P \). For each double point \( s \) of \( P \), we take small neighborhoods \( U_s \) and \( V_s \) satisfying \( U_s \subset V_s \) and assume that \( f_P \) has the form \( (X + Y)(X - Y) \) in \( V_s \). For a sufficiently small \( \eta > 0 \), we define the map \( \theta_{P, \eta} : \{ x \in \mathbb{R}^2 \mid f_P(x) = 0 \} \to \mathbb{C} \) by

\[
\theta_{P, \eta}(x, u) = f_P(x) + \eta df_P(x)(u) - \frac{1}{2} \eta^2 \chi(x) H_{f_P}(x)(u, u),
\]

where \( \chi(x) : D \to [0, 1] \) is a positive \( C^\infty \)-differentiable bump function such that \( \chi(x) = 0 \) outside \( V_s \) and \( \chi(x) = 1 \) inside \( U_s \), and \( H_{f_P} \) is the Hessian of \( f_P \). We then define the map \( \pi_{P, \eta} : ST(\mathbb{R}^2) \setminus L(P) \to S^1 \) by

\[
\pi_{P, \eta}(x, u) := \frac{\theta_{P, \eta}}{|\theta_{P, \eta}|}.
\]

This map constitutes a fibration in the complement of the link \( L(P) \) in \( ST(\mathbb{R}^2) \).

**Theorem 3.2** ([AC3, AC4]). For a connected divide \( P \), the map \( \pi_{P, \eta} : ST(\mathbb{R}^2) \setminus L(P) \to S^1 \) is a locally trivial fibration over \( S^1 \). Moreover, the monodromy of the fibration is represented by a product of right-handed Dehn twists and the closed curve for each Dehn twist corresponds to a quadratic singularity of \( \theta_{P, \eta} \).

**Remark 3.3.** (i) Hirasawa proved in [Hi2] that the fiber surface of the fibration of a connected divide is a stable, positive Hopf plumbing. The first assertion in Theorem 3.2 can be derived by combining it with Theorem 2.4. In the present paper we will prove that the fiber surface is a positive Hopf plumbing. This derives the whole assertion in Theorem 3.2 including that concerning monodromies.

(ii) The author introduced in [I3] a divide immersed into a compact, oriented, smooth 2-dimensional manifold, possibly with boundary, and proved a generalized version of the fibration theorem of a connected divide stated in Theorem 3.2. In the proof, he used a handlebody decomposition of a Lefschetz fibration (cf. [Kas]). This proof includes an alternative proof of Theorem 3.2.
A subdisk replacement is the following operation for connected divides: Let $P$ be a connected divide with an associated Morse function $f_P : D \rightarrow \mathbb{R}$. A subdisk $\Delta \subset D$ is the closure of an open subset of $D$ homeomorphic to an open disk. We remark that, from a technical reason, $\Delta$ may not be homeomorphic to a disk though it is named a subdisk. Let $P'$ be another divide in $D$ which coincides with $P$ in $D \setminus \Delta$, where $D \setminus \Delta$ is the closure of $D \setminus \Delta$, and let $f_{P'}$ be a Morse function associated with $P'$ which also coincides with $f_P$ in $D \setminus \Delta$. The local replacement $(P,f_P) \rightarrow (P',f_{P'})$ of the pair of the divide and its associated Morse function is called a subdisk replacement.

We will deal with only subdisk replacements such that the subdisk $\Delta$ satisfies that $\partial \Delta \cap \partial D$ is connected and $\dim(\partial \Delta \cap \partial D) = 1$. Our main tools are the following U-subdisk and X-subdisk replacements.

**Definition 4.1.** A U-subdisk replacement is a subdisk replacement $(P,f_P) \rightarrow (P',f_{P'})$ which satisfies the following: Let $\Delta$ be the subdisk of $D$. Then

(i) $\partial \Delta \cap \partial D$ is connected and $\dim(\partial \Delta \cap \partial D) = 1$.

(ii) $P \cap \Delta$ consists of two disjoint embedded intervals, each of which connects $\partial \Delta \setminus \partial D$ with $\partial \Delta \setminus (\partial \Delta \setminus \partial D)$ and intersects $\partial \Delta$ transversely, where $\partial \Delta \setminus \partial D$ is the closure of $\partial \Delta \setminus \partial D$.

(iii) $P' \cap \Delta$ consists of an embedded interval which connects the two points $P \cap (\partial \Delta \setminus \partial D)$ in $\Delta \setminus \partial D$.

(iv) At each point of $(\partial \Delta \setminus \partial D) \cap \partial D$, a level-curve of $f_P$ intersects $\partial D$ tangentially.

A typical figure is shown in Figure 4. There are two cases, either $(\partial \Delta \setminus \partial D) \cap \partial D$ consists of two points or it consists of one point. The latter is the case where the subdisk $\Delta$ is not homeomorphic to a disk. In this case we need to modify the level-curves of $f_{P'}$ in the right figure so that the level-curve containing that point lies on the boundary $\partial D$ as required in Definition 3.1 (5). Then the level-curves of $f_{P'}$ described in the figure show that there always exists a Morse function $f_{P'}$ associated with $P'$ which coincides with $f_P$ in $D \setminus \Delta$.

**Definition 4.2.** An X-subdisk replacement is a subdisk replacement $(P,f_P) \rightarrow (P',f_{P'})$ which satisfies the following: Let $\Delta$ be the subdisk of $D$. Then

(i) $\partial \Delta \cap \partial D$ is simply-connected and $\dim(\partial \Delta \cap \partial D) = 1$.

(iii') $P \cap \Delta$ consists of an embedded interval which connects $\partial \Delta \setminus \partial D$ with $\partial \Delta \setminus (\partial \Delta \setminus \partial D)$ and intersects $\partial \Delta$ transversely.

(iii') $P' \cap \Delta$ consists of two embedded intervals intersecting each other at one point transversely.

A typical figure is shown in Figure 5. The level-curves in the figure again show that there always exists a Morse function $f_{P'}$ associated with $P'$ which coincides with $f_P$ in $D \setminus \Delta$. 

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Fig. 4. U-subdisk replacement. The foliations represent the level-curves of \( f_P \) and \( f_{P'} \).

Fig. 5. An X-subdisk replacement. The foliations represent the level-curves of \( f_P \) and \( f_{P'} \).

The next proposition asserts that the U- and X-subdisk replacements are fundamental operations of subdisk replacements, though we do not use it in the rest of this paper.

**Proposition 4.3.** Let \( \Delta \) be a subdisk in \( D \) satisfying that \( \partial \Delta \cap \partial D \) is non-empty and simply connected. Then, any subdisk replacement with the subdisk \( \Delta \) can be realized by a combination of U- and X-subdisk replacements and their inverses.

**Proof.** The point of this proof is that we have to perform the inverses of U-subdisk replacements without breaking the 'connected' condition of divides. Let \( (P, f_P) \to (P', f_{P'}) \) be a subdisk replacement of the subdisk \( \Delta \). Denote by \( \ell_1, \ldots, \ell_m \) parallel arcs connecting \( P \cap (\partial \Delta \setminus \partial D) \) and \( \partial \Delta \setminus (\partial \Delta \setminus \partial D) \) in \( \Delta \) without any double points, and define \( \ell_0 \) to be an embedded interval in \( \Delta \) which intersects each arc \( \ell_i \), \( i = 1, \ldots, m \), at one point transversely and whose two endpoints lie on \( \partial \Delta \setminus (\partial \Delta \setminus \partial D) \). Let \( P'' \) be a divide which coincides with \( P \) in \( D \setminus \Delta \) and is given by \( \cup_{i=0}^{m} \ell_i \) in \( \Delta \). For such a divide \( P'' \), it is easy to show that

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the subdisk replacement \((P, f_P) \rightarrow (P'', f_{P''})\) can be realized by a combination of U- and X-subdisk replacements and their inverses, and the subdisk replacement \((P', f_{P'}) \rightarrow (P'', f_{P''})\) can be also. Thus the proof is completed. □

5. PROOF OF THEOREM 1.1

**Lemma 5.1** Let \((P, f_P) \rightarrow (P', f_{P'})\) be a U- or X-subdisk replacement. Then the fiber surface of \(\pi_{P',y'}\) is a plumbing of the fiber surface of \(\pi_{P,y}\) and a positive Hopf band.

**Proof.** Since \(f_P\) and \(f_{P'}\) are the same in \(D \setminus A\), their corresponding fibrations are the same in the set \(\{(x,u) \in ST(\mathbb{R}^2) \mid x \in D \setminus \Delta\}\). So, it is enough to study the fibrations in the remainder set

\[
B := \{(x,u) \in ST(\mathbb{R}^2) \mid x \in \Delta\}.
\]

When \((\partial \Delta \setminus \partial D) \cap \partial D\) consists of two points, since \(\dim(\partial \Delta \cap \partial D) = 1\), \(B\) is homeomorphic to a ball. When it consists of one point, \(B\) is homeomorphic to the set obtained from a ball by identifying its north and south poles. The proof for the latter case is the same as the former case except the shape of \(B\), so we will give only the proof of the former case, that is, from now on we assume that \((\partial \Delta \setminus \partial D) \cap \partial D\) consists of two points. Since \(\partial \Delta \cap \partial D\) is non-empty and simply connected, \(B\) is homeomorphic to a ball.

Suppose that \(\Delta\) is sufficiently small so that it does not contain the positive \(x_1\)-axis \(\{(x_1,x_2) \in D \mid x_1 \geq 0, x_2 = 0\}\). We now choose coordinates \((w_1, w_2)\) in \(\Delta\) as follows:

\[
w_1 := \sqrt{1 - (x_1^2 + x_2^2)},
\]
\[
w_2 := \pi - \arg(x_1, x_2).
\]

The ball \(B\) is given, with the coordinates \((w_1, w_2)\), by

\[
B = \{(w_1, w_2, u_1, u_2) \in T(\mathbb{R}^2) \mid (w_1, w_2) \in \Delta, |(u_1, u_2)| = w_1\}.
\]

Setting new coordinates as

\[
(5.1) \quad (X, Y, Z) := (w_1 \cos(\arg(u_1, u_2)), w_1 \sin(\arg(u_1, u_2)), w_2) \in \mathbb{R}^3,
\]

we can embed the ball \(B\) in \(\mathbb{R}^3\) (cf. [11]).

First we prove the assertion for a U-subdisk replacement. Let \((P, f_P) \rightarrow (P', f_{P'})\) be a U-subdisk replacement. We will study the shapes of the links \(L(P)\) and \(L(P')\) and their fiber surfaces in \(B\), and show that the replacement induces a plumbing of a positive Hopf band.

With the new coordinates \((5.1)\), the links \(L(P)\) and \(L(P')\) can be described as shown on the right in Figure 6 and Figure 7 respectively. Let \(P_+\) be the regions in \(D\) given by \(\{x \in D \mid f_P > 0\}\). We now assume that \(P_+ \cap \Delta\) is connected and study the fiber surface \(S_P\) of \(\pi_{P,y}\) over \(1 \in S^1\). (Otherwise \(P_- \cap \Delta := \{x \in \Delta \mid f_P < 0\}\) is connected. In this case we need to study the fiber surface
over $-1 \in S^1$. The proof of this case is analogous to that of the case where $P_+ \cap \Delta$ is connected, so we omit it here.) Recall that the fiber surface $S_P$ over $1 \in S^1$ is given by

$$S_P = \{(x, u) \in ST(\mathbb{R}^2) \mid \arg(\theta_{P,n}(x,u)) = 0\}.$$ 

Since $\Delta$ does not contain any critical points of $f_P$, the fiber surface $S_P \cap B$ in $B$ is given by

$$S_P \cap B = \{(x, u) \in ST(\mathbb{R}^2) \mid x \in P_+ \cap (\Delta \setminus M), df_P(x)(u) = 0\}.$$ 

This set except the points over $\partial D$ consists of all tangent vectors to the foliation of level-curves of $f_P$ in $P_+ \cap \Delta$. Hence, using the new coordinates (5.1), we can describe the surface $S_P \cap B$ as shown on the right in Figure 6.

Fig. 6. The fiber surface $S_P$ and the link $L(P)$ in $B$.

For the divide $P'$, the fiber surface $S_{P'} \cap B$ in $B$ except over the maximum $M$ indicated on the left in Figure 7 is given by

$$\{(x, u) \in ST(\mathbb{R}^2) \mid x \in P'_+ \cap (\Delta \setminus M), df_{P'}(x)(u) = 0\}.$$ 

It was observed in [AC4] that the pull-back $F_M$ of the maximum $M$ is a circle which connects two cells corresponding to two lifts of the inside region containing $M$. Then, a direct inspection shows that the surface $S_{P'} \cap B$ is described as shown on the right in Figure 7.
In the figure, the surface $S' \cap B$ constitutes a positive Hopf band. Now perform a deplumbing to the positive Hopf band with respect to the square $a_1, b_1, a_2, b_2$ indicated in Figure 8. (The four vertices indicated in the figure represent the corners of the square. The edges $a_1$ and $a_2$ lie on the fiber surface, and the union of the edges $b_1$ and $b_2$ constitutes the intersection of the fiber surface with the boundary $\partial B$.) After removing the positive Hopf band according to the deplumbing operation, we obtain the fiber surface as shown on the right in Figure 8, which is ambient isotopic in $B$ to the surface described on the right in Figure 6. Thus we conclude that the U-subdisk replacement $(P, f_P) \to (P', f_{P'})$ induces a plumbing of a positive Hopf band.
Next we prove the assertion for an X-subdisk replacement. Let \( (P, f_P) \to (P', f_{P'}) \) be an X-subdisk replacement. It has been observed in [AC4] that such a replacement corresponds to a connected sum of the link \( L(P) \) and a positive Hopf link. Note that the link of a divide consisting of two embedded intervals with one double point is a positive Hopf link. Since a connected sum can be realized by a plumbing of their Seifert surfaces, we conclude that the X-subdisk replacement induces a plumbing of a positive Hopf band. 

Proof of Theorem 1.1. Let \( P \) be a connected divide. By using the inverses of U-subdisk replacements, \( P \) can be replaced by a connected divide \( P' \) without inside regions. Furthermore, since \( P' \) constitutes an embedded tree in \( D \), it can be replaced by a trivial divide by using the inverses of X-subdisk replacements. Here the trivial divide means a divide consisting of only one immersed interval without double points. It can be easily checked that the link of a trivial divide is a trivial knot. The inverse of the sequence of the above subdisk replacements concludes, with Lemma 5.1, that the fiber surface of the fibration of \( P \) can be obtained from a disk, which is a fiber surface of the trivial knot, by plumbing positive Hopf bands corresponding to the X- and U-subdisk replacements.

6. PLUMBING CONSTRUCTIONS OF THE MILNOR FIBERS OF ISOLATED, COMPLEX PLANE CURVE SINGULARITIES

In this section, we briefly review the real morsification theory of isolated, complex plane curve singularities and show a geometric understanding of plumbing constructions of the Milnor fibers of these singularities.

Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be a polynomial map and suppose that \( f \) has a singular point at the origin \( (0, 0) \) with critical value \( f(0, 0) = 0 \). Following [AC1], we assume that

(IS) the singularity is isolated, and

(RB) for each locally irreducible component \( f_i, i = 1, \ldots, r \), of \( f \), the branch \( \{(x_1, x_2) \in \mathbb{R}^2 \mid f_i(x_1, x_2) = 0\}, i = 1, \ldots, r \), is a 1-dimensional arc connected with \((0, 0)\).

A real morsification of the isolated, complex plane curve singularity is a family \( f_t, t \in [0, 1] \), of polynomial maps such that

1) \( f_0 = f \),

2) in a small neighborhood \( \mathcal{U} \) of \((0, 0) \) in \( \mathbb{R}^2 \), for \( t > 0 \) the curve \( \{(x_1, x_2) \in \mathcal{U} \mid f_t(x_1, x_2) = 0\} \) consists of a union of immersed intervals with only double points, and

3) the number \( \delta \) of double points satisfies \( \mu = 2\delta + r - 1 \) for \( t > 0 \), where \( \mu \) is the Milnor number of the singularity (that is, the first Betti number of the Milnor fiber).

Remark that all the integers \( \mu, \delta \) and \( r \) are topological invariants of isolated, complex plane curve singularities.

The next is a basic fact in the real morsification theory.
Theorem 6.1 ([AC1,GZ3]). Any complex plane curve singularity satisfying (IS) and (RB) has a real morsification.

The real morsified curve has several properties which are useful to understand the geometry of its Milnor fibration.

The curve of a real morsification has $\delta$ double points in the small neighborhood $\mathcal{U}$ and $\delta + r - 1$ inside regions in $\mathcal{U}$. Each of double points and inside regions corresponds to a first homology cycle of the Milnor fiber. The curve of the real morsification clarifies the mutual positions of the first homology cycles, and this gives us a geometric understanding of the monodromy of the Milnor fibration. These studies were developed in the 1970s by A'Campo and Gusein-Zade. For details, see [AC1,AC2,GZ1,GZ2,GZ3].

In [AC3] and [AC4], it was proved that the Milnor fibration of an isolated, complex plane curve singularity is topologically equivalent to the fibration of a connected divide consisting of immersed intervals with the same configuration as the curve of a real morsification of the singularity. In particular, the link of the singularity is ambient isotopic to the link of the divide. In consequence, we can regard the link of the singularity as the set of tangent vectors to the curve of its real morsification. Furthermore, the study of the monodromies of the fibrations of connected divides, developed in [AC4], gives us an explicit description of the monodromies of the Milnor fibrations in terms of the tangent bundle to the unit disk.

In the proof of Theorem 1.1, the plumbing construction of the fiber surface of the fibration of a connected divide was described in terms of U- and X-subdisk replacements. Now we assume that the divide consists of a real morsified curve of an isolated, complex plane curve singularity. Then, the plumbing construction of the Milnor fiber, stated in Corollary 1.2, can also be understood in terms of subdisk replacements of the curve of the real morsification. From the definitions of U- and X-subdisk replacements, it is clear that each U- (respectively X-) subdisk replacement $(P, f_P) \to (P', f_{P'})$ corresponds to an addition of a saddle (respectively a maximum or a minimum) to $f_P$. This implies that there is a one-to-one correspondence between the positive Hopf bands of the plumbing construction of the Milnor fiber and the quadratic singularities of the real morsified function. The plumbing of the Milnor fiber consists of $\delta$ X-subdisk replacements and $\mu - \delta$ U-subdisk replacements, where $\delta$ is the number of double points of the real morsified curve and $\mu$ is the Milnor number of the singularity. In particular, the number of these subdisk replacements (that is, the number of positive Hopf bands) is equal to the Milnor number of the singularity.

We conclude the present paper with showing an example of plumbing constructions of the Milnor fibrations of isolated, complex plane curve singularities. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the polynomial map defined by $f(z_1, z_2) = -z_1^4 + z_2^3$, which has an isolated singularity at $(0, 0)$. Figure 9 shows a real morsification of the singularity. The coordinates $(x_1, x_2)$ in the figure represent the real parts of $(z_1, z_2)$. The left figure is described by plotting the curve $(t^3, t^4)$, while the
right is described by plotting the curve \((T(3, t), T(4, t))\), where \(T(d, t)\) is the Chebyshev polynomial of degree \(d\). This method for producing a real morsification was introduced by Gusein-Zade, see [GZ1, GZ2, GZ3].

![Diagram of morsification](image)

Fig. 9. A real morsification of the isolated singularity of \(f(z_1, z_2) = -z_1^4 + z_2^3\) at \((0, 0)\).

The plumbing construction of the Milnor fiber of this isolated singularity is described in Figure 10. In the figure, 6 subdisk replacements are described, and thus we conclude that the link can be obtained from a disk by plumbing 6 positive Hopf bands. Furthermore, we can see that the plumbing construction consists of \(\delta (= 3)\) X-subdisk replacements and \(\mu - \delta (= 6 - 3 = 3)\) U-subdisk replacements, where \(\delta (= 3)\) is the number of double points of the real morsified curve and \(\mu (= 6)\) is the Milnor number of the singularity.

![Diagram of plumbing construction](image)

Fig. 10. The plumbing construction of the Milnor fiber of the isolated singularity of \(f(z_1, z_2) = -z_1^4 + z_2^3\) at \((0, 0)\).
REFERENCES


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