Characterizations of closed classes of Boolean functions in terms of forbidden subfunctions and Post classes

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Abstract

We characterize Post classes of Boolean functions (also known as clones) in terms of forbidden subfunctions that allows one to give a comparably short proof of the classical Post theorem.

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1. Introduction

Two Boolean functions \( f(x_1, x_2, \ldots, x_n) \) and \( g(y_1, y_2, \ldots, y_n) \) are congruent if there exists a bijection \( \phi : \{x_1, x_2, \ldots, x_n\} \rightarrow \{y_1, y_2, \ldots, y_n\} \) such that \( f(\phi(x_1), \phi(x_2), \ldots, \phi(x_n)) = g(y_1, y_2, \ldots, y_n) \). Two Boolean functions \( f \) and \( g \) are similar if deleting all inessential variables from \( f \) and \( g \) produces congruent functions. We distinguish Boolean functions (and classes of Boolean functions) up to similarity.

Definition 1. Let \( \Phi \) be a class of Boolean functions. The closure \( [\Phi] \) is the class of all functions that can be obtained by substitutions of functions from \( \Phi \) and/or by identifica-
tion of variables, i.e., $[\Phi]$ consists of all functions that can be obtained by the following rules:

(CLO1): $\Phi \subseteq [\Phi]$, and

(CLO2): if $f(x_1, x_2, \ldots, x_n) \in [\Phi]$ and each of $X_1, X_2, \ldots, X_n$ is either a member of $[\Phi]$ or a Boolean variable, then $f(X_1, X_2, \ldots, X_n) \in [\Phi]$.

We say that the class $[\Phi]$ is generated by $\Phi$. A class $\Phi$ is called a closed class (also, a clone) if $\Phi = [\Phi]$.

**Proposition 1.** Intersection of any number of closed classes is a closed class.

**Proof.** Straightforward. □

A set $\Psi$ of Boolean functions is called complete in a closed class $\Phi$ if $[\Psi] = \Phi$. For a class $X$ of Boolean functions, we denote $\overline{X} = \{ \overline{f} : f \in X \}$ where $\overline{f}(\bar{x}) = \overline{f(x)}$ for every $x$. By $f^d$ we denote the function which is dual to $f$, i.e., $f^d(x) = \overline{f(\bar{x})}$ for every $x$. Also, $\Phi^d = \{ f^d : f \in \Phi \}$. The number of essential variables of a function $f$ is denoted by $\text{ess}(f)$.

**Definition 2.** The following classes of Boolean functions are called Post classes:

Classes of type $E$: $E = \{0, 1\}$, $E_0 = \{0\}$, $E_1 = \{1\}$, and $E_{01} = \emptyset$.

Classes of type $O$: $O = \{0, 1, x, \overline{x}\}$, $O_0 = \{0, x\}$, $O_1 = \{1, x\}$, $O_{01} = \{x\}$, $OS = \{x, \overline{x}\}$, and $OM = \{0, 1, x\}$.

Classes of type $T$: $T$ is the class of all Boolean functions, $T_0 = \{ f : f(0, 0, \ldots, 0) = 0 \}$ is the class of functions preserving 0, $T_1 = \{ f : f(1, 1, \ldots, 1) = 1 \}$ is the class of functions preserving 1, and $T_{01} = T_0 \cap T_1$.

Classes of type $P$: $P = \{0, 1, x_1x_2 \cdots x_n : n \geq 1 \text{ is not fixed}\}$, $P_0 = P \cap T_0$, $P_1 = P \cap T_1$, and $P_{01} = P \cap T_{01}$.

Classes of type $P^d$: $P^d = \{0, 1, x_1 \lor x_2 \lor \cdots \lor x_n : n \geq 1 \text{ is not fixed}\}$, $P_{0}^d = P^d \cap T_0$, $P_1^d = P^d \cap T_1$, and $P_{01}^d = P^d \cap T_{01}$. Note that the classes $P^d$, $P_{0}^d$, $P_1^d$, and $P_{01}^d$ are dual to $P$, $P_0$, $P_1$, and $P_{01}$, respectively.

Classes of type $M$: $M = \{ f : \alpha \leq \beta \text{ implies } f(\alpha) \leq f(\beta) \}$ is the class of monotone Boolean functions \{ for $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$, $\alpha \leq \beta$ means that $a_i \leq b_i$ for all $i = 1, 2, \ldots, n\}$. $M_0 = M \cap T_0$, $M_1 = M \cap T_1$, and $M_{01} = M \cap T_{01}$.

Classes of type $S$: $S = \{ f : f^d = f \}$ is the class of all self-dual functions, $S_{01} = M \cap T_{01}$, and $SM = S \cap M$.

Classes of type $L$: $L = \{a_0 \oplus a_1x_1 \oplus \cdots \oplus a_nx_n : n \geq 0 \text{ is not fixed}, a_i \in \{0, 1\}\}$ is the class of all linear Boolean functions, $L_0 = L \cap T_0$, $L_1 = L \cap T_1$, $L_{01} = L \cap T_{01}$, and $LS = L \cap S$.

Classes of type $A^k (k \geq 2)$: $A^k = \{ f : f(a_1) = f(a_2) = \cdots = f(a_k) = 1 \text{ then } a_1, a_2, \ldots, a_k \text{ have common coordinate } 1 \}$ (a common coordinate is a coordinate with the same index), $A_{1}^{k} = A^{k} \cap T_{1}$, $MA^{k} = A^{k} \cap M$, and $MA_{1}^{k} = A^{k} \cap T_{1} \cap M$.

Classes of type $A^\infty$: $A^\infty = \bigcap_{k=2}^{\infty} A^k$, i.e., $A^\infty$ is the class of all Boolean functions $f$ such that all sequences over the set $\{x : f(x) = 1\}$ have a common 1, $A_{1}^{\infty} = A^{\infty} \cap T_{1}$, $MA^{\infty} = A^{\infty} \cap M$, $MA_{1}^{\infty} = A^{\infty} \cap T_{1} \cap M$.

Classes of type $a^k (k \geq 2)$: $a^k = \{ f : f(a_1) = f(a_2) = \cdots = f(a_k) = 0 \text{ then } a_1, a_2, \ldots, a_k \text{ have a common coordinate } 0\}$, $a_{0}^{k} = a^{k} \cap T_{0}$, $Ma^{k} = a^{k} \cap M$, $Ma_{0}^{k} = a^{k} \cap T_{0} \cap M$. 


Classes of type $a^\infty$: $a^\infty = \bigcap_{k=2}^\infty a^k$, i.e., $a^\infty$ is the class of all Boolean functions $f$ such that all sequences over the set $\{x : f(x) = 0\}$ have a common 0, $a_0^\infty = a^\infty \cap T_0$, $Ma^\infty = a^\infty \cap M$, and $Ma_0^\infty = a^\infty \cap T_0 \cap M$.

**Proposition 2.** Each Post class is closed.

**Proof.** It is sufficient to note that $E$, $O$, $P$, $P^d$, $T_0$, $T_1$, $M$, $S$, $L$, $A^k(k \geq 2)$, and $a^k(k \geq 2)$ are closed and apply Proposition 1. □

2. Hereditary classes

A Boolean function $g$ is called a subfunction of a Boolean function $f$ if $g$ can be obtained by identification of variables of $f$. We denote by $\text{Sub}(f)$ the set of all subfunctions of $f$ considered up to similarity.

**Definition 3.** A class $\Phi$ of Boolean functions is called hereditary if $\text{Sub}(f) \subseteq \Phi$ for each function $f \in \Phi$.

**Proposition 3.** Every closed class of Boolean functions is hereditary.

**Proof.** It follows directly from the definitions. □

Let $Z$ be an arbitrary set of Boolean functions. We put $\text{FS}(Z) = \{f : \text{Sub}(f) \cap Z = \emptyset\}$, a hereditary class defined by $Z$ as a set of forbidden subfunctions.

**Theorem 1.** (i) A class $\Phi$ of Boolean functions is hereditary if and only if $\Phi = \text{FS}(Z)$ for some set $Z$ of Boolean functions.

(ii) The inclusion-wise minimal set $Z$ satisfying (i) is uniquely defined (up to similarity).

**Proof.** (i) Let $Z = T \setminus \Phi$, where $T$ is the class of all Boolean functions. Let $f \in \Phi$. By hereditariness, $\text{Sub}(f) \subseteq \Phi$ and therefore $\text{Sub}(f) \cap Z = \emptyset$, i.e., $f \in \text{FS}(Z)$. Thus, $\Phi \subseteq \text{FS}(Z)$. Conversely, if $f \in \text{FS}(Z)$ then $\text{Sub}(f) \cap Z = \emptyset$. In particular, $f \notin Z = T \setminus \Phi$. Hence $f \in \Phi$, and $\text{FS}(Z) \subseteq \Phi$. (ii) A function $f \notin \Phi$ is called a minimal forbidden subfunction for $\Phi$ if $\text{Sub}(f) \setminus \{f\} \subseteq \Phi$. Let $Z^0$ be the set of all minimal functions in $Z$. We show that $\Phi = \text{FS}(Z^0)$. Since $Z^0 \subseteq Z$, we have $\Phi = \text{FS}(Z) \subseteq \text{FS}(Z^0)$.

We show that $\text{FS}(Z^0) \subseteq \Phi$. Let $f \in \text{FS}(Z^0)$. If $f \notin \Phi$ then $f \in Z$. By finiteness of $\text{Sub}(f)$, it contains a minimal function $g$. Since $g \in Z^0 \cap \text{Sub}(f)$, we have $f \notin \text{FS}(Z^0)$, a contradiction.

Now we prove the uniqueness of the set $Z^0$ by showing that $\Phi = \text{FS}(Z')$ implies $Z^0 \subseteq Z'$. Let $f \in Z^0$ and $f \notin Z'$. It follows from $\Phi = \text{FS}(Z^0)$ and $f \in Z^0$ that $f \notin \Phi$. By minimality, $\text{Sub}(f) \setminus \{f\} \subseteq \Phi$. Since $\Phi = \text{FS}(Z')$, we have $\text{Sub}(f) \setminus \{f\} \cap Z' = \emptyset$. Also, $f \notin Z'$. Hence $\text{Sub}(f) \cap Z' = \emptyset$. It implies that $f \in \text{FS}(Z') = \Phi$, a contradiction. □

**Proposition 4.** If $\Phi_1 = \text{FS}(Z_1)$ and $\Phi_2 = \text{FS}(Z_2)$ are two hereditary classes of Boolean functions, then $\Phi_1 \cap \Phi_2 = \text{FS}(Z_1 \cup Z_2)$. 

Theorem 2

See also Table 1 for alternative definitions.

Proof. If \( f \in \Phi_1 \cap \Phi_2 \) then \( Sub(f) \cap Z_i = \emptyset \) for \( i = 1, 2 \). Hence \( Sub(f) \cap (Z_1 \cup Z_2) = \emptyset \) and \( f \in FS(Z_1 \cup Z_2) \). Conversely, let \( f \in FS(Z_1 \cup Z_2) \). Since \( Z_i \subseteq Z_1 \cup Z_2, i = 1, 2 \), we have \( FS(Z_1 \cup Z_2) \subseteq FS(Z_i) \) and \( f \in FS(Z_i) = \Phi_i \). Thus, \( f \in \Phi_1 \cap \Phi_2 \). \( \square \)

Note that the set \( Z_1 \cup Z_2 \) in Proposition 4 may be redundant, so non-minimal forbidden subfunctions for \( \Phi_1 \cap \Phi_2 \) can be deleted from \( Z_1 \cup Z_2 \).

A Boolean function \( g \) is called a strong subfunction (respectively, a 0-subfunction; a 1-subfunction) of a Boolean function \( f \) if \( g \) can be obtained from \( f \) by substituting of constants 0, 1 (respectively, by substituting of 0; by substituting of 1) and/or by identification of variables. We denote by \( Sub_{01}(f) \), \( Sub_{02}(f) \) and \( Sub_1(f) \) the set of all strong subfunctions, 0-subfunctions and 1-subfunctions of \( f \), respectively. Note that a strong subfunction is not necessarily a subfunction (but the other way round).

Definition 4. A class \( \Phi \) of Boolean functions is called strongly-hereditary (respectively, 0-hereditary; 1-hereditary) if \( Sub_{01}(f) \subseteq \Phi \) (respectively, \( Sub_{02}(f) \subseteq \Phi \); \( Sub_1(f) \subseteq \Phi \)) for each function \( f \in \Phi \).

Let \( Z \) be a set of Boolean functions. We put \( FS_{01}(Z) = \{ f : Sub_{01}(f) \cap Z = \emptyset \} \), a strong-hereditary class defined by the set \( Z \) of forbidden strong subfunctions. In a similar way we define classes \( FS_{02}(Z) \) and \( FS_1(Z) \). For these modified concepts of subfunction and hereditary classes, analogs of Theorem 1 and Proposition 4 are valid.

3. Characterizations of Post classes in terms of forbidden subfunctions

We introduce the following Boolean functions:

- \( \phi_0(x, y, z) = xy \oplus xz \oplus yz \),
- \( \phi_1(x, y, z) = x \oplus y \oplus z \),
- \( \phi_2(x, y, z) = xy \oplus xz \oplus yz \oplus x \oplus y \),
- \( \phi_3(x, y, z) = x(y \sim z) [\phi_3^{d}(x, y, z) = xy \oplus xz \oplus x \oplus y \oplus z] \),
- \( \phi_4(x, y, z) = x(y \lor z) [\phi_4^{d}(x, y, z) = xyz \oplus xy \oplus yz \oplus x \oplus y] \),
- \( \phi_5(x, y, z) = xy \oplus x \oplus z \),
- \( \phi_6(x, y, z) = xy \oplus xz \oplus y \),
- \( \phi_7(x, y, z) = xyz \oplus x \oplus y [\phi_7^{d}(x, y, z) = xyz \oplus xy \oplus xz \oplus yz \oplus z] \),
- \( \phi_8(x, y, z) = xyz \oplus xy \oplus z [\phi_8^{d}(x, y, z) = xz \oplus yz \oplus x \oplus y] \),
- \( \phi_9(x, y, z) = x(y \lor z) \),
- \( \phi_{10}(x, y, z) = xy \oplus xz \oplus y \oplus z \),
- \( \phi_{11}(x, y, z) = xyz \oplus x \oplus y \oplus z \), and
- \( \phi_{12}(x, y, z) = xyz \oplus xy \oplus xz \oplus yz \oplus x \oplus y \).

See also Table 1 for alternative definitions.

Theorem 2 (Characterizations of Post classes). The following statements hold:
Table 1

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Theorem 3. The following statements hold for all $k \geq 2$ and $k = \infty$:
(H25) \( A^k = \text{FS}(Z_A \cup Z^k) = \text{FS}_0(\{1, \bar{x}, x \oplus y, x \lor y\} \cup Z^k) \), where \( Z^k = A^2 \setminus A^k \) and 
\( Z_A = \{1, \bar{x}, x \oplus y, x \lor y, \phi_1, \phi_2, \phi_8, x, \phi_{10}, \phi_{11}, \phi_{12}\}; \)

(H26) \( A^k = \text{FS}(Z_{A_1} \cup (Z^k \cap T_1)) \), where \( Z_{A_1} = \{0, 1, \bar{x}, x \lor y, \phi_1, \phi_2, \phi_8\}; \)

(H27) \( MA^k = \text{FS}(Z_{MA} \cup (Z^k \cap M)) = \text{FS}_0(\{1, \bar{x}, x \oplus y, x \lor y\} \cup (Z^k \cap M)) \), where 
\( Z_{MA} = \{1, \bar{x}, x \oplus y, x \lor y, \bar{x}, \phi_1, \phi_2, \phi_3, \phi_4, \phi_8\}; \)

(H28) \( MA^k = \text{FS}(Z_{MA_1} \cup (Z^k \cap M_1)) \), where \( Z_{MA_1} = \{0, 1, \bar{x}, x \lor y, \bar{x}, \phi_1, \phi_2, \phi_3, 0\}; \)

(H25d) \( a^k = \text{FS}(Z^d \cup Z^k) = \text{FS}_1(\{0, \bar{x}, x \lor y\} \cup Z^k) \), where \( Z^k = a^2 \setminus a^k; \)

(H26d) a^k = \text{FS}(\{Z_A\}^d \cup (Z^k \cap T_0));

(H27d) \( a^k = \text{FS}(Z_{MA} \cup \{Z^k \cap M\}) = \text{FS}_1(\{0, \bar{x}, x \lor y\} \cup (Z^k \cap M)); \)

(H28d) \( a^k = \text{FS}(Z_{MA_1} \cup \{Z^k \cap M_0\}) = xy. \)

Proof. It is enough to prove the statement for the classes \( E, O, P, P^d, T_0, T_1, M, S, L, A^k \), and \( a^k \) only, since the other classes are intersections of these (see Definition 2), and we may use Proposition 4. Also, every statement (H\#d) is dual to (H\#). Proofs of the dual statements are straightforward and hence omitted.

Suppose that we have a characterization \( \Phi = \text{FS}(Z) \) of a hereditary class \( \Phi \). Let \( Z_0 \) be the set of all functions \( f \in Z \) that cannot be obtained from a function in \( Z \setminus \{f\} \) by substitution of 0’s. Then \( \Phi = \text{FS}_0(Z_0). \) Thus, it is easy to produce an \( \text{FS}_0 \)-characterization (similarly for an \( \text{FS}_1 \)-characterization and an \( \text{FS}_{01} \)-characterization) of \( \Phi \) from a given characterization \( \Phi = \text{FS}(Z) \). See our proofs of (H22) and (H23) for illustrations.

(H1) We can obtain either 1 or \( \bar{x} \) from any Boolean function \( f \notin T_0 \) by identification of all its variables. Hence \( T_0 = \text{FS}(1, \bar{x}) \) and \( T_0 = \text{FS}_0(1). \)

(H3) Let \( f \notin M \). By definition, there exist \( a = (a_1, a_2, \ldots, a_n) \prec \beta = (b_1, b_2, \ldots, b_n) \) (i.e., \( a \preceq \beta \) and \( a \neq \beta \) such that \( f(a) = 1 \) and \( f(\beta) = 0. \) We put \( x = x_i \) if \( a_i = 0 \) and \( b_i = 1 \); \( y = x_i \) if \( a_i = b_i = 0 \); \( z = x_i \) if \( a_i = b_i = 1. \) We have constructed a Boolean function \( g(x, y, z) \) such that \( g(0, 0, 1) = f(x) = 1 \) and \( g(1, 0, 1) = f(\beta) = 0. \)

- If \( g(0, 0, 0) = g(1, 1, 1) = 0 \) then \( g(x, x, z) \in \{\bar{x}z, x \oplus z\} \subseteq Z_M. \)
- If \( g(0, 0, 0) = g(1, 1, 1) = 0 \) then \( g(x, x, x) \in \{x \lor y, \bar{x} \lor y\} \subseteq Z_M. \)
- If \( g(0, 0, 0) = 1 \) and \( g(1, 1, 1) = 0 \) then \( g(x, x, x) = \bar{x} \in Z_M. \)
- If \( g(0, 0, 0) = 0 \) and \( g(1, 1, 1) = 1 \) then we represent \( g \) in polynomial form:

\[
g = Axyz \oplus Bxy \oplus Cxz \oplus Dyz \oplus Ex \oplus Fy \oplus Gz \oplus H.
\]

- Since \( g(0, 0, 0) = 0 \), we have \( H = 0. \)
- Since \( g(0, 0, 1) = 1 \) and \( H = 0 \), we have \( G = 1. \)
- Since \( g(1, 0, 1) = 0 \) and \( G \oplus H = 1 \), \( C \oplus E = 1. \)
- Since \( g(1, 1, 1) = 1 \) and \( C \oplus E \oplus G \oplus H = 0 \), \( A \oplus B \oplus D \oplus F = 1. \)

When \( C = 0 \) and \( E = 1 \) we obtain the functions \( \phi_1, \phi_5, \phi_7, \phi_8, \phi_9 \) and \( \phi_{10} \). When \( C = 1 \) and \( E = 0 \) we obtain the functions \( \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7 \) and \( \phi_9 \).

To prove the other equalities in (H3) we use the method described above, that is we delete from \( Z_M \) all functions that are reduced to some other functions in \( Z_M \) by substituting 0 (respectively, 1; either 0 or 1).
To prove the FS 01-characterization, we use the general method.

If \( g(x, y, z, t) \) produces a function \( g(x, y) \) such that \( g(0, 1) = g(1, 0) \). If \( g(0, 0) = g(1, 1) \) then \( g(x, x) \in \{0, 1\} \subseteq FS(Z_S \cup Z_S) \). Otherwise \( g \in \{xy, x \vee y, xy, x \lor y\} \subseteq FS(Z_S \cup Z_S) \).

(H9) Let \( f \notin L \) be a minimal function, i.e., \( Sub(f) \setminus \{f\} \subseteq L \). Since \( f \) is not linear, there is a term \( K \) with more than one variable in the polynomial \( p \) of \( f \).

First suppose that \( K \) contains at least three variables, say \( x, y, z \). By minimality of \( f \), identification of \( x \) and \( y \) results in a linear function. So \( p \) contains a term \( K' \) which is obtained from \( K \) by deleting some \( t \in \{x, y\} \). Similar statements are valid for pairs \( x, z \) and \( y, z \). Therefore, at least two of the three terms \( K^x, K^y, K^z \) are contained in \( p \). By symmetry, we may assume that \( p \) contains terms \( K^x \) and \( K^y \). Putting \( x = y \), we see that the terms \( K, K^x, K^y \) only are transformed into \( K^x \), i.e., we obtain a non-linear function, a contradiction to minimality of \( f \). Therefore \( p \) contains a term \( K = xy \) and there are no terms with more than two variables. Clearly, if \( f \) has exactly two variables then it is a function of the form (1).

If \( ess(f) \geq 4 \) then identification of variables distinct from \( x \) and \( y \) produces a non-linear function, a contradiction to minimality of \( f \). Finally, if \( f \) has exactly three essential variables \( x, y, \) and \( z \), then both \( f(x, y, x) \) and \( f(x, y, y) \) do not have a term \( K = xy \) (by minimality of \( f \)). So \( p \) has terms \( xz, yz \) and \( f \) is of the form (2). By substitution of either \( z = 0 \) or \( 1 \) a function of the form (2) is transformed to a function of the form (1), i.e., to the set \( Z_L' \cup Z_L' \).

To prove the FS01-characterization, we use the general method.

(H13) Since \( O \subseteq L \), it is sufficient to forbid \( Z_L \) and functions \( x \oplus y \oplus a, x \oplus y \oplus z \oplus a \), where \( a \in \{0, 1\} \). It remains to delete all non-minimal functions.

(H18) Since \( E \subseteq O \), it is sufficient to forbid \( Z_O \cup \bar{Z}_O \) and functions \( x, \bar{x} \). It remains to delete all nonminimal functions.

(H21) Since \( P \subseteq M \), all functions in \( Z_M \) are forbidden for \( P \). It is easy to check that \( x, x \oplus y, x \sim y, x \overline{y}, x \vee \ar{y}, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \) are minimal forbidden for \( P \), while the others have a subfunction \( x \lor y \).

Now let \( f \in M \setminus P \). As usual, \( \min T(f) = \{z : f(z) = 1 \text{ and } f^z = 0 \text{ for each } z < x\} \) is the set of all \textit{minimal true points} of \( f \). Since \( f \notin P \), there are distinct points \( z = (a_1, a_2, \ldots, a_n) \in \min T(f) \) and \( \beta = (b_1, b_2, \ldots, b_n) \in \min T(f) \). We put

\[
\begin{align*}
x_i &= \begin{cases} 
  x & \text{if } a_i = 0 \text{ and } b_i = 1, \\
y & \text{if } a_i = 1 \text{ and } b_i = 0, \\
z & \text{if } a_i = 0 \text{ and } b_i = 0, \\
t & \text{if } a_i = 1 \text{ and } b_i = 1 
\end{cases}
\end{align*}
\]

and obtain a monotone function \( g(x, y, z, t) \) such that \( g(0, 1, 0, 1) = f(z) = 1 \) and \( g(1, 0, 0, 1) = f(z) = 1 \). Since \( z \in \min T(f) \), \( g(0, 0, 0) = g(0, 0, 0, 1) = g(0, 1, 0, 0) = 0. \) Since \( g(1, 0, 0, 1) = 1 \) and \( g \) is a monotone function, we have \( g(1, 1, 0, 1) = g(1, 1, 0, 1) = 1. \) If \( g(1, 0, 1, 0) = 1 \) then \( g(x, y, x, y) = x \lor y \in Z_P \). If \( g(1, 0, 1, 0) = 0 \) then

- \( g(x, y, x, t) = t(x \lor y) \) when \( g(1, 1, 1, 0) = 0 \) (\( g \) is congruent to \( \phi_0 \in Z_P \)),
- \( g(x, y, x, t) = xy \oplus xt \oplus yt \) when \( g(1, 1, 1, 0) = 1 \) (\( g \) is congruent to \( \phi_0 \in Z_P \)).
For the FS$_{01}$-characterization, the set $Z_P$ can be reduced by deleting all functions which are either $x$ or $x \lor y$ after substitution of constants.

(H22) We apply Proposition 4 to $P = \text{FS}(Z_P)$ and $T_0 = \text{FS}(1, x)$. Deleting the functions $x \sim y$ and $x \lor y$ reducible to 1, we obtain $P_0 = \text{FS}(1, x, x \lor y, x \oplus y, x \overline{y}, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_8, \phi_9)$. To obtain the FS$_{01}$-characterization, we delete the function $\overline{x}$ (reducible to 1), $\phi_1, \phi_8$ (reducible to $x \lor y$), $\phi_2$ (reducible to $x \lor y$), and $\phi_3, \phi_4$ (reducible to $x \overline{y}$).

(H23) We apply Proposition 4 to $P = \text{FS}(Z_P)$ and $T_1 = \text{FS}(0, x)$. Deleting the functions $x \oplus y, x \overline{y},$ and $\phi_8$ reducible to 0, we obtain $P_1 = \text{FS}(1, x \lor y, x \sim y, x \lor \overline{y}, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4).$ To obtain the FS$_{1}$-characterization, we delete the function $\overline{x}$ (reducible to 0), $\phi_0, \phi_9$ (reducible to $x \lor y$), $\phi_1, \phi_2$ (reducible to $x \sim y$), and $\phi_2, \phi_4$ (reducible to $x \lor \overline{y}$).

(H25) It is sufficient to consider the case $k = 2$. Let $f \notin A^2$. By the definition, there exist $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ such that $f(x) = f(\beta) = 1$, but $x$ and $\beta$ have no common coordinate 1. Putting $x = x_i$ if $a_i = 1$ and $b_j = 0; y = x_i$ if $a_i = 0$ and $b_j = 1; z = x_i$ if $a_i = b_i = 0$, we obtain a function $g(x, y, z)$ such that $g(1, 0, 0) = f(x) = 1$ and $g(0, 1, 0) = f(\beta) = 1$.

- If $g(0, 0, 0) = 1$ then $g(x, x, x) \in \{1, \overline{x}\} \subseteq Z_A$. Let $g(0, 0, 0) = 0.$
- If $g(1, 0, 1) = 1$ then $g(x, x, y) \in \{x \oplus y, x \lor y\} \subseteq Z_A$. Let $g(1, 0, 1) = 0.$
- If $g(0, 1, 1) = 1$ then $g(x, y, y) \in \{x \oplus y, x \lor y\} \subseteq Z_A$. Let $g(0, 1, 1) = 0.$
- If $g(0, 0, 1) = g(1, 1, 0) = 1$ then $g(x, x, z \oplus z, x \lor z) \subseteq Z_A.$
- If $g(0, 0, 1) = g(1, 1, 0) = 0$ then $g \in \{\phi_8, \phi_9, \phi_{10}\} \subseteq Z_A,$ where $\phi_{10}$ is congruent to $\phi_{10}.$
- If $g(0, 0, 1) = 0$ and $g(1, 1, 0) = 1$ then $g \in \{\phi_2, \phi_{12}\} \subseteq Z_A.$
- If $g(0, 0, 1) = 1$ and $g(1, 1, 0) = 0$ then $g \in \{\phi_1, \phi_{11}\} \subseteq Z_A.$

To prove the second equality we note that it is possible to obtain either $x \oplus y$ or $x \lor y$ from functions belonging to $Z_A$ and having three essential variables by substitution of 0.

(H27) We apply Proposition 4 to $M$ and $A^k$. For the FS$_0$-characterization, we delete redundant functions $\overline{x}, \phi_1, \phi_2, \phi_3, \phi_4$, and $\phi_7$. $\square$

4. Criteria of completeness in Post classes

In the following theorem we omit trivial criteria for classes of types $E, O, P,$ and $P^d$.

**Theorem 3** (Criteria of completeness). Let $\Psi$ be a Post class. A set of Boolean functions $\Phi \subseteq \Psi$ is complete in $\Psi$ if and only if $\Phi$ contains the following functions:

(C1) for $\Psi = T$: $f_0 \notin T_0, f_1 \notin T_1, f_L \notin L, f_S \notin S$, and $f_M \notin M$;
(C2) for $\Psi = T_0$: $f_1 \notin T_1, f_L \notin L, f_M \notin M$, and $f_A \notin A^2$;
(C2d) for $\Psi = T_1$: $f_0 \notin T_0, f_L \notin L, f_M \notin M$, and $f_A \notin A^2$;
(C3) for $\Psi = T_0$: $f_L \notin S, f_M \notin M, f_A \notin A^2$, and $f_A \notin A^2$;
(C4) for $\Psi = L$: $f_0 \notin T_0, f_1 \notin T_1, f_S \notin S$, and $f \notin O$;
(C5) for $\Psi = L_0$: $f_1 \notin T_1$ and $f \notin O$;
(C5d) for $\Psi = L_1$: $f_0 \notin T_0$ and $f \notin O$;
Thus, we have constructed one of the sets: y, x, y, x

The statements (C# d) are dual to (C#), hence their proofs are omitted.

Note that criteria of completeness in classes of type $A_k$ and $A^\infty$ are dual to corresponding criteria in classes of type $A^k$ and $A^\infty$.

**Proof.** The statements (C#d) are dual to (C#), hence their proofs are omitted.

Necessity can be directly checked by choosing required Boolean functions from Table 2, using the fact that $A^k \supset A^\infty$ and the function

$$h_k(x_1, x_2, \ldots, x_{k+1}) = \bigvee_{i=1}^{k+1} x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_{k+1} \in MA^k_1 \setminus A^k \quad (k \geq 2). \quad (3)$$

**Sufficiency.** (C1) Since $T_0 = FS(1, \overline{x})$ and $T_1 = FS(0, \overline{x})$, from $f_0$ and $f_1$ we obtain either $\{0, 1\}$ or $\{\overline{x}\}$. Since $S = FS(Z_S \cup \overline{Z_S})$, where $Z_S = \{0, xy, x \lor y\}$, using $\overline{x}$ and $f_S$ it is possible to obtain either $\{0, 1\}$, or $\{\overline{x}, xy\} = T$, or $\{\overline{x}, x \lor y\} = T$. The last two equalities are implied by representation of an arbitrary Boolean function in a disjunctive normal form and De Morgan’s laws: $x \lor y = \overline{x} \overline{y}$, $xy = \overline{x} \overline{y}$. Using $f_M \notin M = FS_{01}(\overline{x})$, 0 and 1, we construct $\overline{x}$. Thus, we have $\{0, 1, \overline{x}\}$.

Since $L = FS_{01}(Z'_L \cup \overline{Z'_L})$, where $Z'_L = \{xy, x\overline{y}, x \lor y\}$, using 0, 1, $\overline{x}$ and $f_L$ we can construct one of the sets (which are complete in $T$): $\{\overline{x}, xy\}$ or $\{\overline{x}, x \lor y\}$. Indeed, $xy$ is generated by $x\overline{y}$: $x\overline{y} = xy$.

(C2) Using $f_1 \notin T_1 = FS(0, \overline{x})$, we obtain 0, since $\overline{x} \notin T_0$. Using 0 and $f_L \notin L = FS_0(xy, x\overline{y}, x \lor y, y\overline{x}, x \lor \overline{y}, x \lor y)$, it is possible to obtain either $xy$ or $x \lor y$. Indeed, $\overline{x}xy, x \lor \overline{y}, x \lor y \notin T_0$, and using $x\overline{y}$ we can construct $xy$: $x\overline{xy} = xy$.

From 0 and $f_M \notin M = FS(0, x \oplus y, x\overline{y})$ we obtain either $x \oplus y$ or $x\overline{y}$, since $\overline{x} \notin T_0$. Using 0 and $f_A \notin A^2 = FS_0(1, \overline{x}, x \oplus y, x \lor y)$ we can obtain either $x \oplus y$ or $x \lor y$, since $1, \overline{x} \notin T_0$. Thus, we have constructed one of the sets: $\{x \oplus y, xy\}, \{x \oplus y, x \lor y\}$ or $\{x\overline{y}, x \lor y\}$.

A function belongs to $T_0$ if and only if its polynomial has constant coefficient 0. Any such polynomial is generated by the functions $x \oplus y$ and $xy$. Therefore $[x \oplus y, xy] = T_0$. 

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<tr>
<th>Boolean function</th>
<th>$O$</th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$P$</th>
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The set \( \{x \oplus y, x \lor y\} \) is complete in \( T_0 \), since from \( x \oplus y \) it is possible to obtain \( x \oplus y \oplus z \). Further, using \( x \lor y \), it is possible to construct \( xy: x \oplus y \oplus (x \lor y) = xy \). The set \( \{x \bar{y}, x \lor y\} \) is complete in \( T_0 \), since its closure contains \( x \oplus y; x \bar{y} \lor \bar{x} y = x \lor y \).

(C3) Using \( f_S \not\in S = FS(0, 1, xy, x \lor y, \bar{x}y, \bar{x} \lor y) \) we can obtain either \( xy \) or \( x \lor y \) (since \( 0, 1, \bar{x}y, \bar{x} \lor y \not\in T_0 \)). Using \( f_M \not\in M = FS(Z_M) \) we can construct one of the functions: \( \phi_1, \phi_2, \ldots, \phi_8, \phi_3^d, \phi_4^d, \phi_5^d \) or \( \phi_6^d \), since the other functions in \( Z_M \) do not belong to \( T_0 \).

A function belongs to \( T_0 \) if and only if its polynomial has constant coefficient 0 and the total number of terms is odd. Therefore the set \( \Phi = \{xy, \phi_1 = x \oplus y \oplus z\} \) is complete in \( T_0 \). Since \( T_0^d = T_0 \), the dual set \( \Phi^d = \{x \lor y, \phi_1\} \) is complete in \( T_0 \).

Since \( \phi_1 = \phi_2(\phi_2(x, z, y), \phi_2(y, z, x), z) \), we have \( [xy, \phi_2] = T_0 \) and \( [x \lor y, \phi_2] = [xy, \phi_2]^d = T_0 \). From \( \phi_3 = xy \oplus x \oplus z \) (by substitution of \( x \oplus y \oplus z \) for \( z \)) we can obtain \( \phi_1 \). Besides, \( \phi_5(x, y, x) = xy \). Therefore \([\phi_5] = T_0 \). From \( \phi_6 = xy \oplus xz \oplus y \) (by substitution of \( x \oplus y \oplus z \) for \( x \)) we can obtain a function congruent to \( \phi_2 \). Also, \( \phi_6(x, y, x) = x \lor y \).

Further, we consider the functions \( \phi_3, \phi_4 \in A^2 \). From \( f_A \not\in A^2 \) it is possible to obtain either \( \phi_1, \phi_2, \phi_3 \) (they are already considered), or \( x \lor y \). From \( x \lor y \) and \( \phi_3 \) we obtain a function congruent to \( \phi_2 = \phi_3(x \lor y, y, z) \). Hence \([x \lor y, \phi_3] = T_0 \). Substitution of \( x \lor y \) for \( x \) in \( \phi_3 \) gives either \( \phi_6 \) or \( \phi_8 \) (since from \( x \lor y \) we can obtain any linear function with constant coefficient 0). Then, by substitution of 1, we may obtain any linear function. Finally, \([0, x \lor y \oplus 1] = [1, x \lor y]^d = L^d = L \).

Further, we consider the functions \( \phi_3, \phi_4 \in A^2 \) may be considered in the dual way.

(C4) Since \( S = FS(0, 1, xy, x \lor y, \bar{x}y, \bar{x} \lor y) \) and \( f_S \not\in S \) is a linear function, we can obtain either \( 0 \) or \( 1 \) from \( f_S \). If we have 0 then from \( f_0 \not\in T_0 = FS_0(1) \) we construct 1. If we have 1 then from \( f_1 \not\in T_1 = FS_1(0) \) we construct 0. Thus, from \( f_0, f_1 \) and \( f_S \) it is possible to obtain \([0, 1]\). Substitution of 0 in the linear function \( f \not\in O \) gives either \( x \oplus y \) or \( x \oplus y \oplus 1 \). The set \([1, x \lor y]\) is complete in \( L \), since from \( x \lor y \) we can obtain any linear function with constant coefficient 0. Then, by substitution of 1, we may obtain any linear function. Finally, \([0, x \lor y \oplus 1] = [1, x \lor y]^d = L^d = L \).

(C5) From \( f_1 \not\in T_1 = FS(0, \bar{x}) \) we can obtain 0, since \( \bar{x} \not\in L_0 \). From \( f \not\in O \) by substitution 0 we can obtain a linear function of \( f \), namely, \( x \oplus y \) (since \( x \oplus y \oplus 1 \not\in L_0 \)). It is easy to see that \([x \oplus y] = L_0 \).

(C6) From \( f \neq x \) by identifying variables we can obtain \( x \oplus y \oplus z \), which generates \( L_{01} \).

(C7) A linear function belongs to \( L_S \) if and only if the number of its essential variables is odd. From \( f_0 \not\in T_0 = FS(1, \bar{x}) \) we obtain \( \bar{x}, \) since \( 1 \not\in L_2 \). From \( f \not\in O \) by substitution of \( \bar{x} \) and identification of variables we construct \( x \oplus y \oplus z \oplus 1 \). It is easy to show that \([x \oplus y \oplus z \oplus 1] = L_S \).

(C8) Since \( T_0 = FS(1, \bar{x}) \) and \( 1 \not\in S, \bar{x} \in [f_0] \). The set \( Z_L \) contains exactly four self-dual functions, namely, \( \bar{\phi}_0, \bar{\phi}_2, \bar{\phi}_0 \) and \( \bar{\phi}_2 \). Therefore from \( \bar{x} \) and \( f_L \) it is possible to obtain \( \bar{\phi}_0 \) or \( \bar{\phi}_2 \). Since \( \bar{\phi}_2 = \bar{\phi}_0(x, y, \bar{z}) \), it is sufficient to show that \([\bar{\phi}_2] = S \) each function \( f \in S \) can be represented in the following form:

\[
f(x_1, x_2, \ldots, x_n) = x_1 f(1, x_2, \ldots, x_n) \lor \bar{x}_1 \bar{f}(1, \bar{x}_2, \ldots, \bar{x}_n).
\]

Indeed, it holds for \( x_1 = 1 \). If \( x_1 = 0 \) then \( f(0, x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n) = \bar{f}(1, \bar{x}_2, \ldots, \bar{x}_n) \).
By the criterion of completeness in $T$, any Boolean function $f(1, x_2, \ldots, x_n)$ is generated by $p(x, y) = \overline{xy}$. Let $E(p)$ be an expression for $f(1, x_2, \ldots, x_n)$ in terms of $p$. ($E(p)$ is determined by a rooted tree $T_E$ directed from the root. Pendant nodes are labelled with Boolean variables. Each non-pendant node corresponds to an occurrence of $p(x, y)$, and it is incident to two outgoing arcs labelled with $x$ and $y$. Given a function $p' = p'(\ldots, x, \ldots, y, \ldots)$, we may use $T_E$ to construct the corresponding expression $E(p')$ just considering non-pendant nodes of $T_E$ as corresponding to $p'$. Note that in $E(p')$, we may substitute subexpressions for $x$ and/or $y$ only.) We denote $q(x, y) = \overline{p(x, y)} = \overline{xy}$. Clearly, $E(q)$ is an expression for $\overline{f}(1, x_2, \ldots, x_n)$. We define $r(x_1, x, y) = x_1(xy) \lor x_1(xy) = \overline{p2}(x, y, x_1)$. Since $E(r) = E(p)$ if $x_1 = 0$, and $E(r) = E(q)$ if $(x_1 = 1, (4)$ implies that $E(r)$ is an expression for $f$. Thus, $f$ is generated by $r(x_1, x, y) = \overline{p2}(x, y, x_1)$.

(C9) From $f_M \notin M$ we obtain either $\phi_1$ or $\phi_2$, both are in $Z_M \cap S_{01}$. From $f_L \notin L$ we obtain either $\phi_0$ or $\phi_2$; both are in $Z_L \cap S_{01}$. Thus, we have constructed either $\{\phi_0, \phi_1\}$ or $\{\phi_2\}$. We represent an arbitrary Boolean function $f \in S_{01}$ in the form (4). By the criterion of completeness in $T_1$, any function $f(1, x_2, \ldots, x_n) \in T_1$ is generated by $p_1(x, y) = x \lor y$ and $p_2(x, y) = x \land y$. Let $E(p_1, p_2)$ be an expression for $f(1, x_2, \ldots, x_n)$ in terms of $p_1$ and $p_2$. We denote $q_1(x, y) = \overline{p_1(x, y)} = xy$ and $q_2(x, y) = \overline{p_2(x, y)} = x \lor y$. Clearly, $E(p_1, p_2)$ is an expression for $f(1, x_2, \ldots, x_n)$. We define $r_1(x_1, x, y) = x_1(x \lor y) \lor x_1x_2$ and $r_2(x_1, x, y) = x_1(x \land y) \lor x_1x_3$. Since $E(r_1, r_2) = E(p_1, p_2)$ if $x_1 = 0$, and $E(r_1, r_2) = E(q_1, q_2)$ if $x_1 = 1$, (4) implies that $E(r_1, r_2)$ is an expression for $f$. Thus, $f$ is generated by $r_1(x_1, x, y) = \phi_0(x_1, x, y)$ and $r_2(x_1, x, y) = \phi_1(x_1, x, y)$, i.e., $[\phi_0, \phi_1] = S_{01}$. Finally, $\phi_0 = \phi_2(x, y, \phi_2(x, y, z))$ and $\phi_1 = \phi_2(x, z, y, \phi_2(y, z, x), z)$ imply $[\phi_2] = S_{01}$.

(C10) First we show that from $f_O \neq x$ it is possible to construct $\phi_0$. If there exists an $x$ in which exactly one coordinate $a_i$ is 0 and such that $f_O(x) = 0$, then $f_O(\beta) = 0$ for each $\beta \prec x$ (by monotonicity). Clearly, the number of such $\beta$ is a half of all $(0, 1)$-points of length $n$. By self-duality, $f_O$ equals 1 on other sequences. But then $f_O = x_i$, a contradiction. Thus, $f_O$ satisfies the following condition II: $f_O(x) = 1$ for each $x$ having exactly one 0. If $1 < \text{ess}(f_O) \leq 3$ then it is easy to check that $f_O = \phi_0$.

Let $\text{ess}(f_O) \geq 4$. If $x$ has at most one 0 then $f_O(x) = 1$. By self-duality, $f_O(x) = 0$ if $x$ has at most one 0. So there exists $x = (a_1, a_2, \ldots, a_n)$ having at least two 0 and at least two 1, and such that $f_O(x) = 1$. We put $x = x_i$ if $a_i = 0$. We obtain a function $g \in SM$ with $\text{ess}(g) < \text{ess}(f_O)$ satisfying II. Therefore $\text{ess}(g) > 1$. Repeating this process, we obtain a function $g'$ with $1 < \text{ess}(g') \leq 3$, i.e., $\phi_0$.

It remains to show that $[\phi_0] = SM$. Let $f \in SM$. When $\text{ess}(f) \leq 3$ there exist exactly two monotone self-dual functions, namely, $x = \phi_0(x, x, x)$ and $\phi_0$. Let $\text{ess}(f) > 4$. Suppose that every Boolean function with a smaller number of essential variables is generated by $\phi_0$. We define $f_1 = f(x_1, x_2, x_3, x_4, x_5)$. $f_2 = f(x_1, x_2, x_3, x_4, x_5)$, and $f_3 = f(x_1, x_2, x_3, x_4, x_5)$. Let us show that $f = \phi_0(f_1, f_2, f_3)$. We consider an arbitrary point $x = (a_1, a_2, \ldots, a_n)$. We denote by $f_i^x$ the corresponding values of $f_i$, $i = 1, 2, 3$.

Among $a_1, a_2, a_3$ at least two are equal, say $a_1 = a_2$. In other words, $f(x) = f_1^x$. If $a_1 = a_2 = 0$ then $f_2^x \leq f(x) \leq f_1^x$ and $\phi_0(f_1^x, f_2^x, f_3^x) = f(x) \\ f_2^x + f(x) \\ f_3^x = f(x)$ (by monotonicity). Indeed, if $f(x) = 0$ then $f_2^x = 0$, while if $f(x) = 1$ then $f_2^x = 1$. If $a_1 = a_2 = 1$ then $f_3^x \leq f(x) \leq f_2^x$ and $\phi_0(f_1^x, f_2^x, f_3^x) = f(x) \\ f_2^x + f(x) \\ f_3^x = f(x) \\ f_2^x + f(x) \\ f_3^x = f(x)$
(by monotonicity). Indeed, if $f(x) = 0$ then $f_x^z = 0$, while if $f(x) = 1$ then $f_x^z = 1$. By inductive hypothesis, $f_1, f_2, f_3$ are generated by $\phi_0$. Thus, $f$ is generated by $\phi_0$.

(C11) From $f_A$ and $f_x$ we obtain $x \lor y$ and $xy$, respectively (the other functions in $Z_A$ and $Z_{la}$ do not belong to $M_{01}$). We show that $[xy, x \lor y] = M_{01}$. Let $f \in M_{01}$. Recall that $\min T(f)$ is the set of all minimal true points of $f$. For $x = (a_1, a_2, \ldots, a_n) \in \min T(f)$, we construct a conjunction $K_x = x_1 x_2 \ldots x_k$ of all variables $x_i$ such that $a_i = 1$. It is well-known and easy to see that $f = \bigvee_{x \in \min T(f)} K_x$. So $M_{01} = [xy, x \lor y]$.

(C12) From $f_1 \notin T_1$, $f_A \notin A^2$ and $f_x \notin P^d$ we can obtain $0, x \lor y$ (the other functions in $Z_A$ do not belong to $M_{01}$) and $xy$ (the other functions in $Z_{pd}$ either do not belong to $M_0$, or they can be transformed to $xy$ by substitution of $z = 0$). Since $M_{01} = [xy, x \lor y]$ and $M_0 = M_{01} \cup \{0\}$, $M_0 = [0, xy, x \lor y]$. 

(C13) From $f_0 \notin T_0 = FS(1, \bar{x})$, $f_1 \notin T_1 = FS(0, \bar{x})$, $f_2 \notin P = FS(1, x \lor y)$ and $f_3 \notin P^d = FS(0, x \lor y)$, we can obtain $1, 0, x \lor y$ and $xy$, respectively, because $\bar{x} \notin M$. Since $M_{01} = [xy, x \lor y]$, we have $M = M_{01} \cup \{0, 1\} = [0, 1, xy, x \lor y]$. 

(C14) From $f_1 \notin T_1 = FS(0, \bar{x})$ it is possible to obtain $0$, because $\bar{x} \notin A^\infty$. From $f_M \notin M = FS(\bar{x}, x \lor y, xy)$ it is possible to obtain $x \lor y$, because $x, x \lor y \notin A^\infty$. We show that $[x \lor y] = A^\infty$. Clearly, $f \in A^\infty$ if and only if $f$ is of the form $f = xig$, where $g \in T$. The criterion of completeness in $T$ implies that $g$ is generated by $p(x, y) = \bar{x}y$. Let $E(p)$ be an expression for $g$ in terms of $p$. We define $r(x_1, x, y) = x_1(\bar{x}y)$. Since $E(r) \in E(\bar{p})$ if $x_1 = 1$, and $E(r) = 0$ if $x_1 = 0$, we see that $E(r)$ is an expression for $f$. Thus, $f$ is generated by $x_1(\bar{x}y)$. It remains to note that $x \lor y$ generates $xy = x \lor y$ and $x_1 y$.

(C15) From $f_M \notin M$ we can obtain either $\phi_3$ or $\phi_4$, since $Z_M \cap A^\infty = \{\phi_3, \phi_4\}$. An arbitrary function $f \in A^\infty$ is of the form $f = x_1g$, where $g \in T_1$. By the criterion of completeness in $T_1$, $T_1 = [x \sim y, x \lor \bar{y}]$. Using arguments as in (C9), we can show that $f$ is generated by $x_1(x \sim y)$ and $x_1(x \lor \bar{y})$, i.e., by $\phi_3$ and $\phi_4$. Since $\phi_4 = \phi_3(x, \phi_3(y, y, z), z)$ and $\phi_3 = \phi_4(\phi_4(x, y, z), z, y)$,

$$[\phi_3] = [\phi_4] = A^\infty_1. \tag{5}$$

(C16) From $f_P \notin P$ we can obtain $\phi_9$, since $Z_P \cap MA^\infty = \{\phi_9\}$. An arbitrary function $f \in MA^\infty$ is of the form $f = x_1g$, where $g \in M_1$. According to the criterion of completeness in $M$, $M = \{1, xy, x \lor y\}$. Using arguments as in (C9), we can show that $f$ is generated by the functions $x_i, x_i xy$ and $x_1(x \lor y)$ which can be obtained from $\phi_9$, since $xy = \phi_9(x, y, y)$.

(C17) From $f_1 \notin T_1 = FS(0, \bar{x})$ we can obtain $0$, since $\bar{x} \notin MA^\infty$. From $f_P \notin P$ it is possible to obtain $\phi_9$, since $Z_P \cap MA^\infty = \{\phi_9\}$. Since $MA^\infty = MA^\infty_1 \cup \{0\}$ and $[\phi_9] = MA^\infty_1$, $[0, \phi_9] = MA^\infty$.

(C18) From $f_M \notin M$ we can obtain either $\phi_3$ or $\phi_4$, since $Z_M \cap A^\infty_1 = \{\phi_3, \phi_4\}$. By (5), $[\phi_3] = [\phi_4] = A^\infty_1$. Since $f_A \notin A^{k+1}$, there exist $x_i = (a_{i1}, a_{i2}, \ldots, a_{in}), i = 1, 2, \ldots, k + 1$, such that $f(x_i) = 1$, and for each $j = 1, 2, \ldots, n$ the sequence $(a_{ij}, a_{2j}, \ldots, a_{k+1j})$ contains 0. For every $j = 1, 2, \ldots, n$ we define $g_j(x_1, x_2, \ldots, x_{k+1})$ which

- equals 0 on any point with at least two 0’s,
- equals $a_{ij}$ on any point with exactly one 0 (in the $i$th position), and
- equals 1 on 1 = (1, 1, \ldots, 1).
Thus, putting $x(y \lor z), h_k = h_k$, see (3). If $z$ has at least one 0 then $g_j(z) = 0$. We have $f_A(0) = 0$ (since $f_A \in A^k \subseteq T_0$) and $h_k(z) = 0$. Also, $f_A(1) = h_k(1) = 1$. If $z$ has exactly one 0 (say, in the $i$th position), then $g_j(z) = a_{ij}$ and $f_A(a_{i1}, a_{i2}, \ldots, a_{in}) = 1 = h_k(z)$.

The set $\{f_M, f_A\}$.

We show that both Boolean functions $f \in A^k$ is generated by $\phi_4$ and $h_k$. Let $m$ be the number of points $z$ such that $f(z) = 1$. If $m \leq k + 1$ then $f \in A^\infty$ and $f$ is generated by $\phi_4$. Let $m \geq k + 2$. Suppose that the statement holds for smaller $m$. We denote by $\{x_1, x_2, \ldots, x_m\}$ the set of all true points of $f$. We may assume that $x_1 = 1$. For $i = 2, 3, \ldots, m$, let $\psi_i = f$ on all $x_i$, except $\psi_i(x_i) = 0$. Clearly, $\psi_i \in A^{k+1}$. Obviously, $h_k(\psi_2, \psi_3, \ldots, \psi_{k+2}) = f$.

We show that $\psi_i \in [\phi_4, h_k]$. Thus, $f \in [\phi_4, h_k]$.

We construct 0 from $f_M \notin M = FS_0(\overline{x}, x \oplus y, x \overline{y})$, since $\overline{x}, x \oplus y \notin A^k$. As for $A^k$, from $f_A$ we obtain $h_k$, changing values of $g_j$ on 1 only: $g_j(1) = a_{ij}$. By the criterion of completeness in $A^\infty$, $g_j \in A^\infty = [x \overline{y}]$.

Thus, $[x \overline{y}, h_k] = A^k$. Let $f \in A^k$. If $g \in A^k$, then $f$ is generated by $x(y \lor z)$ and $h_k$. Since $x(y \lor z) = x\overline{y} \lor \overline{z}$, $f \in [x \overline{y}, h_k]$. Let $f \notin A^k$. If $f = 0$ then $f = x\overline{y}$. If $f \neq 0$ then we construct a function $\psi(x_1, x_2, \ldots, x_{n+1}) = x_1 x_2 \cdots x_{n+1} \lor f(x_1, x_2, \ldots, x_n)$. It follows from $\psi \in A^{k+1}$ that $\psi$ is generated by $x\overline{y}$ and $h_k$. We substitute $0 = x\overline{y}$ for $x_{n+1}$ in $\psi$ and obtain $f$. Again, $f \in [x \overline{y}, h_k]$.

We show that $\psi_i$ satisfies the property $II$: $f_A(x) = 1$ for each $x = (a_1, a_2, \ldots, a_n)$ having exactly one 0. Indeed, if $a_1 = 0$ and $f_A(x) = 0$ then, by monotonicity, $f_A(\beta) = 0$ for any $\beta = (b_1, b_2, \ldots, b_n)$ with $b_1 = 0$. But then $f_A \in A^\infty \subseteq A^{k+1}$, a contradiction to $f_A \notin A^{k+1}$.

Since $f_A \notin A^{k+1}$, there exist $x_i = (a_{i1}, a_{i2}, \ldots, a_{in}), i = 1, 2, \ldots, k+1$, such that $f_A(x_i) = 1$ and there are no all-1 sequences among $\gamma_j = (a_{j1}, a_{j2}, \ldots, a_{j+1}), j = 1, 2, \ldots, n$. Since $f_A \in A^k$, among $\gamma_j$ there are all points of length $k + 1$ with exactly one 0. Hence $n \geq k + 1$.

Suppose that there exists $\delta = (d_1, d_2, \ldots, d_n)$ such that $f_A(\delta) = 1$ and $|\delta| < k$, where $I = \{i : d_i = 1\}$. We consider points $d_i, i \in I$, having exactly one 0 (in the $i$th position). The points $\delta$ and $d_i$, for all $i \in I$, must have a common 1, since $f_A \in A^k$. By the construction, it does not hold, a contradiction. Hence $f_A(\delta) = 0$ for any $\delta$ having less than $k$ coordinates 1. It follows that if $n = k + 1$ then $f_A = h_k$.

Let $n \geq k + 2$. Since $f_A \notin A^{k+1}$, there exists $\delta = (d_1, d_2, \ldots, d_n)$ such that $\delta$ has at least two coordinates 0, at least $k$ coordinates 1, and $f_A(\delta) = 1$. Putting $x_i = x$ if $d_i = 0$, we obtain a function $g_A$ satisfying $II$. Since $\delta$ has at least $k$ coordinates 1, $g_A$ has at least $k + 1$ variables.

Continuing this process, we construct a function $g$ of $k + 1$ variables which satisfies $II$, i.e., $g = h_k$. Thus, $h_k \in [f_A]$.

When $k = 2$, from $f_A \notin S = FS(0, 1, x\overline{y}, x \lor y, \overline{x \lor y})$ it is possible to obtain $xy$, since $0, 1, x\overline{y}, x \lor y, \overline{x \lor y} \notin MA_1^k$. By substituting $xz$ for $z$ in $h_2 = xy \lor xz \lor yz$, we construct $x(y \lor z)$. Putting $x = x_1$, $y = x_2$ and $z = x_3 = x_4 = \cdots = x_n$ in $h_k$, we obtain $x(y \lor z)$.

We show that $[x(y \lor z), h_k] = MA_1^k$. Let $f \in MA_1^k$. Since $f \in M_{k+1}$, $f = K_1 \lor K_2 \lor \cdots \lor K_m$, where $K_i$ consists of conjunctions corresponding to all minimal true points of $f$. If $m \leq k$
then it follows from \( f \in A^k \) that all minimal true points have a common 1. By the criterion of completeness in \( MA_1^\infty \), \( f \in MA_1^\infty = [x(y \lor z)] \).

We show that the statement holds for \( f \) with \( m \geq k + 1 \) minimal true points. Suppose that it holds for all functions with smaller values of \( m \). We define \( f_j = K_1 \lor K_2 \lor \cdots \lor K_{j-1} \lor K_{j+1} \lor \cdots \lor K_m \) for any \( j = 1, 2, \ldots, m \). Clearly, \( f_j \in M_1 \). If \( f_j(x) = 1 \) then \( f(x) = 1 \). So \( f_j \in A^k \). We obtain \( f_j \in MA_1^k \). By the inductive hypothesis, \( f_j \in [x(y \lor z), h_k] \). Since \( f = h_k(f_1, f_2, \ldots, f_{k+1}) \), we have \( f \in [x(y \lor z), h_k] \).

(C21) From \( f_j \notin T_1 \) we obtain 0. Since \( MA^k = MA_1^k \cup \{0\} \) and \( f_A \neq 0 \), \( f_A \in MA_1^k \). As we have seen, \( h_k \in [f_A] \). When \( k = 2 \), from \( h_2 \) it is possible to construct \( xy \notin S \) (by substitution of 0). By (C20), \( MA_1^k = [xy, h_k] \). Thus, \( MA^k = [0, xy, h_k] \). □

5. Post theorem

Characterizations of Post classes in terms of forbidden subfunctions (Theorem 2) are useful for proving criteria of completeness in these classes (Theorem 3). In turn, it gives a possibility to easily prove the following Post theorem.

**Theorem 4** (Post [26,25]). A class of Boolean functions is closed if and only if it is a Post class.

**Proof.** Necessity. Let \( \Phi \) be a closed class. It is easy to see that if either \( \Phi \subseteq O \) or \( \Phi \subseteq P \), or \( \Phi \subseteq P^d \) then \( \Phi \) is a Post class of type either \( E \), \( O \), \( P \) or \( P^d \). Therefore we assume that \( \Phi \nsubseteq O \), \( P \), \( P^d \). Now we check that \( \Phi \) coincides with a Post class using criteria of completeness (Theorem 3).

**Case 1.** \( \Phi \subseteq L \).

- If \( \Phi \subseteq T_0 \) and \( \Phi \subseteq T_1 \), then \( \Phi = L_{01} \).
- If \( \Phi \subseteq T_0 \) and \( \Phi \nsubseteq T_1 \), then \( \Phi = L_0 \).
- If \( \Phi \nsubseteq T_0 \) and \( \Phi \subseteq T_1 \), then \( \Phi = L_1 \).
- If \( \Phi \nsubseteq T_0 \) and \( \Phi \nsubseteq T_1 \) and \( \Phi \subseteq S \), then \( \Phi = L_S \).
- If \( \Phi \nsubseteq T_0 \) and \( \Phi \nsubseteq T_1 \) and \( \Phi \nsubseteq S \), then \( \Phi = L \).

**Case 2.** \( \Phi \nsubseteq L \) and \( \Phi \subseteq S \).

- If \( \Phi \subseteq M \) then \( \Phi = SM \).
- If \( \Phi \nsubseteq M \) and \( \Phi \subseteq T_0 \), then \( \Phi = S_{01} \) (\( \Phi \subseteq T_0 \cap S \) implies \( \Phi \subseteq T_1 \)).
- If \( \Phi \nsubseteq T_0 \) then \( \Phi = S \) (\( \Phi \nsubseteq T_0 \) and \( \Phi \subseteq S \) imply \( \Phi \nsubseteq M \)).

**Case 3.** \( \Phi \nsubseteq L \), \( S \) and \( \Phi \subseteq A^2 \).

- If \( \Phi \subseteq M \) and \( \Phi \subseteq T_1 \), then either \( \Phi = MA_1^k \), \( k \geq 2 \), or \( \Phi = MA_1^\infty \).
- If \( \Phi \subseteq M \) and \( \Phi \nsubseteq T_1 \), then either \( \Phi = MA^k \), \( k \geq 2 \), or \( \Phi = MA^\infty \).
- If \( \Phi \nsubseteq M \) and \( \Phi \subseteq T_1 \), then either \( \Phi = A_1^k \), \( k \geq 2 \), or \( \Phi = A_1^\infty \).
If $\Phi \not\subseteq M$ and $\Phi \not\subseteq T_1$, then either $\Phi = A^k$, $k \geq 2$, or $\Phi = A^\infty$.

**Case 4.** $\Phi \not\subseteq L, S$ and $\Phi \subseteq a^2$. It is dual to Case 3.

**Case 5.** $\Phi \not\subseteq L, S, a^2$ and $\Phi \subseteq M$.

- If $\Phi \subseteq T_0$ and $\Phi \subseteq T_1$, then $\Phi = M_{01}$.
- If $\Phi \subseteq T_0$ and $\Phi \not\subseteq T_1$, then $\Phi = M_0$.
- If $\Phi \not\subseteq T_0$ and $\Phi \subseteq T_1$, then $\Phi = M_1$.
- If $\Phi \not\subseteq T_0$ and $\Phi \not\subseteq T_1$, then $\Phi = M$.

**Case 6.** $\Phi \not\subseteq L, S, a^2, M$.

- If $\Phi \subseteq T_0$ and $\Phi \subseteq T_1$, then $\Phi = T_{01}$.
- If $\Phi \subseteq T_0$ and $\Phi \not\subseteq T_1$, then $\Phi = T_0$.
- If $\Phi \not\subseteq T_0$ and $\Phi \subseteq T_1$, then $\Phi = T_1$.
- If $\Phi \not\subseteq T_0$ and $\Phi \not\subseteq T_1$, then $\Phi = T$.

Sufficiency is stated in Proposition 2.

Different proofs of the Post theorem were proposed by Kuntzmann [16], Yablonsky et al. [36], Mukhopadhyay [22], Ugolnikov [33], Marchenkov and Ugol’nikov [21], Reschke et al. [28], Reschke and Denecke [27], Pelletier and Martin [23], and Marchenkov [19,20]; see also discussions on the topic in the following monographs: Davio et al. [4], Gavrilov and Sopozhenko [8], Gindikin [9], Glushkov et al. [10], Karpov and Moshchenskii [12], Pippenger [24], and Yablonsky [35]. Ugolnikov [33] used “finite basibility” structure that was first developed by Gavrilov [6,7] and Marchenkov [17]. The method of Reschke and Denecke [27] is based on universal algebra results.

Benzaken [1–3] considered Post classes of monotone Boolean functions and their relation to hypergraph colorings. Many interesting properties of clones were investigated in Shestopal [31,30], Gorlov [11], Rosenberg [29], Yablonsky [34], Gavrilov [7,6], Marchenkov [17,18], Korshunov [13–15], Stetsenko [32], and Foldes and Pogosyan [5]. There is also an extensive bibliography on closed classes in multi-valued logic which has a gap of difficulty even for 3-valued logic.

### References


