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A maximum principle for combinatorial Yamabe flow

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Abstract

This article studies a discrete geometric structure on triangulated manifolds and an associated curvature flow (combinatorial Yamabe flow). The associated evolution of curvature appears to be like a heat equation on graphs, but it can be shown to not satisfy the maximum principle. The notion of a parabolic-like operator is introduced as an operator which satisfies the maximum principle, but may not be parabolic in the usual sense of operators on graphs. A maximum principle is derived for the curvature of combinatorial Yamabe flow under certain assumptions on the triangulation, and hence the heat operator is shown to be parabolic-like. The maximum principle then allows a characterization of the curvature as well was a proof of long term existence of the flow.

Keywords: Curvature flow; Maximum principle; Yamabe flow; Sphere packing; Laplacians on graphs; Discrete Riemannian geometry

1. Introduction

In [9] we introduced the combinatorial Yamabe flow on three-dimensional piecewise linear complexes as an analogue of the smooth Yamabe flow (see [11,20] for the Yamabe flow and [13] for a look at the Yamabe problem). The complexes are geometric in the sense that each Euclidean tetrahedron is given a metric structure by having the edge lengths determined by weights r_i defined at each vertex i. The length of an edge $\{i, j\}$ is defined to be $r_i + r_j$; all such structures are called a conformal class as they are a conformal deformation of the triangulation where all edges are of the same length. For each

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(nondegenerate) tetrahedron, we can think of the structure as coming from four mutually tangent spheres such that the centers are connected by the edges of the tetrahedron and the ith sphere has radius r_i. We showed that under sufficient long term existence conditions, the flow converges to constant curvature.

The three-dimensional combinatorial Yamabe flow was inspired by the work of Chow and Luo [3] (see also [14]). They looked at a combinatorial Ricci flow on two-dimensional simplicial complexes and showed that the equation satisfies a maximum principle. The maximum principle, which says that the maximum of a solution decreases and the minimum increases, is one of the most useful concepts in the study of the heat equation and other parabolic partial differential equations. Maximum principle techniques have been used to great benefit in the smooth category. We are especially inspired by Hamilton's work on the Ricci flow (see [12,2]). It is in general very difficult to prove a maximum principle, or even to prove short term existence of solutions, when you do not have a strictly parabolic equation.

In this paper we investigate an analytic result about the flow which leads to a long-term existence result for some structures. We find that the evolution of curvature admits a maximum principle under certain assumptions on the triangulation. It is especially interesting that we are able to derive a maximum principle even in a situation where the evolution is not parabolic in the usual sense of graph Laplacians. We thus introduce the notion of parabolic-like operators which satisfy the maximum principle for a given function. We can then show that under sufficient assumptions the evolution of curvature is parabolic-like.

2. Parabolic-like operators

The weighted (unnormalized) Laplacian on a graph $G=(V,E)$, where V are the vertices and E are the edges, is defined as the operator

$$(\Delta f)_i = \sum_{\{i,j\}\in E} a_{ij}(f_j - f_i)$$

for each $i \in V$, where the coefficients satisfy $a_{ij}=a_{ji}$ and $a_{ij}\geqslant 0$ (see, for instance, [4]). The operator Δ takes functions on V to functions on V. The coefficients depend on the edge $\{i,j\}$ (compare [5]). The symmetry condition is simply self-adjointness with respect to the Euclidean metric, so we can replace it with another self-adjointness condition. That is, we can define an inner product $\langle f,g\rangle_b=\sum_i b_i f_i g_i$ if we are given coefficients $\{b_i\}$. An operator S is self-adjoint with respect to b if

$$\langle Sf, g\rangle_b = \langle f, Sg\rangle_b.$$

It is clear that symmetry corresponds to being self-adjoint with respect to the inner product determined by $b_i=1$ for all i. We shall call $\{b_i\}$ a (positive definite) metric if $b_i>0$ for all i. In order to match with the notation in [9], we shall let $\mathscr{S}_0$ denote the vertices and $\mathscr{S}_1$ denote the edges. In later sections we will usually consider the inner product coming from the metric $\{r_i\}_{i\in\mathscr{S}_0}$.

We define a (discrete) parabolic operator on functions

$$f : \mathscr{S}_0 \times [A,Z) \to \mathbb{R}, \tag{1}$$

where $\mathscr{S}_0$ is a discrete set of vertices and $[A,Z)\subset\mathbb{R}$, as follows. First we shall call $\mathscr{F}$ the class of functions of the form (1). We write the evaluation of f at the point (i,t) as $f_i(t)$.

Definition 1. An operator

$$P : \mathscr{F} \to \mathscr{F}$$

of the form

$$(Pf)_i(t) = \frac{\mathrm{d} f_i(t)}{\mathrm{d}t} - \sum_{\{i,j\}\in\mathscr{S}_1} a_{ij}(t)(f_j(t) - f_i(t)),$$

where $a_{ij} : [A, Z) \to \mathbb{R}$ are self-adjoint with respect to some metrics $\{b_i(t)\}_{i\in\mathscr{S}_0}$, is called *parabolic* if $a_{ij}(t)\geqslant 0$ for all $i, j \in \mathscr{S}_0$ and for all $t \in [A, Z)$.

Parabolic operators are of the form $(\mathrm{d}/\mathrm{d}t) - \varDelta$ if $\varDelta$ is defined to be an appropriate Laplacian with weights. We note that it is easy to prove a maximum principle for these operators as follows.

Proposition 2. *If P is parabolic and f is a solution to* $Pf = 0$, *then f satisfies*

$$\frac{\mathrm{d} f_M}{\mathrm{d}t}(t)\leqslant 0,$$
$$\frac{\mathrm{d} f_m}{\mathrm{d}t}(t)\geqslant 0,$$

where $M, m \in \mathscr{S}_0$ *such that*

$$f_M(t)\doteqdot \max_{i\in\mathscr{S}_0}\{f_i(t)\},$$
$$f_m(t)\doteqdot \min_{i\in\mathscr{S}_0}\{f_i(t)\}.$$

Proof. Since $Pf = 0$ we have, for a given t,

$$\frac{\mathrm{d} f_M}{\mathrm{d}t} = \sum_{\{M,j\}\in\mathscr{S}_1} a_{Mj}(t)(f_j(t) - f_M(t))$$

and since $a_{Mj}\geqslant 0$ and $f_j(t)\leqslant f_M(t)$ for all $i \in \mathscr{S}_0 - \{M\}$ we see that

$$\frac{\mathrm{d} f_M}{\mathrm{d}t}(t)\leqslant 0.$$

The argument for $f_m(t)$ is similar. $\square$

We may find operators that are of the form

$$\sum_{\{i,j\}\in\mathscr{S}_1} a_{ij}(t)(f_j(t) - f_i(t))$$

but some coefficients are negative. The argument above does not work, but it is possible that a maximum principle still holds if the sum is positive when f_i is minimal and the sum is negative when f_i is maximal even though each term is not positive or negative, respectively. This motivates our definition of parabolic-like operators as operators which satisfy the maximum principle for some function.

Definition 3. An operator

$$P : \mathscr{F} \to \mathscr{F}$$

of the form

$$(Pf)_i(t) = \frac{\mathrm{d}f_i(t)}{\mathrm{d}t} - \sum_{\{i,j\}\in\mathscr{S}_1} a_{ij}(t)(f_j(t) - f_i(t)),$$

where $a_{ij} : [A, Z) \to \mathbb{R}$ are self-adjoint with respect to some metrics $\{b_i(t)\}_{i\in\mathscr{S}_0}$, is called *parabolic-like* for a function g if $Pg = 0$ implies

$$\frac{\mathrm{d}g_M}{\mathrm{d}t} \leqslant 0$$
$$\frac{\mathrm{d}g_m}{\mathrm{d}t} \geqslant 0$$

for all $t \in [A, Z)$.

Parabolic-like operators formally look the same as parabolic operators, but some of the coefficients a_{ij} may be negative. We shall show that our discrete curvature flow equation is parabolic-like for the curvature function in a large subset of the domain. The hope is that this will be enough to prove pinching and convergence theorems.

We note that the maximum and minimum may be done separately, defining *upper parabolic-like* and *lower parabolic-like* in the obvious ways. In some situations we may be interested in only one of these.

3. Combinatorial Yamabe flow

Here, we reintroduce the concepts of combinatorial Yamabe flow, based on the work on the combinatorial Ricci flow in two-dimensions by Chow–Luo [3] and the combinatorial scalar curvature by Cooper–Rivin [6]. Further details can be found in [9]. Recall that if $\mathscr{S} = \{\mathscr{S}_0, \mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3\}$ is a simplicial complex of dimension 3, where $\mathscr{S}_i$ is the i-dimensional skeleton, we define the metric structure as a map

$$r : \mathscr{S}_0 \to (0, \infty)$$

such that for every edge $\{i, j\} \in \mathscr{S}_1$ between vertices i and j, the length of the edge is $\ell_{ij} = r_i + r_j$. The set of all such metrics is called the conformal class since rescaling the r_i will deform the structure to the metric structure with all edges of the same length. We shall use $\mathscr{T} = \{\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3\}$ to denote the triangulation of one tetrahedron. Recall that in order for four positive numbers r_i, r_j, r_k, r_ℓ to define a nondegenerate tetrahedron, they must satisfy the Descartes inequality

$$Q_{ijk\ell} = \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_\ell^2}\right) > 0.$$

We refer to $Q_{ijk\ell}$ as the nondegeneracy quadratic. $Q_{ijk\ell}$ is related both to the volume of the tetrahedron and the radius of the circumscripted sphere, i.e. the sphere which is tangent to all six edges of the tetrahedron. For a given tetrahedron, the existence of a circumscripted sphere is equivalent to being able to define the lengths by assigning weights r_i to the vertices.

The curvature K_i associated to a vertex i is defined as

$$K_i = 4\pi - \sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} \alpha_{ijk\ell},$$

where $\alpha_{ijk\ell}$ is the solid angle vertex i in the tetrahedron $\{i, j, k, \ell\}$. The solid angle is the area of the triangle on the unit sphere cut out by the planes determined by $\{i, j, k\}$, $\{i, j, \ell\}$, $\{i, k, \ell\}$ where i is the center of the sphere. Note that $\alpha_{ijk\ell}$ is symmetric in all permutations of the last three indices; when it is clear which tetrahedron we are working with, we will use the simplified notation α_i. The combinatorial Yamabe flow is defined to be

$$\frac{\mathrm{d}r_i}{\mathrm{d}t} = -K_i r_i.$$

Careful calculation shows that curvature satisfies the following evolution:

$$\frac{\mathrm{d}K_i}{\mathrm{d}t} = \sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} [\Omega_{ijk\ell}(K_j - K_i) + \Omega_{ikj\ell}(K_k - K_i) + \Omega_{i\ell jk}(K_\ell - K_i)].$$

This form is gotten by using the Schläfli formula (see, for instance [17]) which can be written in the following way in this case:

$$r_i \frac{\partial\alpha_{ijk\ell}}{\partial r_i} + r_j \frac{\partial\alpha_{ijk\ell}}{\partial r_j} + r_k \frac{\partial\alpha_{ijk\ell}}{\partial r_k} + r_\ell \frac{\partial\alpha_{ijk\ell}}{\partial r_\ell} = 0. \tag{2}$$

We computed the partial derivatives of the angles to be

$$\begin{aligned}\frac{\partial\alpha_{ijk\ell}}{\partial r_i} = -\frac{8r_j^2r_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}P_{ik\ell}V_{ijk\ell}}&\left[\left(\frac{2}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)\right.\\ &+\frac{r_j}{r_i}\left(\frac{1}{r_i}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)+\frac{r_k}{r_i}\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_\ell}\right)\\ &\left.+\frac{r_\ell}{r_i}\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}\right)+(2r_i+r_j+r_k+r_\ell)Q_{ijk\ell}\right]\end{aligned} \tag{3}$$

and

$$\begin{aligned}\frac{\partial\alpha_{ijk\ell}}{\partial r_j} = \frac{4r_ir_jr_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}V_{ijk\ell}}&\left(\frac{1}{r_i}\left(\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)+\frac{1}{r_j}\left(\frac{1}{r_i}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)\right.\\ &\left.-\left(\frac{1}{r_k}-\frac{1}{r_\ell}\right)^2\right),\end{aligned} \tag{4}$$

where P_{ijk} is the perimeter of the triangle $\{i, j, k\}$ and $V_{ijk\ell}$ is the volume of the tetrahedron $\{i, j, k, \ell\}$. The coefficients $\Omega_{ijk\ell}$ are

$$\Omega_{ijk\ell} = \frac{\partial \alpha_{ijk\ell}}{\partial r_j} r_j$$

and are thus easily computed to be

$$\Omega_{ijk\ell} = \frac{4 r_i r_j^2 r_k^2 r_\ell^2}{3 P_{ijk} P_{ij\ell} V_{ijk\ell}} \left(\frac{1}{r_i} \left(\frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell} \right) + \frac{1}{r_j} \left(\frac{1}{r_i} + \frac{1}{r_k} + \frac{1}{r_\ell} \right) - \left(\frac{1}{r_k} - \frac{1}{r_\ell} \right)^2 \right). \tag{5}$$

Geometrically, we find that

$$\sum_{\{i,j,k,\ell\} \in \mathscr{S}_3} \Omega_{ijk\ell} = \frac{\ell_{ij}^*}{r_i \ell_{ij}},$$

where ℓ_{ij}^* is the area of the dual faces and the sum is over all k and ℓ, where duality comes from assigning the geometric dual to a tetrahedron to be the center of the circumscripted sphere. The evolution of curvature can be written compactly as

$$\frac{\mathrm{d}K_i}{\mathrm{d}t} = \Delta K_i$$

if we define the operator Δ as

$$\Delta f_i = \frac{1}{r_i} \sum_{\{i,j\} \in \mathscr{S}_1} \frac{\ell_{ij}^*}{\ell_{ij}} (f_j - f_i). \tag{6}$$

Note that this operator looks like

$$\Delta f_i = \sum_{\{i,j\} \in \mathscr{S}_1} a_{ij} (f_j - f_i),$$

which is a Laplacian on the graph $(\mathscr{S}_0, \mathscr{S}_1)$ with weights a_{ij}, except that it is possible for a_{ij} to be negative. It can be argued that Δ is a discrete analogue of the Laplace–Beltrami operator. We shall see that since $\Omega_{ijk\ell}$ are not always positive, the maximum principle is not ensured. In the next section we explore the maximum principle for this Laplacian.

4. Parabolicity of $(\partial/\partial t) - \Delta$

The operator $(\partial/\partial t) - \Delta$, where the Laplace–Beltrami operator Δ is defined by (6), may not form a parabolic operator as defined in Definition 1. This is because the coefficients $\Omega_{ijk\ell} = (\partial \alpha_i / \partial r_j) r_j$ may, in fact be negative. We can see this by using our explicit calculation for the coefficients in (5). If we choose, for instance, $r_1 = r_2 = r_3 = 1$ and $r_4 = \frac{1}{5}$ then we see that the tetrahedron is not degenerate since

$$Q_{1234} = 8 > 0$$

and that one coefficient is negative,

$$\Omega_{1234} = -\frac{8}{75 P_{123} P_{124} V_{1234}} < 0.$$

This indicates that the maximum principle will not hold in general. However, there are several indications that this operator is a good operator and that it might be parabolic-like for the curvature in the sense of Definition 3. First is that the matrix $\partial\alpha_i/\partial r_j$ is negative semidefinite, where the nullspace is spanned by the vector (r_1, r_2, r_3, r_4). This is shown in Appendix A. It was originally stated in [6] but the proof there is incorrect. It is interesting to note that the incorrect proof tried to show that the coefficients of the Laplace–Beltrami operator are all positive, which we have shown to be false by a difficult calculation. Clearly the case when the coefficients are all positive is a large set.

We pursue a different understanding of why the maximum principle should hold. The first evidence is that the Schläfli formula tells us that

$$r_i \frac{\partial\alpha_{ijk\ell}}{\partial r_i} + r_j \frac{\partial\alpha_{jik\ell}}{\partial r_i} + r_k \frac{\partial\alpha_{kij\ell}}{\partial r_i} + r_\ell \frac{\partial\alpha_{\ell ijk}}{\partial r_i} = 0$$

while formula (3) for $\partial\alpha_{ijk\ell}/\partial r_i$ calculated in [9] shows that

$$\frac{\partial\alpha_{ijk\ell}}{\partial r_i} < 0$$

for any tetrahedron. Together with the following lemma, we shall see that there are large restrictions on when coefficients $\Omega_{ijk\ell}$ can be negative.

Lemma 4. *If $\Omega_{abcd} < 0$ then $r_a, r_b > \min\{r_c, r_d\}$.*

Proof. Suppose $\Omega_{abcd} < 0$, then

$$\frac{1}{r_a}\left(\frac{1}{r_b} + \frac{1}{r_c} + \frac{1}{r_d}\right) + \frac{1}{r_b}\left(\frac{1}{r_a} + \frac{1}{r_c} + \frac{1}{r_d}\right) - \left(\frac{1}{r_c} - \frac{1}{r_d}\right)^2 < 0.$$

Eliminating the denominators and regrouping terms we get

$$r_d r_c[(r_d + r_b)r_c + (r_b + r_c)r_d] + r_a(r_d r_c(r_d + r_c) - r_b(r_d - r_c)^2) < 0,$$

so in particular we need

$$r_d r_c(r_d + r_c) - r_b(r_d - r_c)^2 < 0.$$

Solving for r_b we get

$$r_b > \frac{r_d r_c(r_d + r_c)}{(r_d - r_c)^2} \geqslant \frac{r_d r_c}{|r_d - r_c|} \geqslant \min\{r_d, r_c\}$$

since $r_d + r_c \geqslant |r_d - r_c|$ and $\max\{r_c, r_d\} \geqslant |r_d - r_c|$. Since the initial expression is symmetric in a and b, we get $r_a \geqslant \min\{r_d, r_c\}$ too. $\square$

Corollary 5. *If* $\{i, j, k, \ell\} \in \mathscr{S}_3$ *and* $r_i = \min\{r_i, r_j, r_k, r_\ell\}$ *then*

$$\Omega_{ijk\ell} \leqslant 0 \quad \textit{and} \quad \Omega_{jik\ell} \geqslant 0$$

or, equivalently,

$$\frac{\partial \alpha_{ijk\ell}}{\partial r_j} \leqslant 0 \quad \textit{and} \quad \frac{\partial \alpha_{jik\ell}}{\partial r_i} \geqslant 0.$$

5. Monotonicity

We need some way to relate the coefficients $\Omega_{ijk\ell}$ and the curvatures. We attempt this by trying to prove that the K's are monotonic as functions of the r's. This will turn out to be true for the two cases of the double tetrahedron and the boundary of a 4-simplex, though not in general. We consider a tetrahedron determined by $\{r_1, r_2, r_3, r_4\}$. First we prove a lemma about the kinds of degeneracies that can develop.

Proposition 6. *If* $Q_{ijk\ell} \to 0$ *without any of the* r_i *going to* 0, *then one solid angle goes to* 2π *and the others go to* 0. *The solid angle* α_i *which goes to* 2π *corresponds to* r_i *being the minimum.*

Proof. Rewrite $Q_{ijk\ell}$ as

$$\begin{aligned} Q_{ijk\ell} &= \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_\ell^2}\right) \\ &= \frac{1}{r_i}\left(\frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell} - \frac{1}{r_i}\right) + \frac{1}{r_j}\left(\frac{1}{r_i} + \frac{1}{r_k} + \frac{1}{r_\ell} - \frac{1}{r_j}\right) \\ &\quad + \frac{1}{r_k}\left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_\ell} - \frac{1}{r_k}\right) + \frac{1}{r_\ell}\left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} - \frac{1}{r_\ell}\right). \end{aligned} \tag{7}$$

If r_i is the minimum, then

$$\begin{aligned} &\frac{1}{r_i} + \frac{1}{r_k} + \frac{1}{r_\ell} - \frac{1}{r_j} > 0, \\ &\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_\ell} - \frac{1}{r_k} > 0, \\ &\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} - \frac{1}{r_\ell} > 0. \end{aligned}$$

Hence if $Q_{ijk\ell} = 0$ then

$$\frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell} - \frac{1}{r_i} < 0.$$

Now we look at the partial derivative

$$\frac{\partial}{\partial r_i} Q_{ijk\ell} = -\frac{2}{r_i^2}\left(\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}-\frac{1}{r_i}\right) \geqslant 0.$$

So if $Q_{ijk\ell}=0$, we can always increase r_i to make the tetrahedron nondegenerate.

Now we can categorize the degenerations. Notice that by the formula for volume,

$$V_{ijk\ell} = \frac{2A_{ijk}A_{ij\ell}\sin\beta_{ijk\ell}}{3\ell_{ij}},$$

we must have that if the tetrahedron degenerates, $\sin\beta_{ijk\ell} \to 0$ for all dihedral angles. Hence $\beta_{ijk\ell}$ goes to 0 or to π. Since

$$\frac{\partial \beta_{ijk\ell}}{\partial r_i} = \frac{2r_ir_jr_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}V_{ijk\ell}}\left[-\frac{1}{r_k^2}-\frac{1}{r_\ell^2}-2\frac{r_j}{r_i}\left(\frac{1}{r_ir_k}+\frac{1}{r_ir_\ell}+\frac{1}{r_kr_\ell}\left(2+\frac{r_j}{r_i}\right)\right)\right.$$
$$\left.+\left(\frac{1}{r_j}-\frac{1}{r_i}\right)\left(\frac{2}{r_i}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)\right]$$

(see [9]), if r_i is the minimum, then $\partial\beta_{ijk\ell}/\partial r_i < 0$. When the tetrahedron first becomes degenerate, we can increase r_i to become nondegenerate again. But this indicates that in this case $\beta_{ijk\ell}$ would decrease, so if $\beta_{ijk\ell}=0$, $\beta_{ijk\ell}$ would become negative in a nondegenerate tetrahedron. This is a contradiction, so we cannot have $\beta_{ijk\ell}=0$ in the limit. Hence

$$\beta_{ijk\ell} = \beta_{ikj\ell} = \beta_{i\ell jk} = \pi$$

and $\alpha_{ijk\ell}=2\pi$. Since $\alpha_{ijk\ell}+\alpha_{jik\ell}+\alpha_{kij\ell}+\alpha_{\ell ijk} \leqslant 2\pi$ in any tetrahedron (see proof in [8]), we must have

$$\alpha_{jik\ell} = \alpha_{kij\ell} = \alpha_{\ell ijk} = 0. \qquad \square$$

Now we can prove a monotonicity formula for angles in a given simplex as follows.

Lemma 7. *$r_i \leqslant r_j$ if and only if $\alpha_{ijk\ell} \geqslant \alpha_{jik\ell}$.*

Proof. It is equivalent to prove that the strict inequality $r_i<r_j$ implies $\alpha_{ijk\ell}>\alpha_{jik\ell}$ and that $r_i=r_j$ implies $\alpha_{ijk\ell}=\alpha_{jik\ell}$. The second statement is clear. Since we are only looking at one tetrahedron, we can use α_i instead of $\alpha_{ijk\ell}$ without causing confusion.

Consider the path $\sigma(s)=(\sigma_1(s),\sigma_2(s),\sigma_3(s),\sigma_4(s))$ defined by $\sigma(s)=(r_i,r_j,(1-s)r_k+sr_\ell, sr_k+(1-s)r_\ell)$, where $r_k<r_\ell$. We can think of α as a function of four variables, where $\alpha(r_i,r_j,r_k,r_\ell)=\alpha_i$. Let $\alpha_3(x,y,z,w)=\alpha(z,x,y,w)$ and $\alpha_4(x,y,z,w)=\alpha(w,x,y,z)$.

Consider the function

$$A \doteqdot \sum_{i\in\mathcal{T}_0} r_i\alpha_i. \tag{8}$$

By the Schläfli formula, we find that

$$\mathrm{d}A = \sum_{i\in\mathcal{T}_0} \alpha_i\,\mathrm{d}r_i$$

so

$$\frac{\partial A}{\partial r_i} = \alpha_i,$$
$$\frac{\partial^2 A}{\partial r_i \partial r_j} = \frac{\partial \alpha_j}{\partial r_i} = \frac{\partial \alpha_i}{\partial r_j}.$$

More details can be found in [9]. Now, consider

$$D(s) \doteq \frac{\mathrm{d}}{\mathrm{d}s} A(\sigma(s)) = [\alpha_3(\sigma(s)) - \alpha_4(\sigma(s))](r_\ell - r_k).$$

The path is constructed so that $D(\frac{1}{2}) = 0$. Since the solid angles are between 0 and 2π, $D(s) \leqslant 2\pi(r_\ell - r_k)$ if the tetrahedron is nondegenerate for $\sigma(s)$. Consider the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} D(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} A(\sigma(s)) = [\partial_3 \alpha_3(\sigma(s)) - 2\partial_4 \alpha_3(\sigma(s)) + \partial_4 \alpha_4(\sigma(s))](r_\ell - r_k)^2.$$

If the tetrahedron is nondegenerate, then $(\mathrm{d}/\mathrm{d}s)D(s) < 0$ since the Hessian of A is negative definite (see Appendix A) and none of the σ_i are equal to zero. Now, as we move along the path starting at $s = 0$, either the tetrahedron degenerates for some $s_0 \in (0, \frac{1}{2})$ or the tetrahedron is nondegenerate up to $s = \frac{1}{2}$. Suppose the first degeneracy is at $s = s_0$. Then either

$$\sigma_3(s_0) = \min_{i=1,\dots,4} \{\sigma_i(s_0)\}$$

or not. If it is the minimum, then at the degeneracy, $\alpha_3 = 2\pi$ and the other angles are 0. This cannot happen, however, since then $D(s_0) = 2\pi(r_\ell - r_k)$, the maximum possible, but the derivative is negative, since $(\mathrm{d}/\mathrm{d}s)D(s_0) \leqslant 0$ implies that $D(s)$ must have been larger than its maximum for some $s < s_0$, a contradiction. If $\sigma_3(s)$ is not the minimum, then at a degenerate point, $\alpha_3 = \alpha_4 = 0$. So there exists a first point $s_1 \in (0, \frac{1}{2}]$ where $\alpha_k = \alpha_\ell$ such that the tetrahedron is nondegenerate for $s \in [0, s_1)$. Hence $D(s_1) = 0$. Since the tetrahedron is nondegenerate on $[0, s_1)$, we have $(\mathrm{d}/\mathrm{d}s)D < 0$ for $s \in [0, s_1)$. Together with $D(s_1) = 0$ we have $D(s) > 0$ for $s \in [0, s_1)$. In particular, $D(0) > 0$, i.e. $\alpha_k > \alpha_\ell$.

The reverse inequality must be true as well since the argument is symmetric. □

The lemma has the following interesting geometric consequence for a conformal tetrahedron which is an analogue of the fact that in any triangle the longest side is opposite the largest angle. The author does not know if this statement is true for a general tetrahedron.

Corollary 8. *For a conformal tetrahedron with metric structure $\{r_1, r_2, r_3, r_4\}$, the side with the largest area is opposite the largest solid angle, and the side with the smallest area is opposite the smallest solid angle*.

Proof. This follows from the fact that the angle $\alpha_{ijk\ell}$ is opposite the side with area $\sqrt{r_j r_k r_\ell (r_j + r_k + r_\ell)}$. □

The lemma can be used to show monotonicity of curvature for small triangulations as follows.

Corollary 9. *In the case of the double tetrahedron, $r_i \leqslant r_j$ if and only if $K_i \leqslant K_j$.*

Proof. This follows from the fact that $K_i = 4\pi - 2\alpha_{ijk\ell}$. $\square$

Corollary 10. *In the case of the boundary of a* 4*-simplex, $r_i \leqslant r_j$ if and only if $K_i \leqslant K_j$.*

Proof. Once again it is sufficient to show that $r_i < r_j$ implies $K_i < K_j$ and $r_i = r_j$ implies $K_i = K_j$. The latter is trivial. Let us number the vertices $\{1, \ldots, 5\}$. Suppose $r_1 < r_2$. Then

$$K_1 = 4\pi - \alpha_{1234} - \alpha_{1235} - \alpha_{1245} - \alpha_{1345},$$
$$K_2 = 4\pi - \alpha_{2134} - \alpha_{2135} - \alpha_{2145} - \alpha_{2345}.$$

We know by Lemma 7 that $\alpha_{2134} < \alpha_{1234}$, $\alpha_{2135} < \alpha_{1235}$ and $\alpha_{2145} < \alpha_{1245}$. We thus need only show $\alpha_{2345} < \alpha_{1345}$. Consider the path

$$\sigma(s) = \left(\frac{r_1 r_2}{(1-s)r_2 + s r_1}, r_3, r_4, r_5 \right).$$

We notice that the nondegeneracy quadratic $Q(\sigma(s))$ is a polynomial in s with highest term

$$-\left(\frac{1}{r_1} - \frac{1}{r_2} \right)^2 s^2.$$

Since the quadratic $Q(\sigma(s))$ is concave the minimum for $s \in [0, 1]$ must occur at $s = 0$ or $s = 1$. But since $Q(\sigma(0)) = Q_{1345} > 0$ and $Q(\sigma(1)) = Q_{2345} > 0$, $Q(\sigma(s)) > 0$ for all $s \in [0, 1]$. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}s}\alpha(\sigma(s)) = \partial_1 \alpha(\sigma(s)) \frac{r_1 r_2 (r_2 - r_1)}{((1-s)r_2 + r_1 s)^2}$$

which is negative for all $s \in [0, 1]$ by formula (3) since $r_2 > r_1$. Hence $\alpha(\sigma(0)) > \alpha(\sigma(1))$, or $\alpha_{1245} > \alpha_{2145}$. $\square$

Proving a similar statement about larger complexes would be much harder, since we cannot pair up angles which are in the same or bordering tetrahedra as we do here. We shall call this condition the *monotonicity condition* for a tetrahedron $\{i, j, k, \ell\}$:

$$r_i \leqslant r_j \text{ if and only if } K_i \leqslant K_j. \tag{MC}$$

The condition is true for an open set of triangulations. Unfortunately, MC is not necessarily preserved under the flow. For instance, if $r_i = r_j$ but $K_i < K_j$ then $\mathrm{d}r_i/\mathrm{d}t = -K_i r_i > -K_j r_j = \mathrm{d}r_j/\mathrm{d}t$. The monotonicity will counteract some of the potential degenerations of the flow, and thus allows proof of the maximum principle and long term estimates.

6. Proof of the maximum principle

Suppose we have a complex such that each tetrahedron $\{i, j, k, \ell\}$ satisfies the monotonicity condition MC. Assume that $r_i \leqslant r_j$, $r_k \leqslant r_\ell$. We shall first look at the minimum. By Corollary 5, $\Omega_{ijk\ell}$, $\Omega_{ikj\ell}$, $\Omega_{i\ell jk}$

are all nonnegative, so since K_i is the minimum curvature among $\{i, j, k, \ell\}$,

$$\Omega_{ijk\ell}(K_j - K_i) + \Omega_{ikj\ell}(K_k - K_i) + \Omega_{i\ell jk}(K_\ell - K_i) \geqslant 0.$$

Now for the entire triangulation, we get that if $K_m = \min_{i \in \mathscr{S}_0}\{K_i\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} K_m = \sum_{\{m,j,k,\ell\} \in \mathscr{S}_3} [\Omega_{mjk\ell}(K_j - K_m) + \Omega_{mkj\ell}(K_k - K_m) + \Omega_{i\ell jk}(K_\ell - K_m)]$$

must be a sum of nonnegative numbers since in any tetrahedron containing m, $r_i = r_m$ is the smallest weight.

We now look at the maximum. We want to show that

$$\Omega_{\ell ijk}(K_i - K_\ell) + \Omega_{\ell jik}(K_j - K_\ell) + \Omega_{\ell kij}(K_k - K_\ell) \leqslant 0.$$

This is certainly true if $\Omega_{\ell ijk}, \Omega_{\ell jik}, \Omega_{\ell kij}$ are all nonnegative since K_ℓ is the largest curvature. Again using Lemma 4 we see that if $\Omega_{abcd} < 0$ then $b \neq i$, so $\Omega_{\ell ijk} \geqslant 0$. We are then left with the case that both $\Omega_{\ell jik}, \Omega_{\ell kij}$ are negative or only one is negative. First consider the case when both are negative. In this case it is sufficient to show that

$$\Omega_{\ell ijk} + \Omega_{\ell jik} + \Omega_{\ell kij} \geqslant 0$$

since in this case we have

$$\Omega_{\ell ijk} \geqslant -\Omega_{\ell jik} - \Omega_{\ell kij},$$

$$\begin{aligned} \Omega_{\ell ijk}(K_i - K_\ell) &\leqslant -(\Omega_{\ell jik} + \Omega_{\ell kij}) \min\{(K_j - K_\ell), (K_k - K_\ell)\}, \\ &\leqslant -\Omega_{\ell jik}(K_j - K_\ell) - \Omega_{\ell kij}(K_k - K_\ell) \end{aligned}$$

since

$$\begin{aligned} (K_i - K_\ell) &\leqslant \min\{(K_j - K_\ell), (K_k - K_\ell)\} \\ &\leqslant \max\{(K_j - K_\ell), (K_k - K_\ell)\} \end{aligned}$$

which is nonpositive and $\Omega_{\ell ijk}, -\Omega_{\ell jik}, -\Omega_{\ell kij} \geqslant 0$. So

$$\Omega_{\ell ijk}(K_i - K_\ell) + \Omega_{\ell jik}(K_j - K_\ell) + \Omega_{\ell kij}(K_k - K_\ell) \leqslant 0$$

and we are done. The inequality $\Omega_{\ell ijk} + \Omega_{\ell jik} + \Omega_{\ell kij} \geqslant 0$ follows from the Schläfli formula, since

$$\begin{aligned} \Omega_{\ell ijk} + \Omega_{\ell jik} + \Omega_{\ell kij} &= \frac{\partial \alpha_{\ell ijk}}{\partial r_i} r_i + \frac{\partial \alpha_{\ell ijk}}{\partial r_j} r_j + \frac{\partial \alpha_{\ell ijk}}{\partial r_k} r_k \\ &= -\frac{\partial \alpha_{\ell ijk}}{\partial r_\ell} r_\ell. \end{aligned}$$

This is nonnegative by formula (3) for $\partial \alpha_{\ell ijk} / \partial r_\ell$.

Now suppose that only one is negative, say $\Omega_{\ell kij} < 0$, then similarly it is enough to show

$$\Omega_{\ell ijk} + \Omega_{\ell kij} \geqslant 0$$

since then

$$\Omega_{\ell ijk} \geqslant -\Omega_{\ell kij},$$
$$\Omega_{\ell ijk}(K_i - K_\ell) \leqslant -\Omega_{\ell kij}(K_k - K_\ell)$$

and $\Omega_{\ell jik}(K_j - K_\ell) \leqslant 0$. We argue from our explicit calculations. The sum $\Omega_{\ell ijk} + \Omega_{\ell kij}$ is equal to

$$\frac{4 r_\ell r_j^2 r_k^2 r_i^2}{3 P_{\ell ik} P_{\ell ij} V_{ijk\ell}} \left(\frac{1}{r_\ell}\left(\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_i}\right) + \frac{1}{r_i}\left(\frac{1}{r_\ell}+\frac{1}{r_j}+\frac{1}{r_k}\right) - \left(\frac{1}{r_j}-\frac{1}{r_k}\right)^2 \right)$$
$$+ \frac{4 r_\ell r_j^2 r_k^2 r_i^2}{3 P_{\ell jk} P_{\ell ik} V_{ijk\ell}} \left(\frac{1}{r_\ell}\left(\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_i}\right) + \frac{1}{r_k}\left(\frac{1}{r_\ell}+\frac{1}{r_j}+\frac{1}{r_i}\right) - \left(\frac{1}{r_j}-\frac{1}{r_i}\right)^2 \right)$$

which, when simplifying and multiplying by a positive factor, has the same sign as

$$(P_{\ell jk} + 2P_{\ell ij})\frac{1}{r_i r_j} + (P_{\ell jk} + P_{\ell ij})\frac{1}{r_i r_k} + (P_{\ell ij} + 2P_{\ell jk})\frac{1}{r_\ell r_i}$$
$$+ (2P_{\ell jk} + P_{\ell ij})\frac{1}{r_j r_k} + (P_{\ell ij} + P_{\ell jk})\frac{1}{r_\ell r_j} + (2P_{\ell ij} + P_{\ell jk})\frac{1}{r_\ell r_k}$$
$$- P_{\ell ij}\frac{1}{r_i^2} - (P_{\ell jk} + P_{\ell ij})\frac{1}{r_j^2} - P_{\ell jk}\frac{1}{r_k^2}.$$

This is greater than or equal to

$$P_{\ell ij} Q_{ijk\ell} + P_{\ell jk}\left(\frac{1}{r_i r_j} - \frac{1}{r_j^2}\right) + (P_{\ell jk} - P_{\ell ij})\left(\frac{1}{r_i r_k} - \frac{1}{r_k^2}\right)$$

which is nonnegative since $r_i \leqslant r_j$ and $\{i, j, k, \ell\}$ is nondegenerate.

Thus we have proven the following.

Theorem 11. *On a complex with a metric structure which satisfies the monotonicity condition* MC, *if* K_m *is the minimum curvature,* K_M *is the maximum curvature, and we satisfy the combinatorial Yamabe flow*

$$\frac{\mathrm{d}}{\mathrm{d}t} r_i = -K_i r_i$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t} K_m \geqslant 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} K_M \leqslant 0,$$

i.e. the combinatorial Yamabe flow is parabolic-like for K.

Corollary 12. *On the double tetrahedron and the boundary of a* 4*-simplex the combinatorial Yamabe flow is parabolic-like for the curvature function K, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} K_M \leqslant 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} K_m \geqslant 0.$$

Proof. This follows from Corollaries 9 and 10 which say that the monotonicity condition is satisfied in these cases. □

The maximum principle has been used by Hamilton and others to prove many pinching results for geometric evolution equations (e.g. [10,12]). The most basic use is to show preservation of positive or negative curvature, which is an easy corollary.

Corollary 13. *On a complex with a metric structure which satisfies the monotonicity condition* MC *for* $t \in [0, T)$*, then nonnegative curvature is preserved, i.e. if* $K_i \geqslant 0$ *for all i for* $t = 0$*, then* $K_i \geqslant 0$ *for all i for all* $t \in [0, T)$*. Similarly, nonpositive curvature is preserved.*

Proof. If $K_i \geqslant 0$ for all i, then in particular the minimum is nonnegative and increasing. □

7. Long term existence

The monotonicity condition (MC) also gives us a way to show long-term existence. The maximum principle will allow us to bound the growth or decay of the lengths r_i.

In order to show long-term existence, we must show that $Q_{ijk\ell} > 0$ for every $\{i, j, k, \ell\} \in \mathscr{S}_3$ and that r_i does not go to zero or infinity in finite time for any $i \in \mathscr{S}_0$. Since we are only working with one tetrahedron, we can use $Q = Q_{ijk\ell}$ without fear of confusion. We calculate

$$\frac{\partial Q}{\partial r_i} = -\frac{2}{r_i^2}\left(\frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell} - \frac{1}{r_i}\right) \tag{10}$$

and hence using formula (7) we see

$$2Q = -\left(\frac{\partial Q}{\partial r_i} r_i + \frac{\partial Q}{\partial r_j} r_j + \frac{\partial Q}{\partial r_k} r_k + \frac{\partial Q}{\partial r_\ell} r_\ell\right).$$

If $Q = 0$, then we have

$$\frac{\partial Q}{\partial r_i} r_i + \frac{\partial Q}{\partial r_j} r_j + \frac{\partial Q}{\partial r_k} r_k + \frac{\partial Q}{\partial r_\ell} r_\ell = 0. \tag{11}$$

Write the evolution of Q as

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= -\frac{\partial Q}{\partial r_i} K_i r_i - \frac{\partial Q}{\partial r_j} K_j r_j - \frac{\partial Q}{\partial r_k} K_k r_k - \frac{\partial Q}{\partial r_\ell} K_\ell r_\ell \\ &= -\frac{\partial Q}{\partial r_j} r_j (K_j - K_i) - \frac{\partial Q}{\partial r_k} r_k (K_k - K_i) - \frac{\partial Q}{\partial r_\ell} r_\ell (K_\ell - K_i)\end{aligned}$$

using (11). Notice that if r_i is the minimum, then $\partial Q/\partial r_j \leqslant 0$ for $j \neq i$ by (10). If the monotonicity condition MC is satisfied, then we must have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \geqslant 0$$

if $Q = 0$ since $K_j \geqslant K_i$ and hence the tetrahedra do not degenerate.

Now we can show that the maximum principle gives bounds on growth and decay of the r_i.

Proposition 14. *If* $\mathrm{d}K_M/\mathrm{d}t \leqslant 0$ *and* $\mathrm{d}K_m/\mathrm{d}t \geqslant 0$ *and then there is a constant C such that*

$$r_i(0)\mathrm{e}^{-Ct} \leqslant r_i(t) \leqslant r_i(0)\mathrm{e}^{Ct}.$$

Proof. Let $C(t) = \max\{K_M(t), -K_m(t)\} \geqslant 0$. Then

$$-Cr_i \leqslant -K_i r_i \leqslant Cr_i$$

for each t. Since $\mathrm{d}K_M/\mathrm{d}t \leqslant 0$ and $\mathrm{d}K_m/\mathrm{d}t \geqslant 0$, we must have $C(t) \leqslant C(0)$. So if we look at the evolution

$$-C(0)r_i \leqslant \frac{\mathrm{d}r_i}{\mathrm{d}t} \leqslant C(0)r_i$$

we get

$$r_i(0)\mathrm{e}^{-C(0)t} \leqslant r_i(t) \leqslant r_i(0)\mathrm{e}^{C(0)t}. \qquad \square$$

Thus the solution exists for all time. Convergence to constant curvature now follows from [9].

8. Further remarks

In this paper we have seen two large sets of possible metric structures within which the maximum principle holds: the set where the coefficients $\Omega_{ijk\ell}$ are positive and the set where the monotonicity condition MC is satisfied. Unfortunately, neither of these conditions is obviously preserved by the flow. It would be highly desirable to find a set which is preserved by the flow within which the curvature satisfies the maximum principle.

Numerical data suggests that the maximum principle holds in much greater generality, even for large simplicial complexes that do not satisfy monotonicity. Numerical simulation of the flow requires a true simplicial complex; a CW decomposition will not work because there are not enough vertices to allow the different tetrahedra in the complex to evolve independently. Thus the current numerical work has been limited to certain small triangulations (fewer than 15 vertices) of the 3-sphere, the direct product of the 2-sphere with the circle, the twisted product of the 2-sphere with the circle, and the 3-torus. Some of the small triangulations are due to Lutz (see [16,15]). The condition of monotonicity is not particularly well understood for large complexes either, though it is known not to hold in general even for triangulations of S^3.

The maximum principle is closely connected to the fact that the operator Δ is negative semi-definite in the smooth case, but it is not clear that the definition of maximum principle for graph Laplacians which we use here is the right maximum principle to correspond to the definiteness. This may also be related to the fact that the principle eigenfunction in the smooth case is positive, while in the discrete case here the principle eigenvector usually will not have all positive (or all negative) entries. Perhaps a kind of ‘refined maximum principle’ is needed, as in [1,18]. Also integral type maximum principles have been successful in studying discrete Laplacians as in [7].

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Appendix A

In this appendix we prove that the 4×4 symmetric matrix $(\partial\alpha_i/\partial r_j)$ has three negative eigenvalues and one zero eigenvalue. The proof in [6] is incorrect; Igor Rivin has given a new proof in [19]. We do not understand part of Rivin’s proof, but we can complete Rivin’s arguments by a calculation given below. The complete argument is given here.

The zero eigenvalue comes from the vector (r_1, r_2, r_3, r_4) by the Schläfli formula (2). Recall that the $(n-1)$-dimensional minor $M(i, j)$ of a matrix M is the matrix with the ith row and the jth column removed. Let A be the matrix of partial derivatives, so $A_{ij} = \partial\alpha_{ijk\ell}/\partial r_j$. We take as the domain of A the set of (r_1, r_2, r_3, r_4) such that the associated tetrahedron is nondegenerate, i.e. $r_i > 0$ for $i = 1, \ldots, 4$ and $Q_{ijk\ell} > 0$.

Proposition A1. *The minors $A(i, j)$ have determinant*

$$\det A(i, j) = (-1)^{i+j+1} \frac{288 V_{ijk\ell}}{r_i r_j P_{ijk} P_{ij\ell} P_{ik\ell} P_{jk\ell}}$$

and hence is nonzero if the tetrahedron $\{i, j, k, \ell\}$ is nondegenerate.

Proof. We can do a rather lengthy calculation using (3) and (4). Note that we need only compute the minors $A(i, j)$ where $i \neq j$ since $\det A = 0$. The minors $A(i, i)$ on the diagonal are slightly more difficult to calculate because there are three entries of the more complicated form $\partial\alpha_i/\partial r_i$ instead of only two for the off-diagonal $A(i, j)$ where $i \neq j$. □

Corollary A2. *The matrix A is negative semidefinite, rank* 3, *and the nullspace is the span of the vector* (r_1, r_2, r_3, r_4).

Proof. The rank follows from Proposition A1. The nullspace condition follows from the Schläfli formula (2). Since the domain is connected and the rank is always 3, the eigenvalues must always have the same sign. We need only compute the matrix at one point, say $r_i = 1$ for $i = 1, \ldots, 4$. The matrix A at this point is easily computed to be

$$\frac{1}{3\sqrt{2}}\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix},$$

which has eigenvalues $0, -\frac{2}{3}\sqrt{2}, -\frac{2}{3}\sqrt{2}, -\frac{2}{3}\sqrt{2}$. $\square$

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