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# BRIDGES OF LONGEST CYCLES

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This paper is concerned with bridges of longest cycles in 3-connected non-hamiltonian graphs. Let $G$ be such a graph and let

$$d(u)+d(v)\geqslant m$$

for each pair of non-adjacent vertices $u$ and $v$. Let the length of its longest cycle $C$ be $r$. Then the length of any bridge of $G$ is at most $r-m+2$.

## 1. Introduction

Some graphs contain hamilton cycles and some do not. How long is a longest cycle in non-hamiltonian graphs? What can be said about the structure of the subgraph outside a longest cycle? These are two problems among many interesting similar problems. Some results about the structure of the subgraph outside a longest cycle have been found by Nash–Williams [5], Bondy [1] and Boss [6]. It is obvious that the length of a longest cycle and the structure of the subgraph outside a longest cycle are not independent. This paper will establish a result which gives a relation between the length of a longest cycle and its bridges.

**Definitions.** Let $C$ be a subgraph of $G$. A *bridge* of $C$ is either a component of $G\setminus V(C)$ together with its attachments on $C$ or a chord of $C$. A *$C$-path* is a path of $G$ such that only its endvertices are on $C$. If $B$ is a bridge of $C$, let $P$ be a longest $C$-path contained in $B$. Then *the length of the bridge $B$ is defined as the length of $P$.*

**Theorem 1.** *Let $G$ be a 3-connected non-hamiltonian graph and*

$$d(x)+d(y)\geqslant m$$

*for each pair of non-adjacent vertices $x$ and $y$. Let the length of any longest cycle $C$ be $r$. Then the length of any bridge of $C$ is at most $r-m+2$.*

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In other words, let $C$ be a longest cycle of $G$ and let $p$ be the length of the longest bridge of $C$. Then the length of $C$ is at least $m+p-2$. Hence, the shorter a longest cycle is, the shorter the bridges of the cycle are.

Some examples will show that this theorem is the best possible result. The condition of 3-connectivity cannot be reduced, for example, $3K_t+K_2$ is a 2-connected graph which is constructed by joining all vertices of three vertex disjoint $K_t$'s to two new vertices $x$ and $y$. This graph contains a longest cycle of length $2t+2$ with a bridge of length $t+1$, but $m=2t+2$. The inequality of the theorem cannot be reduced, either. One example is the complete bipartite graph $K_{t,t+1}$ which is 3-connected (if $t \geq 3$) and contains a longest cycle of length $2t$ with a bridge of length 2, but $m=2t$. Another example is $4K_t+K_3$ which is also 3-connected and contains a longest cycle of length $3t+3$ with a bridge of length $t+1$, but $m=2t+4$.

Theorem 1 generalizes a result found by Linial for 3-connected graphs.

**Theorem 2** (Linial [4]). *Let $G$ be a 2-connected graph, and*

$$d(x)+d(y) \geq m$$

*for each pair of non-adjacent vertices $x$ and $y$. Then $G$ contains either a hamilton cycle or a cycle of length at least $m$.*

In [6], Voss obtained a result about the lengths of longest cycles in graphs.

**Theorem 3** (Voss [6]). *Let $G$ be a $k$-connected graph with minimum degree $\delta$ and $r$ be the length of a longest cycle in $G$. Then either $r \geq k(\delta-k+2)$ or every bridge of a longest cycle is of order at most $k-2$.*

Voss' result can be applied to find the relation between the length of a longest cycle and its bridge. If $G$ is 4-connected and some bridge of a longest cycle is not short enough, then the length of the longest cycle is at least $4(\delta-2)$. Since the length $p$ of a bridge of the longest cycle is at most $r/2$, $p \leq r/2 \leq r-2(\delta-2) = r-2\delta+4$. In this sense, Theorem 1 is a generalization of Theorem 3 for 3-connected graph with Ore-type condition.

## 2. Terminology

Let $C=v_1 \ldots v_r v_1$. The path $v_i v_{i+1} \ldots v_{j-1} v_j$ will be denoted by $v_i C v_j$ and the path $v_i v_{i-1} \ldots v_{j+1} v_j$ will be denoted by $v_i \bar{C} v_j$ where $v_{r+1}$ is taken to be $v_1$.

Denote

$$N_D(x)=\{y \mid (x,y) \in E(G), y \in V(D)\}$$

where $D$ is a subgraph of $G$. When $V(D)=V(G)$, we simply write $N_D(x)=N(x)$.

Denote

$$d_D(x) = |N_D(x)| \quad \text{and} \quad d(x) = |N(x)|.$$

If $P = u_1 \ldots u_h$ is a path and $T$ is a subset of its vertices, let

$$T_P^{+1} = \{u_{k+1} \in P \mid u_k \in T \cap P\}, \quad \text{and} \quad T_P^{-1} = \{u_{k-1} \in P \mid u_k \in T \cap P\}.$$

Sometimes we simply write $T^{+1}$ if no confusion will occur.

Let $D$ be a subgraph of $G$, $v$ be a vertex of $D$ and $LP(v, D)$ be the collection of all longest paths in $G\backslash[V(D)\backslash\{v\}]$ with one specified endvertex $v$. Note that for any path $Q = v \ldots u$ of $LP(v, D)$, the neighbours of $u$ are contained in $V(Q) \cup V(D)$. The collection $LP(v, D)$ may contain more than one path. But we only need to consider one of them. Take one path $v \ldots u$ of $LP(v, D)$ and denote the endvertex $u$ by $w(v, D)$.

Let $h(a, D) = |N(b) \cap [G\backslash(D\backslash a)]|$ where $b = w(a, D)$. Note that if $h(a, D) = 0$, then $w(a, D) = a$ and $a$ is an isolated vertex in $G\backslash[V(D)\backslash\{a\}]$.

Let

$$M(a, D) = \{v \in V(D)\backslash a \mid \text{there is a } D\text{-path joining } a \text{ and } v \text{ with length at least } h(a, D) + 1\}.$$

Let

$$N(a, D) = \{v \in V(D)\backslash a \mid \text{there is a } D\text{-path joining } a \text{ and } v\}.$$

Obviously, $M(a, D) \subseteq N(a, D)$. Note that if $h(a, D) = 0$, then $N(a) \subseteq V(D)$ and, hence, $M(a, D) = N(a, D) = N(a)$.

By $a^{***}c$ denote a $D$-path $a \ldots c$ of $D$, where $a, c \in V(D)$. Note that a single edge in $D$ is also a $D$-path according to the definition in Section 1, because the two endvertices are in $D$.

## 3. Lemmas

**Lemma 1** (Dirac [2], Fournier and Fraisse [3]). *Let $D$ be a subgraph of a 2-connected $G$ with $|V(D)| \geq 2$, and $P = x \ldots y$ be a longest path in $G\backslash[D\backslash\{x\}]$ starting at $x$. Then there is a $D$-path starting at $x$ that contains $y$ and all its neighbours in $G\backslash V(D)$.*

In other words, if $G$ is 2-connected and $D$ is any subgraph of $G$ satisfying $|V(D)| \geq 2$ and $a \in V(D)$, then $M(a, D) \neq \emptyset$.

**Lemma 2.** *If $G$ is 3-connected, then $|M(a, D)| \geq 2$ for any subgraph $D$ of $G$ with $|V(D)| \geq 3$ and $a \in V(D)$.*

**Proof.** By Lemma 1, there is $b \in M(a, D)$. Since $G\backslash\{b\}$ is 2-connected, by Lemma 1 we have $|M(a, D\backslash b)| \geq 1$. □

**Lemma 3.** *Let $P = x_1 \dots x_t$ be a path and let $y, z \notin V(P)$. If $N_P(y) \cap N_P^{+1}(z) = \emptyset$, then $|N_I(y)| + |N_I(z)| \leq |I| + 1$ for any interval $I = x_i \dots x_j \subseteq P$.*

**Proof.** Since $N_I(y) \cap N_I^{+1}(z) = \emptyset$ and $|N_I(z)| \leq |N_I^{+1}(z)| + 1$,

$$|I| \geq |N_I(y)| + |N_I^{+1}(z)| \geq |N_I(y)| + |N_I(z)| - 1. \quad \square$$

## 4. Proof of Theorem 1

Let $C = v_1 \dots v_r v_1$ be a longest cycle of $G$ and $p$ be the length of a longest bridge of $C$. We will prove the theorem by contradiction. Assume that $r \leq m + p - 3$.

*Part A.*

In this part, we will obtain some general propositions which will be used frequently during the proof.

Let $B = v_r{*}{*}{*}v_t$ be a longest $C$-path. Note that it contains $p - 1$ vertices not in $C$.

For the sake of convenience, denote $w(v_i, C)$, $h(v_i, C)$, $M(v_i, C)$ and $N(v_i, C)$ by $w(i)$, $h(i)$, $M(i)$ and $N(i)$, respectively, for $i = 1, 2, \dots, r$.

Since $p$ is the length of a longest bridge of $C$, by Lemma 1, we must have that

$$h(i) \leq p - 1, \quad \text{for any } i. \tag{4.1}$$

And

$$d(w(i)) \leq h(i) + |M(i)|, \quad \text{for any } i. \tag{4.2}$$

**Proposition 1.** *We have $M(i) \cap \{v_{i-h(i)}, \dots, v_{i+h(i)}\} = \emptyset$, for any $i$.*

**Proof.** Otherwise, let $v_j \in M(i)$ and $i - h(i) \leq j \leq i - 1$. Then $v_j{*}{*}{*}v_i C v_j$ would be a cycle longer than $C$. A similar argument works if $i + 1 \leq j \leq i + h(i)$. $\square$

**Proposition 2.** *We have $t \geq p$ and $r - t \geq p$.*

**Proof.** If $t \leq p - 1$, the cycle $v_r B v_t C v_r$ is longer than $C$. A similar contradiction arises when $r - t < p$. $\square$

**Proposition 3.** *We have $m \geq p + 3$.*

**Proof.** If $m \leq p + 2$, then $r \leq m + p - 3 \leq 2p - 1$. It then follows that either $t \leq p - 1$ or $r - t \leq p - 1$, both of which contradict Proposition 2. $\square$

**Definition.** The pair $(i, j)$ is called a *summable pair* on $C$ if $v_i$ and ·· are not joined by a $C$-path (which implies that $(w(i), w(j)) \notin E(G)$) and either $M(i) \cap M^{+1}(j) = \emptyset$ or $M(j) \cap M^{+1}(i) = \emptyset$ on any interval of $C \setminus \{v_i, v_j\}$.

During the proof, the basic method will be to get a summable pair $(i, j)$ and to check the sum of $d(w(i))$ and $d(w(j))$. So we need some propositions about summable pairs and the sums of the appropriate degrees.

**Proposition 4.** *The pairs* $(1, t+1)$ *and* $(t-1, r-1)$ *are summable.*

**Proof.** Obviously, $v_1 \notin N(t+1)$. Otherwise, the cycle $v_1 C v_t B v_r \bar{C} v_{t+1}{}^{***} v_1$ would be longer than $C$.

Moreover,

$$M(1) \cap M^{+1}(t+1) = \emptyset \quad \text{in } \{v_2, \ldots, v_t\}$$

and

$$M(t+1) \cap M^{+1}(1) = \emptyset \quad \text{in } \{v_{r+2}, \ldots, v_r\}.$$

Otherwise, without loss of generality, let $v_i \in M(1) \cap M^{+1}(t+1)$, $2 \leq i \leq t+1$. Then the cycle $v_1 C v_{i-1}{}^{***} v_{t+1} C v_r B v_t \bar{C} v_i{}^{***} v_1$ would be longer than $C$.

The pair $(t-1, r-1)$ is symmetric to $(1, t+1)$. □

**Proposition 5.** *Let* $\{J_\mu \mid \mu \in I\}$ *be a collection of pairwise vertex-disjoint intervals of* $C \setminus \{v_i, v_j\}$, $(i, j)$ *be a summable pair on* $C$, *and* $M(i) \cup M(j) \subseteq \bigcup_{\mu \in I} J_\mu$. *Let*

$$I' = \{\mu \in I \mid M(i) \cap J_\mu \neq \emptyset \quad \text{and} \quad M(j) \cap J_\mu \neq \emptyset\}$$

*and* $J = C \setminus [(\bigcup_{\mu \in I} J_\mu) \cup \{v_i, v_j\}]$. *Then* $|J| \leq h(i) + h(j) + p - 5 + |I'| \leq h(i) + h(j) + p - 5 + |I|$.

**Proof.** Since $w(i)$ and $w(j)$ are non-adjacent, $m \leq d(w(i)) + d(w(j))$ by the hypotheses of Theorem 1. By (4.2), it follows that

$$\begin{aligned} m &\leq h(i) + h(j) + |M(i)| + |M(j)| \\ &= h(i) + h(j) + \sum_{\mu \in I} [|J_\mu \cap M(i)| + |J_\mu \cap M(j)|] \\ &\leq h(i) + h(j) + \sum_{\mu \in I'} [|J_\mu| + 1] + \sum_{\mu \in I \setminus I'} |J_\mu| \quad \text{(by Lemma 3)} \\ &= h(i) + h(j) + \left| \bigcup_{\mu \in I} J_\mu \right| + |I'|. \end{aligned}$$

Since $r \leq m + p - 3$ and $|J| + |\bigcup_{\mu \in I} J_\mu| = r - 2$, $|J| \leq p - 5 + h(i) + h(j) + |I'|$. □

The following proposition is the main result of this section. It is a very important part of the proof of the theorem.

**Proposition 6.** *We have*

$$M(1) \subseteq \{v_{r-1}, v_r, v_{2+h(1)}, \ldots, v_t\},$$

$$M(t-1) \subseteq \{v_r, v_1, \ldots, v_{t-2-h(t-1)}, v_t, v_{t+1}\},$$

$$M(t+1) \subseteq \{v_{t-1}, v_t, v_{t+2+h(t+1)}, \ldots, v_r\} \quad \text{and}$$

$$M(r-1) \subseteq \{v_t, \ldots, v_{r-2-h(r-1)}, v_r, v_1\}.$$

*That is, $M(1)$ does not intersect with $\{v_{t+1}, \ldots, v_{r-2}\}$, and so on.*

**Proof.** Without loss of generality, we may consider $M(t+1)$ and assume that $M(t+1) \cap \{v_1, \ldots, v_{t-2}\} \neq \emptyset$. Choose $v_k$ to be the vertex in this intersection with $k$ as large as possible.

*I. Case 1: $M(t+1) \cap \{v_1, \ldots, v_{1+h(1)}\} = \emptyset$.*

(i) We claim that if $2+h(1) \leq i < j \leq t$, it is impossible that

$$v_i \in M(t+1) \quad \text{and} \quad v_j \in N(1).$$

Prove this claim by contradiction, so let

$$v_i \in M(t+1) \quad \text{and} \quad v_j \in N(1)$$

and choose $j-i$ as small as possible.

Since the cycle $v_1Cv_i{}^{***}v_{t+1}Cv_rBv_t\bar{C}v_j{}^{***}v_1$ is not longer than $C$, $\{v_{i+1}, \ldots, v_{j-1}\}$ must contain at least $p-1+h(t+1)$ vertices. This follows because the $C$-path $v_i{}^{***}v_{t+1}$ contains at least $h(t+1)$ vertices not in $C$ and $v_rBv_t$ contains $p-1$ vertices not in $C$.

Let

$$J_1 = \{v_{2+h(1)}, \ldots, v_i\}, \qquad J_2 = \{v_j, \ldots, v_t\} \quad \text{and} \quad J_3 = \{v_{t+2}, \ldots, v_r\}.$$

Here,

$$M(1) \cup M(t+1) \subseteq J_1 \cup J_2 \cup J_3 \quad \text{and} \quad I = \{1, 2, 3\}.$$

Let

$$J = \{v_2, \ldots, v_{1+h(1)}, v_{i+1}, \ldots, v_{j-1}\}$$

when $h(1) > 0$, or

$$J = \{v_{i+1}, \ldots, v_{j-1}\}$$

when $h(1) = 0$, which contains at least $h(1) + h(t+1) + p - 1$ vertices. This is a contradiction of Proposition 5.

(ii) By (i) and the assumption of Case 1, $v_{t-1} \notin N(1)$. Hence, $w(1)$ and $w(t-1)$ are a pair of non-adjacent vertices.

We shall consider this pair of vertices. First of all, we wish to show that $(1, t-1)$ is a summable pair on $C$.

Assume that $v_i \in M(1) \cap M^{+1}(t-1)$. If $t \leq i \leq r$, then the cycle $v_1 C v_{t-1}{}^{***} v_{i-1} \bar{C} v_t B v_r \bar{C} v_i{}^{***} v_1$ would be longer than $C$. If $2 \leq i \leq t-2$, then $i \leq k$ by (i). The fact that the cycle $v_1 C v_{i-1}{}^{***} v_{t-1} v_t B v_r \bar{C} v_{t+1}{}^{***} v_k \bar{C} v_i{}^{***} v_1$ is not longer than $C$ implies that

$$J = \{v_{k+1}, \ldots, v_{t-2}\}$$

must contain at least $p - 1 + h(1) + h(t-1) + h(t+1)$ vertices and $J$ does not intersect with $M(1)$ or $M(t+1)$ by the choice of $k$. Consider the summable pair $(1, t+1)$. Let

$$J_1 = \{v_2, \ldots, v_k\}, \qquad J_2 = \{v_{t-1}, v_t\} \quad \text{and} \quad J_3 = \{v_{t+2}, \ldots, v_r\}.$$

Here

$$M(1) \cup M(t+1) \subseteq J_1 \cup J_2 \cup J_3, \qquad I = \{1, 2, 3\}$$

which leads to a contradiction of Proposition 5. Thus $(1, t-1)$ is a summable pair.

(iii) If $1 \leq i < j \leq t-1$, then it is impossible that

$$v_i \in M(t-1) \quad \text{and} \quad v_j \in M(1).$$

We prove this claim by contradiction. Choose $j - i$ as small as possible. (The proof of this claim is quite similar to parts of ii.)

By (i), $j \leq k$ and by the choice of $k$,

$$J = \{v_{i+1}, \ldots, v_{j-1}, v_{k+1}, \ldots, v_{t-2}\}$$

will not intersect with $M(1)$ and $M(t+1)$. Since the cycle

$$v_1 C v_i{}^{***} v_{t-1} v_t B v_r \bar{C} v_{t+1}{}^{***} v_k \bar{C} v_j{}^{***} v_1$$

is not longer than $C$, $J$ must contain at least $p - 1 + h(1) + h(t-1) + h(t+1)$ vertices.

But consider the summable pair $(1, t+1)$. Let

$$J_1 = \{v_2, \ldots, v_i\}, \qquad J_2 = \{v_j, \ldots, v_k\},$$

$$J_3 = \{v_{t-1}, v_t\} \quad \text{and} \quad J_4 = \{v_{t+2}, \ldots, v_r\}.$$

Here, $I = \{1, 2, 3, 4\}$ and $I' \subseteq \{1, 2, 4\}$ because $M(1) \cap J_3 = \emptyset$ by (i) and (ii). This leads to a contradiction of Proposition 5.

(iv) If $t \leq i < j \leq r$, then it is impossible that

$$v_i \in M(1) \quad \text{and} \quad v_j \in M(t-1).$$

We prove this claim by contradiction. Choose $j - i$ as small as possible. Then

$$J = \{v_{i+1}, \ldots, v_{j-1}\}$$

would not intersect with $M(1)$ and $M(t-1)$. Since the cycle

$$v_1 C v_{t-1}{}^{***} v_j C v_r B v_t C v_i{}^{***} v_1$$

is not longer than $C$, $J$ must contain at least $p - 1 + h(1) + h(t-1)$ vertices. Now consider the summable pair $(1, t-1)$. Let

$$J_1 = \{v_2, \ldots, v_{t-2}\}, \qquad J_2 = \{v_t, \ldots, v_i\} \quad \text{and} \quad J_3 = \{v_j, \ldots, v_r\}.$$

Here, $I = \{1, 2, 3\}$ and again it leads to a contradiction of Proposition 5.

(v) By (iii) and (iv), there are integers $a$ and $b$ such that $2 \leqslant a \leqslant t-2$, $t \leqslant b \leqslant r$,

$$M(1) \subseteq \{v_b, \ldots, v_r, v_2 v_a\} \setminus \{v_{r+1-h(1)}, \ldots, v_{1+h(1)}\},$$
$$M(t-1) \subseteq \{v_a, \ldots, v_{t-2}, v_t, \ldots, v_b\} \setminus \{v_{t-1-h(t-1)}, \ldots, v_{t-1+h(t-1)}\}.$$

We now have enough information to get the final contradiction for this case.

Choose $i$ and $j$ such that $t \leqslant i < j \leqslant r$, $v_i \in M(t-1) \cup \{v_t\}$, $v_j \in M(1) \cup \{v_r\}$, and $j - i$ is as small as possible. Obviously, $i \leqslant b \leqslant j$. Since the cycle $v_1 C v_{t-1}{}^{**} v_i \bar{C} v_t B v_r \bar{C} v_j{}^{***} v_1$ is not longer than $C$,

$$|\{v_{i+1}, \ldots, v_{j-1}\}| \geqslant p - 1 + (|v_1{}^{***} v_j| - 2) + (|v_{t-1}{}^{***} v_i| - 2).$$

($\alpha$). If $v_j \in M(1)$ and $v_i \in M(t-1)$, let

$$J = \{v_{i+1}, \ldots, v_{j-1}\}.$$

Then $|J| \geqslant p - 1 + h(1) + h(t-1)$. If we let

$$J_1 = \{v_2, \ldots, v_{t-2}\}, \qquad J_2 = \{v_t, \ldots, v_i\}, \qquad J_3 = \{v_j, \ldots, v_r\} \quad \text{and} \quad I = \{1, 2, 3\},$$

we again contradict Proposition 5.

($\beta$) If $v_i \notin M(t-1)$ and $v_j \in M(1)$, that is, $v_i = v_t$, then

$$M(t-1) \subseteq \{v_b, v_a, \ldots, v_{t-2-h(t-1)}\}.$$

(By Lemma 2, $|M(t-1)| \geqslant 2$ which implies that $t - 2 - h(t-1) \geqslant a$.) Let

$$J = \{v_{t-1-h(t-1)}, \ldots, v_{t-2}, v_t, \ldots, v_{j-1}\}$$

when $h(t-1) > 0$, or

$$J = \{v_i, \ldots, v_{j-1}\}$$

when $h(t-1) = 0$. Note that $|J| \geqslant p + h(1) + h(t-1)$ because $|\{v_{t+1}, \ldots, v_{j-1}\}| \geqslant p - 1 + h(1)$. If

$$J_1 = \{v_2, \ldots, v_{t-2-h(t-1)}\}, \qquad J_2 = \{v_j, \ldots, v_r\} \quad \text{and} \quad I = \{1, 2\},$$

we again contradict Proposition 5.

Via a symmetric argument, a contradiction follows for $v_i \in M(t-1)$ and $v_j \notin M(1)$.

($\gamma$). So we consider $v_i \notin M(t-1)$ and $v_j \notin M(1)$, that is, $v_i = v_t$ and $v_j = v_r$. Let $J = K_1 \cup K_2 \cup K_3$ where $K_1 = \{v_2, \ldots, v_{1+h(1)}\}$ when $h(1) > 0$ or the empty set when $h(1) = 0$, $K_2 = \{v_{t-1-h(t-1)}, \ldots, v_{t-2}\}$ when $h(t-1) > 0$ or the empty set when $h(t-1)=0$ and $K_3=\{v_{t+1},\ldots,v_{r-1}\}$. By Proposition 2, $|J|\geq p-1+h(1)+h(t-1)$. Since $|M(1)|\geq 2$ and $|M(t-1)|\geq 2$, $2+h(1)\leq a\leq t-2-h(t-1)$. Letting

$$J_1=\{v_{2+h(1)},\ldots,v_{t-2-h(t-1)}\},\qquad J_2=\{v_t\}\quad\text{and}\quad J_3=\{v_r\}$$

with $I=\{1,2,3\}$, we again contradict Proposition 5.

The first case of Proposition 6 has now been proved.

*II. Case 2.* $M(t+1)\cap\{v_1,\ldots,v_{1+h(1)}\}\neq\emptyset$.

Let $v_i$ be a vertex of this intersection.

(i) Since the cycle $v_iCv_tBv_r\bar{C}v_{t+1}{}^{***}v_i$ is not longer than $C$, $i\geq h(t+1)+p$. By (4.1), $h(1)\leq p-1$. So $p\geq 1+h(1)\geq i\geq h(t+1)+p$ implies that $h(t+1)=0$, $h(1)=p-1$ and $v_i=v_{1+h(1)}\in M(t+1)$.

(ii) Since Case 1 of Proposition 6 has been proved, we have a symmetric result for $M(1)$ which is

$$M(1)\cap\{v_{t+1},\ldots,v_{r-2}\}=\emptyset\quad\text{if}\quad M(1)\cap\{v_{t+1},\ldots,v_{t+1-h(t+1)}\}=\emptyset.$$

By (i), $h(t+1)=0$ and we have that

$$\{v_{t+1},\ldots,v_{t+1-h(t+1)}\}=\{v_{t+1}\}$$

with which $M(1)$ does not intersect. Hence,

$$M(1)\cap\{v_{t+1},\ldots,v_{r-2}\}=\emptyset.$$

(iii) Since $h(1)=p-1\geq 1$ and $v_r\notin M(1)$, $M(1)\cap\{v_{2+h(1)},\ldots,v_t\}\neq\emptyset$ because $|M(1)|\geq 2$ and by Proposition 1.

Since $v_i\in M(t+1)\cap\{v_1,\ldots,v_{1+h(1)}\}$ and $M(1)\cap\{v_{2+h(1)},\ldots,v_t\}\neq\emptyset$, there are integers $k$ and $j$, with $j-k$ as small as possible, such that $2\leq k<j\leq t$, $v_j\in M(1)$ and $v_k\in M(t+1)$. Let $J=\{v_{k+1},\ldots,v_{j-1}\}$ with which neither $M(1)$ nor $M(t+1)$ intersects or else $j-k$ could be chosen smaller. Since the cycle $v_1Cv_k{}^{***}v_{t+1}Cv_rBv_t\bar{C}v_j{}^{***}v_1$ is not longer than $C$, $J$ contains at least $p-1+h(1)+h(t+1)$ vertices. On the other hand, letting

$$J_1=\{v_2,\ldots,v_k\},\qquad J_2=\{v_j,\ldots,v_t\}\quad\text{and}\quad J_3=\{v_{t+2},\ldots,v_r\},$$

$M(1)\cup M(t+1)\subseteq J_1\cup J_2\cup J_3$. With $I=\{1,2,3\}$, Proposition 5 is contradicted and the proof of Proposition 6 is complete. □

**Proposition 7.** *We have*

$$M(1)\cap\{v_2,\ldots,v_t\}\neq\emptyset,\qquad M(t-1)\cap\{v_r,v_1,\ldots,v_{t-2}\}\neq\emptyset,$$
$$M(t+1)\cap\{v_{t+2},\ldots,v_r\}\neq\emptyset\quad\textit{and}\quad M(r-1)\cap\{v_t,\ldots,v_{r-2}\}\neq\emptyset.$$

**Proof.** Without loss of generality, we consider $M(1)$. If $h(1)=0$, $v_2\in M(1)$. If $h(1)\geq 1$, $v_r\notin M(1)$. Since $M(1)\subseteq\{v_{r-1},v_r,v_2,\ldots,v_t\}$, by the previous proposition, and $|M(1)|\geq 2$, $M(1)\cap\{v_2,\ldots,v_t\}\neq\emptyset$. □

**Proposition 8.** *We have $t \geq 3$ and $r-t \geq 3$.*

**Proof.** If $t \leq 2$, $p=2$ and $t=2$ by Proposition 2. By Proposition 6, $M(1) \subseteq \{v_{r-1}, v_r, v_2, \ldots, v_t\} = \{v_{r-1}, v_r, v_2\}$ and $v_3 = v_{t+1} \notin M(1)$. Since $v_1 = v_{t-1}$, $M(1) = M(t-1) \subseteq \{v_r, \ldots, v_{t-2}, v_t, v_{t+1}\} = \{v_r, v_2, v_3\}$ and $v_{r-1} \notin M(t-1) = M(1)$. So $M(1) = \{v_r, v_2\}$ because $|M(1)| \geq 2$. Now $h(1) = 0$, otherwise, $v_r, v_2 \notin M(1)$. But then $v_1$ is a vertex of degree two which contradicts the 3-connectivity of the graph. Thus $t \geq 3$ and by symmetry $r-t \geq 3$. □

Now we can get into the main part of the theorem's proof. First, we define a $Y$-bridge of a longest cycle $C$.

**Definition.** If $D$ is a bridge of $C$ and vertices $v_r$, $v_{t'}$, $v_{t''}$ of $C$ are distinct attachments of $D$ such that there are two $C$-paths $v_r{*}{*}{*}v_{t'}$ and $v_r{*}{*}{*}v_{t''}$ of length $p$ contained in $D$, then $D$ is called a *$Y$-bridge* of $C$.

We shall consider two cases in the proof, namely, with a $Y$-bridge (Part B) and without a $Y$-bridge (Part C).

*Part B. Case one. C has a Y-bridge*

Propositions 5 and 6 will be the keys to the proof in this case.

Let $B' = v_r{*}{*}{*}v_{t'}$ and $B'' = v_r{*}{*}{*}v_{t''}$ be two $C$-paths of length $p$ contained in a $Y$-bridge of $C$, $t'' > t'$. Obviously, $t'' \geq t'+2$. The index $t$ in all propositions of Part $A$ can be replaced by both $t'$ and $t''$.

*I. We claim $t''-t' \leq p-2$, that is, $1 \leq |\{v_{t'+1}, \ldots, v_{t''-1}\}| \leq p-3$.*

Let us consider the summable pair $(r-1, t'-1)$. Let

$$K_1 = \{v_{t'-1-h(t'-1)}, \ldots, t_{t'-2}\}$$

when $h(t'-1) > 0$ or the empty set when $h(t'-1) = 0$,

$$K_2 = \{v_{r-1-h(r-1)}, \ldots, v_{r-2}\}$$

when $h(r-1) > 0$ or the empty set when $h(r-1) = 0$, and

$$K_3 = \{v_{t'+2}, \ldots, v_{t''-1}\}$$

when $t'' \geq t'+3$ or the empty set when $t'' = t'+2$. Let

$$J = K_1 \cup K_2 \cup K_3.$$

By (4.1) and Proposition 2, $r - t'' \geq p \geq h(r-1)+1$ implies that $r-1-h(r-1) \cdot \geq t''$. Hence, $K_1$, $K_2$ and $K_3$ are pairwise disjoint. Let

$$J_1 = \{v_r, \ldots, v_{t'-2-h(t'-1)}\}, \qquad J_2 = \{v_{t'}, v_{t'+1}\} \quad \text{and}$$
$$J_3 = \{v_{t''}, \ldots, v_{r-2-h(r-1)}\}.$$

By Proposition 7, $M(t'-1) \cap \{v_r, v_1, \ldots, v_{t'-2-h(t'-1)}\} \neq \emptyset$ and $t'-2-h(t'-1) \geq 0$.

When $t'-2-h(t'-1)\geqslant 1$, $\{v_r, v_1\}\subseteq J_1$. Hence, by Proposition 6, $M(r-1)\subseteq J_1\cup J_3$ and $M(t'-1)\subseteq J_1\cup J_2$, $I=\{1,2,3\}$ and $I'\subseteq\{1\}$.

When $t'-2-h(t'-1)=0$, $v_r$ is the single vertex in $M(t'-1)\cap\{v_r,\ldots,v_{t'-2}\}$ by Proposition 7. If $v_1=v_{t'-1-h(t'-1)}\in M(r-1)$, then $v_1Cv_{t'-1}{}^{***}v_rBv_{t'}Cv_{r-1}{}^{***}v_1$ would be a cycle longer than $C$. Hence, $v_1\notin M(r-1)$. So we still have that $M(r-1)\subseteq J_1\cup J_3$, $M(t'-1)\subseteq J_1\cup J_2$, $I=\{1,2,3\}$ and $I'\subseteq\{1\}$. By Proposition 5,

$$|K_1|+|K_2|+|K_3|=|J|\leqslant h(r-1)+h(t'-1)+p-5+|I'|.$$

Since $|K_1|=h(t'+1)$, $|K_2|=h(r-1)$ and $|K_3|=t''-t'-2$, $t''-t'-2=|K_3|\leqslant p-4$.

*II. We claim $v_{r-1}$ and $v_{t'+1}$ are not joined by a C-path.*

Otherwise, the cycle $v_rCv_{t'+1}{}^{***}v_{r-1}\bar{C}v_{t''}B''v_r$ would be longer than $C$ because $|\{v_{t'+2},\ldots,v_{t''-1}\}|\leqslant p-4$. Hence, $(w(r-1), w(t'+1))\notin E(G)$.

*III. We claim $(r-1, t'+1)$ is summable.*

By Proposition 6 and II, we only need to consider the intervals $\{v_{t'+2},\ldots,v_{r-2}\}$ and $\{v_r, v_1\}$.

If $v_i\in M^{+1}(r-1)\cap M(t'+1)$, $t'+2\leqslant i\leqslant r-2$. (Note that $v_{i-1}\in M(r-1)$ implies $i-1\geqslant t''$ by Proposition 6). Then the cycle

$$v_rCv_{t'+1}{}^{***}v_iCv_{r-1}{}^{***}v_{i-1}\bar{C}v_{t''}B''v_r$$

would be longer than $C$ because of *I*.

If $v_1\in M^{+1}(t'+1)\cap M(r-1)$, then the cycle $v_{r-1}{}^{***}v_1Cv_{t'}B'v_r{}^{***}v_{t'+1}Cv_{r-1}$ would be longer than $C$. Finally, $v_r\notin M^{+1}(t'+1)$ by *II*.

*IV. If $t''\leqslant i<j\leqslant r-1$, it is impossible that $v_j\in M(t'+1)$ and $v_i\in M(r-1)$.*

Otherwise, choose $j-i$ as small as possible. Since the cycle $v_rCv_{t'+1}{}^{***}v_jCv_{r-1}{}^{***}v_i\bar{C}v_{t''}B''v_r$ is not longer than $C$, $\{v_{i+1},\ldots,v_{j-1}\}\cup[\{v_{t'+1},\ldots,v_{t''-1}\}\setminus\{v_{t'+1}\}]$ must contain at least $p-1+h(t'+1)+h(r-1)$ vertices. By *I*, $\{v_{i+1},\ldots,v_{j-1}\}$ contains at least $h(t'+1)+h(r-1)+3$ vertices. Let

$$J_1=\{v_r, v_1\},\qquad J_2=\{v_{t'-1}, v_{t'}\}$$

$$J_3=\{v_{t'+2},\ldots,v_i\},\qquad J_4=\{v_j,\ldots,v_{r-2}\},$$

(note, by Proposition 8, $J_1\cap J_2=\emptyset$),

$$J=\{v_2,\ldots,v_{t'-2}, v_{i+1},\ldots,v_{j-1}\}\quad\text{and}\quad I=\{1,2,3,4\}.$$

From above

$$|J|\geqslant(t'-3)+[h(t'+1)+h(r-1)+3]$$

and by Proposition 2,

$$|J|\geqslant p+h(t'+1)+h(r-1).$$

This contradicts Proposition 5.

*V*. By IV and Propositions 6 and 7, there is an integer $k$ such that

$$t'+2+h(t'+1)\leqslant k\leqslant r-2-h(r-1)$$

with

$$M(t'+1)\subseteq\{v_{t'-1}, v_{t'}, v_{t'+2+h(t'+1)}, \ldots, v_k\} \quad \text{and}$$
$$M(r-1)\subseteq\{v_k, \ldots, v_{r-2-h(r-1)}, v_r, v_1\}.$$

Let

$$J_1=\{v_r, v_1\}, \qquad J_2=\{v_{t'-1}, v_{t'}\},$$
$$J_3=\{v_{t'+2+h(t'+1)}, \ldots, v_{r-2-h(r-1)}\}$$

where $I=\{1, 2, 3\}$ and $I'\subseteq\{3\}$. Let $J=K_1\cup K_2\cup K_3$ where

$$K_1=\{v_2, \ldots, v_{t'-2}\}, \qquad K_2=\{v_{t'+2}, \ldots, v_{t'+1+h(t'+1)}\}$$

when $h(t'+1)>0$ or the empty set when $h(t'+1)=0$, and

$$K_3=\{v_{r-1-h(r-1)}, \ldots, v_{r-2}\}$$

when $h(r-1)>0$ or the empty set when $h(r-1)=0$. Here,

$$|J|=(t'-3)+h(t'+1)+h(r-1)\geqslant p-3+h(t'+1)+h(r-1)$$

which contradicts Proposition 5.

Case One now has been proved.

*Part C. Case Two, C has no Y-bridge*

Let $B=v_ru_1\ldots u_{p-1}v_t$ be a longest $C$-path of $C$.

*I*. Since $G$ is 3-connected, $G\setminus\{v_r, v_t\}$ is still connected. Let $\Phi=\{q \mid$ there is a $(B\cup C)$-path $P=u_q\ldots v_{q'}$ joining $B$ and $C$ in $G\setminus\{v_r, v_t\}\}$. Obviously, $\Phi\subseteq\{2, \ldots, p-2\}$. Otherwise, there would be a $Y$-bridge of $C$.

*II*. In the proof of the previous case, we paid more attention to the cycle $C$. In the proof of this case, we will pay more attention to the bridge $B$.

For the sake of convenience, denote $w(u_i, B\cup C)$, $h(u_i, B\cup C)$ and $M(u_i, B\cup C)\cap V(B)$ by $w_i$, $h_i$ and $M_i$, respectively. Here, we have that

$$d(w_i)\leqslant h_i+d_C(w_i)+|M_i|$$

by Lemma 1.

*III*. *Subcase 1. Assume there is a $q$ in $\Phi$ such that*

$$d_C(w_1)+d_C(w_{q+1})\leqslant 3 \quad \text{or} \quad d_C(w_{p-1})+d_C(w_{q-1})\leqslant 3.$$

Without loss of generality, let $q\in\Phi$ and $d_C(w_1)+d_C(w_{q+1})\leqslant 3$. And let $P=v_{q'}{}^{***}u_q$ be a $(B\cup C)$-path joining $u_q$ and $v_{q'}$ $(q'\neq r, t)$.

The pair of vertices $w_1$, $w_{q+1}$ have some properties similar to a *summable pair* on $B$ which was considered in Case one. This similarlity will be considered and exploited in this subcase.

(i) We claim that there is no $(B \cup C)$-path joining $v_{q'}$ and $u_1$. Otherwise, either the $C$-path $v_{q'}{}^{***}u_1Bu_{p-1}v_t$ would be longer than $B$ or $C$ would have a $Y$-bridge.

We claim that there is no $(B \cup C)$-path joining $u_1$ and $u_{q+1}$. Otherwise, the $(B \cup C)$-path $u_1{}^{***}u_{q+1}$ would not intersect with $P$, and then either the $C$-path $v_{q'}Pu_q\bar{B}u_1{}^{***}u_{q+1}Bu_{p-1}v_t$ would be longer than $B$ or $C$ would have a $Y$-bridge. Hence, $w_1$ and $w_{q+1}$ are a pair of non-adjacent vertices.

We claim that $u_q$ and $u_{q+1}$ is not joined by a $(B \cup C)$-path of length at least 2. Otherwise, $B$ would not be a longest $C$-path.

Hence, if $u_i \in N(u_1, B \cup C)$ and $u_j \in N(u_q, B \cup C)$, then the three $(B \cup C)$-paths $u_i{}^{***}u_1$, $u_j{}^{***}u_{q+1}$ and $P$ are internally disjoint.

(ii) *We claim* $M_1 \cap M_{q+1}^{+1} = \emptyset$ *in* $\{u_2, \ldots, u_q\}$.

Otherwise, let $u_i$ be a vertex in this intersection. By *i*,

$$v_{q'}Pu_q\bar{B}u_i{}^{***}u_1Bu_{i-1}{}^{***}u_{q+1}Bu_{p-1}v_t$$

is a $C$-path. This path is either longer than $B$ or else there is a $Y$-bridge of $C$ both of which are contradictions.

Similarly, $M_{q+1} \cap M_1^{+1} = \emptyset$ in $\{u_{q+2}, \ldots, u_{p-1}\}$. Now we can use Lemma 3 on $M_1$ and $M_{q+1}$.

(iii) Let's get a general inequality similar to Proposition 5 for $M_1$ and $M_{q+1}$. Let

$$M_1 \cup M_{q+1} \subseteq \bigcup_{\mu \in I} J_\mu$$

where $\{J_\mu \mid \mu \in I\}$ is a collection of pairwise-disjoint subintervals of $\{u_2, \ldots, u_q\}$ or $\{u_{q+2}, \ldots, u_{p-1}\}$. By Lemma 3,

$$|M_1 \cap J_\mu| + |M_{q+1} \cap J_\mu| \leq |J_\mu| + 1$$

for any $\mu \in I$. So

$$|M_1| + |M_{q+1}| \leq \sum_{\mu \in I} |J_\mu| + |I|.$$

Let

$$J = [\{u_2, \ldots, u_q\} \cup \{u_{q+2}, \ldots, u_{p-1}\}] \Big/ \left[\bigcup_{\mu \in I} J_\mu\right].$$

We have that

$$|M_1| + |M_{q+1}| \leq p - 3 - |J| + |I|.$$

Hence, by Proposition 3,

$$\begin{aligned} p + 3 &\leq m \\ &\leq d(w_1) + d(w_{q+1}) \\ &\leq h_1 + h_{q+1} + d_C(w_1) + d_C(w_{q+1}) + |M_1| + |M_{q+1}| \\ &\leq h_1 + h_{q+1} + 3 + (p - 3 - |J| + |I|), \end{aligned}$$

that is,

$$|J| \leq h_1 + h_{q+1} - 3 + |I|. \tag{4.3}$$

This inequality will be used frequently in this subcase.

(iv) *If $2 \leq i < j \leq q$, it is impossible that $u_i \in M_{q+1}$ and $u_j \in M_1$.*

If not, choose $j - i$ as small as possible. By i,

$$v_{q'} P u_q \bar{B} u_j {}^{***} u_1 B u_i {}^{***} u_{q+1} B u_{p-1} v_t$$

is a $C$-path and is either longer than $B$ or produces a $Y$-bridge if

$$J = \{u_{i+1}, \ldots, u_{j-1}\}$$

contains fewer than $h_1 + h_{q+1} + 1$ vertices. Hence, let

$$J_1 = \{u_2, \ldots, u_i\}, \qquad J_2 = \{u_j, \ldots, u_q\} \quad \text{and} \quad J_3 = \{u_{q+2}, \ldots, u_{p-1}\}$$

with $I = \{1, 2, 3\}$ which leads to a contradiction of (4.3) of iii.

(v) *If $q + 2 \leq i < j \leq p - 1$, it is impossible that $u_j \in M_{q+1}$ and $u_i \in M_1$.*

If not, choose $j - i$ as small as possible. As in the previous cases, $v_{q'} P u_q \bar{B} u_1 {}^{***} u_i \bar{B} u_{q+1} {}^{***} u_j B u_{p-1} v_t$ is a $C$-path. This path is either longer than $B$ or $C$ has a $Y$-bridge unless

$$J = \{u_{i+1}, \ldots, u_{j-1}\}$$

contains at least $h_1 + h_{q+1} + 1$ vertices. Hence, letting

$$J_1 = \{u_2, \ldots, u_q\}, \qquad J_2 = \{u_{q+2}, \ldots, u_i\}, \qquad J_3 = \{u_j, \ldots, u_{p-1}\}$$

and $I = \{1, 2, 3\}$, we contradict (4.3) of iii.

(vi) By iv and v, there are integers $a$ and $b$ such that, $2 \leq a \leq q$, $q + 2 \leq b \leq p - 1$, $M_1 \subseteq \{u_2, \ldots, u_a, u_b, \ldots, u_{p-1}\}$ and $M_{q+1} \subseteq \{u_a, \ldots, u_b\}$. Because the maximum length of a $C$-path is $p$, we must have that

$$M_1 \subseteq \{u_2, \ldots, u_a, u_b, \ldots, u_{p-1}\} \setminus \{u_1, \ldots, u_{1+h_1}\}$$

and

$$M_{q+1} \subseteq \{u_a, \ldots, u_b\} \setminus \{u_{q+1-h_{q+1}}, \ldots, u_{q+1+h_{q+1}}\}.$$

(vii) *We claim $M_1 \neq \emptyset$ and $M_{q+1} \neq \emptyset$.*

If $h_1 = 0$, $u_2 \in M_1$. If $h_1 \geq 1$, $C \cap M(u_1, B \cup C) \subseteq \{v_t\}$ since $B$ is a longest $C$-path and $C$ has no $Y$-bridges. So $|M_1| \geq 1$ because $|M(u_1, B \cup C)| \geq 2$ by Lemma 2.

If $M_{q+1} = \emptyset$, then $m(u_{q+1}, B \cup C)$ contains at least two vertices of $C$. Let

$$i = \min\{\mu \geq q + 2 \mid u_\mu \in M_1\}$$

when $M_1 \cap \{u_{q+2}, \ldots, u_{p-1}\} \neq \emptyset$, or

$$i = \min\{\mu \mid u_\mu \in M_1\}$$

when $M_1 \cap \{u_{q+2}, \ldots, u_{p-1}\} = \emptyset$.

When $i \geq q+2$, let $v_s \in M(u_{q+1}, B \cup C) \setminus \{v_t\}$. Since the $C$-path $v_s{}^{***}u_{q+1}\bar{B}u_1{}^{***}u_iBu_{p-1}v_t$ is not longer than $B$,

$$J = V(B) \setminus [\{u_1, \ldots, u_{q+1}\} \cup \{u_i, \ldots, u_{p-1}\}]$$

must contain at least $h_{q+1} + h_1$ vertices. Letting

$$J_1 = \{u_2, \ldots, u_q\}, \qquad J_2 = \{u_i, \ldots, u_{p-1}\}$$

and $I = \{1, 2\}$ we contradict (4.3) of iii.

When $i \leq q$, let $v_s \in M(u_{q+1}, B \cup C) \setminus \{v_r\}$. Since the $C$-path

$$v_s{}^{***}u_{q+1}\bar{B}u_i{}^{***}u_1v_r$$

is not longer than $B$,

$$J = \{u_2, \ldots, u_{i-1}, u_{q+2}, \ldots, u_{p-1}\}$$

must contain at least $h_1 + h_{q+1}$ vertices. (Note that $i \geq 2$ because $M_1 \neq \emptyset$.) Letting

$$J_1 = \{u_i, \ldots, u_q\} \quad \text{and} \quad I = \{1\},$$

we again contradict (4.3) of iii.

(viii) Suppose that $M_1 \cap \{u_2, \ldots, u_q\} \neq \emptyset$.

If $M_{q+1} \cap \{u_2, \ldots, u_q\} \neq \emptyset$, let

$$J = \{u_1, \ldots, u_{1+h_1}, u_{q+1-h_{q+1}}, \ldots, u_{q+1}\} \setminus \{u_1, u_{q+1}\}$$

which contains $h_1 + h_{q+1}$ vertices. (By vi, $1 + h_1 < a < q + 1 - h_{q+1}$ because both $M_1$ and $M_{q+1}$ are not empty in $\{u_2, \ldots, u_q\}$.) Let

$$J_1 = \{u_{2+h_1}, \ldots, u_{q-h_{q+1}}\}, \qquad J_2 = \{u_{q+2}, \ldots, u_{p-1}\}$$

and $I = \{1, 2\}$. Since $M_1 \cup M_{q+1} \subseteq J_1 \cap J_2$, we have a contradiction of (4.3) of iii.

If $M_{q+1} \cap \{u_2, \ldots, u_q\} = \emptyset$, then by v and vi, $M_1$ and $M_{q+1}$ would not interesect with $\{u_{q+1}, \ldots, u_{q+1+h_{q+1}}\}$ and

$$M_{q+1} \subseteq \{u_{q+2+h_{q+1}}, \ldots, u_{p-1}\}.$$

(Note that $M_{q+1} \neq \emptyset$ implies that $p - 1 \geq q + 2 + h_{q+1}$.) Let

$$J = \{u_1, \ldots, u_{1+h_1}, u_{q+1}, \ldots, u_{q+1+h_{q+1}}\} \setminus \{u_1, u_{q+1}\},$$

$$J_1 = \{u_{2+h_1}, \ldots, u_q\}, \qquad J_2 = \{u_{q+2+h_{q+1}}, \ldots, u_{p-1}\}$$

and $I = \{1, 2\}$. Here, $M_1 \cup M_{q+1} \subseteq J_1 \cup J_2$ from which follows a contradiction of (4.3) of iii.

So we will assume that $M_1 \cap \{u_2, \ldots, u_q\} = \emptyset$, that is, $M \subseteq \{u_{q+2}, \ldots, u_{p-1}\}$.

(ix) *Let* $i = \min\{\mu \mid u_\mu \in M_{q+1}\}$ *and* $j = \max\{\mu \mid u_\mu \in M_1\}$.

Recall that $M_1 \notin \emptyset$, $M_{q+1} \neq \emptyset$ by vii. By vi and viii, $M_1 \cup M_{q+1} \subseteq \{u_i, \ldots, u_j\}$.

When $i \leq q$, let

$$J_1 = \{u_i, \ldots, u_q\}, \qquad J_2 = \{u_{q+2}, \ldots, u_j\}$$

and $I = \{1, 2\}$. Since the $C$-path $v_r u_1 {}^{***} u_j \bar{B} u_{q+1} {}^{***} u_i B u_q \bar{P} v_{q'}$ is not longer than $B$ and $C$ has no $Y$-bridge,

$$J = V(B) \backslash [J_1 \cup J_2 \cup \{u_1, u_{q+1}\}]$$

must contain at least $h_1 + h_{q+1} + 1$ vertices. This again contradicts (4.3) of iii.

When $i \geq q + 2$, then $M_{q+1} \cap \{u_2, \ldots, u_q\} = \emptyset$ and $i \geq q + 2 + h_{q+1}$ so that

$$J_1 = \{u_{q+2+h_{q+1}}, \ldots, u_j\},$$

contains all vertices of $M_1$ and $M_{q+1}$. Since the $C$-path $v_r u_1 {}^{***} u_j \bar{B} u_q \bar{P} v_{q'}$ is not longer than $B$ and $C$ has no $Y$-bridge, $\{u_2, \ldots, u_q, u_{j+1}, \ldots, u_{p-1}\}$ must contain at least $h_1 + 1$ vertices. Hence, $J = V(B) \backslash [J_1 \cup \{u_1, u_{q+1}\}]$ contains at least $h_1 + h_{q+1} + 1$ vertices. We again contradict (4.3) of iii.

This completes the proof of Subcase 1.

*IV. Subcase 2. We may assume*

$$d_C(w_1) + d_C(w_{q+1}) \geq 4 \quad \textit{and} \quad d_C(w_{p-1}) + d_C(w_{q-1}) \geq 4, \quad \textit{for any} \quad q \in \Phi.$$

(i) In order to avoid a $Y$-bridge,

$$V(C) \cap M(u_1, B \cup C) \subseteq \{v_t, v_r\} \quad \textit{and} \quad V(C) \cap M(u_{p-1}, B \cup C) \subseteq \{v_t, v_r\}.$$

Hence, $d_C(w_1)$, $d_C(w_{p-1}) \leq 2$, and therefore, $d_C(w_{q+1})$ and $d_C(w_{q-1}) \geq 2$ for any $q \in \Phi$.

(ii) If $h_1 \geq 1$, then $M(u_1, B \cup C) \subseteq V(B) \cup \{v_t\}$ in order to avoid a $C$-path of length greater than $p$ joining $v_r$ and $v_t$. So $d_C(w_1) \leq 1$. But we can choose $q$ as the greatest element of $\Phi$. Then

$$M(u_{q+1}, B \cup C) \cap [V(C) \backslash \{v_r, v_t\}] = \emptyset,$$

and hence, $d_C(w_{q+1}) \leq 2$. Thus $d_C(w_1) + d_C(w_{q+1}) \leq 3$ which contradicts the hypotheses of Subcase 2.

So we conclude that $h_1 = 0$ and symmetrically, that $h_{p-1} = 0$.

(iii) Choose $q$ as the greatest element in $\Phi$. Since $q + 1 \notin \Phi$,

$$M(u_q, B \cup C) \cap V(C) = M(u_{q+1}, B \cup C) \cap V(C) = \{v_r, v_t\}.$$

Let $v_{q'} \in N(u_1, B \cup C)$ such that $q' \neq r, t$. Since $B$ is a longest $C$-path, $u_q {}^{***} v_{q'}$ and $u_{q+1} {}^{***} v_r$ are disjoint $(B \cup C)$-paths.

(iv) *We claim* $(u_1, u_{p-1}) \notin E(G)$. Otherwise, the $C$-path

$$v_r {}^{***} u_{q+1} B u_{p-1} u_1 B u_q {}^{***} v_{q'}$$

either is longer than $B$ or has the same length as $B$ and $C$ would have a $Y$-bridge.

(v) Since $h_1 = h_{p-1} = 0$, $N_B(u_1) = M_1$ and $N_B(u_{p-1}) = M_{p-1}$. We have that

$$M_1 \cap M_{p-1}^{+1} \cap \{u_2, \ldots, u_{p-2}\} = \emptyset$$

If not, let $u_i$ be in this set. If $i \neq q + 1$, then without loss of generality assume $i \leq q$. The $C$-path $v_r {}^{***} u_{q+1} B u_{p-1} u_{i-1} \bar{B} u_1 u_i B u_q {}^{***} v_{q'}$ either is longer than $B$ or there is a $Y$-bridge of $C$. If $i = q+1$, then the $C$-path $v_i u_{p-1}\bar{B}u_{q+1}u_1Bu_q{}^{***}v_{q'}$ either is longer than $B$ or there is a $Y$-bridge of $C$.

(vi) By iv and v, the pair of vertices $u_1$, $u_{p-1}$ behaves similar to a "summable pair" on $B$. We have that

$$\begin{aligned} m &\leq d(u_1) + d(u_{p-1}) \\ &\leq d_C(u_1) + d_C(u_{p-1}) + |M_1| + |M_{p-1}| \\ &\leq 4 + (|\{u_2, \ldots, u_{p-2}\}| + 1) \quad \text{(by Lemma 3)} \\ &= p + 2, \end{aligned}$$

which contradicts Proposition 3.

This completes the proof of the theorem. □


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