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Equality of Lifshitz and van Hove exponents on amenable Cayley graphs

Tonći Antunović [a], Ivan Veselić [b,*]

[a] *Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA*

[b] *Emmy-Noether-Programme of the Deutsche Forschungsgemeinschaft & Fakultät für Mathematik, 09107 TU Chemnitz, Germany*

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Abstract

We study the low energy asymptotics of periodic and random Laplace operators on Cayley graphs of amenable, finitely generated groups. For the periodic operator the asymptotics is characterised by the van Hove exponent or zeroth Novikov–Shubin invariant. The random model we consider is given in terms of an adjacency Laplacian on site or edge percolation subgraphs of the Cayley graph. The asymptotic behaviour of the spectral distribution is exponential, characterised by the Lifshitz exponent. We show that for the adjacency Laplacian the two invariants/exponents coincide. The result holds also for more general symmetric transition operators. For combinatorial Laplacians one has a different universal behaviour of the low energy asymptotics of the spectral distribution function, which can be actually established on quasi-transitive graphs without an amenability assumption. The latter result holds also for long range bond percolation models.

Résumé

Nous étudions l'asymptotique spectrale des laplaciens périodiques et aléatoires sur des graphes de Cayley associés à un groupe moyennable de type fini. Pour un opérateur périodique l'asymptotique est caractérisée par l'exposant de van Hove ou l'invariant de Novikov–Shubin d'ordre zéro. Le modèle aléatoire que nous considérons est donné en termes d'opérateur d'adjacence sur un sous-graphe aléatoire d'un graphe de Cayley. Le graphe aléatoire est engendré par un processus sous-critique de percolations par site ou par lien. La fonction de répartition spectrale est exponentiellement décroisante au voisinage de la limite inférieure du spectre. Le comportement exponentiel est caracterisé par l'exposant de Lifshitz. Nous montrons que pour le laplacien d'adjacence les deux exposants coïncident. Les résultats sont vrais pour une classe plus générale d'opérateurs symmétriques. Pour les laplaciens combinatoires on trouve un comportement asymptotique différent de la fonction de répartition spectrale. Cette étude est assez générale et s'applique aussi aux processus souscritiques de percolations sur un graphe quasitransitif moyennable ou non. Le dernier résultat est aussi vrai pour des percolations de type longue portée.

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* Corresponding author.
E-mail address: ivan.veselic@mathematik.tu-chemnitz.de (I. Veselić).
URL: http://www.tu-chemnitz.de/mathematik/enp (I. Veselić).

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1. Introduction

Operators on Euclidean space which are invariant under a group action have a well defined *integrated density of states* (IDS), also known as the *spectral distribution function*. Prominent examples are Laplace and Schrödinger operators. Their IDS exhibits a *van Hove singularity* at the bottom of the spectrum. This means that it vanishes polynomially as the energy parameter approaches the lowest spectral edge, the exponent being equal to the space dimension divided by two. The factor one half is due to the fact that the considered operators are elliptic of second order.

The IDS can be defined also for operators having a more general type of equivariance property, namely for ergodic operators. Two prominent classes of such operators are random and almost periodic ones. Among the pioneering works which have studied the IDS of such models are [35], respectively [37].

Several well-studied types of random operators on $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)$ exhibit a Lifshitz tail at the bottom of the spectrum, meaning that the IDS vanishes exponentially fast. In particular, the spectral density is very sparse in this region and spectral values are created only by extremely rare configurations of the randomness. Hence such spectral edges are called *fluctuation boundaries*. In Euclidean space the Lifshitz exponent is quite universal. In particular, for Laplacians with a variety of random i.i.d. non-negative perturbations it equals $d/2$, cf. for instance the survey [21] and the references therein.

Historically, physicists have introduced the IDS as a limit of spectral distribution functions of finite volume operators. For this approximation to converge, the underlying space or group needs to have some amenability property. However, for the purposes of the present paper the approximation property is not relevant and we may rather consider the IDS as given by a *Shubin–Pastur trace formula* (2).

In the present paper we want to analyse whether the Lifshitz exponent equals the van Hove exponent for operators on more general geometries as well. Of course, for this to hold a proper relation between the considered periodic and random operator is necessary, in the sense that the random operator results from its periodic counterpart by addition of stochastically independent, positive perturbations. The periodic objects we study are Laplace operators on Cayley graphs. We consider two different types of random perturbations thereof: the adjacency and the combinatorial Laplacians on random subgraphs generated by a subcritical percolation process. While the first type of operators indeed shows a coincidence of van Hove and Lifshitz exponents, the second ones exhibit a different type of universal behaviour, the reason being, that the random perturbation is not positive in this case.

Our motivation to study this question is threefold: firstly, to extend the results of [24] and [22] concerning lattice bond percolation models; secondly, to study the relation between van Hove and Lifshitz exponents, as done at internal spectral edges of random perturbations of periodic Schrödinger operators e.g. in [23,25], and finally to clarify some of the links between geometric L^2-invariants and the IDS, see e.g. [29,12]. Note in particular that the van Hove exponent equals the Novikov–Shubin invariant of order zero, cf. [33,18]. Our strategy of proof is coined after the one in [22]. The description of the asymptotic behaviour of the IDS at spectral boundaries of random operators plays a key role in the proof of Anderson localisation, see e.g. [15]. For more background on the IDS of percolation Hamiltonians on Cayley graphs see the discussion in [4].

In the next section we state our theorems. Thereafter, in Section 3 we present abstract upper and lower bounds on the IDS. Section 4 is devoted to eigenvalue inequalities. Section 5 contains the proofs of the theorems in the case of adjacency Laplacians on groups with polynomial growth and combinatorial Laplacians on general quasi-transitive graphs. In Section 6 we prove the statements concerning Lamplighter groups. The last section is devoted to the extension of our results to some related models: we derive there the low energy spectral asymptotics of percolation Hamiltonians associated to general symmetric transition operators on discrete, finitely generated, amenable groups. Furthermore we study combinatorial Laplacians on long range edge percolation graphs, and on an abstract ensemble of percolation graphs satisfying certain conditions.

2. Definitions and results

We describe the type of graphs, the percolation process and the operators we will be considering.

Let Γ be a discrete, finitely generated group, S a finite, symmetric set of generators not containing the unit element ι of Γ and $G = (V, E)$ the associated Cayley graph. It is k-regular with $k = |S|$. The ball around ι of radius n is denoted by $B(n)$ and its volume by $V(n)$. From [6,17,38] it is known that either there are $d \in \mathbb{N}$, $a, b > 0$ such that $an^d \leqslant V(n) \leqslant bn^d$, in which case Γ is called to be of *polynomial growth of order* d; or for every $d \in \mathbb{N}$ and every $b \in \mathbb{R}$ there exist only finitely many integers n such that $V(n) \leqslant bn^d$, in which case Γ is called to be of *superpolynomial growth*. The growth type depends only on the group and not on the choice of the set of generators used to define the Cayley graph. Cayley graphs are a particular case of *quasi-transitive* graphs, i.e. graphs whose vertex set decomposes under the action of the automorphism group into finitely many orbits. Most of our results are valid only for Cayley graphs. An exception are the theorems which concern the combinatorial Laplacian (e.g. Theorem 14) which apply to general quasi-transitive graphs with finite vertex degree.

Next we introduce site percolation on infinite, connected, quasi-transitive graphs. For $p \in [0, 1]$, let ω_x, $x \in V$, be an i.i.d. sequence of Bernoulli random variables each taking the value 1 with probability p and the value 0 with probability $1 - p$. The set of possible configurations $\omega = (\omega_x)_{x \in V}$ is denoted by Ω and the corresponding product probability measure with $\mathbb{P}$. We call $V(\omega) := \{x \in V \mid \omega_x = 1\}$ the set of *open sites*. The induced subgraph of G with vertex set $V(\omega)$ is denoted by G_ω and called the *percolation subgraph* in the configuration ω. The connected components of G_ω are called *clusters*. For a fixed vertex $o \in V$ we denote by $C_o(\omega)$ the connected component which contains it. The bond percolation process is defined analogously. In this case the percolation subgraph G_ω is the graph whose edge set $E(\omega)$ is the set of all $e \in E$ with $\omega_e = 1$ and whose vertex set $V(\omega)$ consist of all vertices in V which are incident to an element of $E(\omega)$. For both site and bond percolation there exists a *critical parameter* $0 < p_c \leqslant 1$ such that for $p < p_c$ there is no infinite cluster almost surely and for $p > p_c$ there is an infinite cluster almost surely. The first case is called the *subcritical phase* and the second *supercritical phase*. The theorems of this paper concern only the subcritical percolation phase. We will denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}\{\ldots\}$.

In the following we assume throughout that G is an infinite, countable quasi-transitive graph with bounded vertex degree and that there exists a group of automorphisms acting freely and cofinitely on G. In particular it may be a Cayley graph. Let $G' = (V', E')$ be an arbitrary subgraph of G, possibly G itself. Note that even if G is regular, G' need not be. We denote the degree of the vertex $x \in V'$ in G' by $\deg_{G'}(x)$. If two vertices $x, y \in V'$ are adjacent in the subgraph G' we write $y \sim_{G'} x$.

For G and G' as above we define the following operators on $\ell^2(G') := \ell^2(V')$.

Definition 1.

(a) The identity operator on $\ell^2(V')$ is denoted by Id.

(b) The *degree operator* acts on $\varphi \in \ell^2(V')$ according to

$$\big[D(G')\varphi\big](x) := \deg_{G'}(x)\varphi(x).$$

(c) The *adjacency operator* is defined as

$$\big[A(G')\varphi\big](x) := \sum_{y \in V',\ y \sim_{G'} x} \varphi(y).$$

(d) The *combinatorial Laplacian* is defined as

$$H^N(G') := D(G') - A(G').$$

If G is a k-regular graph we define additionally:

Definition 2.

(e) The *adjacency Laplacian* on G' is defined as

$$H^A(G') := k\,\mathrm{Id} - A(G').$$

(f) The *boundary potential* is the multiplication operator,

$$W^{b.c.}(G') = k\,\mathrm{Id} - D(G').$$

(g) The *Dirichlet Laplacian* is defined as

$$H^D(G') := H^A(G') + W^{b.c.}(G') = 2k\,\mathrm{Id} - D(G') - A(G').$$

Note that $H^N(G') = H^A(G') - W^{b.c.}(G')$. Of course, it is possible to define the operators (e)–(g) also for non-regular graphs, but then there is no canonical choice for the value k which would give them a geometric meaning.

It follows that the quadratic forms of the combinatorial, Dirichlet, and adjacency Laplacian are given by:

$$\begin{aligned}
\langle H^N(G')\phi, \phi\rangle &= \sum_{(x,y)\in E'} \left|\phi(x) - \phi(y)\right|^2, \\
\langle H^D(G')\phi, \phi\rangle &= 2\sum_{x\in V'} \left(k - \deg_{G'}(x)\right)\left|\phi(x)\right|^2 + \sum_{[x,y]\in E'} \left|\phi(x) - \phi(y)\right|^2, \\
\langle H^A(G')\phi, \phi\rangle &= \sum_{x\in V'} \left(k - \deg_{G'}(x)\right)\left|\phi(x)\right|^2 + \sum_{[x,y]\in E'} \left|\phi(x) - \phi(y)\right|^2,
\end{aligned} \tag{1}$$

and satisfy $H^N(G') \leqslant H^A(G') \leqslant H^D(G')$ in the sense of quadratic forms.

Remark 3 *(Terminology).* If $G' = G$ and G is regular then the operators H^A, H^N, H^D coincide and we denote them simply by H. If G is the Cayley graph of an amenable group the spectral bottom of H equals zero. Usually in the graph theory literature the adjacency matrix and the combinatorial Laplacian are the objects of study. For the first operator one is (among others) interested in the properties related to the upper edge of the spectrum, whereas for the second operator one considers the low-lying spectrum. In order to be able to treat both operators in parallel it is convenient to consider H^A rather than A. Of course, spectral properties of H^A directly translate to those of A.

Motivated by the Dirichlet–Neumann bracketing for Laplacians in the continuum, in [36] the terminology of Neumann H^N and Dirichlet H^D Laplacians was introduced. This is the reason why we use the superscript N for the combinatorial Laplacian. While in the continuum the boundary conditions are necessary to define a selfadjoint operator, in the discrete setting they correspond to a boundary potential $W^{b.c.}$, which is either added or subtracted to/from the Laplacian without boundary term, i.e. the adjacency Laplacian H^A. Note however, that the term Neumann Laplacian is sometimes, e.g. in [9], used for a different operator. Likewise, the operator H^A is often called Dirichlet Laplacian, e.g. in [8], while in [22] it is called Pseudo-Dirichlet Laplacian.

Given a (site or bond) percolation subgraph $G_\omega \subset G$ we use the following abbreviations for operators on $\ell^2(V(\omega))$: $\deg_\omega(x) = \deg_{G_\omega}(x)$, $A_\omega = A(G_\omega)$, $H^A_\omega = H^A(G_\omega)$, $H^N_\omega = H^N(G_\omega)$, $H^D_\omega = H^D(G_\omega)$, $W^{b.c.}_\omega = W^{b.c.}(G_\omega)$. Any one of the operators $H^\#_\omega$, $\# \in \{A, N, D\}$ will be called a percolation Laplacian. If G is a Cayley graph we consider all three types H^A, H^N, H^D, while in the case of a quasi-transitive graph we will derive results only for the combinatorial Laplacian H^N.

Next we define the IDS. Let G be a quasi-transitive graph equipped with a subgroup Γ of its automorphism group which acts freely and cofinitely on G. Denote by $\mathcal{F}$ an arbitrary, but fixed Γ-fundamental domain, i.e. a subset of G, which contains exactly one element of each Γ-orbit. The IDS of the random operator $(H^\#_\omega)_\omega$ may be defined by the following trace formula:

$$N^\#(E) := \frac{1}{|\mathcal{F}|}\mathbb{E}\big\{\mathrm{Tr}\big[\chi_{\mathcal{F}}\chi_{]-\infty,E]}\big(H^\#_\omega\big)\big]\big\}. \tag{2}$$

Here $\chi_{\mathcal{F}}$ is understood to be a multiplication operator. If Γ acts transitively on G the expression (2) simplifies to $\mathbb{E}\{\langle\delta_x, \chi_{]-\infty,E]}(H^\#_\omega)\delta_x\rangle\}$, where x denotes an arbitrary vertex in G and δ_x its characteristic function. If moreover $p = 1$, i.e. we consider the IDS of the Laplacian H on G itself, the formula simplifies further to $N_{\mathrm{per}}(E) = \langle\delta_x, \chi_{]-\infty,E]}(H)\delta_x\rangle$. We denote by N_{per} the IDS of the periodic operator H, while $N^\#$ is reserved for the IDS of the random operator $H^\#$.

Remark 4. Several properties of the random family $(H^\#_\omega)_\omega$ of operators play a role in the definition of the IDS. These hold for any of the boundary types $\# \in \{A, N, D\}$. Firstly, $(H^\#_\omega)_\omega$ is a measurable family of operators in the sense of [26] (which extends the notion introduced in [20]). Secondly, each operator is bounded, selfadjoint and non-negative.

If the group Γ is amenable, it is possible to approximate the IDS by its analogs associated to operators restricted to finite graphs along a Følner (van Hove) sequence. This has been shown for periodic operators in [14,30] and for site percolation Hamiltonians in [40]. For bond percolation Hamiltonians the same proof applies. For bond percolation on the lattice $\mathbb{Z}^d$ these results were proven in [22].

A finitely generated, discrete group is amenable if and only if it contains an increasing Følner sequence, i.e. an increasing sequence of finite subsets $I_n \subset \Gamma$ such that

$$\lim_{j\to\infty} \frac{|I_j \triangle F \cdot I_j|}{|I_j|} = 0, \quad \text{for any finite } F \subset \Gamma.$$

Any increasing Følner sequence induces a monotone exhaustion $\Lambda_n, n \in \mathbb{N}$ consisting of finite subsets Λ_n of the vertex set of G, such that if we denote by $H_\omega^{\#,n}$ the restriction of $H_\omega^\#$ to $\ell^2(\Lambda_n \cap V(\omega))$ the convergence

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \operatorname{Tr}\bigl[\chi_{]-\infty,E]}\bigl(H_\omega^{\#,n}\bigr)\bigr] = N^\#(E) \tag{3}$$

holds for almost all ω and all continuity points E of $N^\#$. Actually, for this convergence one has to assume that the Følner sequence $(I_j)_j$ is tempered, cf. the ergodic theorem in [28]. This is no loss of generality, since every amenable group contains a tempered Følner sequence. Note that since $H_\omega^{\#,n}$ is a finite dimensional operator its spectrum consists entirely of eigenvalues and hence $\operatorname{Tr}[\chi_{]-\infty,E]}(H_\omega^{\#,n})]$ equals the number of eigenvalues of $H_\omega^{\#,n}$ not exceeding E. Using the inequalities for the quadratic forms (1) and Weyl's monotonicity principle it follows that $\operatorname{Tr}[\chi_{]-\infty,E]}(H_\omega^{N,n})] \geqslant \operatorname{Tr}[\chi_{]-\infty,E]}(H_\omega^{A,n})] \geqslant \operatorname{Tr}[\chi_{]-\infty,E]}(H_\omega^{D,n})]$. Passing to the limit $n \to \infty$ one obtains $N^N \geqslant N^A \geqslant N^D$. Recently it turned out that the statement in (3) can be strenghtened: in [27] it was shown that the convergence holds *uniformly* with respect to the energy parameter E.

There are other important properties of $(H_\omega^\#)_\omega$ which are appropriate to mention here although they are not necessary for the formulation of our definitions or theorems. The spectrum of $H_\omega^\#$ is almost surely ω-independent, cf. [26,40]. We denote it by $\Sigma^\#$ in the sequel. The same holds for the measure-theoretic components of the spectrum. The topological support of the measure whose distribution function is $N^\#$ coincides with $\Sigma^\#$. Using the same arguments as in [22] one can show that $\Sigma^\# \supset \sigma(H)$. The IDS of percolation Hamiltonians has a rich set of discontinuities [7], and a characterisation of this set is given in [41]. It is also possible to extend the percolation Hamiltonians to the removed vertices $V \setminus V(\omega)$ by a constant. This is just a matter of convention and does not alter the results essentially. For a broader discussion of the above facts see [4].

The next statement characterises the asymptotic behaviour of the IDS of the periodic Laplacian H at the spectral bottom and can be inferred from [29,39]. For groups of polynomial growth it exhibits a van Hove singularity, while in the case of superpolynomial growth one encounters a different type of asymptotics which may be interpreted as corresponding to a van Hove exponent equal to infinity.

Theorem 5. *Let Γ be an infinite, finitely generated, amenable group, H the Laplace operator on a Cayley graph of Γ and N_{per} the associated IDS. If Γ has polynomial growth of order d, then*

$$\lim_{E\searrow 0} \frac{\ln N_{\mathrm{per}}(E)}{\ln E} = \frac{d}{2}, \tag{4}$$

and if Γ has superpolynomial growth, then

$$\lim_{E\searrow 0} \frac{\ln N_{\mathrm{per}}(E)}{\ln E} = \infty. \tag{5}$$

Next we state our result about the low energy asymptotics of $(H_\omega^A)_\omega$ and $(H_\omega^D)_\omega$ and compare it with the asymptotic behaviour of the Laplacian H on the full Cayley graph. Here and in the sequel we restrict ourselves to the subcritical phase of (site or bond) percolation, i.e. we consider a percolation parameter $p < p_c$. The asymptotic behaviour of the IDS of the adjacency and the Dirichlet percolation Laplacian on a Cayley graph at low energies is as follows:

Theorem 6. *Let G be a k-regular Cayley graph of an amenable, finitely generated group Γ. Let $(H_\omega^A)_\omega$ and $(H_\omega^D)_\omega$ be the adjacency, respectively the Dirichlet percolation Laplacian for subcritical site or bond percolation on G.*

Then there is a positive constant a_p such that for all positive E small enough we have:

$$N^D(E) \leqslant N^A(E) \leqslant \exp\left(-\frac{a_p}{2} V\left(\frac{1}{8\sqrt{2}k} E^{-1/2} - 1\right)\right). \tag{6}$$

Assume that G has polynomial growth and $V(n) \sim n^d$. Then there are positive constants $\alpha_D^+(p)$ and $\alpha_D^-(p)$ such that for all positive E small enough,

$$e^{-\alpha_D^-(p)E^{-d/2}} \leqslant N^D(E) \leqslant N^A(E) \leqslant e^{-\alpha_D^+(p)E^{-d/2}}. \tag{7}$$

Assume that G has superpolynomial growth. Then

$$\lim_{E\searrow 0} \frac{\ln|\ln N^D(E)|}{|\ln E|} = \lim_{E\searrow 0} \frac{\ln|\ln N^A(E)|}{|\ln E|} = \infty. \tag{8}$$

Remark 7. The inequality $N^D(E) \leqslant N^A(E)$ in (6) and (7) is deduced from the convergence of the finite volume eigenvalue counting functions to the IDS which is explained in Remark 4. This is slightly inconsistent with our approach that we want to deduce the asymptotic behaviour of the IDS from the trace formula (2) alone. Note however that our proof of Theorem 6 shows that even *without the use of the finite volume approximation* the estimate $N^D(E) \leqslant \exp(-\frac{a_p}{2} V((8k\sqrt{2E})^{-1} - 1))$ holds. In the case of polynomial growth of order d we have the two-sided bounds:

$$e^{-\alpha_D^-(p)E^{-d/2}} \leqslant N^D(E) \leqslant e^{-\tilde{\alpha}_D^+(p)E^{-d/2}}$$

and

$$e^{-\tilde{\alpha}_D^-(p)E^{-d/2}} \leqslant N^A(E) \leqslant e^{-\alpha_D^+(p)E^{-d/2}},$$

with some positive constants $\tilde{\alpha}_D^+, \tilde{\alpha}_D^-$. Thus even without the knowledge that the IDS has finite volume approximations the correct asymptotic behaviour of the IDS may be deduced. An analogous remark applies to Eq. (8).

Remark 8. Theorem 6 is a generalisation of the results in [24] and [22] on subcritical bond percolation on the lattice. Actually, [24] treats random hopping models, of which the edge percolation model is just a special case; moreover it covers supercritical bond percolation on the lattice as well. However, [24] gives only the upper bound on the IDS (which, in Euclidean geometries, is considered the harder inequality), but does not supply a lower bound. In [22] upper and lower bounds are given using an independent proof. Theorem 6 is consistent with the Lifshitz asymptotics for various other types of random Schrödinger operators in Euclidean space, cf. e.g. [21]. In particular, Lifshitz tails have been proven for the Anderson model, i.e. the discrete random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ with an i.i.d. potential. The first proofs of this result were given in [31,36]. Let us note that [24] shows that eigenvalue inequalities for some models with off-diagonal disorder can be reduced to the analogous inequalities for the Anderson model. Thus, Lifshitz type estimates for Anderson models imply those also for bond (and site) percolation models. This suggests that one should derive Lifshitz asymptotics for the Anderson model on general graphs, rather than study percolation models. However, it is not clear whether the proof of [31,36] can be adapted to general amenable Cayley graphs. One obstacle for this extension is the fact one has to bound the IDS in terms of the eigenvalues of the random operator restricted to a finite graph. In Euclidean space this can be established using the fact that cubes are a very neat Følner sequence which are at the same time fundamental domains of sublattices. For general amenable groups such sequences do not exist necessarily. The other reason is that the eigenvalue estimates for the Anderson models restricted on finite graphs are established using the Temple inequality, which in turn to be applied efficiently needs lower bounds on the distance between the two lowest eigenvalues. This lower bound on the spectral gap is immediate in the Euclidean case, while for more general transitive graphs it may be inferred from a strengthened version of the Cheeger inequality. Taking these considerations into account one may hope that the proof of Lifshitz tails for the Anderson model on $\mathbb{Z}^d$ can be adapted for Cayley graphs of polynomial growth. They are, apart from being amenable, residually finite and thus admit an approximation by finite transitive graphs.

Remark 9. Actually the statements of Theorems 5 and 6 hold not only for Laplacians but also for more general symmetric transition operators associated to Markov chains on the group Γ. A precise formulation of these results is presented in Section 7.

Remark 10 *(Test functions and lower bounds on the IDS).* Let us comment on the fact that in (6) a lower bound of the same type as the upper bound is missing. For random operators in Euclidean space the upper bound on the IDS is considered the non-trivial part of the Lifshitz asymptotics, while the lower bound can be obtained by a natural choice of test functions for the Rayleigh quotient. For operators on Cayley graphs the situation is similar, if we restrict ourselves to polynomial volume growth. Namely, in that case we can match the upper bound with a lower bound of the same type, by the use of an appropriate test function. For groups of super-polynomial volume growth it is not clear whether this can be achieved in general. In that situation the choice of a test function becomes intricate since the leading contribution to the Rayleigh quotient may come from the boundary. It turns out that Lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$ have certain special properties which enable one to find effective test functions and thus establish a proper lower bound on the IDS. These groups are amenable, but of exponential growth.

Theorem 6 implies in particular that the IDS is very sparse near the bottom of the spectrum $E = 0$ and consequently zero is a fluctuation boundary. Relation (7) implies that in the case of polynomial growth the Lifshitz exponent coincides with the van Hove exponent of the Laplacian on the full Cayley graph. In particular, we have:

$$\lim_{E \searrow 0} \frac{\ln|\ln N^D(E)|}{|\ln N_{\mathrm{per}}(E)|} = \lim_{E \searrow 0} \frac{\ln|\ln N^A(E)|}{|\ln N_{\mathrm{per}}(E)|} = 1.$$

In the case of superpolynomial growth we have that both exponents are infinite. One may ask whether the sequences defining them diverge at the same rate and whether the relation,

$$\lim_{E \searrow 0} \frac{\ln\ln|\ln N^D(E)|}{\ln|\ln N_{\mathrm{per}}(E)|} = \lim_{E \searrow 0} \frac{\ln\ln|\ln N^A(E)|}{\ln|\ln N_{\mathrm{per}}(E)|} = 1,$$

holds under appropriate conditions, for instance, assuming exponential volume growth. We are not able prove this for arbitrary groups of exponential growth, but at least for the case of the Lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$.

Theorem 11. *Let G be a Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$. There are positive constants a_1^+ and a_2^+ such that*

$$N_{\mathrm{per}}(E) \leqslant a_1^+ e^{-a_2^+ E^{-1/2}}, \quad \textit{for all } E \textit{ small enough.}$$

Moreover for every $r > 1/2$ there are positive constants $a_{r,1}^-$ and $a_{r,2}^-$ such that

$$N_{\mathrm{per}}(E) \geqslant a_{r,1}^- e^{-a_{r,2}^- E^{-r}}, \quad \textit{for all } E \textit{ small enough.}$$

Thus we have an exponential behaviour of the IDS at the bottom of the spectrum, in particular:

$$\lim_{E \searrow 0} \frac{\ln|\ln N_{\mathrm{per}}(E)|}{|\ln E|} = \frac{1}{2}. \tag{9}$$

Now we turn to random operators on Lamplighter groups.

Theorem 12. *Let G be an arbitrary Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$. For every $p < p_c$ there are positive constants b_1, b_2, c_1, c_2, such that the IDS of the adjacency and Dirichlet* (*site or bond*) *percolation Laplacian satisfies the following inequality*:

$$e^{-c_1 e^{c_2 E^{-1/2}}} \leqslant N^D(E) \leqslant N^A(E) \leqslant e^{-b_1 e^{b_2 E^{-1/2}}}, \quad \textit{for all } E > 0 \textit{ small enough.} \tag{10}$$

Remark 13. Our proofs show that the lower bounds $e^{-\alpha_D^-(p) E^{-d/2}} \leqslant N^D(E) \leqslant N^A(E)$ in Theorem 6 and $e^{-c_1 e^{c_2 E^{-1/2}}} \leqslant N^D(E) \leqslant N^A(E)$ in Theorem 12 are valid for all values of the percolation parameter $p \in]0, 1]$.

Let us now turn to combinatorial Laplacians $(H_\omega^N)_\omega$, i.e. Laplacians with the third type of boundary term which we did not discuss yet. In the case of Neumann boundary conditions the energy zero is not a fluctuation boundary. The IDS has a discontinuity at zero, thus one may say that the zeroth L^2-Betti number of the random operator $(H_\omega^N)_\omega$ does not vanish. For the combinatorial Laplacian we are able to treat general quasi-transitive graphs. In particular,

G does not need to be neither amenable nor a Cayley graph. The following generalises a result of [22] on $\mathbb{Z}^d$-bond percolation.

Theorem 14. *Let G be a infinite graph with bounded vertex degree and Γ a group of automorphisms acting freely and cofinitely on G. Consider the IDS of the Neumann percolation Hamiltonian $(H_\omega^N)_\omega$ of a subcritical site or bond percolation process. There exist positive constants $\alpha_N^+(p)$ and $\alpha_N^-(p)$ such that for all positive E small enough,*

$$e^{-\alpha_N^-(p)E^{-1/2}} \leqslant N^N(E) - N^N(0) \leqslant e^{-\alpha_N^+(p)E^{-1/2}}. \tag{11}$$

The value $N^N(0)$ coincides with the average number of clusters per vertex in the random graph G_ω. After subtracting this value we can speak of (11) as a kind of 'renormalised' Lifshitz asymptotics with exponent $1/2$.

Remark 15. Again, Theorem 14 can be extended to more general models. More precisely, one can replace the Laplacian H_ω^N by a regularised Markov transition operator in which case the estimates (11) still hold. Furthermore, it is possible to establish the same result for random combinatorial Laplacians generated by a long range bond percolation process on a quasi-transitive graph G. Both generalisations are presented in Section 7.

On an abstract level Theorem 14 and its proof show that the low energy asymptotics of the combinatorial Laplacian does not depend on geometric properties of G, but only on the rate at which the linear clusters are produced by the percolation process, see Section 7.3 for a discussion of this phenomenon. In this context let us note that Müller and Richard [32] have obtained results on the low energy asymptotics of combinatorial percolation Laplacians on certain Delone sets in $\mathbb{R}^d$.

3. Abstract upper and lower bounds on the IDS

To obtain upper bounds for the integrated density of states near the lower spectral edge, we have to prove that the spectrum is relatively scarce in this area. In the subcritical phase the spectrum is only pure point and consists of the eigenvalues of the operators $H^\#(G')$, where G' goes over the set of all finite subgraphs. So what one really needs are certain lower bounds for the eigenvalues of the operators $H^\#(G')$, $\# \in \{N, A, D\}$. Vice versa for lower bounds for the IDS we shall need upper bounds for these eigenvalues in some neighbourhood of the lower spectral edge. In this spirit we present Propositions 16 and 17, which are generalisations of Lemmata 2.7 and 2.9 in [22]. Denote with $\lambda^\#(G')$ the lowest non-zero eigenvalue of the operator $H^\#(G')$, $\# \in \{N, A, D\}$.

Proposition 16. *Let G be a quasi-transitive graph and $\# \in \{A, D, N\}$. Assume that there is a continuous strictly decreasing function $f : [1, \infty[\to \mathbb{R}^+$ such that $\lim_{s\to\infty} f(s) = 0$ and $\lambda^\#(G') \geqslant f(|G'|)$ for any finite subgraph G'. Then, for every $0 < p < p_c$ there is a positive constant a_p such that*

$$N^\#(E) - N^\#(0) \leqslant e^{-a_p f^{-1}(E)}, \tag{12}$$

for all E from the interval $]0, f(1)[$ on which the inverse function f^{-1} is well defined.

Proof. Fix $\# \in \{N, A, D\}$ and $0 < E < f(1)$. Since the subspace $\ell^2(C_x(\omega))$ is invariant for the operator $H_\omega^\#$ and the restriction on this subspace is exactly $H^\#(C_x(\omega))$ we can write:

$$N^\#(E) - N^\#(0) = \frac{1}{|\mathcal{F}|} \sum_{x\in\mathcal{F}} \mathbb{E}\big(\langle \delta_x, \chi_{]0,E]}\big(H^\#\big(C_x(\omega)\big)\big)\delta_x\rangle\big). \tag{13}$$

Now $\chi_{]0,E]}(H^\#(C_x(\omega)))$ is the zero operator if $E < \lambda^\#(C_x(\omega))$, in particular in the case $|C_x(\omega)| < f^{-1}(E)$. Since $\langle \delta_x, \chi_{]0,E]}(H^\#(C_x(\omega)))\delta_x\rangle \leqslant 1$ for any E and ω we can write:

$$\begin{aligned} N^\#(E) - N^\#(0) &= \frac{1}{|\mathcal{F}|} \sum_{x\in\mathcal{F}} \mathbb{E}\big(\langle \delta_x, \chi_{]0,E]}\big(H^\#\big(C_x(\omega)\big)\big)\delta_x\rangle \chi_{\{|C_x(\omega)|\geqslant f^{-1}(E)\}}(\omega)\big) \\ &\leqslant \frac{1}{|\mathcal{F}|} \sum_{x\in\mathcal{F}} \mathbb{P}\big(\big|C_x(\omega)\big| \geqslant f^{-1}(E)\big). \end{aligned}$$

Now the result follows from the fact that the probabilities of large subcritical clusters in quasi-transitive graphs decay exponentially, i.e. $\mathbb{P}(|C_x(\omega)| \geqslant n) \leqslant e^{-a_p n}$ for all positive integers n, all vertices x and all $p < p_c$, where a_p is a positive constant depending only on the value of the parameter p. This fact is established for general quasi-transitive graphs in [3] using the methods developed in [1,2]. $\square$

Proposition 17. *Let G be a graph with bounded vertex degree and Γ a group of automorphisms acting cofinitely on G and $\# \in \{A, D, N\}$. Suppose that there is a sequence of connected subgraphs $(G'_n)_n$ and a sequence $(c_n)_n$ in $\mathbb{R}^+$ such that*

(i) $\lim_{n\to\infty} |G'_n| = \infty$,
(ii) $\lim_{n\to\infty} c_n = 0$,
(iii) $\lambda^\#(G'_n) \leqslant c_n$.

For every $E > 0$ small enough define $n(E) := \min\{n;\ c_n \leqslant E\}$. Then for every $0 < p < 1$ there is a positive constant b_p such that the following inequality holds for all $E > 0$ small enough,

$$N^\#(E) - N^\#(0) \geqslant \frac{1}{|\mathcal{F}|}\mathbb{P}\big(G'_{n(E)} \textit{ is a cluster in } G_\omega\big) \geqslant e^{-b_p|G'_{n(E)}|}. \tag{14}$$

Proof. Fix $\# \in \{N, A, D\}$ and $E > 0$ small enough so that $n(E)$ is well defined. Define $\mathcal{S}_x(E) := \{\tau \in \Gamma;\ x \in \tau G'_{n(E)}\}$ where $\tau G'_{n(E)}$ is the translation of the subgraph $G'_{n(E)}$ obtained by mapping each vertex of the subgraph $G'_{n(E)}$ by the automorphism τ. On the set $\mathcal{S}_x(E)$ define the equivalence relation $\simeq$ in the following way $\tau_1 \simeq \tau_2 :\Leftrightarrow \tau_1 G'_{n(E)} = \tau_2 G'_{n(E)}$. Now take $\mathcal{T}_x(E)$, a subset of $\mathcal{S}_x(E)$, which contains exactly one element from each equivalence class. Formula (13) implies immediately:

$$\begin{aligned}|\mathcal{F}|\big(N^\#(E) - N^\#(0)\big) &\geqslant \sum_{x\in\mathcal{F}} \mathbb{E}\big(\big\langle\delta_x, \chi_{]0,E]}\big(H^\#\big(C_x(\omega)\big)\big)\delta_x\big\rangle \chi_{\{\exists\tau\in\mathcal{T}_x(E):\ C_x(\omega)=\tau G'_{n(E)}\}}\big)\\ &\geqslant \sum_{x\in\mathcal{F}} \mathbb{E}\big(\big\langle\delta_x, \chi_{]0,\lambda^\#(G'_{n(E)})]}\big(H^\#\big(C_x(\omega)\big)\big)\delta_x\big\rangle \chi_{\{\exists\tau\in\mathcal{T}_x(E):\ C_x(\omega)=\tau G'_{n(E)}\}}\big),\end{aligned} \tag{15}$$

since $\lambda^\#(G'_{n(E)}) \leqslant c_{n(E)} \leqslant E$. By definition of $\mathcal{T}_x(E)$, the last expression in (15) equals:

$$\begin{aligned}&\sum_{x\in\mathcal{F}}\sum_{\tau\in\mathcal{T}_x(E)} \mathbb{E}\big(\big\langle\delta_x, \chi_{]0,\lambda^\#(G'_{n(E)})]}\big(H^\#\big(\tau G'_{n(E)}\big)\big)\delta_x\big\rangle \chi_{\{C_x(\omega)=\tau G'_{n(E)}\}}\big)\\ &\quad= \sum_{x\in\mathcal{F}}\sum_{\tau\in\mathcal{T}_x(E)} \big\langle\delta_x, U_\tau^{-1}\chi_{]0,\lambda^\#(G'_{n(E)})]}\big(H^\#\big(G'_{n(E)}\big)\big)U_\tau\delta_x\big\rangle \mathbb{P}\big(C_x(\omega) = \tau G'_{n(E)}\big)\\ &\quad= \sum_{x\in\mathcal{F}}\sum_{\tau\in\mathcal{T}_x(E)} \big\langle\delta_{\tau^{-1}x}, \chi_{]0,\lambda^\#(G'_{n(E)})]}\big(H^\#\big(G'_{n(E)}\big)\big)\delta_{\tau^{-1}x}\big\rangle \mathbb{P}\big(C_{\tau^{-1}x}(\omega) = G'_{n(E)}\big).\end{aligned}$$

Here we used the fact that for any subgraph G' and any element τ of the group Γ, $H^\#(\tau G') = U_\tau^{-1} H^\#(G') U_\tau$, where U_τ is a unitary operator on $\ell^2(G)$ defined by $U_\tau f(x) := f(\tau x)$. The operators U_τ have the property $U_\tau \delta_x = \delta_{\tau^{-1}x}$. In the following we introduce a sum over all possible values of $\tau^{-1}x$ and see that (15) is equal to

$$\sum_{y\in G'_{n(E)}} \big\langle\delta_y, \chi_{]0,\lambda^\#(G'_{n(E)})]}\big(H^\#\big(G'_{n(E)}\big)\big)\delta_y\big\rangle \mathbb{P}\big(C_y(\omega) = G'_{n(E)}\big) \sum_{x\in\mathcal{F}} \sum_{\substack{\tau\in\mathcal{T}_x(E)\\ y=\tau^{-1}x}} 1.$$

Note that for each vertex y in $G'_{n(E)}$ there exist a vertex $x \in \mathcal{F}$ and an automorphism $\tau \in \mathcal{T}_x(E)$ which maps y to x, thus $\sum_{x\in\mathcal{F}}\sum_{\tau\in\mathcal{T}_x(E),\ y=\tau^{-1}x} 1 \geqslant 1$. It follows that the last displayed expression can be bounded below by:

$$\mathbb{P}\big(G'_{n(E)} \text{ is a cluster in } G_\omega\big) \sum_{y\in G'_{n(E)}} \big\langle\delta_y, \chi_{]0,\lambda^\#(G'_{n(E)})]}\big(H^\#\big(G'_{n(E)}\big)\big)\delta_y\big\rangle \geqslant \mathbb{P}\big(G'_{n(E)} \text{ is a cluster in } G_\omega\big).$$

In the last step we used the fact that $\chi_{]0,\lambda^{\#}(G'_{n(E)})]}(H^{\#}(G'_{n(E)}))$ is a non-trivial projection and its trace is equal to the dimension of its range which is thus greater or equal than one. Since we are considering independent percolation on a graph of uniformly bounded vertex degree we can find a positive constant b_p depending only on p, such that

$$\frac{1}{|\mathcal{F}|}\mathbb{P}(G' \text{ is a cluster in } G_\omega) \geqslant e^{-b_p|G'|},$$

holds for any finite subgraph G'. $\square$

4. Bounds on eigenvalues

As we have seen in the previous section, for good upper and lower bounds for the IDS we need to estimate $\lambda^{\#}(G')$. Lower bounds for eigenvalues (which give upper bounds for IDS) which are sufficient for our purposes can be given in terms of the growth rate of the group. Recall that $B(n)$ denotes the ball in a Cayley graph G, of radius n around the unit element ι and $V(n)$ stands for the volume (the number of vertices) of $B(n)$. Also define $\phi(t) := \min\{n \geqslant 0;\ V(n) > t\}$.

Proposition 18. *Let $G = (V, E)$ be a Cayley graph of a finitely generated group Γ. For every finite connected subgraph G'*:

$$\lambda^A(G') \geqslant \frac{1}{128}\frac{1}{k^2\phi(2|G'|)^2}. \tag{16}$$

Proof. If we prove that every non-zero φ satisfies $\frac{\langle\varphi, H^A(G')\varphi\rangle}{\|\varphi\|^2} \geqslant \frac{(128k^2)^{-1}}{\phi(2|G'|)^2}$, the inequality will follow by the mini-max principle, after taking the infimum over all non-zero φ. The above inequality follows from results in [10] and [42]. Namely in the course of the proof of Proposition 14.1 in [42] one proves that for any $\varphi \in \ell^2(G)$ with finite support we have:

$$\frac{D_P(\varphi)}{\|\varphi\|^2} \geqslant \frac{1}{2\kappa^2\mathfrak{f}(|\operatorname{supp}\varphi|)^2}, \tag{17}$$

where κ is a positive constant and $\mathfrak{f}:\mathbb{N}\to\mathbb{R}$ is non-decreasing and such that $\kappa|\partial_E A| \geqslant \frac{|A|}{\mathfrak{f}(|A|)}$ for all finite subsets of vertices A. Here $\partial_E A$ is the edge boundary, i.e. the set of edges which have one end-vertex in A and the other outside A and D_P is the Dirichlet sum, which in the special case when P defines the nearest neighbour simple random walk satisfies $D_P(\varphi) = \sum_{x\sim_G y}|\varphi(x) - \varphi(y)|^2$. Note that (1) implies the fact that for any finite subgraph G' of a Cayley graph G and any $\zeta \in \ell^2(G')$ we have $\langle H^A(G')\zeta, \zeta\rangle = \sum_{x\sim_G y}|\tilde{\zeta}(x) - \tilde{\zeta}(y)|^2$, where $\tilde{\zeta}$ is an extension of ζ in $\ell^2(G)$ defined by setting $\tilde{\zeta}(x)$ to be equal to 0 for every $x \notin G'$. Thus the Dirichlet sum considered in [42] satisfies,

$$D_P(\tilde{\zeta}) = \langle H^A(G')\zeta, \zeta\rangle, \tag{18}$$

in the special case where the transition matrix P corresponds to a simple nearest neighbour random walk on G. On the other hand Theorem 1 in [10] shows that for any Cayley graph of a finitely generated group,

$$8k|\partial_V A| \geqslant \frac{|A|}{\phi(2|A|)}, \tag{19}$$

holds for all finite subsets of vertices A. (Here $\partial_V A$ is the inner vertex boundary of A, i.e. the set of vertices in A which have a neighbour outside A.) Since $|\partial_E A| \geqslant |\partial_V A|$, the conditions of Proposition 14.1 in [42] are satisfied with $\mathfrak{f}(n) = \phi(2n)$ and so (17) and (18) together imply the desired inequality. $\square$

The role of the subgraphs G'_n from Proposition 17 will be played by the balls $B(n)$. As for the sequence c_n from the same proposition, the next proposition will give us a candidate.

Proposition 19. *Let $G = (V, E)$ be a Cayley graph of a finitely generated group with polynomial growth. Then there exists a positive constant β_D^+ such that for every positive integer n we have*:

$$\lambda^D\big(B(n)\big) \leqslant \frac{\beta_D^+ k}{n^2}. \tag{20}$$

Proof. From the mini-max principle we know

$$\lambda^D\big(B(n)\big) \leqslant \frac{\langle \varphi, H^D(B(n))\varphi\rangle}{\|\varphi\|^2}, \tag{21}$$

for every $\varphi \in \ell^2(B(n))$. For a test function φ use the radially symmetric function defined in the following way:

$$\varphi(x) := \begin{cases} n - d(\iota, x), & \text{if } d(\iota, x) \in \{\lfloor n/2\rfloor, \dots, n\}, \\ \lceil n/2\rceil, & \text{if } d(\iota, x) < \lfloor n/2\rfloor, \\ 0, & \text{else.} \end{cases}$$

Now we have:

$$\big\langle \varphi, H^D\big(B(n)\big)\varphi\big\rangle = \sum_{\substack{[x,y]\in E \\ x,y\in B(n)}} \big|\varphi(x) - \varphi(y)\big|^2 + 2 \sum_{\substack{[x,y]\in E \\ x\in B(n),\ y\notin B(n)}} \big|\varphi(x) - \varphi(y)\big|^2 \leqslant kV(n),$$

$$\|\varphi\|^2 = \sum_{x\in B(n)} \big|\varphi(x)\big|^2 \geqslant \lceil n/2\rceil^2 V\big(\lfloor n/2\rfloor\big).$$

Inserting these two inequalities into (21) and using the fact that $V(n)$ grows polynomially one easily obtains (20). □

Now we give bounds for the eigenvalues for the combinatorial Laplacian on quasi-transitive graphs. The first one is a variant of the Cheeger inequality.

Proposition 20. *Let $G = (V, E)$ be a quasi-transitive graph with vertex degree bounded by $\tilde{k}$. For a finite subgraph $G' = (V', E')$ denote by* $\operatorname{diam} G' := \max_{x,y\in V'} d(x, y)$ *its diameter. Then we have*:

$$\lambda^N(G') \geqslant \frac{2}{|G'|\operatorname{diam} G'} \geqslant \frac{2}{|G'|^2}. \tag{22}$$

For the proof see Lemma 1.9 in [9] or Lemma A.1 in [19].

The role of the subgraphs G'_n from Proposition 17, in the case of the Neumann Laplacian, will be played by linear subgraphs. A linear subgraph $L_n \subset G$ of length n is the subgraph induced by a path $v_1, v_2, \dots, v_{n+1}$ in the graph G, such that the distance between v_i and v_j is equal to $|j - i|$, for every $1 \leqslant i, j \leqslant n+1$. Notice that for every connected infinite graph G and every $n \in \mathbb{N}$ there exists a linear subgraph of length n in G. To see this fix an arbitrary vertex w_0 and take any vertex w_n on the sphere of radius n with centre in w_0 (this sphere is obviously non-empty). Now take a shortest path $(w_0, w_1, \dots, w_{n-1}, w_n)$ between the vertices w_0 and w_n. Clearly, the vertices $\{w_0, w_1, \dots, w_n\}$ are vertices of a linear subgraph L_n.

Proposition 21. *Let $G = (V, E)$ be a quasi-transitive graph with bounded vertex degree. For any integer n we have*:

$$\lambda^N(L_n) \leqslant \frac{12}{n^2}. \tag{23}$$

Proof. We will again use the mini-max principle, i.e. $\lambda^N(L_n) \leqslant \frac{\langle \varphi, H^N(L_n)\varphi\rangle}{\|\varphi\|^2}$, for all $\varphi \in \ell^2(L_n)$, which are orthogonal to the kernel of the operator $H^N(L_n)$. Since the kernel is one-dimensional and contains only constant functions, the condition that φ is orthogonal to the kernel is equivalent to $\sum_{x\in L_n} \varphi(x) = 0$. One obtains (23) by inserting the function which grows linearly along L_n having the value $-n/2$ on one end-vertex and $n/2$ on the other, see Lemma 2.6 in [22]. □

5. Proofs of the theorems for groups of polynomial growth and quasi-transitive graphs

We insert the eigenvalue bounds from the previous section into Propositions 16 and 17 to obtain the estimates on the IDS stated in Theorems 6 and 14.

Proof of Theorem 6. First we prove the general upper bound. Define the function $g(s) := \frac{1}{128k^2\phi(2s)^2}$, which is a right-continuous, non-increasing function which converges to 0 as s approaches to ∞. Now define $g^*(E) := \min\{s;\ g(s) \leqslant E\}$. We note that $\phi(V(n)) = n+1$ for $n \in \mathbb{N}$ and estimate:

$$g^*(E) = \min\left\{s; \phi(2s) \geqslant \frac{1}{8\sqrt{2}k}E^{-1/2}\right\} = \frac{1}{2}V\left(\left\lceil \frac{1}{8\sqrt{2}k}E^{-1/2}\right\rceil - 1\right) \geqslant \frac{1}{2}V\left(\frac{1}{8\sqrt{2}k}E^{-1/2} - 1\right). \tag{24}$$

Now take a sequence of continuous decreasing functions $(f_n)_n$ converging pointwise to g such that for every integer n we have:

$$f_n(s) \leqslant g(s) \quad \text{for every positive } s, \quad \text{and}$$
$$f_n(s) = g(s) \quad \text{for every } s \text{ at which } g \text{ is not continuous.}$$

Clearly $f_n^{-1}(E) = g^*(E)$ for every E in the image of g and $\lim_{n\to\infty} f_n^{-1}(E) = g^*(E)$ if E is not in the image of g. Having in view Proposition 18, every f_n satisfies the conditions of Proposition 16. Now (6) follows.

For the first inequality in (7) use Proposition 17 with $G'_n := B(n)$ and $c_n := \frac{\beta_D^+ k}{n^2}$, where β_D^+ is the constant from Proposition 19. When E approaches 0 from above, $n(E)E^{1/2}$ is bounded from above by a constant and thus the same is true for $|G'_{n(E)}|E^{d/2}$. Now using the fact that $N^D(0) = 0$ the result follows directly from Proposition 17.

For the second inequality in (7) we refer to Remark 4. By the polynomial growth of $V(n)$ the third inequality follows directly from (6).

Now we prove (8). By $N^D \leqslant N^A$ the divergence in (8) has to be proven only for the case of the adjacency Hamiltonian. Using (6) we get:

$$\frac{\ln|\ln N^A(E)|}{|\ln E|} \geqslant \frac{\ln\frac{a_p}{2}}{|\ln E|} + \frac{\ln V(\frac{1}{8\sqrt{2}k}E^{-1/2} - 1)}{|\ln E|}. \tag{25}$$

In the case of superpolynomial growth we have $\lim_{n\to\infty} \frac{\ln V(cn)}{\ln n} = \infty$ for any $c > 0$, and thus

$$\lim_{E\searrow 0} \frac{\ln V(\frac{1}{8\sqrt{2}k}E^{-1/2} - 1)}{|\ln E|} = \infty.$$

Now the claim follows from (25). □

Proof of Theorem 14. By Proposition 20 we see that the assumptions of Proposition 16 are satisfied with $f(s) := \frac{2}{s^2}$. Moreover, by Proposition 21 we can use Proposition 17 with $G'_n = L_n$ and $c_n = \frac{12}{n^2}$. Now the bounds in (11) follow directly. □

As for the periodic case, the formulae for the limits in Theorem 5 are not new, see for instance Lemma 2.46 in [29]. There the operators under consideration are introduced using the language of homological algebra. The idea of the proof is to use a Tauber-type lemma to turn the return probability estimates from [39] into bounds for the IDS. Let us be a bit more detailed: Consider the scaled adjacency operator $\frac{1}{k}A(G)$ and its integrated density of states $N_{\frac{1}{k}A}$. Denote by $\mathbb{P}_o(X_n = o)$ the return probability after n steps of the simple nearest neighbour random walk (X_n) which started at o. It follows that $\mathbb{P}_o(X_n = o) = \int_{\mathbb{R}} t^n\, dN_{\frac{1}{k}A}(t)$. Now it is possible to give sharp bounds on the behaviour of $N_{\frac{1}{k}A}$ near the upper spectral edge (i.e. $E = 1$) using estimates of the return probabilities of the simple random walk. These arguments have the flavour of a Tauberian theorem. In the case of Cayley graphs of groups with polynomial growth the probabilities $\mathbb{P}_o(X_n = o)$ behave like $n^{-d/2}$ (see [39] or Corollary 14.5 and Theorems 14.12 and 14.19 in [42]). Now the desired bounds for N_{per} follow.

The idea to relate the IDS with the return probabilities of the simple random walk will be important for studying the same problem in the case of Lamplighter groups. Here we shall refer to results in [34].

6. Estimates for Lamplighter groups

In this section we derive upper and lower bounds on the IDS for a particular class of amenable groups of super-polynomial growth, namely for Lamplighter groups.

Fix a positive integer $m \geqslant 2$. The Lamplighter group is defined as the wreath product $\mathbb{Z}_m \wr \mathbb{Z}$. In other words, elements of the group are ordered pairs (φ, x), where φ is a function $\varphi: \mathbb{Z} \to \mathbb{Z}_m$ with finite support and $x \in \mathbb{Z}$. The multiplication is given by $(\varphi_1, x_1) * (\varphi_2, x_2) := (\varphi_1 + \varphi_2(\cdot - x_1), x_1 + x_2)$. We shall use the following notation. For $x \in \mathbb{Z}$ let δ_x denote the function which has value 1 at x and 0 everywhere else. The zero function will be denoted by $\mathbf{0}$.

Lamplighter groups are examples of amenable groups with exponential growth. It suffices to prove these two properties for some Cayley graph of the Lamplighter group. Consider the Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$, defined with respect to the set of generators $\{(\mathbf{0}, \pm 1), (k\delta_0, 0);\ k \in \mathbb{Z}_m \backslash \{0\}\}$. To prove amenability one only has to notice that the sequence of sets,

$$\big(\big\{(\varphi, x);\ \operatorname{supp}\varphi \subseteq \{-n, \dots, n\},\ x \in \{-n, \dots, n\}\big\}\big)_n,$$

is a Følner sequence. Exponential growth follows directly from the fact that for any function φ with support in $\{1, 2, \dots, n\}$ one is able reach the vertex (φ, n), from the zero element in at most $2n$ steps, and so ball of radius $2n$ has at least m^n elements.

Using the same ingredients as in the case of groups with polynomial growth we now prove the upper bound in (10).

Proof of the upper bound from Theorem 12. Using the fact that the growth of the Lamplighter group is exponential, the upper bound from (10) follows directly from (6). $\square$

The lower bound in Theorem 12 requires an additional step. In the proof we shall first prove the claimed estimate in the case of a particular generator set and then we shall show how to generalise the result to arbitrary Cayley graphs.

For an arbitrary generator set S of $\mathbb{Z}_m \wr \mathbb{Z}$ denote by $(\mathbb{Z}_m \wr \mathbb{Z})_S$ the Cayley graph induced by the generator set S. Also if V' is a subset of $\mathbb{Z}_m \wr \mathbb{Z}$ denote by $G(V', S)$ the subgraph of $(\mathbb{Z}_m \wr \mathbb{Z})_S$ induced by the vertex set V'.

Define the following symmetric set of generators:

$$S_0 := \big\{(l \cdot \delta_1, 1),\ l \in \mathbb{Z}_m\big\} \cup \big\{(l \cdot \delta_0, -1),\ l \in \mathbb{Z}_m\big\}. \tag{26}$$

As explained in Section 2 of [43] the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$ is the horocyclic product of two $(m+1)$-regular trees. We will briefly sketch the necessary definitions and results. For a comprehensive introduction and a graphical illustration of horocyclic products of trees we refer to [5].

Let $T = (V, E)$ be a $(m+1)$-regular rooted tree with graph metric d. Let ξ be an arbitrary but fixed end. (In the case of trees, an end is an infinite path from the root o in which vertices do not repeat.) For each vertex x there is the unique path γ_x from o to x. Denote the intersection of the paths γ_x and ξ, that is the sequence of edges which lie both in γ_x and ξ, by $\gamma_x \cap \xi$. Now the *Busemann function* of the tree T (with respect to the root o and the end ξ) is defined as $\mathfrak{h}: V \to \mathbb{Z}$, $\mathfrak{h}(x) := |\gamma_x| - 2|\gamma_x \cap \xi|$. For two vertices x and y which satisfy $\mathfrak{h}(y) \geqslant \mathfrak{h}(x)$ and $d(x, y) = \mathfrak{h}(y) - \mathfrak{h}(x)$ we shall write $x \leqslant y$.

Assume now that we are given two $(m+1)$-regular trees T_1 and T_2 with Busemann functions $\mathfrak{h}_1$ and $\mathfrak{h}_2$ respectively. The *horocyclic product* of the trees T_1 and T_2 is defined as the graph whose vertex set is given by $\{(x_1, x_2);\ x_i \in T_i,\ \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$, with two vertices (x_1, x_2) and (x_1', x_2') adjacent if x_i and x_i' are adjacent in T_i for $i = 1, 2$. The choice of a root and an end in the definition is irrelevant since all horocyclic product of two given trees are mutually isomorphic. As we mentioned before, the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$ is isomorphic to the horocyclic product of two $(m+1)$-regular trees.

The spectrum of the full Laplace operator on the graph $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$ is pure point, with eigenfunctions having only finite support. This was shown for the Lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ in [16] and for more general wreath products in [11]. Here we shall follow the methods from [5] where the same facts are proven for Diestel–Leader graphs, which include certain Cayley graphs of the Lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$ as a particular case. Moreover, there the spectrum of the Laplace operator restricted to certain subgraphs called tetrahedrons is calculated. This is where the representation of $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$ as a horocyclic product becomes essential.

Assume we are given a horocyclic product of two $(m+1)$-regular trees T_1 and T_2 with Busemann functions $\mathfrak{h}_1$ and $\mathfrak{h}_2$ and graph metrics d_1 and d_2 respectively. Fix a positive integer n and take two vertices $x_1 \in T_1$ and $x_2 \in T_2$ such that $\mathfrak{h}_2(x_2) = -\mathfrak{h}_1(x_1) - n$. Now the *tetrahedron* K_n with height n is defined as the subgraph of the horocyclic product of T_1 and T_2 induced by the set of vertices $\{(x_1', x_2') \in T_1 \times T_2;\ \mathfrak{h}_1(x_1') + \mathfrak{h}_2(x_2') = 0, x_i \leqslant x_i', 1 = 1, 2\}$. Note that we do not need to specify the vertices x_1 and x_2 in the definition of the tetrahedron, since all tetrahedra with height n are isomorphic.

Corollary 1 and Proposition 1 from [5] specify certain eigenvalues for the Laplacian restricted to tetrahedron with height n among which is $2m(1-\cos\frac{\pi}{n})$. Moreover there exist an eigenfunction corresponding to this eigenvalue which vanishes on the inner vertex boundary of the tetrahedron, so $2m(1-\cos\frac{\pi}{n})$ is an eigenvalue of the operators $H^{\#}(K_n)$ for $\#=N, A, D$. This gives us upper bounds on the lowest eigenvalue of $H^D(K_n)$ which are precise enough to lead to the lower bounds for the IDS given in (10).

Proof of the lower bound from Theorem 12. First we shall consider the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$. Again we shall use Proposition 17 for $\# = D$. We set $G'_n = K_n$. It is easy to see that $|K_n| = (n+1)m^n$. Moreover $\lambda^D(K_n) \leqslant 2m(1-\cos\frac{\pi}{n}) \leqslant \frac{m\pi^2}{n^2}$ and thus we can set $c_n = \frac{m\pi^2}{n^2}$. Proposition 17 now gives the desired result.

Now take an arbitrary generator set S and consider the corresponding Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_S$. Let V_n be a set of vertices which induces a tetrahedron with height n in the Cayley graph $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$. The same set of vertices need not be connected in $(\mathbb{Z}_m \wr \mathbb{Z})_S$ and thus the induced subgraph in $(\mathbb{Z}_m \wr \mathbb{Z})_S$ will not be a good candidate for G'_n in Proposition 17. For this reason we consider a thickening of this set defined by $V_{n,R} := \bigcup_{x\in V_n} B_S(x,R)$, where $B_S(x,R)$ is the ball in $(\mathbb{Z}_m \wr \mathbb{Z})_S$ of radius R with centre in x. Here R is a positive integer, large enough so that the set $V_{n,R}$ is connected in $(\mathbb{Z}_m \wr \mathbb{Z})_S$. (We can take R equal to the maximal distance in $(\mathbb{Z}_m \wr \mathbb{Z})_S$ between vertices which were neighbours in $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$.) The set $V_{n,R}$ induces a connected subgraph $G(V_{n,R}, S)$ of $(\mathbb{Z}_m \wr \mathbb{Z})_S$. The volume of $G(V_{n,R}, S)$ is bounded above by a constant times $|V_n| = (n+1)m^n$, where for the constant we can take the volume of $B_S(x,R)$.

Next we will prove that

$$\lambda^D\big(G(V_{n,R}, S)\big) \leqslant \varrho\lambda^A(K_n), \tag{27}$$

for all n and some positive constant ϱ. Having in mind that $2m(1-\cos\frac{\pi}{n})$ is in the spectrum of $H^A(K_n)$ the desired estimate will follow with the choice $G'_n = G(V_{n,R}, S)$ and $c_n = \varrho\frac{m\pi^2}{n^2}$.

For each function $\varphi \in \ell^2(V_n)$ define the extension $\tilde{\varphi}$ to $V_{n,R}$ by setting $\tilde{\varphi}(x) = 0$ for all $x \in V_{n,R}\setminus V_n$. Theorem 3.2 in [42] implies:

$$\big\langle H^D\big(G(V_{n,R}, S)\big)\tilde{\varphi}, \tilde{\varphi}\big\rangle = \big\langle H^A\big(G(V_{n,R}, S)\big)\tilde{\varphi}, \tilde{\varphi}\big\rangle \leqslant \varrho\big\langle H^A\big(G(V_{n,R}, S_0)\big)\tilde{\varphi}, \tilde{\varphi}\big\rangle, \tag{28}$$

for some positive constant ϱ. (To see this consider the special case of Theorem 3.2 in [42] where the supporting graph is $(\mathbb{Z}_m \wr \mathbb{Z})_{S_0}$ and the transition matrix P defines the nearest neighbour simple random walk on $(\mathbb{Z}_m \wr \mathbb{Z})_S$ and use (18).)

From (1) and the fact that $K_n = G(V_n, S_0)$ it follows that

$$\big\langle H^A\big(G(V_{n,R}, S_0)\big)\tilde{\varphi}, \tilde{\varphi}\big\rangle = \big\langle H^A\big(G(V_n, S_0)\big)\varphi, \varphi\big\rangle = \big\langle H^A(K_n)\varphi, \varphi\big\rangle. \tag{29}$$

Now, having in mind $\|\varphi\| = \|\tilde{\varphi}\|$, (27) follows from (28) and (29) and the proof is finished. □

Now we are left to consider the case of the full Laplacian on the Lamplighter group, i.e. to prove Theorem 11. As we have said before we shall use the relation between the integrated density of states and return probabilities of the simple random walk. To simplify expressions we shall use the following notation. If f and g are two functions $f, g : \mathbb{R}^+ \to \mathbb{R}$, we shall write $f \preccurlyeq g$ if there exist an $\varepsilon > 0$ and positive constants A and B such that $f(x) \leqslant Ag(Bx)$ for every $x \in]0, \varepsilon[$.

Theorem 22. *Let G be a Cayley graph of a finitely generated amenable group and $(X_n)_n$ the simple random walk on G, started at o. Let $\mathbb{P}_o(X_n = o)$ be the return probability of the simple random walk after n steps.*

(i) *Assume that there is a constant $0 < b < 1$ such that for every positive integer n we have $\mathbb{P}_o(X_{2n} = o) \preccurlyeq e^{-(2n)^b}$. Then the integrated density of the full Laplace operator N_{per} satisfies*:

$$N_{\mathrm{per}}(E) \preccurlyeq e^{-E^{-\frac{b}{1-b}}}.$$

(ii) *Assume that there is a constant $0 < b < 1$ such that $e^{-(2n)^b} \preccurlyeq \mathbb{P}_o(X_{2n} = o)$. Then, for every $r > \frac{b}{1-b}$ we have*:

$$e^{-E^{-r}} \preccurlyeq N_{\mathrm{per}}(E).$$

Proof. The proof of both parts is a minor modification of the proof of Theorem 4.4 (parts (ii) and (iii)) in [34]. Using the notation in [34] we shall explain the adjustments which are needed to obtain Theorem 22 from the proof of [34, Theorem 4.4]. The results in [34] are formulated in terms of a certain distribution function F. First note that the value $F(\lambda)$, for any given positive λ, is nothing but $1 - \lim_{s \nearrow 1-\lambda} N_{\frac{1}{k}A}(s)$, where $N_{\frac{1}{k}A}$ is the IDS of the rescaled adjacency operator $\frac{1}{k}A$. Here k is the vertex degree in the graph. From the relation $N_{\mathrm{per}}(\lambda) = 1 - \lim_{s \nearrow 1-\frac{1}{k}\lambda} N_{\frac{1}{k}A}(s)$ it is clear that $N_{\mathrm{per}}(\lambda) = F(\lambda/k)$. Thus it is sufficient to prove the desired inequalities for the function F.

In the proof of the part (ii) we choose,

$$n_\lambda := \left[\left[\left(\frac{Cb}{\ln(\frac{1}{1-\lambda})}\right)^{1/(1-b)}\right]\right],$$

which replaces the choice,

$$n_\lambda := \left[\left[\left(\frac{1}{\lambda}\right)^{1/(1-b+\varepsilon)}\right]\right],$$

in [34]. This enables us to eliminate the variable ε from the calculations and to prove the wanted upper bound for $F(\lambda)$.

For the lower bounds notice that our assumptions are somewhat different than those in the part (iii) of Theorem 4.4 in [34]. Namely we assume uniform lower bounds for the return probabilities. Following the steps of the cited proof, one can prove the same inequalities for all positive λ small enough (i.e. we do not need to define the sets Λ_C). This is exactly what we wanted. □

Proof of Theorem 11. Since the return probabilities of the simple random on any Cayley graph of the Lamplighter group $\mathbb{Z}_m \wr \mathbb{Z}$ satisfy the conditions from both parts of the preceding theorem with $b = 1/3$ (see Theorem 15.15 in [42]), the proof is straightforward from Theorem 22. □

7. Related models

In this last section we shall present several generalisations of the theorems in Section 2. The first one concerns the case where the adjacency operator on G is replaced by a general symmetric transition operator P. It corresponds to a Markov chains whose state space is the vertex set of a Cayley graph. The percolation process on G leads then to a collection H_ω^P, $\omega \in \Omega$ of random operators for which we characterise the low energy asymptotics. We consider also a regularised version H_ω^R, $\omega \in \Omega$ of the transition operator restricted to the percolation subgraph. In the case of the Laplacian this regularisation corresponds to Neumann boundary conditions.

The second generalisation concerns combinatorial Laplacians on random sub-graphs generated by a long range bond percolation process on a quasi-transitive graph.

Finally we discuss the spectral asymptotics of combinatorial Laplacians on an abstract ensemble of percolation graphs.

7.1. General symmetric transition operators

We consider now operators which correspond to general transition operators on Γ, respectively to Cayley graphs with weights on the edges.

Let Γ be a discrete, finitely generated group and $\mathcal{P}$ a matrix indexed by $\Gamma \times \Gamma$ whose coefficients $\mathcal{P}(x,y)$ are non-negative and satisfy:

(a) the set $S := \{x \in \Gamma;\ \mathcal{P}(\iota, x) \neq 0\}$ is a finite symmetric set of generators of the group Γ, which does not contain ι,
(b) for all pairs of group elements (x,y) we have $\mathcal{P}(y,x) = \mathcal{P}(x,y)$,
(c) for all pairs of group elements (x,y) we have $\mathcal{P}(x,y) = \mathcal{P}(\iota, x^{-1}y)$.

Remark 23. Note that by (a) and (c) there exists a constant $M \in \mathbb{R}$ such that $\sum_{y \in \Gamma} \mathcal{P}(x,y) = M$ for all $x \in \Gamma$. Since the spectral properties of the matrix $\mathcal{P}(x,y)$, $x, y \in \Gamma$, can be recovered from those of $\frac{1}{M}\mathcal{P}(x,y)$, $x, y \in \Gamma$, we assume in the sequel without loss of generality that $\sum_{y\in\Gamma}\mathcal{P}(x,y)=1$. Thus the linear map $P:\ell^2(\Gamma)\to\ell^2(\Gamma)$ defined by,

$$(P\varphi)(x)=\sum_y \mathcal{P}(x,y)\varphi(y),$$

is a *transition operator* whose matrix of transition probabilities is given by the coefficients $\mathcal{P}(x,y), x,y\in\Gamma$. The Laplace operator corresponding to P is defined as $H^P:=\mathrm{Id}-P$.

The symmetry of the transition probabilities (b) implies the reversibility of the Markov chain associated to P. More explicitly, there exists a positive function $m:\Gamma\to\mathbb{R}$ such that $m(x)\mathcal{P}(x,y)=m(y)\mathcal{P}(y,x)$ for all pairs of elements (x,y).

We construct a graph G_P whose vertex set equals Γ and such that two vertices x and y are adjacent if and only if $\mathcal{P}(x,y)\neq 0$. Notice that this graph is actually a Cayley graph of the group Γ with respect to the generator set S. In the particular case in which all probabilities $\mathcal{P}(\iota,x)$, $x\in S$, are the same, the operator H^P is actually equal to $\frac{1}{|S|}H^A(G_P)$.

If $G'=(V',E')$ is a subgraph of G_P we shall denote by $H^P(G')$ the restriction of the operator H^P to $\ell^2(V')$. In other words $H^P(G')$ is defined on $\ell^2(V')$ and satisfies $\langle\delta_x,H^P(G')\delta_y\rangle=\langle\delta_x,H^P\delta_y\rangle$ for every two vertices x and y in V'.

Now we can run the nearest neighbour independent bond percolation process on the graph G_P. Each percolation subgraph G_ω will induce a perturbation of the operator H^P. Namely we define the operators $H^P_\omega:=H^P(G_\omega)$. In this way we obtain a family of bounded selfadjoint operators, indexed by the set of all possible percolation configurations Ω.

Now we introduce the analog of a percolation Laplacian with Neumann boundary conditions which corresponds to a general transition operator P. For a subgraph $G'=(V',E')$ of G_P we define the *regularised Laplacian* as

$$\big(H^R(G')\varphi\big)(x)=\sum_{y\in G';\ y\sim_{G'}x}\mathcal{P}(x,y)\big(\varphi(x)-\varphi(y)\big),\quad \text{for every } x\in\ell^2(V').$$

Now the *regularised percolation Laplacian* is defined as $H^R_\omega:=H^R(G_\omega)$.

The quadratic forms of the two operators $H^P(G')$ and $H^R(G')$ are given by

$$\big\langle\varphi,H^P(G')\varphi\big\rangle=\sum_{[x,y]\in E'}\mathcal{P}(x,y)\big|\varphi(x)-\varphi(y)\big|^2+\sum_{x\in V',\ [x,y]\notin E'}\mathcal{P}(x,y)\big|\varphi(x)\big|^2,\tag{30}$$

and

$$\big\langle\varphi,H^R(G')\varphi\big\rangle=\sum_{[x,y]\in E'}\mathcal{P}(x,y)\big|\varphi(x)-\varphi(y)\big|^2.\tag{31}$$

The integrated density of states is again defined as

$$N^\#(E):=\mathbb{E}\big\{\big\langle\delta_x,\chi_{]-\infty,E]}\big(H^\#_\omega\big)\delta_x\big\rangle\big\},$$

for an arbitrary $x\in\Gamma$ and $\#\in\{P,R\}$. Clearly the IDS does not depend on the choice of the vertex x in the definition. The IDS of the deterministic operator on the full graph is defined as $N^P_{\mathrm{per}}(E):=\langle\delta_x,\chi_{]-\infty,E]}(H^P)\delta_x\rangle$.

Now we state the results of this section. They concern the low energy asymptotics of N^P_{per}, N^P, and N^R. Recall that $B(n)$ denotes the ball of radius n around ι in the graph G_P and $V(n)$ the number of vertices in $B(n)$.

As in the special case of the Laplacian the first result can be inferred from [42] or [29].

Theorem 24. *Let Γ be a finitely generated amenable group. If Γ has polynomial growth of order d, then*

$$\lim_{E\searrow 0}\frac{\ln N^P_{\mathrm{per}}(E)}{\ln E}=\frac{d}{2}.$$

If Γ has superpolynomial growth, then

$$\lim_{E\searrow 0}\frac{\ln N^P_{\mathrm{per}}(E)}{\ln E}=\infty.$$

Proof. As explained in Section 5 one can relate the return probabilities of the simple random walk and the moments of the measure induced by the IDS. For general transition operators P as above the relation between the return probabilities of the Markov chain X and the moments of the measure induced by the IDS is the same. Thus, just like before, we only have to prove that the return probabilities $\mathbb{P}(X_n = \iota)$ behave like $n^{-d/2}$. This actually follows from the same results in [42] as in Section 5. □

Theorem 25. *Let Γ be an amenable, finitely generated group.*

Assume that Γ has a polynomial growth, i.e. there exists a positive integer d such that $V(n) \sim n^d$. Then for every $p < p_c$ there are positive constants $\alpha_P^+(p)$ and $\alpha_P^-(p)$ such that for all positive E small enough,

$$e^{-\alpha_P^-(p)E^{-d/2}} \leqslant N^P(E) \leqslant e^{-\alpha_P^+(p)E^{-d/2}}. \tag{32}$$

Assume that Γ has superpolynomial growth. Then

$$\lim_{E \searrow 0} \frac{\ln|\ln N^P(E)|}{|\ln E|} = \infty. \tag{33}$$

Theorem 26. *Let Γ be a finitely generated group. Then for every $p < p_c$ there exist positive constants $\alpha_R^+(p)$ and $\alpha_R^-(p)$ such that*

$$e^{-\alpha_R^-(p)E^{-1/2}} \leqslant N^R(E) - N^R(0) \leqslant e^{-\alpha_R^+(p)E^{-1/2}}. \tag{34}$$

Proofs of Theorems 25 and 26. For both types of the Laplacian Propositions 16 and 17 are trivially extended. Thus the problem of finding upper and lower bounds for the IDS is again reduced to the problem of finding lower and upper bounds for the lowest non-zero eigenvalue on finite subgraphs. On the other hand from (30) and (31) one directly obtains:

$$\Bigl(\min_{[x,y]\in G_P} \mathcal{P}(x,y)\Bigr) H^A(G') \leqslant H^P(G') \leqslant \Bigl(\max_{[x,y]\in G_P} \mathcal{P}(x,y)\Bigr) H^A(G'),$$

and

$$\Bigl(\min_{[x,y]\in G_P} \mathcal{P}(x,y)\Bigr) H^N(G') \leqslant H^R(G') \leqslant \Bigl(\max_{[x,y]\in G_P} \mathcal{P}(x,y)\Bigr) H^N(G').$$

Since $\min_{[x,y]\in G_P} \mathcal{P}(x,y) = \min_{y\in S} \mathcal{P}(\iota,y)$ this term is strictly positive. By the invariance under the group Γ the term $\max_{[x,y]\in G_P} \mathcal{P}(x,y)$ is finite.

Now the bounds for eigenvalues from Propositions 18, 19, Theorems 6 and 14 (with additional positive multiplication factors $\min_{[x,y]\in G_P} \mathcal{P}(x,y)$ and $\max_{[x,y]\in G_P} \mathcal{P}(x,y)$) transfer directly to this generalised setting. Using these bounds, the proofs of Theorems 25 and 26 are completed in the same way as the proofs of Theorems 6 and 14 in Section 5. □

7.2. Laplacians on long range percolation graphs

The long range percolation model is a generalisation of the nearest neighbour model. In this model one allows any pair of vertices to be directly connected, i.e. adjacent, in the percolation graph. However, to control the size of the percolation clusters, the probabilities that two vertices are directly connected must decay as the distance between them converges to infinity. More precisely we take an arbitrary quasi-transitive graph G with finite vertex degrees and a fundamental domain $\mathcal{F}$. We construct the graph $\overline{G}$ by connecting each pair of vertices in G. For the metric on the set of vertices of $\overline{G}$ we will take the graph metric d in G. In particular, two vertices x, y of $\overline{G}$ may be adjacent (directly connected) although $d(x,y) > 1$.

For each pair of vertices x and y we take a positive real number $J_{[x,y]}$ such that

- $J_{[\gamma x,\gamma y]} = J_{[x,y]}$, for all vertices x and y and all graph automorphisms γ,
- $J_x := \sum_{y\in G} J_{[x,y]} < \infty$ for all vertices x (we define $J := \max_{x\in\mathcal{F}} J_x$).

Now for each edge e in $\overline{G}$, one declares e to be open with probability $1 - e^{-\beta J_e}$, for some positive parameter β, independently of all other edges in $\overline{G}$. The percolation subgraph $G_\omega = (V_\omega, E_\omega)$ is defined as the subgraph spanned by the set of open edges. G_ω contains arbitrary long edges almost surely. Notice that the probability that certain edge is open is increasing in β. Thus, the subcritical phase, in which all clusters are almost surely finite corresponds to small values of the parameter β and the supercritical phase in which there exists an infinite cluster corresponds to large values of the parameter β. Just like in the case of the nearest neighbour percolation model these two phases are separated by a single value of the parameter β. This value will be denoted by β_c. The cluster containing an arbitrary vertex x will again be denoted by C_x. In [3] it is proven that the probabilities $\mathbb{P}(|C_x| \geqslant n)$ decay exponentially in the subcritical phase, i.e. $\beta < \beta_c$, of the long range model (see [2] for the case $G = \mathbb{Z}^d$).

The percolation Laplacian is defined as the combinatorial Laplacian on the percolation subgraph. More precisely we define the operator $H^{N,L}$ on $\ell^2(V_\omega)$ for all φ with finite support by:

$$\big(H_\omega^{N,L}\varphi\big)(x) = \sum_{y \in G_\omega;\ y \sim_{G_\omega} x} \big(\varphi(x) - \varphi(y)\big).$$

Since we have no upper bound on the vertex degrees any more, this operator is not bounded almost surely. It is still self-adjoint on its maximal domain $\mathcal{D}(H_\omega^{N,L}) := \{\varphi \in \ell^2(V_\omega);\ H_\omega^{N,L}\varphi \in \ell^2(V_\omega)\}$.

The integrated density of states is again defined as

$$N^{N,L}(E) := \mathbb{E}\big\{\mathrm{Tr}\big[\chi_{\mathcal{F}}\chi_{]-\infty,E]}\big(H_\omega^{N,L}\big)\big]\big\}.$$

It exhibits the same asymptotics as the combinatorial Laplacian in the nearest neighbour percolation model.

Theorem 27. *Let G be a quasi-transitive graph with finite vertex degrees. For every subcritical parameter β there exist positive constants $\alpha_{N,L}^-(\beta)$ and $\alpha_{N,L}^+(\beta)$ such that for all positive E small enough,*

$$e^{-\alpha_{N,L}^-(\beta)E^{-1/2}} \leqslant N^{N,L}(E) - N^{N,L}(0) \leqslant e^{-\alpha_{N,L}^+(\beta)E^{-1/2}}.$$

Proof. Similarly as in Sections 3 and 4 we are able to prove the following statements:

(1) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, strictly decreasing function, such that $\lim_{s\to\infty} f(s) = 0$ and $\lambda^N(G') \geqslant f(|G'|)$ holds for every finite subgraph G' of $\overline{G}$. Then, for each $\beta < \beta_c$, the inequality $N^{N,L}(E) - N^{N,L}(0) \leqslant e^{-a_\beta^L f^{-1}(E)}$ holds for some positive constant a_β^L and all positive E small enough.

(2) Assume that there is a sequence of connected subgraphs $(G_n')_n$ in $\overline{G}$, with the property $\lim_{n\to\infty} |G_n'| = \infty$ and a sequence $(c_n)_n$ in $\mathbb{R}^+$ which converges to 0 such that $\lambda^N(G_n') \leqslant c_n$, for all n. Furthermore assume there is a positive integer k such that for any $n \in \mathbb{N}$ and any two directly connected vertices x and y in G_n' we have $d(x, y) \leqslant k$. Again define $n(E) := \min\{n;\ c_n \leqslant E\}$. Then for every $\beta \in \mathbb{R}^+$ there is a positive constant b_β^L such that for all positive E small enough we have:

$$N^{N,L}(E) - N^{N,L}(0) \geqslant \frac{1}{|\mathcal{F}|}\mathbb{P}\big(G_{n(E)}' \text{ is a cluster in } G_\omega\big) \geqslant e^{-b_\beta^L |G_{n(E)}'|}.$$

(3) For every finite subgraph G' of G we have $\lambda^N(G') \geqslant \frac{1}{|G'|^2}$.

(4) For any linear graph L_n in $\overline{G}$ (i.e. a subgraph of $\overline{G}$ with $n+1$ vertices $v_1, \ldots, v_{n+1}$, such that $d(v_i, v_j) = |j - i|$ and edge set $\{[v_i, v_{i+1}], 1 \leqslant i \leqslant n\}$) we have $\lambda^N(L_n) \leqslant \frac{12}{n^2}$.

From this results our claim follows like in the proof of Theorem 14. Statements (1) and (4) are proven in the same way as Propositions 16 and 21. For statement (3) see Proposition 20. As for statement (2), the proof proceeds along the same lines as the proof of Proposition 17. However, to bound from below the probability that $G_{n(E)}'$ is a cluster in G_ω, we need the additional condition that there are no edges longer than k in G_n' and the following lemma. □

Lemma 28. *For an arbitrary $k \in \mathbb{N}$ and an arbitrary positive β there exists a positive constant ς_k such that the following statement is true:*

For every connected subgraph G' of $\overline{G}$ such that the distance between any two directly connected vertices in G' is less or equal than k we have:

$$\mathbb{P}(G' \textit{ is a cluster of } G_\omega) \geqslant e^{-\varsigma_k |G'|}. \tag{35}$$

Proof. Let G' be an arbitrary subgraph which satisfies the assumptions of the lemma. We partition the set of edges in G adjacent to x into two disjoint subsets: in I_x we put those which are edges of G' and in O_x others. It is clear that the probability $\mathbb{P}(G'$ is a cluster of $G_\omega)$ can be estimated by the product of the probability that all edges in I_x, $x \in G'$, are open and the probability that all edges in O_x, $x \in G'$, are closed. Therefore we can write:

$$\begin{aligned}
\mathbb{P}(G' \text{ is a cluster of } G) &= \mathbb{P}\biggl(\bigcap_{x\in G'}\bigcap_{e\in I_x}\{e \text{ is open}\}\biggr)\mathbb{P}\biggl(\bigcap_{x\in G'}\bigcap_{e\in O_x}\{e \text{ is closed}\}\biggr) \\
&\geqslant \prod_{x\in G'}\biggl(\prod_{e\in I_x}\mathbb{P}(e \text{ is open})\prod_{e\in O_x}\mathbb{P}(e \text{ is closed})\biggr) \\
&= \prod_{x\in G'}\biggl(e^{-\beta J}\prod_{e\in I_x}\bigl(e^{\beta J_e}-1\bigr)\biggr) \\
&\geqslant \bigl(e^{-\beta J}c\bigr)^{|G'|}.
\end{aligned}$$

Here c is defined as $c := \min_{x\in\mathcal{F}} \min_{A;\, A\subset(B(x,k)\setminus\{x\})} \prod_{y\in A}(e^{\beta J_{[x,y]}}-1)$, where $B(x,k)$ is a ball of radius k around x. Obviously c is positive and because of the invariance of the parameters $J_{[x,y]}$ under the automorphisms of G we have $\prod_{e\in I_x}(e^{\beta J_e}-1) \geqslant c$, for all $x \in G'$. Since the constant c does not depend on G', the claim follows. □

7.3. An abstract result

We have encountered the phenomenon, that in the case of the combinatorial Laplacian the low energy asymptotics is independent of the volume growth behaviour of the graph. This is consistent with the results on Erdös–Rényi random graphs obtained in [19].

In the following we present an abstract result which tries to capture this phenomenon and to single out properties which the stochastic process which generates the random graphs needs to satisfy to obtain a low energy asymptotics as in Theorem 14.

Let $G = (V, E)$ be a graph with countable vertex set V and vertex degree bounded by $\tilde{k}$. Let an independent (site or bond) percolation process on G be given and denote the percolation subgraph of G associated to the configuration $\omega \in \Omega$ by G_ω. Fix a finite subset $\mathcal{F}$ of V and assume that there exists a doubly infinite path $\mathfrak{P}$ in G which contains a vertex $o \in \mathcal{F}$. In other words $\mathfrak{P}: \mathbb{Z} \to V$ is injective and contains o in its image. Denote by H_ω^N the combinatorial Laplacian on G_ω and define the monotone function,

$$N^N(E) := |\mathcal{F}|^{-1}\mathbb{E}\bigl\{\mathrm{Tr}\bigl[\chi_{\mathcal{F}}\chi_{]-\infty,E]}\bigl(H_\omega^N\bigr)\bigr]\bigr\},$$

which in many situations can be interpreted as the IDS. Assume that there is no infinite cluster in the graph G_ω almost surely and that the cluster size distribution decays exponentially, more precisely,

$$\mathbb{P}\bigl\{\bigl|C_x(\omega)\bigr| \geqslant n\bigr\} \leqslant e^{-an}, \tag{36}$$

for some $a > 0$ and all $x \in \mathcal{F}$. In the case of site percolation assume furthermore that $p_a := \inf_{x\in V}\mathbb{P}\{x \text{ is open}\}$ and $p_d := \inf_{x\in V}\mathbb{P}\{x \text{ is closed}\}$ are strictly positive. Similarly in the case of bond percolation assume that $\inf_{e\in E}\mathbb{P}\{e \text{ is open}\}$ and $\inf_{e\in E}\mathbb{P}\{e \text{ is closed}\}$ are strictly positive.

Theorem 29. *Assume the setting described in this paragraph. Then there exist constants $\alpha_-, \alpha_+ > 0$ such that for all $E > 0$ sufficiently small*:

$$e^{-\alpha_- E^{-1/2}} \leqslant N^N(E) - N^N(0) \leqslant e^{-\alpha_+ E^{-1/2}}. \tag{37}$$

Proof. For the proof of the upper bound one uses the same inequalities as in the proof of Proposition 16, together with the eigenvalue estimate in Proposition 20 and the exponential decay assumption (36).

For the lower bound one uses that

$$N^N(E) - N^N(0) \geqslant |\mathcal{F}|^{-1} \sum_{j=0}^{n} \mathbb{E}\bigl\{\chi_{\Omega_{n,j}} \langle \delta_o, \chi_{]-\infty,E]}\bigl(H_\omega^N\bigr)\delta_o\rangle\bigr\}, \tag{38}$$

where n is chosen such that $\frac{12}{n^2} \leqslant E$ and $\Omega_{n,j} \subset \Omega$ denotes the set of configurations where the cluster $C_o(\omega)$ is a linear cluster L_n and o is the vertex at the jth position of L_n. By the assumption on the existence of the infinite path $\mathfrak{P}$ such configurations exist and by the independence assumption we estimate the probability of $\Omega_{n,j}$ from below by $p_a^n \cdot p_d^{\tilde{k}n}$.

Now let ϕ_n be a normalised eigenfunction associated to the eigenvalue $\lambda^N(L_n) \leqslant \frac{12}{n^2}$. Since

$$\sum_{j=0}^{n} \mathbb{E}\bigl\{\chi_{\Omega_{n,j}} \langle \delta_o, \chi_{]-\infty,E]}\bigl(H_\omega^N\bigr)\delta_o\rangle\bigr\} \geqslant \sum_{j=0}^{n} \mathbb{E}\bigl\{\chi_{\Omega_{n,j}} \bigl|\phi_n(o)\bigr|^2\bigr\} = \sum_{j=0}^{n} \mathbb{E}\bigl\{\chi_{\Omega_{n,j}} \bigl|\phi_n(j)\bigr|^2\bigr\},$$

we have $N^N(E) - N^N(0) \geqslant |\mathcal{F}|^{-1} p_a^n \cdot p_d^{\tilde{k}n}$. This completes the proof. □

References

[1] M. Aizenman, C.M. Newman, Tree graph inequalities and critical behavior in percolation models, J. Statist. Phys. 36 (1–2) (1984) 107–143.

[2] M. Aizenman, D.J. Barsky, Sharpness of the phase transition in percolation models, Comm. Math. Phys. 108 (3) (1987) 489–526.

[3] T. Antunović, I. Veselić, Sharpness of the phase transition and exponential decay of the subcritical cluster size for percolation on quasi-transitive graphs, J. Statist. Phys. 130 (5) (2008) 983–1009.

[4] T. Antunović, I. Veselić, Spectral asymptotics of percolation Hamiltonians on amenable Cayley graphs, Operator Theory: Advances and Applications 186 (2008) 1–29. http://arxiv.org/abs/0707.4292.

[5] L. Bartholdi, W. Woess, Spectral computations on lamplighter groups and Diestel–Leader graphs, J. Fourier Anal. Appl. 11 (2) (2005) 175–202.

[6] H. Bass, The degree of polynomial growth of finitely generated nilpotent groups, Proc. London Math. Soc. 25 (1972) 603–614.

[7] J.T. Chayes, L. Chayes, J.R. Franz, J.P. Sethna, S.A. Trugman, On the density of states for the quantum percolation problem, J. Phys. A 19 (18) (1986) L1173–L1177.

[8] F. Chung, A. Grigor'yan, S.-T. Yau, Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs, Comm. Anal. Geom. 8 (5) (2000) 969–1026.

[9] F.R.K. Chung, Spectral Graph Theory, CBMS Regional Conference Series in Mathematics, vol. 92, Conference Board of the Mathematical Sciences, Washington, DC, 1997.

[10] T. Coulhon, L. Saloff-Coste, Isopérimétrie pour les groupes et les variétés, Rev. Mat. Iberoamericana 9 (2) (1993) 293–314.

[11] W. Dicks, T. Schick, The spectral measure of certain elements of the complex group ring of a wreath product, Geom. Dedicata 93 (2002) 121–134. http://www.arxiv.org/math/0107145.

[12] J. Dodziuk, N. Lenz, D. Peyerimhoff, T. Schick, I. Veselić (Eds.), L^2-Spectral Invariants and the Integrated Density of States, Oberwolfach Rep., vol. 3, 2006. http://www.mfo.de/programme/schedule/2006/08b/OWR_2006_09.pdf.

[13] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, S. Yates, Approximating L^2-invariants, and the Atiyah conjecture, Comm. Pure Appl. Math. 56 (7) (2003) 839–873.

[14] J. Dodziuk, V. Mathai, S. Yates, Approximating L^2 torsion on amenable covering spaces, math.DG/0008211 on arxiv.org, see also [13].

[15] F. Germinet, A. Klein, Explicit finite volume criteria for localization in continuous random media and applications, Geom. Funct. Anal. 13 (6) (2003) 1201–1238. http://www.ma.utexas.edu/mp_arc/c/02/02-375.ps.gz.

[16] R.I. Grigorchuk, A. Żuk, The lamplighter group as a group generated by a 2-state automaton, and its spectrum, Geom. Dedicata 87 (2001) 209–244.

[17] M. Gromov, Groups of polynomial growth and expanding maps, Inst. Hautes Études Sci. Publ. Math. 53 (1981) 53–73.

[18] M. Gromov, M.A. Shubin, Von Neumann spectra near zero, Geom. Funct. Anal. 1 (4) (1991) 375–404.

[19] O. Khorunzhiy, W. Kirsch, P. Müller, Lifshitz tails for spectra of Erdös–Rényi random graphs, Ann. Appl. Probab. 16 (1) (2006) 295–309.

[20] W. Kirsch, F. Martinelli, On the density of states of Schrödinger operators with a random potential, J. Phys. A: Math. Gen. 15 (1982) 2139–2156.

[21] W. Kirsch, B. Metzger, The integrated density of states for random Schrödinger operators, in: Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, in: Proceedings of Symposia in Pure Mathematics, vol. 76, AMS, 2007, pp. 649–698. http://www.arXiv.org/abs/math-ph/0608066.

[22] W. Kirsch, P. Müller, Spectral properties of the Laplacian on bond-percolation graphs, Math. Z. 252 (4) (2006) 899–916. http://www.arXiv.org/abs/math-ph/0407047.

[23] F. Klopp, Internal Lifshits tails for random perturbations of periodic Schrödinger operators, Duke Math. J. 98 (2) (1999) 335–396.

[24] F. Klopp, S. Nakamura, A note on Anderson localization for the random hopping model, J. Math. Phys. 44 (11) (2003) 4975–4980.
[25] F. Klopp, T. Wolff, Lifshitz tails for 2-dimensional random Schrödinger operators, J. Anal. Math. 88 (2002) 63–147 (dedicated to the memory of Tom Wolff).
[26] D. Lenz, N. Peyerimhoff, I. Veselić, Von Neumann algebras, groupoids and the integrated density of states, Math. Phys. Anal. Geom. 10 (1) (2007) 1–41. http://arXiv.org/abs/math-ph/0203026.
[27] D. Lenz, I. Veselić, Hamiltonians on discrete structures: jumps of the integrated density of states and uniform convergence, Math. Z. (2008) (Online first). http://www.arxiv.org/abs/0709.2836.
[28] E. Lindenstrauss, Pointwise theorems for amenable groups, Invent. Math. 146 (2) (2001) 259–295.
[29] W. Lück, L^2-Invariants: Theory and Applications to Geometry and K-Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 44, Springer-Verlag, Berlin, 2002.
[30] V. Mathai, S. Yates, Approximating spectral invariants of Harper operators on graphs, J. Funct. Anal. 188 (1) (2002) 111–136. http://arXiv.org/math.FA/0006138.
[31] G.A. Mezincescu, Bounds on the integrated density of electronic states for disordered Hamiltonians, Phys. Rev. B 32 (1985) 6272–6277.
[32] P. Müller, Ch. Richard, Random colourings of aperiodic graphs: ergodic and spectral properties, http://www.arxiv.org/abs/0709.0821.
[33] S.P. Novikov, M.A. Shubin, Morse inequalities and von Neumann II_1-factors, Dokl. Akad. Nauk SSSR 289 (2) (1986) 289–292.
[34] S.-I. Oguni, The secondary Novikov–Shubin invariants of groups and quasi-isometry, J. Math. Soc. Japan 59 (1) (2007) 223–237. http://www.math.kyoto-u.ac.jp/preprint/preprint2005.html.
[35] L.A. Pastur, Selfaverageability of the number of states of the Schrödinger equation with a random potential, Mat. Fiz. Funkcional Anal. (Vyp. 2) (1971) 111–116, 247.
[36] B. Simon, Lifschitz tails for the Anderson model, J. Stat. Phys. 38 (1985) 65–76.
[37] M.A. Shubin, Almost periodic functions and partial differential operators, Uspekhi Mat. Nauk 33 (2(200)) (1978) 3–47, 247.
[38] L. van den Dries, A. Wilkie, Gromov's theorem on groups of polynomial growth and elementary logic, J. Algebra 89 (1984) 349–374.
[39] N. Varopoulos, Random walks and Brownian motion on manifolds, Symp. Math. 29 (1987) 97–109.
[40] I. Veselić, Quantum site percolation on amenable graphs, in: Proceedings of the Conference on Applied Mathematics and Scientific Computing, Springer, Dordrecht, 2005, pp. 317–328. http://arXiv.org/math-ph/0308041.
[41] I. Veselić, Spectral analysis of percolation Hamiltonians, Math. Ann. 331 (4) (2005) 841–865. http://arXiv.org/math-ph/0405006.
[42] W. Woess, Random Walks on Infinite Graphs and Groups, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000.
[43] W. Woess, Lamplighters, Diestel–Leader graphs, random walks, and harmonic functions, Combin. Probab. Comput. 14 (3) (2005) 415–433.