Interpolation theorems for domination numbers of a graph

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Received 10 May 1996; revised 6 January 1997; accepted 23 December 1997

Abstract

An integer-valued graph function \( \pi \) is an interpolating function if for every connected graph \( G \), \( \pi(\mathcal{T}(G)) \) is a set of consecutive integers, where \( \mathcal{T}(G) \) is the set of all spanning trees of \( G \). The interpolating character of a number of domination related parameters is considered. \( \text{©} \ 1998 \) Elsevier Science B.V. All rights reserved

Keywords: Interpolation; Domination; Spanning tree

1. Introduction and preliminary results

In 1980, Chartrand \cite{4} raised the following problem: If a graph \( G \) possesses a spanning tree having \( m \) end vertices and another having \( M \) end vertices, where \( M > m \), does \( G \) possess a spanning tree having \( k \) end vertices for every \( k \) between \( m \) and \( M \)? This question was answered affirmatively in \cite{22,1} and it led to a number of papers studying the interpolation properties of parameters of spanning trees of a given graph. In \cite{13}, the various known interpolation results are examined and classified on the basis of the proof techniques used in establishing them. Motivated by results of the papers \cite{2,13,15}, we investigate interpolation properties of domination related parameters of a graph. For the sake of completeness we give a few definitions here. For a connected graph \( G \), let \( \mathcal{T}(G) \) be the set of all spanning trees of \( G \). Let \( T \) be a spanning tree of \( G \) and let \( e \) be an edge of \( G \) which is not in \( T \). If \( f \) is an edge which belongs to the unique cycle of \( T + e \), then \( T + e - f \) is a spanning tree of \( G \) and the transformation of \( T \) into \( T + e - f \) is called a simple edge-exchange. If \( e \) and \( f \) are adjacent edges of \( G \), then the transformation of \( T \) into \( T + e - f \) is called an adjacent edge-exchange. An adjacent edge-exchange of \( T \) into \( T + e - f \) is called an end

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\textit{PII S0012-365X(98)00108-3}
edge-exchange if \( e \) and \( f \) are incident with a common end vertex of \( T \) (and then also of \( T + e - f \)). It is well known that any spanning tree \( T \in \mathcal{T}(G) \) can be transformed into any other spanning tree \( T^* \in \mathcal{T}(G) \) by a sequence of adjacent edge-exchanges. Lovász [19, p. 269] and Harary et al. [12] have proved that if \( G \) is a 2-connected graph, then any \( T \in \mathcal{T}(G) \) can be transformed into any other \( T^* \in \mathcal{T}(G) \) by a sequence of end edge-exchanges.

An integer-valued graph function \( \pi \) is said to interpolate over (the spanning trees of) a connected graph \( G \) if the set \( \pi(\mathcal{T}(G)) = \{ \pi(T) : T \in \mathcal{T}(G) \} \) consists of consecutive integers, i.e. \( \pi(\mathcal{T}(G)) \) is an integer interval. We shall call \( \pi \) an interpolating function if \( \pi \) interpolates over each connected graph.

The interpolating character of different graph parameters was investigated in a number of papers: the number of end vertices in [1, 2, 18, 22], diameter in [12], covering numbers in [14, 16], domination and independence numbers in [11, 13, 15, 23], to quote a few. In this paper we establish interpolating theorems for various types of independence, domination and irredundance numbers of a graph. We also give a coherent and simplified exposition of earlier results. In addition we present several related open questions.

Proving that the number of end vertices in a graph is an interpolating function, Lin [18] observed that it is a consequence of the fact that the number of end vertices interpolates over every unicyclic graph; a unicyclic graph is a connected graph having exactly one cycle. Our first theorem, which is an important tool in this paper, generalizes that observation and it indicates that unicyclic graphs play a significant role when we investigate the interpolation character of integer-valued functions.

**Theorem 1.1.** An integer-valued graph function \( \pi \) is an interpolating function if and only if \( \pi \) interpolates over every unicyclic graph.

**Proof.** The necessity of the condition is clear. To prove the sufficiency, assume that \( \pi \) interpolates over every unicyclic graph. Let \( G \) be any connected graph. It suffices to show that \( \pi(\mathcal{T}(G)) \) is an integer interval. Let \( m \) and \( M \) be the smallest and largest integer of \( \pi(\mathcal{T}(G)) \), respectively. Let \( T_0, T^* \in \mathcal{T}(G) \) be such that \( \pi(T_0) = m \) and \( \pi(T^*) = M \), and let \( T_0, T_1, \ldots, T_n = T^* \) be a sequence of adjacent edge-exchanges transforming \( T_0 \) into \( T^* \). For \( i = 0, 1, \ldots, n-1 \), let \( e_i \) and \( f_i \) be the edges of \( G \) such that \( T_{i+1} = T_i + e_i - f_i \). Since \( T_i + e_i \) is a unicyclic graph, according to our hypothesis \( \pi(\mathcal{T}(T_i + e_i)) \) is an integer interval, \( 0 \leq i \leq n-1 \). Moreover, since \( T_i \) and \( T_{i+1} \) both belong to \( \mathcal{T}(T_i + e_i) \), the integer intervals \( \pi(\mathcal{T}(T_i + e_i)) \) and \( \pi(\mathcal{T}(T_{i+1} + e_{i+1})) \) share a common element \( \pi(T_{i+1}) \) and therefore their union is an integer interval. Consequently, the union \( \bigcup_{i=0}^{n-1} \pi(\mathcal{T}(T_i + e_i)) \) is an integer interval. Finally, \( \{m, m+1, \ldots, M\} \subseteq \bigcup_{i=0}^{n-1} \pi(\mathcal{T}(T_i + e_i)) \subseteq \pi(\mathcal{T}(G)) \subseteq \{m, m+1, \ldots, M\} \) and therefore \( \pi(\mathcal{T}(G)) = \{m, m+1, \ldots, M\} \) is an integer interval. \( \square \)

It follows from Theorem 1.1 that if an integer-valued graph function \( \pi \) is not an interpolating function, then there exists a unicyclic graph \( G \) such that \( \pi \) does not...
interpolate over $G$. The following corollary gives a useful sufficient condition for an integer-valued graph function to be an interpolating function. This corollary was observed by Harary and Plantholt [13] and it follows immediately from Theorem 1.1.

**Corollary 1.2.** An integer-valued graph function $\pi$ is an interpolating function if one of the conditions is satisfied:

1. For every unicyclic graph $H$ and every edge $vu$ of $H$, $\pi(H) \leq \pi(H-vu) \leq \pi(H)+1$.
2. For every unicyclic graph $H$ and every edge $vu$ of $H$, $\pi(H)-1 \leq \pi(H-vu) \leq \pi(H)$.

2. Independence, domination and irredundance numbers of a graph

The **line graph** of a graph $G$ is the graph $L(G)$ with $V(L(G)) = E(G)$ in which two vertices are adjacent if they are adjacent edges of $G$. The **total graph** of $G$ is the graph $T(G)$ with $V(T(G)) = V(G) \cup E(G)$ in which two vertices $x$ and $y$ are adjacent if and only if either $x$ and $y$ are adjacent vertices of $G$, or $x$ and $y$ are adjacent edges of $G$, or $x$ and $y$ are an incident vertex and edge of $G$. The **subdivision graph** of $G$ is the graph $S(G)$ with $V(S(G)) = V(G) \cup E(G)$ and two vertices $x$ and $y$ are adjacent in $S(G)$ if they are an incident vertex and edge of $G$. If $x$ and $y$ are two elements of a graph $G$, i.e., $x, y \in V(G) \cup E(G)$ with $x \neq y$, then the distance $d_G(x, y)$ between $x$ and $y$ is the number of edges in any shortest path that contains both $x$ and $y$ in the total graph $T(G)$; if no such path exists, the distance $d_G(x, y)$ is defined to be $\infty$. Moreover, for $x \in V(G) \cup E(G)$ and a non-empty subset $X \subseteq V(G) \cup E(G)$, $d_G(x, X)$ denotes $\min\{d_G(x, y) : y \in X\}$. For a positive integer $k$, the $k$th **power** of a graph $G$ is the graph $G^k$ with $V(G^k) = V(G)$ and two vertices $x$ and $y$ are adjacent in $G^k$ whenever $0<d_G(x, y)\leq k$. Certainly, for every two positive integers $m$ and $k$, $(G_m)^k = G^{mk}$. It is also easy to observe that $(S(G))^2 = T(G)$ for any graph $G$. A graph $G$ is said to be **chordal** if every cycle of $G$ of length four or more contains an edge joining two non-consecutive vertices of a cycle. A **block graph** is a connected graph in which every block is a clique. It easily follows from [3, Theorem 2.2] (see also [9, Corollary 6.9]) that if $H$ is a block graph, then $H^k$ is a chordal graph for every $k \geq 1$. In particular, if $H$ is a tree, then $H$, $L(H)$ and $S(H)$ are block graphs and we have the following lemma.

**Lemma 2.1.** If $H$ is a tree, then $H^k$, $(L(H))^k$ and $(T(H))^k$ = (($S(H))^2)^k = (S(H))^{2k}$ are chordal graphs for every positive integer $k$.

For a positive integer $k$ and a vertex $x$ of a graph $G$, $N_G^k(x)$ and $N_G^k[x]$ denote $\{y \in V(G) : 0 < d_G(x, y) \leq k\}$ and $N_G^k(x) \cup \{x\}$, respectively. More generally, for a subset $X \subseteq V(G)$, $N_G^k(X)$ and $N_G^k[X]$ denote $\bigcup_{x \in X} N_G^k(x)$ and $\bigcup_{x \in X} N_G^k[x]$, respectively. For a vertex $x \in V(G)$ and a subset $X \subseteq V(G)$, we define $I_G^k[x, X] = N_G^k[x] - N_G^k[X - \{x\}]$. 
We write $N_G(x), N_G[x], N_G(X), N_G[X]$ and $I_G[x,X]$ instead of $N^3_G(x), N^3_G[x], N^3_G(X), N^3_G[X]$ and $I^3_G[x,X]$, respectively.

In this section we consider vertex, edge and mixed (here called total) versions of independent, dominating and irredundant sets in a graph. More precisely, a subset $I$ of $V(G)$ ($E(G)$, $V(G) \cup E(G)$, resp.) is said to be vertex (edge, total, resp.) distance $k$-independent (shortly VD$k$I (ED$k$I, TD$k$I, resp.)) in $G$ if $N^k_G(I) \cap I = \emptyset$ ($N^k_G(I) \cap I = \emptyset$, resp.). A subset $D$ of $V(G)$ ($E(G)$, $V(G) \cup E(G)$, resp.) is said to be vertex (edge, total, resp.) distance $k$-dominating (shortly VD$k$D (ED$k$D, TD$k$D, resp.)) in $G$ if $N^k_G[D] = V(G)$ ($N^k_{L(G)}[D] = E(G)$, $N^k_{I(G)}[D] = V(G) \cup E(G)$, resp.). A subset $X$ of $V(G)$ ($E(G)$, $V(G) \cup E(G)$, resp.) is said to be vertex (edge, total, resp.) distance $k$-irredundant (shortly VD$k$Ir (ED$k$Ir, TD$k$Ir, resp.) in $G$ if $I^k_G[x,X] \neq \emptyset$ ($I^k_{L(G)}[x,X] \neq \emptyset$, $I^k_{I(G)}[x,X] \neq \emptyset$, resp.) for every $x \in X$. A vertex distance 1-independent (1-dominating, 1-irredundant, resp.) set in a graph $G$ is shortly said to be independent (dominating, irredundant, resp.) in $G$. The lower vertex (edge, total, resp.) distance $k$-independence number $i_k(G)$ ($i'_k(G)$, $i''_k(G)$, resp.) of a graph $G$ is defined to be the cardinality of a minimum maximal VD$k$I (ED$k$I, TD$k$I, resp.) set of $G$. The upper vertex (edge, total, resp.) distance $k$-independence number $\gamma_k(G)$ ($\gamma'_k(G)$, $\gamma''_k(G)$, resp.) of $G$ is the cardinality of a maximum VD$k$I (ED$k$I, TD$k$I, resp.) set of $G$. The lower vertex (edge, total, resp.) distance $k$-domination number $\Gamma_k(G)$ ($\Gamma'_k(G)$, $\Gamma''_k(G)$, resp.) of $G$ is the cardinality of a minimum VD$k$D (ED$k$D, TD$k$D, resp.) set of $G$. The upper vertex (edge, total, resp.) distance $k$-domination number $\Gamma_k(G)$ ($\Gamma'_k(G)$, $\Gamma''_k(G)$, resp.) of $G$ is the cardinality of a maximum VD$k$D (ED$k$D, TD$k$D, resp.) set of $G$. It is clear from the above definitions that if $\pi_k$ is one of the six vertex parameters $i_k, \gamma_k, i'_k, \gamma'_k, i''_k, \gamma''_k$ and if $i_k$ and $i''_k$ are, respectively, the edge and total versions of the parameter $\pi_k$, then for any graph $G$,

$$\pi'_k(G) = \pi_k(L(G)) \quad \text{and} \quad \pi''_k(G) = \pi_k(T(G)).$$

(1)

It is also easy to observe that a set $S$ of vertices of a graph $G$ is a VD$k$I (VD$k$D, VD$k$Ir, resp.) set of $G$ if and only if $S$ is a VD$k$I (VD$k$D, VD$k$Ir, resp.) set of $G^k$. Consequently, for any graph $G$,

$$\pi_k(G) = \pi_1(G)^k, \quad \pi'_k(G) = \pi_1((L(G))^k) \quad \text{and} \quad \pi''_k(G) = \pi_1((T(G))^k),$$

(2)

where $\pi_k \in \{i_k, \alpha_k, \gamma_k, \Gamma_k, i'_k, I'_k\}$ and again $\pi'_k$ and $\pi''_k$ are respectively the edge and total versions of $\pi_k$. The parameters $i_1$, $\alpha_1$, $\gamma_1$, $\Gamma_1$, $i'_1$ and $I'_1$ are well known and it is clear (see [7]) that for any graph $H$,

$$ir_1(H) \leq \gamma_1(H) \leq i_1(H) \leq \alpha_1(H) \leq \gamma_1(H) \leq I_1(H).$$

(3)
Now it is clear from (1)-(3) that for any graph $G$,
\begin{align*}
ir_k(G) &\leq \gamma_k(G) \leq \iota_k(G) \leq \kappa_k(G) \leq \Gamma_k(G) \leq IR_k(G), \\
ir'_k(G) &\leq \gamma'_k(G) \leq \iota'_k(G) \leq \kappa'_k(G) \leq \Gamma'_k(G) \leq IR'_k(G), \\
ir''_k(G) &\leq \gamma''_k(G) \leq \iota''_k(G) \leq \kappa''_k(G) \leq \Gamma''_k(G) \leq IR''_k(G).
\end{align*}

Jacobson and Peters proved in [17] that $\alpha_1(G) = \Gamma_1(G) = IR_1(G)$ for any chordal graph $G$. Consequently, by (2) and Lemma 2.1, we have the following useful corollary.

**Corollary 2.2.** If $H$ is a tree, then for any positive integer $k$, $\alpha_k(H) = \Gamma_k(H) = IR_k(H)$. $\alpha'_k(H) = \Gamma'_k(H)$ and $\alpha''_k(H) = \Gamma''_k(H) = IR''_k(H)$.

Harary and Schuster proved in [15] that $\alpha_1$ and $\alpha'_1$ are interpolating functions. The next theorem shows that the upper distance $k$-independence, $k$-domination and $k$-irredundance numbers $\alpha_k$, $\alpha'_k$, $\alpha''_k$, $\Gamma_k$, $\Gamma'_k$, $\Gamma''_k$, $IR_k$, $IR'_k$ and $IR''_k$ are interpolating functions for every positive integer $k$. In the proof, we use the following lemma.

**Lemma 2.3.** For any positive integer $k$ and any edge $vu$ of a graph $G$,
\begin{enumerate}
\item $\alpha_k(G) \leq \alpha_k(G - vu) \leq \alpha_k(G) + 1$;
\item $\alpha'_k(G) - 1 \leq \alpha'_k(G - vu) \leq \alpha'_k(G) + 1$;
\item $\alpha''_k(G) - 1 \leq \alpha''_k(G - vu) \leq \alpha''_k(G) + 1$.
\end{enumerate}

**Proof.** Since every VD$k$I set of $G$ is also VD$k$I in $G - vu$, so $\alpha_k(G) \leq \alpha_k(G - vu)$. In order to prove the inequality $\alpha_k(G - vu) \leq \alpha_k(G) + 1$, let $I$ be a maximum VD$k$I set of $G - vu$. If $I$ is also VD$k$I in $G$, then $\alpha_k(G - vu) = |I| \leq \alpha_k(G) \leq \alpha_k(G) + 1$. Thus assume that $I$ is not VD$k$I in $G$. Then the set $I_0 = \{x \in I : d_G(x, I - \{x\}) \leq k\}$ contains at least two vertices. Note that if $x$ and $y$ are distinct vertices of $I_0$ and $d_G(x, y) \leq k$, then any shortest $x - y$ path passes through $vu$ in $G$. Consequently, the sets $I_r = \{x \in I_0 : d_G(x, v) < d_G(x, u)\}$ and $I_u = \{y \in I_0 : d_G(y, u) < d_G(y, v)\}$ are non-empty and they form a partition of $I_0$. Certainly, if $x$ and $y$ are distinct elements of $I_r$ (or of $I_u$), then $d_G(x, y) > k$. We claim that $|I_r| = 1$ or $|I_u| = 1$. Suppose on the contrary that $|I_r| \geq 2$ and $|I_u| \geq 2$. Let $x_0 \in I_r$ and $y_0 \in I_u$ be such that $d_G(v, x_0) = d_G(v, I_r)$ and $d_G(u, y_0) = d_G(u, I_u)$. Take any $x \in I_r - \{x_0\}$ and any $y \in I_u - \{y_0\}$. Then $d_G(x_0, x) > k$ and $d_G(y, y_0) > k$, while $d_G(x, y_0) \leq k$ and $d_G(y, x_0) \leq k$. Therefore $2k < d_G(x, x_0) + d_G(y, y_0) \leq d_G(x, v) + d_G(v, x_0) + d_G(y, u) + d_G(u, y_0) - (d_G(x, v) + 1 + d_G(u, y_0)) = d_G(x, y_0) + d_G(y, x_0) \leq 2k$, a contradiction which proves the claim. Consequently we may assume that $|I_r| = 1$. Then the set $I - I_r$ is VD$k$I in $G$ and so $\alpha_k(G - vu) - 1 = |I - I_r| \leq \alpha_k(G)$. This completes the proof of (1).

If $I$ is a maximum ED$k$I (TD$k$I, resp.) set of $G$, then $I - \{vu\}$ is ED$k$I (TD$k$I, resp.) in $G - vu$ and therefore $\alpha'_k(G - vu) \geq |I - \{vu\}| \geq \alpha'_k(G) - 1$. $\alpha''_k(G - vu) \geq |I - \{vu\}| \geq \alpha''_k(G) - 1$, resp.). The proofs of the inequalities $\alpha'_k(G - vu) \leq \alpha'_k(G) + 1$ and $\alpha''_k(G - vu) \leq \alpha''_k(G) + 1$ are analogous to that for $\alpha_k(G - vu) \leq \alpha_k(G) + 1$ and are therefore omitted. □
Theorem 2.4. For any positive integer \( k \), the upper distance \( k \)-independence numbers \( \alpha_k \), \( \alpha'_k \) and \( \alpha''_k \) are interpolating functions.

Proof. The fact that \( \alpha_k \) is an interpolating function is obvious from Lemma 2.3(1) and Corollary 1.2(1).

To prove that \( \alpha'_k \) and \( \alpha''_k \) are interpolating functions, after Theorem 1.1, we need only show that \( \alpha'_k \) and \( \alpha''_k \) interpolate over every unicyclic graph. We give the proof for \( \alpha'_k \), the proof for \( \alpha''_k \) is similar. Let \( G \) be a unicyclic graph with \( \alpha'_k(G) = a \) and let \( C \) be the unique cycle of \( G \). By virtue of Lemma 2.3(2), \( \alpha'_k(\mathcal{F}(G)) \subset \{a-1, a, a+1\} \) and to prove that \( \alpha'_k(\mathcal{F}(G)) \) is an integer interval it suffices to show that \( a \in \alpha'_k(\mathcal{F}(G)) \) if \( \{a-1, a+1\} \subset \alpha'_k(\mathcal{F}(G)) \). Suppose on the contrary that \( a \notin \alpha'_k(\mathcal{F}(G)) \) and \( \{a-1, a+1\} \subset \alpha'_k(\mathcal{F}(G)) \). Then \( \alpha'_k(\mathcal{F}(G)) = \{a-1, a+1\} \) and there are adjacent edges \( vu \) and \( uw \) in \( C \) such that \( \alpha'_k(G-vu) = a+1 \) and \( \alpha'_k(G-uw) = a-1 \). Let \( E \) be a maximum \( ED_k \) set of \( G-vu \). Since \( |E| = a+1 > \alpha'_k(G) \), the set \( E \) is not \( ED_k \) in \( G \). Then, as in the proof of Lemma 2.3, the sets \( E_0 = \{e \in E: d_G(e, E - \{e\}) \leq k\} \), \( E_e = \{e \in E: d_G(e, v) < d_G(e, u)\} \) and \( E_u = \{f \in E_0: d_G(f, u) < d_G(f, v)\} \) are non-empty, \( E_e \) and \( E_u \) form a partition of \( E_0 \), \( |E_e| = 1 \) or \( |E_u| = 1 \), and \( E - E_e \) (\( E - E_u \), resp.) is an \( ED_k \) set of \( G \) if \( |E_e| = 1 \) (\( |E_u| = 1 \), resp.). Observe that \( uw \notin E \); otherwise necessarily \( E_u = \{uw\} \) and then \( E - \{uw\} \) would be an \( ED_k \) set of \( G \) or \( G-uw \) which is impossible as \( |E - \{uw\}| = a > \alpha'_k(G-uw) \). But then, if \( |E_e| = 1 \) (\( |E_u| = 1 \), resp.), the set \( E' = E - E_e \) (\( E' = E - E_u \), resp.) is an \( ED_k \) in \( G \) or \( G-uw \) and therefore \( \alpha'_k(G-uw) \geq |E'| = a > a - 1 = \alpha'_k(G-uw) \), a contradiction. This proves that \( \alpha'_k \) is an interpolating function. \( \square \)

Corollary 2.5. For any positive integer \( k \), the upper distance \( k \)-domination numbers \( \Gamma_k \), \( \Gamma'_k \) and \( \Gamma''_k \) and the upper distance \( k \)-irredundance numbers \( IR_k \), \( IR'_k \) and \( IR''_k \) are interpolating functions.

Proof. Since \( \alpha_k \), \( \alpha'_k \), \( \alpha''_k \) are interpolating functions (by Theorem 2.4) and \( \Gamma_k(\mathcal{F}(G)) = IR_k(\mathcal{F}(G)) \), \( \Gamma'_k(\mathcal{F}(G)) = IR'_k(\mathcal{F}(G)) = \alpha'_k(\mathcal{F}(G)) \) and \( \Gamma''_k(\mathcal{F}(G)) = IR''_k(\mathcal{F}(G)) = \alpha''_k(\mathcal{F}(G)) \) for any connected graph \( G \) (by Corollary 2.2), it follows that \( \Gamma_k \), \( IR_k \), \( IR'_k \), \( IR''_k \) are interpolating functions. \( \square \)

A set \( X \) of vertices of a graph \( G \) is a vertex-edge cover in \( G \) if every edge of \( G \) is incident with a vertex of \( X \). Similarly, a set \( Y \) of edges of \( G \) is an edge-vertex cover in \( G \) if every vertex of \( G \) is incident with an edge of \( Y \). The minimum cardinality of a vertex-edge cover in a graph \( G \) is called the vertex-edge covering number of \( G \) and is denoted by \( \alpha_{01}(G) \). The edge-vertex covering number \( \alpha_{10}(G) \) of a graph \( G \) (without isolated vertices) is the minimum cardinality of an edge-vertex cover in \( G \). For any graph \( G \) we have \( \alpha_{01}(G) + \alpha_{10}(G) = |V(G)| \) and Gallai has proved that if \( G \) is a graph without isolated vertices, then \( \alpha_{01}(G) + \alpha_{10}(G) = |V(G)| \) (see [5, Theorem 8.17]). König has proved that \( \alpha_{01}(G) = \alpha'_1(G) \) for every bipartite graph \( G \) (see [5, Theorem 8.18]). Since trees (of order at least 2) are bipartite graphs without
isolated vertices, these facts easily imply that if any of the parameters $x_0, x_1, x_3$ and $x'_1$ is an interpolating function, then each of them is an interpolating function. Thus, since $x_1$ is an interpolating function (by Theorem 2.4), the following result (proved in [14]) is obvious.

**Corollary 2.6.** The vertex-edge covering number $x_0$ and the edge-vertex covering number $x_1$ are interpolating functions.

Meir and Moon [20, Theorem 7] (see also [3, Theorem 4.1] or Domke et al. [8, Theorem 4]) have proved that if $T$ is a tree, then $y_k(T) = x_{2k}(T)$ for any positive integer $k$. Thus it follows readily from Theorem 2.4 that $y_k$ is an interpolating function. Two different proofs of the fact that $y_1'$ is an interpolating function were given by Harary, Schuster and Vestergaard [16]. (In [16], $y_1'$ is called the edge-edge covering number and is denoted by $x_{11}$). We now give a short and self-contained proof of the fact that each of the three lower distance $k$-domination numbers $y_k, y'_k$ and $y''_k$ is an interpolating function. We begin with the lemma which describes how $y_k, y'_k$ and $y''_k$ vary as we delete an edge from a graph.

**Lemma 2.7.** For any positive integer $k$ and any edge $vu$ of a graph $G$,

1. $y_k(G) \leq y_k(G - vu) \leq y_k(G) + 1$;
2. $y'_k(G) - 1 \leq y'_k(G - vu) \leq y'_k(G) + 1$;
3. $y''_k(G) - 1 \leq y''_k(G - vu) \leq y''_k(G) + 1$.

**Proof.** (1) If $C$ is a minimum VDKD set of $G - vu$, then $C$ is VDKD in $G$ and therefore $y_k(G) \leq |C| = y_k(G - vu)$. To prove that $y_k(G - vu) \leq y_k(G) + 1$, let $D$ be a minimum VDKD set of $G$. If $D$ is VDKD in $G - vu$, then $y_k(G - vu) \leq |D| = y_k(G) \leq y_k(G) + 1$. Thus assume that $D$ is not VDKD in $G - vu$. Then the set $V_0 = \{x \in V(G) - D: d_{G - vu}(x, D) > k\}$ is non-empty and every path of length at most $k$ joining a vertex of $V_0$ to a vertex of $D$ (in $G$) contains the edge $vu$. This implies that $d_G(v, D) \neq d_G(u, D)$, say $d_G(v, D) < d_G(u, D)$. Then it is easy to observe that the set $D \cup \{u\}$ is VDKD in $G - vu$ and hence $y_k(G - vu) \leq |D \cup \{u\}| = y_k(G - vu) + 1$.

(2) It is obvious that if $E$ is an EDKD set of $G - vu$, then $E \cup \{vu\}$ is EDKD in $G$. Thus, $y'_k(G) \leq y'_k(G - vu) + 1$. To show that $y'_k(G - vu) \leq y'_k(G) + 1$, let $F$ be a minimum EDKD set of $G$. We consider two cases.

Case 1. $vu \notin F$. Certainly, if $F$ is an EDKD set of $G - vu$, then $y'_k(G - vu) \leq |F| = y'_k(G) \leq y'_k(G) + 1$. Thus assume that $F$ is not EDKD in $G - vu$. Then the set $E_0 = \{e \in E(G - vu) - F: d_{G - vu}(e, F) > k\}$ is non-empty and every path of length at most $k$ joining a vertex of $E_0$ to a vertex of $F$ (in $L(G)$ or $T(G)$) contains $vu$. This implies that $d_G(v, F) \neq d_G(u, F)$, say $d_G(v, F) < d_G(u, F)$. Then, if $f$ is an edge incident with $u$ in $G - vu$, the set $F \cup \{f\}$ is an EDKD set in $G - vu$ and $y'_k(G - vu) \leq |F \cup \{f\}| = y'_k(G) + 1$.

Case 2. $vu \in F$. If $F - \{vu\}$ is an EDKD set of $G - vu$, then $y'_k(G - vu) \leq |F - \{vu\}| = y'_k(G) - 1 \leq y'_k(G) + 1$. If $F - \{vu\}$ is not an EDKD set of $G - vu$, then, similarly as in Case 1, adding to $F - \{vu\}$ one or two edges of $G - vu$ that are incident with $v$
or \( u \), we form an ED\( k \)D set of \( G - vu \) of cardinality at most \(|F| + 1\). Consequently, 
\[ \gamma_k'(G - vu) \leq |F| + 1 = \gamma_k'(G) + 1. \]

(3) If \( D \) is a TD\( k \)D set of \( G - vu \), then the set \( D \cup \{vu\} \) is TD\( k \)D in \( G \) and so 
\[ \gamma_k'(G) \leq \gamma_k'(G - vu) + 1. \]

In a quite similar way as in the proof of (2), we obtain that if \( D \) is a minimum TD\( k \)D set of \( G \), then at least one of the sets \( D \cup \{v\} \), \( D \cup \{u\} \) if \( vu \notin D \) (\( G \), \( D - \{vu\} \) \( \cup \{v\} \), \( (D - \{vu\}) \cup \{u\} \), \( (D - \{vu\}) \cup \{v,u\} \) if \( vu \in D \)) is a TD\( k \)D set of \( G - vu \) which, in turn, implies that 
\[ \gamma_k'(G - vu) \leq \gamma_k'(G) + 1. \]

**Theorem 2.8.** For any positive integer \( k \), the lower distance \( k \)-domination numbers \( \gamma_k, \gamma'_k \) and \( \gamma''_k \) are interpolating functions.

**Proof.** The interpolating character of \( \gamma_k \) follows from Lemma 2.7(1) and Corollary 1.2(1).

We now prove that \( \gamma''_k \) is an interpolating function, the proof for \( \gamma'_k \) is similar and we omit it. By Theorem 1.1, it suffices to show that \( \gamma''_k \) interpolates over every uniciclyclic graph. Therefore, suppose that \( G \) is a uniciclyclic graph with \( \gamma''_k(G) = a \) and let \( C \) be the unique cycle of \( G \). Because of Lemma 2.7(3), the set \( \gamma''_k(\mathcal{F}(G)) \) is a subset of \( \{a - 1, a, a + 1\} \) and therefore the proof that \( \gamma''_k(\mathcal{F}(G)) \) is an integer interval will be finished if we show that \( a \in \gamma''_k(\mathcal{F}(G)) \) whenever \( \{a - 1, a + 1\} \) is a subset of \( \gamma''_k(\mathcal{F}(G)) \). Assuming the contrary, we can find adjacent edges \( vu \) and \( uw \) in \( C \) such that \( \gamma''_k(G - vu) = a - 1 \) and \( \gamma''_k(G - uw) = a + 1 \). Let \( D \) be a minimum TD\( k \)D set of \( G - vu \). Then, since \( D \) is not a TD\( k \)D set of \( G \) (as \( |D| = a - 1 < \gamma''(G) \)), the edge \( vu \) is the only element \( x \) of \( G \) for which \( d_G(x,D) > k \). Consequently, since \( D \) is a TD\( k \)D set of \( G - vu \), \( d_{G - vu}(u,D) = k \) and \( d_{G - vu}(e,D) = k \) for every edge \( e \) incident with \( u \) in \( G - vu \), which, in turn, implies that no path of length at most \( k \) joining \( w \) to an element of \( D \) (in \( T(G - vu) \)) contains \( u \) or \( uw \). Then it is easy to observe that \( D \cup \{u\} \) (as well as \( D \cup \{vu\} \)) is a TD\( k \)D set of \( G - uw \) and so 
\[ \gamma''_k(G - uw) \leq |D| + 1 = a < a + 1 = \gamma''_k(G - uw) \], a contradiction completing the proof.

We now turn our attention to interpolation properties of the lower distance \( k \)-independence numbers \( i_k, i'_k \) and \( i''_k \). One can verify that if \( T \) is a spanning tree of the uniciclyclic graph \( G_k \) given in Fig. 1, then \( i_k(T) = 2 \) if \( T = G - x_kx_{k+1} \), while \( i_k(T) = 4 \) for every other spanning tree \( T \) of \( G \). Thus, \( i_k(\mathcal{F}(G_k)) = \{2, 4\} \) and this example shows that the lower vertex distance \( k \)-independence number \( i_k \) does not necessarily interpolate over an arbitrary connected graph. (For \( i_1 \) this was observed by Harary and Schuster [15].) However, Harary and Plantlolt [13] have proved that \( i_1 \) interpolates over every 2-connected graph. We now prove that \( i_2 \) has the same property.

**Theorem 2.9.** The lower vertex distance 1- and 2-independence numbers \( i_1 \) and \( i_2 \) interpolate over every 2-connected graph.
Proof. We prove that $i_2$ interpolates over every 2-connected graph, the proof for $i_1$ is similar. Assume $G$ is a 2-connected graph and let $m$ and $M$ be respectively the smallest and largest integer of $i_2(\mathcal{F}(G))$. Let $T_0$ and $T^*$ be spanning trees of $G$ such that $i_2(T_0) = m$ and $i_2(T^*) = M$. Since $G$ is 2-connected, there exists a sequence of end edge-exchanges $T_0, T_1, \ldots, T_n = T^*$ transforming $T_0$ into $T^*$ (see [19, p. 269; 12]). To prove that $i_2(\mathcal{F}(G))$ is an integer interval, we need only show that each step of the end edge-exchange may increase the value of $i_2$ by at most one, that is $i_2(T_{l+1}) \leq i_2(T_l) + 1$ for $l = 0, \ldots, n - 1$, which, in turn, implies that the sequence $(i_2(T_0), i_2(T_1), \ldots, i_2(T_n))$ contains $(m, m + 1, \ldots, M)$ as a subsequence and consequently $i_2(\mathcal{F}(G)) = \{m, m + 1, \ldots, M\}$. Let $I$ be a minimum maximal $V_{D2I}$ set of $T_l$ and suppose that $T_{l+1} = T_l - uv + vw$, where $v$ is an end vertex of $T_l$ (and of $T_{l+1}$). We consider two cases.

Case 1. $v \notin I$. Then $I$ is a maximal $V_{D2I}$ set of $T_l$ and $T_{l+1}$ and either $I$ or $I \cup \{v\}$ is a maximal $V_{D2I}$ set of $T_{l+1}$. Thus, $i_2(T_{l+1}) \leq |I| + 1 = i_2(T_l) + 1$.

Case 2. $v \in I$. If $I - \{v\}$ is a maximal $V_{D2I}$ set of $T_l$, then either $I - \{v\}$ is a maximal $V_{D2I}$ set of $T_{l+1}$ and $i_2(T_{l+1}) \leq i_2(T_l) + 1$. If $I - \{v\}$ is not a maximal $V_{D2I}$ set of $T_l$, then the set $N^2_I(v) - N^2_I[I - \{v\}]$ is non-empty and for any $x \in N^2_I(v) - N^2_I[I - \{v\}]$, $I' = (I - \{v\}) \cup \{x\}$ is a maximal $V_{D2I}$ set of $T_l$ and $v \notin I'$. (It is worth pointing out just here that if $k > 2$, then there does not necessarily exist $x$ in $N^2_I(v) - N^2_I[I - \{v\}]$ such that $I' = (I - \{v\}) \cup \{x\}$ is a (maximal) $V_{DkI}$ set of $T_l$.) Consequently, as in Case 1, $i_2(T_{l+1}) \leq i_2(T_l) + 1$. □

It is well known (see [5, p. 249]) and easy to prove that $\gamma'_1(G) = i'_1(G)$ for every graph $G$. Consequently, by Theorem 2.8, we have the following fact.

Corollary 2.10. The lower edge distance 1-independence number $i'_1$ is an interpolating function.

Although $\gamma'_1(G) = i'_1(G)$ for every graph $G$, the graph $G_k$ of Fig. 2 illustrates that the equality $\gamma'_1(G) = i'_1(G)$ is not necessarily true for every graph $G$ if $k \geq 2$. The same example shows that the lower edge distance $k$-independence number $i'_k$ is not an interpolating function for $k \geq 2$. We have, however, the following theorem for $i'_2$.

Theorem 2.11. The lower edge distance 2-independence number $i'_2$ interpolates over every 2-connected graph.
Proof. Let $G$ be a 2-connected graph. As in the proof of Theorem 2.9, it is enough to show that $i'_2(T') \leq i'_2(T) + 1$ for every end edge-exchange of a spanning tree $T$ into a spanning tree $T' = T - vu + uw$, where $u$ is an end vertex of $T$ and of $T'$. Let $I$ be a minimum maximal ED2I set of $T$. We consider two cases.

Case 1. $vu \notin I$. Then $I$ is a maximal ED2I set of $T - vu$ and $I$ or $I \cup \{uw\}$ is a maximal ED2I set of $T'$. Thus, $i'_2(T') \leq |I| + 1 = i'_2(T) + 1$.

Case 2. $vu \in I$. If $I - \{vu\}$ is a maximal ED2I set of $T - vu$, then $I - \{vu\}$ or $(I - \{vu\}) \cup \{uw\}$ is a maximal ED2I set of $T'$ and certainly $i'_2(T') \leq i'_2(T) + 1$. Thus assume that $I - \{vu\}$ is not a maximal ED2I set of $T - vu$. Then the set $E_0 = \{xy \in E(T) - I: d_T(xy, vu) \leq 2 \}$ is non-empty. Now, if every edge of $E_0$ is adjacent to $vu$, then for any $xy \in E_0$, $I' = (I - \{vu\}) \cup \{xy\}$ is a minimum maximal ED2I set of $T - vu$ and, as in Case 1, $i'_2(T') \leq i'_2(T) + 1$. Finally, assume that $E_0$ contains an edge $xy$ such that $d_T(xy, vu) = 2$. Let $x'y'$ be the unique edge of $T$ adjacent to both $xy$ and $vu$ in $T$. Since $d_T(xy, I - \{vu\}) \geq 2$ and $d_T(vu, I - \{vu\}) \geq 2$, necessarily $d_T(x'y', I - \{vu\}) \geq 2$ and again it is easy to observe that $I' = (I - \{vu\}) \cup \{x'y'\}$ is a minimum maximal ED2I set of $T$, $vu \notin I'$ and so $i'_2(T') \leq i'_2(T) + 1$. \]

The graph $G_k$ of Fig. 1 illustrates that the lower total distance $k$-independence number $i''_k$ does not necessarily interpolate over a connected graph as $i''_k(\mathcal{G}(G_k)) = (2, 4)$ for $k \geq 1$. However, for $i''_1$ and $i''_2$ we have the following theorem.

Theorem 2.12. The lower total distance 1- and 2-independence numbers $i''_1$ and $i''_2$ interpolate over any 2-connected graph.

Proof. Let $G$ be a 2-connected graph. As in the proofs of Theorems 2.9 and 2.11, it is enough to show that $i''_1(S) \leq i''_1(R) + 1$ and $i''_2(S) \leq i''_2(R) + 1$ for every end edge-exchange of a spanning tree $R$ into a spanning tree $S = R - vu + uw$ of $G$, where $u$ is an end vertex of $R$ and of $S$. Since the two cases are analogous, we shall prove the inequality $i''_2(S) \leq i''_2(R) + 1$ as an example. Let $J$ be a minimum maximal TD2I set of $R$. We consider three subcases.
Case 1. If $J \cap \{u, uv\} = \emptyset$. Then either $J$ or $J \cup \{u\}$ is a maximal TD2I set of $S$ and therefore $i''_s(S) \leq |J| + 1 = i''_s(R) + 1$.

Case 2. $u \in J$. If $I^2_{R}[u,J] \subseteq \{u, uv\}$, i.e.
the only possible distance-2 private neighbours for $u$ with respect to $J$ in the total graph on $R$, then either $J \setminus \{u\}$ or $J$ is a maximal TD2I set of $S$ and so certainly $i''_s(S) \leq i''_s(R) + 1$. On the other hand, if $I^2_{R}[u,J] \not\subseteq \{u, uv\}$, then for any $x \in I^2_{R}[u,J] \setminus \{u, uv\}$, either $(J \setminus \{u\}) \cup \{x\}$ or $J \cup \{x\}$ is a maximal TD2I set of $S$ and again $i''_s(S) \leq i''_s(R) + 1$.

Case 3. $uv \in J$. In this case $\{u, uv\} \subseteq I^2_{R}[u,J] \subseteq \{v\} \cup N_R(v) \cup \{vx: x \in N_R(v)\} \cup \{xy: x \in N_R(v), y \in N_R(x) \setminus \{v\}\}$. It is easy to observe that if $I^2_{R}[uv,J] = \{u, uv\}$, then either $J \setminus \{uv\}$ or $(J \setminus \{uv\}) \cup \{u\}$ is a maximal TD2I set of $S$ and therefore $i''_s(S) \leq i''_s(R) + 1$. Thus assume that $I^2_{R}[uv,J] \setminus \{u, uv\} \neq \emptyset$. Certainly, if $v \in I^2_{R}[uv,J] \setminus \{u, uv\}$, then either $(J \setminus \{uv\}) \cup \{v\}$ or $(J \setminus \{uv\}) \cup \{v, u\}$ is a maximal TD2I set of $S$ and so $i''_s(S) \leq i''_s(R) + 1$. Similarly, if $x \in I^2_{R}[uv,J]$ for some $x \in N_R(v) \setminus \{u\}$, then either $(J \setminus \{uv\}) \cup \{vx\}$ or $(J \setminus \{uv\}) \cup \{vx, u\}$ is a maximal TD2I set of $S$ and $i''_s(S) \leq i''_s(R) + 1$.

Now, to complete the proof, it suffices to show that if $x \in I^2_{R}[uv,J]$ for some $x \in N_R(v) \setminus \{u\}$ or if $xy \in I^2_{R}[uv,J]$ for some $x \in N_R(v) \setminus \{u\}$ and $y \in N_R(x) \setminus \{v\}$, then $vx \in I^2_{R}[uv,J]$. We show this by contradiction.

Suppose first that there exists $x$ in $I^2_{R}[uv,J] \cap (N_R(v) \setminus \{u\})$ such that $vx \notin I^2_{R}[uv,J]$. Then $vx \in N_R^2(J \setminus \{uv\})$ and therefore there exists $t$ in $J \setminus \{uv\}$ such that $d_R(vx,t) = d_R(vx,J \setminus \{uv\}) \leq 2$. Since $x \in I^2_{R}[uv,J]$, $d_R(x,t) > 2$ which combined with $d_R(vx,t) \leq 2$ implies that $t$ is an edge incident with a vertex of $N_R(v) \setminus \{u, x\}$. Consequently, $d_R(t,uv) = 2$ and $J$ is not a TD2I set of $R$, a contradiction.

Finally, suppose that there are vertices $x \in N_R(v) \setminus \{u\}$ and $y \in N_R(x) \setminus \{v\}$ such that $xy \in I^2_{R}[uv,J]$ and $vx \notin I^2_{R}[uv,J]$. Then again there exists $t$ in $J \setminus \{uv\}$ such that $d_R(vx,t) = d_R(vx,J \setminus \{uv\}) \leq 2$. Since $xy \in I^2_{R}[uv,J]$, $d_R(xy,t) > 2$ which combined with $d_R(vx,t) \leq 2$ implies that either $t \in N_R(v) \setminus \{u, x\}$ or $t$ is an edge incident with a vertex of $N_R(v) \setminus \{u, x\}$ which is not incident with $v$. Then $d_R(t,uv) = 2$ and $J$ is not a TD2I set of $R$, a final contradiction. \(\square\)

3. Domination variants

This section is devoted to establishing the interpolating character of next variants of domination parameters. We begin with the $n$-domination and $n$-dependence numbers of a graph introduced by Fink and Jacobson [10]. Let $n$ be a positive integer. An $n$-dominating set of a graph $G$ is a subset $D$ of $V(G)$ such that $|N_G(v) \setminus D| \geq n$ for every $v \in V(G) - D$. The $n$-domination number of $G$, denoted by $\gamma_n(G)$, is the minimum cardinality of an $n$-dominating set of $G$. An $n$-depending set of a graph $G$ is a set $I \subseteq V(G)$ such that $|N_G(v) \cap I| < n$ for every $v \in I$. The $n$-dependence number of $G$, denoted by $\alpha_n(G)$, is the maximum cardinality of an $n$-depending set of $G$. Certainly, $\gamma_n(G) = \gamma_1(G)$ and $\alpha_n(G) = \alpha_1(G)$. We now show that $\alpha_n$ and $\gamma_n$ are interpolating functions.
Lemma 3.1. For any positive integer any edge \( vu \) of a graph \( G \),

1. \( \alpha(n)(G) \leq \alpha(n)(G - vu) \leq \alpha(n)(G) + 1 \);
2. \( \gamma(n)(G) \leq \gamma(n)(G - vu) \leq \gamma(n)(G) + 1 \).

Proof. (1) Since every \( n \)-depending set of \( G \) is \( n \)-depending in \( G - vu \), we have \( \alpha(n)(G) \leq \alpha(n)(G - vu) \).

On the other hand, if \( J \) is a maximum \( n \)-depending set of \( G - vu \), then it is easy to observe that at least one of the sets \( J, J - \{v\} \) or \( J - \{u\} \) is \( n \)-depending in \( G \) and so \( \alpha(n)(G) \geq |J| - 1 = \alpha(n)(G - vu) - 1 \).

(2) Since every \( n \)-dominating set of \( G - vu \) is also \( n \)-dominating in \( G \), we obtain \( \gamma(n)(G - vu) \geq \gamma(n)(G) \).

To prove the last inequality, let \( D \) be a minimum \( n \)-dominating set of \( G \). If \( |D \cap \{v, u\}| = 2 \) or \( D \cap \{v, u\} = \emptyset \), then \( D \) is an \( n \)-dominating set of \( G - vu \) and therefore \( \gamma(n)(G - vu) \leq \gamma(n)(G) \leq \gamma(n)(G) + 1 \). Finally if \( |D \cap \{v, u\}| = 1 \), say \( D \cap \{v, u\} = \{v\} \), then \( D \cup \{v\} \) is \( n \)-dominating in \( G - vu \) and \( \gamma(n)(G - vu) \leq |D \cup \{v\}| = \gamma(n)(G) + 1 \). 

From Lemma 3.1 and Corollary 1.2 we immediately have the following theorem.

Theorem 3.2. For any positive integer \( n \), the \( n \)-domination number \( \gamma(n) \) and the \( n \)-dependence number \( \alpha(n) \) are interpolating functions.

A set \( S \) of vertices of a graph \( G \) is said to be a global dominating set of \( G \) if \( S \) is a dominating set both of \( G \) and of its complement \( \overline{G} \). The global domination number of \( G \), denoted by \( \gamma_g(G) \), is the minimum cardinality of a global dominating set of \( G \). The global domination number was introduced by Sampathkumar [21]. Obviously, a set \( S \subseteq V(G) \) is a global dominating set of \( G \) if and only if \( N_G(v) \cap S \neq \emptyset \) and \( S - N_G(v) \neq \emptyset \) for each \( v \in V(G) - S \). We now prove that the global domination number is an interpolating function. First we show that deletion of a single edge from a graph may change its global domination number by at most 1.

Lemma 3.3. If \( vu \) is an edge of a graph \( G \), then

\( \gamma_g(G) - 1 \leq \gamma_g(G - vu) \leq \gamma_g(G) + 1 \).

Proof. Let \( S \) be a minimum global dominating set of \( G \). If \( S \) is dominating in \( G - vu \), then \( S \) is also a global dominating set of \( G - vu \) and \( \gamma_g(G - vu) \leq |S| \leq \gamma_g(G) + 1 \). If \( S \) is not a dominating set of \( G - vu \), then \( |S \cap \{v, u\}| = 1 \), say \( v \in S \) while \( u \in V(G) - S \). But now \( S \cup \{u\} \) is a global dominating set of \( G - vu \), so \( \gamma_g(G - vu) \leq |S \cup \{u\}| \leq \gamma_g(G) + 1 \).

In order to prove the remaining inequality, let \( R \) be a minimum global dominating set of \( G - vu \). If \( R \) is a global dominating set of \( G \), then \( \gamma_g(G) \leq |R| + 1 = \gamma_g(G - vu) + 1 \). Finally, if \( R \) is not a global dominating set of \( G \), then \( |R \cap \{v, u\}| = 1 \), say \( v \in R \) and \( u \in V(G) - R \). In this case \( R \cup \{u\} \) is a global dominating set of \( G \) and therefore \( \gamma_g(G) \leq |R \cup \{u\}| = \gamma_g(G - vu) + 1 \). 

\[ \square \]
Theorem 3.4. The global domination number $\gamma_g$ is an interpolating function.

Proof. Suppose that the theorem is false. Then it follows from Theorem 1.1 and Lemma 3.3 that there exists a unicyclic graph $G$ such that $\gamma_g(F(G)) = \{a - 1, a + 1\}$, where $a = \gamma_g(G)$. Moreover, the unique cycle $C$ of $G$ contains adjacent edges $vu$ and $uw$ such that $\gamma_g(G - vu) = a - 1$ and $\gamma_g(G - uw) = a + 1$. Let $S$ be a minimum global dominating set of $G$. Then $S$ is dominating in $G$ but it is not a global dominating set of $G$ as $|S| = a - 1 < \gamma_g(G)$. In addition, $|S \cap \{v, u\}| = 1$ (otherwise $S$ would be a global dominating set of $G$) and the unique vertex of $\{v, u\} - S$ is the only vertex $x$ of $V(G) - S$ for which $S \subseteq N_G(x)$. We consider two cases.

Case 1. $\{v, u\} - S = \{u\}$. Since $S \subseteq N_G(u)$ and $S \not\subseteq N_{G-vu}(y)$ for each $y \in V(G) - S$, $S \cup \{u\}$ is a global dominating set of $G - uw$. Thus, $\gamma_g(G - uw) \leq |S \cup \{u\}| = a < a + 1 = \gamma_g(G - uw)$, a contradiction.

Case 2. $\{v, u\} - S = \{v\}$. In this case $S \subseteq N_G(v)$ and $S \cup \{v\}$ is a global dominating set of $G$. If $(N_G[w] - \{u\}) \cap (S \cup \{v\}) \neq \emptyset$, then $S \cup \{v\}$ is a global dominating set of $G - uw$ and therefore $\gamma_g(G - uw) \leq |S \cup \{v\}| = a < \gamma_g(G - uw)$, a contradiction. Thus assume that $(N_G[w] - \{u\}) \cap (S \cup \{v\}) = \emptyset$. Then $v \not\in N_G(w)$ and the length of $C$ is at most four. But now $S \cup \{w\}$ is a global dominating set of $G - uw$ and $\gamma_g(G - uw) \leq |S \cup \{w\}| = a < \gamma_g(G - uw)$, a final contradiction. $\square$

A set $D$ of vertices in a graph $G$ is a total dominating set if each vertex of $G$ is adjacent to a vertex in $D$. Total dominating sets were first defined and studied by Cockayne et al. [6]. The cardinality of a minimum total dominating set in a graph $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$. This parameter is only defined for graphs without isolated vertices. The total domination number is not an interpolating function. This follows from the counterexample shown in Fig. 3, in which the unicyclic graph $G$ has only two nonisomorphic spanning trees $T_1$ and $T_2$ with $\gamma_t(T_1) = 4$ and $\gamma_t(T_2) = 6$; the solid vertices of each tree indicate a minimum total dominating set in this tree. For 2-connected graphs we have the following theorem.

Theorem 3.5. The total domination number $\gamma_t$ interpolates over any 2-connected graph.

Proof. Let $G$ be a 2-connected graph. As in the proof of Theorem 2.9, it is enough to show that $\gamma_t(T') \leq \gamma_t(T) + 1$ for every end edge-exchange of a spanning tree $T$ into a spanning tree $T' = T - vu + uw$ of $G$, where $u$ is an end vertex of $T$ and of $T'$. To show this, let $D$ be any minimum total dominating set of $T$. Since $T \neq K_2$, we may assume that $u \not\in D$; otherwise it follows from the minimality of $D$ that $N_T[v] \cap D = \{v, u\}$.
and then for any \( x \in N_T(v) - \{u\} \), \( (D - \{u\}) \cup \{x\} \) is a minimum total dominating set of \( T \) with the desired property. It is now easy to observe that \( D \cup \{w\} \) is a total dominating set in \( T' \). Thus, \( \gamma_t(T') \leq |D \cup \{w\}| \leq \gamma_t(T) + 1 \). \( \square \)

4. Conclusion

Theorem 1.1 gives a necessary and sufficient condition for a graphical invariant to be an interpolating function with respect to the family of spanning trees of a graph. It would be interesting to have a counterpart of that theorem for interpolation problems with respect to other families of subgraphs of a graph, see [1,2,13] for examples of such families. In Section 2 we have investigated the interpolating character of distance \( k \)-independence, \( k \)-domination and \( k \)-irredundance numbers. Because some questions which arose from our investigation remain unanswered, we pose questions:

- Do \( i_k \) (\( k \geq 3 \)), \( i'_k \) (\( k \geq 3 \)) and \( i''_k \) (\( k \geq 3 \)) interpolate over every 2-connected graph?
- Do \( tr_k \), \( tr'_k \) and \( tr''_k \) interpolate over every (2-connected) graph?

We do not have answers to these questions, but feel they are worth investigating.

References