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Note

Bounds on the number of complete subgraphs

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Abstract

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Let G be a graph with a clique number w. For $1 \leqslant j \leqslant w$, let k_j be the number of complete subgraphs on j nodes. We show that $k_{j+1} \leqslant \binom{w}{j+1}(k_j/\binom{w}{j})^{(j+1)/j}$. This is exact for complete balanced w-partite graphs and gives Turán' theorem when $j = 1$.

A corollary is $w \geqslant (8k_2^3 - 9k_3^2 + 3k_3\sqrt{16k_2^3 + 9k_3^2})/(4k_2^3 - 18k_3^2)$. This new bound on the clique number supercedes an earlier bound from Turán's theorem.

Let G be a simple graph. Let w (the *clique number*) be the size of the largest complete subgraph in G. For $1 \leqslant j \leqslant w$, let k_j be the number of complete subgraphs on j nodes (so G has k_1 nodes, k_2 edges, k_3 triangles, etc.). The main result of this paper is:

$$\left(\frac{k_1}{\binom{w}{1}}\right)^{1/1} \geqslant \left(\frac{k_2}{\binom{w}{2}}\right)^{1/2} \geqslant \left(\frac{k_3}{\binom{w}{3}}\right)^{1/3} \geqslant \cdots \geqslant \left(\frac{k_w}{\binom{w}{w}}\right)^{1/w}. \tag{1}$$

This is exact for a complete balanced w-partite graph.

The first inequality of (1) can be written as $k_2 \leqslant (w-1)k_1^2/(2w)$. This is a slightly weakened form of Turán's theorem [9]. The second inequality is a new bound on the number of triangles, k_3, in a graph with k_2 edges and clique number, w.

$$k_3 \leqslant \frac{\sqrt{2}(w-2)}{3\sqrt{w^2 - w}} k_2^{3/2}. \tag{2}$$

This bound is shown in Fig. 1. Fisher [4] proved (2) for $w = 3$.

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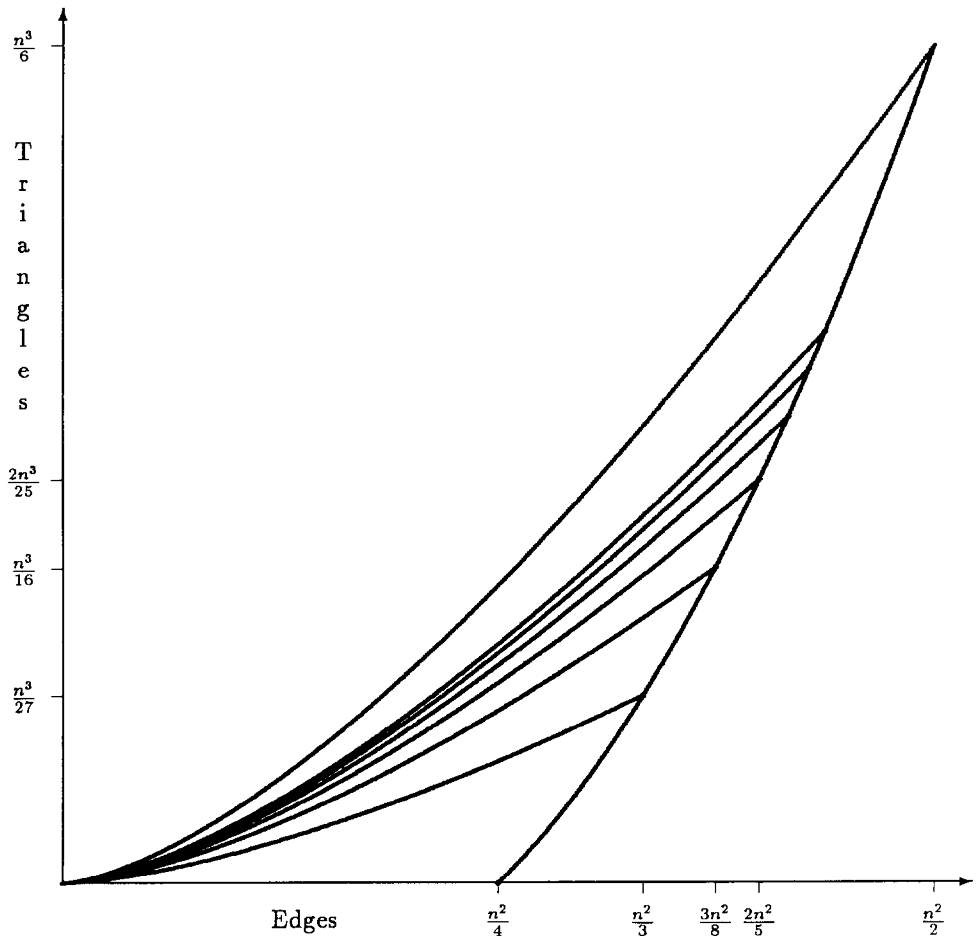

Fig. 1.

Each inequality in (1) is a lower bound on the clique number. Turán's theorem gives the well-known bound, $w \geq k_1^2/(k_1^2 - 2k_2)$. Solving (2) for w gives a new bound:

$$w \geq \frac{8k_2^3 - 9k_3^2 + 3k_3\sqrt{16k_2^3 + 9k_3^2}}{4k_2^3 - 18k_3^2}. \tag{3}$$

We show that (3) always supercedes the bound from Turán's theorem.

1. The main result

This section proves (1) (as Theorem 1) with a sequence of Lemmas. While Lemma 1 is purely analytic, Lemmas 2–5 and Theorem 1 can be described as 'analytic graph theory'.

Definitions. Let G be a simple graph. Let $w(G)$ be the size of the largest complete subgraph in G. Let $K_j(G)$ be the set of j node complete subgraphs in G. Let $\mathcal{K}(G) \equiv \bigcup_{j=1}^{\infty} K_j(G)$ and $k_j(G) = |K_j(G)|$. For each node $a \in G$, let G_a be the subgraph of G induced on the neighbors of a. For all $R \in \mathcal{K}(G)$, let $G_R \equiv \bigcap_{a \in R} G_a$.

When the dependence on the graph suppressed, assume the function depends on the graph (as opposed to its subgraphs). So, for example, $w = w(G)$.

1.1. *A corollary of Hölder's inequality*

Hölder's inequality is: for all nonnegative sequences $\{u_j\}_{i=1}^{m}$ and $\{v_i\}_{i=1}^{m}$, and for all $p \in (0, 1)$.

$$\sum_{i=1}^{m} u_i v_i \leq \left(\sum_{i=1}^{m} u_i^{1/(1-p)}\right)^{1-p} \left(\sum_{i=1}^{m} v_i^{1/p}\right)^{p}.$$

Lemma 1 follows from Hölder's inequality by letting $u_i = 1$ and $v_i = r_i^p$.

Lemma 1. *Let $\{r_i\}_{i=1}^{m}$ be a nonnegative sequence. Then for all $p \in (0, 1)$,*

$$\sum_{i=1}^{m} r_i^p \leq m^{1-p}\left(\sum_{i=1}^{m} r_i\right)^{p}.$$

1.2. *The degree of a complete subgraph*

Here, we define and give several properties of the 'degree of a complete subgraph'. Lemma 5 is the key result needed to prove Theorem 1. Lemma 5 with $j = 1$ is equivalent to Turán's theorem so it also generalizes Turán's theorem. Fig. 2 gives an example of Lemma 5. Lemmas 2–4 are used to prove Lemma 5.

For all $R \in \mathcal{K}$ and for $1 \leq j$, let $\sigma_j(R) = \sum_{a \in R} k_j(G_a)$. Since $\sigma_1(a)$ is the degree of node a, $\sigma_j(R)$ is a generalization of the degee of a node.

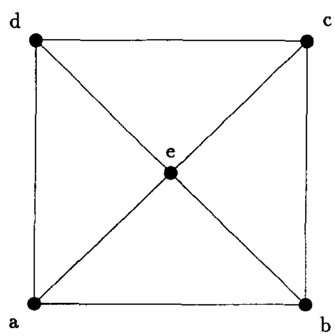

Fig. 2.

Lemma 2. *Let G be a graph and $S \in \mathscr{K}$. Then $\sigma_j(S) = \sum_{R \in K_j} |G_R \cap S|$.*

Proof. If $R \in K_j$, then $a \in G_R$ if and only if $R \in K_j(G_a)$. Thus we may interchange summations to get

$$\sum_{R \in K_j} |G_R \cap S| = \sum_{R \in K_j} \sum_{a \in G_R \cap S} 1 = \sum_{a \in S} \sum_{R \in K_j(G_a)} 1 = \sum_{a \in S} k_j(G_a) = \sigma_j(S). \qquad \square$$

Lemma 3. *Let G be a graph and $S \in \mathscr{K}$. Then $\sigma_j^2(S) \leqslant |S| \sum_{R \in K_j} \sigma_j(G_R \cap S)$.*

Proof. From the definition of $\sigma_j(S)$ and Lemma 1 with $p = \frac{1}{2}$:

$$\sigma_j^2(S) = \Big(\sum_{a \in S} k_j(G_a)\Big)^2 \leqslant |S| \sum_{a \in S} k_j(G_a)^2 = |S| \sum_{a \in S} \sum_{R \in K_j(G_a)} k_j(G_a).$$

Since, for $R \in K_j$, $a \in G_R$ if and only if $R \in K_j(G_a)$, we may exchange summations to get

$$\sigma_j^2(S) \leqslant |S| \sum_{R \in K_j} \sum_{a \in G_R \cap S} k_j(G_a) = |S| \sum_{R \in K_j} \sigma_j(G_R \cap S). \qquad \square$$

Lemma 4. *Let G be a graph. For all $S \in \mathscr{K}$, $\sigma_j(S) \leqslant (w-j)k_j$.*

Proof. Let $R \in K_j$. Since $R \cup (G_R \cap S) \in \mathscr{K}$ and $R \cap G_R = \emptyset$, we have $|R| + |G_R \cap S| \leqslant w$ and hence $|G_R \cap S| \leqslant w - j$. Thus from Lemma 2, $\sigma_j(S) = \sum_{R \in K_j} |G_R \cap S| \leqslant (w-j)k_j$. $\square$

Lemma 5. *Let G be a graph. Then for all $1 \leqslant j \leqslant w$, $\sum_{R \in K_j} \sigma_j(R) \leqslant (j(w-j)/w)k_j^2$.*

Proof. Let x be the largest number such that for all $R \in \mathscr{K}$,

$$\sigma_j(R) \leqslant (w-j)k_j - (w - |R|)x. \tag{4}$$

By Lemma 4, x is well-defined and nonnegative. Also, there is some $S \in \mathscr{K}$ with

$$\sigma_j(S) = (w-j)k_j - (w - |S|)x. \tag{5}$$

Since $R \cup (G_R \cap S) \in \mathscr{K}$ and $G_R \cap R = \emptyset$, we have $|R \cup (G_R \cap S)| = |R| + |G_R \cap S| = j + |G_R \cap S|$ and $\sigma_j(R) = \sigma_j(R \cup (G_R \cap S)) - \sigma_j(G_R \cap S)$. Using these and (4) gives

$$\begin{aligned}\sum_{R \in K_j} \sigma_j(R) &= \sum_{R \in K_j} [\sigma_j(R \cup (G_R \cap S)) - \sigma_j(G_R \cap S)] \\ &\leqslant \sum_{R \in K_j} [(w-j)k_j - x(w - |R \cup (G_R \cap S)|) - \sigma_j(G_R \cap S)] \\ &= (w-j)k_j^2 - xwk_j + xjk_j + x \sum_{R \in K_j} |G_R \cap S| - \sum_{R \in K_j} \sigma_j(G_R \cap S).\end{aligned}$$

We may now use Lemmas 2 and 3 and (5) to get

$$\sum_{R\in K_j} \sigma_j(R) \leq k_j(w-j)(k_j-x) + x\sigma_j(S) - \sigma_j^2(S)/\,|S|$$
$$= k_j(w-j)(k_j-x) + x((w-j)k_j - (w-|S|\,)x)$$
$$-\,[(w-j)k_j - (w-|S|\,)x]^2/|S|$$
$$= k_j(w-j)(k_j-x) - x((w-j)k_j - wx)$$
$$-[(w-j)k_j - wx]^2/\,|S|.$$

Since $|S| \leq w$,

$$\sum_{R\in K_j} \sigma_j(R) \leq k_j(w-j)(k_j-x) - x((w-j)k_j - wx) - \frac{[(w-j)k_j - wx]^2}{w}$$
$$= \frac{j(w-j)}{w} k_j^2. \qquad \square$$

1.3. A proof of (1)

Theorem 1. *Let G be a graph. Then for all $1 \leq j \leq w$, $k_{j+1} \leq \binom{w}{j+1}(k_j/\binom{w}{j})^{(j+1)/j}$.*

Proof. This is by induction on j. For $j = 1$, this is Turán's theorem (or Lemma 5 with $j = 1$). For $j > 1$, let $a \in G$. If G_a had a complete subgraph S with $|S| = w$, then $S + a$ would be a complete subgraph on $w + 1$ nodes. So $w(G_a) = w - 1$. Thus by induction,

$$k_j(G_a) \leq \binom{w-j}{j}\left(\frac{k_{j-1}(G_a)}{\binom{w-1}{j-1}}\right)^{j/(j-1)}. \tag{6}$$

Each $a \in G$ is in $k_j(G_a)$ complete subgraphs on j nodes. Then using (6), we have

$$k_{j+1} = \frac{1}{j+1}\sum_{a\in G} k_j(G_a) = \frac{1}{j+1}\sum_{a\in G} k_j(G_a)^{1/j} k_j(G_a)^{(j-1)/j}$$
$$\leq \frac{1}{j+1}\sum_{a\in G} k_j(G_a)^{1/j}\binom{w-1}{j}^{(j-1)/j}\frac{k_{j-1}(G_a)}{\binom{w-1}{j-1}}$$
$$= \frac{\binom{w-1}{j}^{(j-1)/j}}{(j+1)\binom{w-1}{j-1}}\sum_{a\in G} k_j(G_a)^{1/j} k_{j-1}(G_a).$$

Since node a is in $k_{j-1}(G_a)$ complete subgraphs on j nodes, and using Lemma 1 with $p = 1/j$, we have

$$k_{j+1} \leq \frac{\binom{w-1}{j}^{(j-1)/j}}{(j+1)\binom{w-1}{j-1}} (jk_j)^{(j-1)/j} \Big(\sum_{R \in K_j} \sum_{a \in R} k_j(G_a) \Big)^{1/j}$$

$$= \frac{(jk_j \binom{w-1}{j})^{(j-1)/j}}{(j+1)\binom{w-1}{j-1}} \Big(\sum_{R \in K_j} \sigma_j(R) \Big)^{1/j}.$$

So using Lemma 5,

$$k_{j+1} \leq \frac{(jk_j \binom{w-1}{j})^{(j-1)/j}}{(j+1)\binom{w-1}{j-1}} \Big(\frac{j(w-j)k_j^2}{w} \Big)^{1/j} = \binom{w}{j+1} \Big(\frac{k_j}{\binom{w}{j}} \Big)^{(j+1)/j}. \qquad \square$$

1.4. *How sharp is Theorem* 1?

For all j, Theorem 1 is exact for graphs whose complete subgraphs on j nodes are all in one complete balanced w-partite subgraph. However, not all combinations of k_j, k_{j+1} and w that satisfy Theorem 1 can be achieved. For example, when $j = 2$, $w = 3$ and $k_2 = 9$, Theorem 1 gives $k_3 \leq 5$. But graphs with 9 edges and clique number 3 have at most 4 triangles.

2. Corollaries and related results

Theorem 1 has many implications. This section discusses some of these.

2.1. *Complete subgraph sequences*

Alavi, Malde, Schwenk and Erdős [1] studied the sequence $\{k_1, k_2, \ldots, k_w\}$ (actually they studied independent sets, but the independent sets of a graph corresponds to the complete subgraphs of its complement). Given any permutation of the first w natural numbers, $i_1, i_2, \ldots, i_w$, they exhibited a graph with $k_{i_1} > k_{i_2} > \cdots > k_{i_w}$, i.e., the sequence $\{k_1, k_2, \ldots, k_w\}$ can increase and decrease in an arbitrary fashion.

For $1 \leq j \leq w$, let $s_j \equiv (k_j/\binom{w}{j})^{1/j}$. Theorem 1 brings order into this chaos by requiring $\{s_1, s_2, \ldots, s_w\}$ be monotonically non-increasing.

2.2. *Bounds based on the number of nodes and edges*

A classic problem is: what is the maximum number of j node complete subgraphs in an n node graph with clique number w? Since the first term in (1) is greater than or equal to the jth term, we get the following (this is also a corollary of a result due to Erdős [3] and Sauer [8]).

Corollary 1. *A graph with n nodes and clique number w has at most* $\binom{w}{j} n^j / w^j$ *complete subgraphs on j nodes.*

Similarly, we can find the maximum number of complete subgraphs in a graph with e edges and clique number w. This is a new result.

Corollary 2. *A graph with e edges and clique number w has at most*

$$\binom{w}{j}(2e/(w-1)w)^{j/2}$$

complete subgraphs on $j \geq 2$ nodes.

2.3. Bounds on the clique number

For each j, Theorem 1 gives a lower bound on the clique number. Let h_j be the polynomial

$$h_j(x) \equiv k_j^{j+1}\binom{x}{j+1}^j - k_{j+1}^j\binom{x}{j}^{j+1}.$$

Let w_j be the largest real root of $h_j(w_j) = 0$ (w_j exists because $h_j(j) \leq 0$ and, by Theorem 1, $h_j(w) \geq 0$). Then we have the following.

Corollary 3. *Let G be a graph. Then for all j, $w_j \leq w$.*

We can find explicit expressions for w_1 and w_2. Then Corollary 3 gives the following lower bounds on the clique number:

$$w \geq w_1 = \frac{k_1^2}{k_1^2 - 2k_2} \quad \text{and} \quad w \geq w_2 = \frac{8k_2^3 - 9k_3^2 + 3k_3\sqrt{16k_2^3 + 9k_3^2}}{4k_2^3 - 18k_3^2}.$$

Surprisingly, the second inequality always supercedes the first.

Theorem 2. *For all graphs, $w_1 \leq w_2$.*

Proof. For all graphs, Nordhaus and Stewart [7] showed $k_3 \geq (4k_2 - k_1^2)k_2/(3k_1)$. Thus,

$$\begin{aligned} h_2(w_1) &= k_2^3\binom{w_1}{3}^2 - k_3^2\binom{w_1}{2}^3 \\ &= \frac{k_2^3k_1^6}{(k_1^2-2k_2)^6}\left[\left(\frac{(4k_2-k_1^2)k_2}{3k_1}\right)^2 - k_3^2\right] \leq 0. \end{aligned}$$

Theorem 1 gives $h_2(w) \geq 0$. So we must have $w_1 \leq w_2 \leq w$. □

We believe (see [6]) the bounds on w in Corollary 3 become increasingly better as j increases, i.e., $w_1 \leq w_2 \leq \cdots \leq w_w = w$.

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