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Elliptic problems with discontinuities

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Abstract

In this paper we prove two existence theorems for elliptic problems with discontinuities. The first one is a noncoercive Dirichlet problem and the second one is a Neumann problem. We do not use the method of upper and lower solutions. For Neumann problems we assume that f is nondecreasing. We use the critical point theory for locally Lipschitz functionals.

Keywords: Discontinuous nonlinearities; Countable number of discontinuities; Critical point theory; Locally Lipschitz functionals

1. Introduction

In this paper we study elliptic problems with discontinuous nonlinearities. We use the critical point theory for locally Lipschitz functionals due to Chang [3].

Many authors considered elliptic problems with no Carathéodory right-hand side. For example, Heikkila and Lakshmikantham [7] had used the method of upper and lower solution to obtain existence theorems for certain differential equations with discontinuous nonlinearities involving pseudomonotone operators but they need the existence of upper and lower solutions. On the other hand, many authors established existence results for these problems without upper and lower solutions using the critical point theory for smooth or nonsmooth operators. Hence they need the differential operator to be of variational type. Some characteristic

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papers on this direction is that of Ambrosseti and Badiale [1], Stuart and Tolland [10], and Arcoya and Carahorrano [2] and references therein.

We prove two existence theorems. The first one is for a Dirichlet noncoercive problem. The second one is for a coercive Neumann problem in which we need the right-hand side to be nondecreasing. This result is closely related with the work of Stuart and Tolland [10]. It seems that this is the first result in this direction.

Let $Z \subseteq \mathbf{R}^N$ be a bounded domain with C^1-boundary Γ. The Dirichlet problem under consideration is

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = f\big(z, x(z)\big) & \text{a.e. on } Z, \\ x|_\Gamma = 0, \quad 2 \leqslant p < \infty. \end{cases} \tag{1}$$

The second problem is a Neumann elliptic boundary value problem with multivalued nonlinear boundary conditions. Let $Z \subseteq \mathbf{R}^N$ be a bounded domain with a C^1-boundary Γ:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = f\big(z, x(z)\big) & \text{a.e. on } Z, \\ -\frac{\partial x}{\partial n_p} \in \partial j\big(z, \tau(x)(z)\big) & \text{a.e. on } \Gamma,\ 2 \leqslant p < \infty. \end{cases} \tag{2}$$

Here the boundary condition is in the sense of Kenmochi [9] and the operator τ is the trace operator in $W^{1,p}(Z)$.

In the next section we recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the subdifferential of Clarke.

2. Preliminaries

Let Y be a subset of X. A function $f : Y \to \mathbf{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$\big|f(y) - f(x)\big| \leqslant K\|y - x\|$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x; v)$ is defined as follows:

$$f^o(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^o(x; v)$ is finite, positively homogeneous, subadditive and satisfies $|f^o(x; v)| \leqslant K\|v\|$. In addition f^o satisfies $f^o(x; -v) = (-f)^o(x; v)$. Now we are ready to introduce the generalized gradient which denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \big\{w \in X^*\colon\ f^o(x; v) \geqslant \langle w, v\rangle \text{ for all } v \in X\big\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leqslant K$ for every w in $\partial f(x)$.
(b) For every v in X, one has
$$f^o(x; v) = \max\{\langle w, v\rangle:\ w \in \partial f(x)\}.$$

If f_1, f_2 are locally Lipschitz functions then
$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Let us recall the (PS)-condition introduced by Chang [3].

Definition 2.1. We say that Lipschitz function f satisfies the Palais–Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and $\lambda(x_n) = \mathrm{Min}_{w\in\partial f(x_n)} \|w\|_{X^*} \to 0$ possesses a convergent subsequence.

The (PS)-condition can also be formulated as follows (see Costa and Goncalves [6]):

$(\mathrm{PS})^*_{c,+}$ Whenever $(x_n) \subseteq X, (\varepsilon_n),\ (\delta_n) \subseteq R_+$ are sequences with $\varepsilon_n \to 0$, $\delta_n \to 0$, and such that
$$f(x_n) \to c,$$
$$f(x_n) \leqslant f(x) + \varepsilon_n\|x - x_n\| \quad \text{if } \|x - x_n\| \leqslant \delta_n,$$
then (x_n) possesses a convergent subsequence: $x_{n'} \to \hat{x}$.

Similarly, we define the $(\mathrm{PS})^*_c$ condition from below, $(\mathrm{PS})^*_{c,-}$, by interchanging x and x_n in the above inequality. And finally we say that f satisfies $(\mathrm{PS})^*_c$ provided it satisfies $(\mathrm{PS})^*_{c,+}$ and $(\mathrm{PS})^*_{c,-}$.

Note that these two definitions are equivalent when f is locally Lipschitz functional.

Consider the first eigenvalue λ_1 of $(-\Delta_p, W_o^{1,p}(Z))$. From Lindqvist [8] we know that $\lambda_1 > 0$ is isolated and simple; that is, any two solutions u, v of
$$\begin{cases} -\Delta_p u = -\operatorname{div}(\|Du\|^{p-2}Du) = \lambda_1|u|^{p-2}u \quad \text{a.e. on } Z, \\ u|_\Gamma = 0, \quad 2 \leqslant p < \infty, \end{cases} \tag{3}$$
satisfy $u = cv$ for some $c \in R$. In addition, the λ_1-eigenfunctions do not change sign in Z. Finally, we have the following variational characterization of λ_1 (Rayleigh quotient):
$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p}:\ x \in W_o^{1,p}(Z),\ x \neq 0\right].$$

Let us now recall the two basic theorems that we will use to prove the existence results.

Theorem 2.1. *If a locally Lipschitz functional $f: X \to R$ on the reflexive Banach space X satisfies the* (PS)*-condition and the hypotheses*

(i) *there exist positive constants ρ and a such that*

$$f(u) \geqslant a \quad \text{for all } x \in X \text{ with } \|x\| = \rho;$$

(ii) *$f(0) = 0$ and there a point $e \in X$ such that*

$$\|e\| > \rho \quad \text{and} \quad f(e) \leqslant 0,$$

then there exists a critical value $c \geqslant a$ of f determined by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f\big(g(t)\big),$$

where

$$G = \big\{g \in C\big([0,1], X\big):\ g(0) = 0, g(1) = e\big\}.$$

Theorem 2.2. *Suppose a locally Lipschitz function f defined on a reflexive Banach space, satisfies the* (PS)*-condition and it is bounded from below. Then $c = \inf_X f(x)$ is a critical value of f.*

In what follows we will use the well-known inequality

$$\sum_{j=1}^{N} \big(a_j(\eta) - a_j(\eta')\big)\big(\eta_j - \eta'_j\big) \geqslant C|\eta - \eta'|^p \tag{4}$$

for $\eta, \eta' \in R^N$, with $a_j(\eta) = |\eta|^{p-2}\eta_j$.

3. Dirichlet problems

In this section we prove an existence result for problem (1) using the mountain pass theorem of Chang for locally Lipschitz functionals (i.e., Theorem 2.1).

In the following we will need some definitions. Let

$$f_1(z,x) = \liminf_{x' \to x} f(z,x'), \qquad f_2(z,x) = \limsup_{x' \to x} f(z,x').$$

Definition 3.1. We say that $x \in W_o^{1,p}(Z)$ is a solution of type I of problem (1) if there exists some $w \in W^{-1,q}(Z)$ such that

$$w(z) \in \big[f_1\big(z,x(z)\big),\ f_2\big(z,x(z)\big)\big]$$

and

$$-\operatorname{div}\bigl(\|Dx(z)\|^{p-2}Dx(z)\bigr)=w(z)\quad\text{for almost all } z\in Z.$$

Definition 3.2. We say that $x\in W_o^{1,p}(Z)$ is a solution of type II of problem (1) if

$$-\operatorname{div}\bigl(\|Dx(z)\|^{p-2}Dx(z)\bigr)=f(z,x(z))\quad\text{for almost all } z\in Z.$$

Let us state the hypothesis on the data.

$H(f)_1$ $f: Z\times\mathbf{R}\to\mathbf{R}$ is a N measurable function (i.e., if $x(z)$ is measurable so is $f_1(z,x(z)), f_2(z,x(z)))$ and moreover

(i) for almost all $z\in Z$ and all $x\in\mathbf{R}$, $|f(z,x)|\leqslant c_1|x|^{p-1}+c|x|^{p^*-1}$, with $p^*=Np/(N-p)$;

(ii) there exists $\theta>p$ and $r_o>0$ such that for all $|x|\geqslant r_o$, and all $v\in\partial F(z,x)$ we have $0<\theta F(z,x)\leqslant vx$, and moreover there exists some $a_1\in L^1(Z)$ such that $F(z,x)\geqslant c_3|x|^\theta-a_1(z)$ for every $x\in R$;

(iii) uniformly for all $z\in Z$ we have

$$\limsup_{x\to 0}\frac{pF(z,x)}{|x|^p}\leqslant\theta(z)\leqslant\lambda_1$$

with $\theta(z)\in L^\infty(Z)$ and $\theta(z)<\lambda_1$ on a set of positive measure.

Remark 3.1. It is easy to see that the function $f(z,x)=\theta(z)|x|^{p-2}x+|x|^{p^*-2}x$, with $\theta\in L^\infty$ and $\theta(z)<\lambda_1$ in a set with positive measure, satisfies the above hypotheses.

Theorem 3.1. *If hypotheses $H(f)_1$ holds, then problem* (1) *has a nontrivial solution of type* I.

Proof. Let $\Phi,\psi: W_o^{1,p}(Z)\to\mathbf{R}$ be defined as

$$\Phi(x)=-\int_Z\int_o^{x(z)}f(z,r)\,dr\,dz=-\int_Z F\bigl(z,x(z)\bigr)\,dz$$

with

$$F(z,x)=\int_o^x f(z,r)\,dr\quad\text{and}\quad\psi(x)=\frac{1}{p}\|Dx\|_p^p.$$

Then we set the energy functional $R=\Phi+\psi$. It is clear that R is locally Lipschitz functional.

Claim 1. $R(\cdot)$ satisfies the $(\mathrm{PS})_{c,+}$-condition in the sense of Costa and Goncalves [6].

Indeed, let $\{x_n\}_{n\geqslant 1} \subseteq W_o^{1,p}(Z)$ such that $R(x_n) \to c$ and

$$R(x_n) \leqslant R(x) + \varepsilon_n \|x - x_n\| \quad \text{with } \|x - x_n\| \leqslant \delta_n$$

with $\varepsilon_n, \delta_n \to 0$.

Let $x = x_n + \delta x_n$ with $\delta\|x_n\| \leqslant \delta_n$. First we divide with δ, then in the limit when $\delta \to 0$ we have that

$$\lim_{\delta\to 0} \frac{\Phi(x_n + \delta x_n) - \Phi(x_n)}{\delta} \leqslant \Phi^o(x_n; x_n).$$

Also we have

$$\|Dx_n + \delta Dx_n\|_p^p - \|Dx_n\|_p^p = \frac{1}{p}\|Dx_n\|_p^p\big((1+\delta)^p - 1\big).$$

Now divide this with δ, then in the limit we have that is equal to $\|Dx_n\|_p^p$. Thus, we have

$$\Phi^o(x_n; x_n) + \|Dx_n\|_p^p \geqslant -\varepsilon_n\|x_n\|.$$

Note that there exists some $w_n' \in \partial\Phi(x_n)$ such that $\langle w_n', x_n\rangle = \Phi^o(x_n; x_n)$. This means that

$$\langle w_n, x_n\rangle - \|Dx_n\|_p^p \leqslant \varepsilon_n\|x_n\|, \tag{5}$$

for some $w_n \in \partial(-\Phi(x_n))$. Note that $w_n(z) \in [f_1(z, x_n(z)), f_2(z, x_n(z))]$.

From the choice of the sequence $\{x_n\} \subseteq W_o^{1,p}(Z)$, we have that

$$\theta R(x_n) \leqslant M_1 \quad \text{for some } M_1 > 0. \tag{6}$$

Adding (5) and (6) we have

$$\left(\frac{\theta}{p} - 1\right)\|Dx_n\|_p^p + \int_Z \big(w_n(z)x_n(z) - \theta F\big(z, x_n(z)\big)\big)\, dz \leqslant \varepsilon_n\|x_n\| + M_1. \tag{7}$$

From hypotheses $H(f)_1$(ii) we know that for almost all $z \in Z$ and all $x \in R$ we have $vx - \theta F(z,x) + a(z) \geqslant 0$ for some $a \in L^{q^*}(Z)$ and for every $v \in \partial F(z,x)$.

Suppose now that $\|x_n\| \to \infty$. Inequality (7) then becomes

$$\begin{aligned}&\left(\frac{\theta}{p} - 1\right)\|Dx_n\|_p^p + \int_Z \big(w_n(z)x_n(z) - \theta F\big(z, x_n(z)\big)\big)\, dz + \int_Z a(z)\, dz \\ &\quad\leqslant \varepsilon_n\|x_n\| + \int_Z a(z)\, dz + M_1.\end{aligned}$$

Divide this inequality with $\|Dx_n\|_p^p$; we have in the limit

$$\frac{\theta}{p} - 1 \leqslant 0.$$

Recall that $\|Dx_n\|$ is an equivalent norm in $W_o^{1,p}(Z)$. Since $\theta > p$ we have a contradiction. So $\|x_n\|$ is bounded.

From the properties of the subdifferential of Clarke, we have

$$\begin{aligned}\partial R(x_n) &\subseteq \partial\Phi(x_n) + \partial\psi(x_n)\\ &\subseteq \partial\Phi(x_n) + \partial\left(\frac{1}{p}\|Dx_n\|_p^p\right) \quad \text{(see Clarke [4, p. 83]).}\end{aligned}$$

So, we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle - \int_Z v_n(z)y(z)\,dz$$

with w_n the element with minimal norm of the subdifferential of R (recall that $\|w_n\|_* \to 0$), $v_n \in [f_1(z, x_n(z)), f_2(z, x_n(z))]$ and $A : W_o^{1,p}(Z) \to W^{-1,q}(Z)$ such that

$$\langle Ax, y\rangle = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z)\big)_{R^N}\,dz$$

for all $y \in W_o^{1,p}(Z)$. But $x_n \xrightarrow{w} x$ in $W_o^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W_o^{1,p}(Z) \subseteq L^p(Z)$. Note that v_n is bounded. Choose $y = x_n - x$. Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$. By virtue of the inequality (4) we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W_o^{1,p}(Z)$. The claim is proved. With similar arguments we prove that R satisfies also $(\mathrm{PS})_{c,-}$, thus R satisfies $(\mathrm{PS})_c$.

Now we shall show that there exists $\rho > 0$ such that $R(x) \geqslant \eta > 0$ with $\|x\| = \rho$. To this end we will show that for every sequence $\{x_n\}_{n\geqslant 1} \subseteq W_o^{1,p}(Z)$ with $\|x_n\| = \rho_n \to 0$ we have $R(x_n) \downarrow 0$. Suppose that this is wrong. Then there exists a sequence as above such that $R(x_n) \leqslant 0$. Since $\|x_n\| \to 0$ we have that $x_n(z) \to 0$ a.e. on Z. So we have

$$\|Dx_n\|_p^p \leqslant \int_Z pF\big(z, x_n(z)\big)\,dz.$$

Let $y_n(z) = x_n(z)/\|x_n\|_{1,p}$. Also, from $H(f)_1$(iii) we have uniformly for almost all $z \in Z$ that for all $\varepsilon > 0$ we can find $\delta > 0$ such that for $|x| \leqslant \delta$ we have

$$pF(z, x) \leqslant \theta(z)|x|^p + \varepsilon|x|^p.$$

On the other hand, from hypothesis $H(f)_1$(i) we have that for almost all $z \in Z$ and all $x \in R$ we have that there exists some c_1, c_2 such that $pF(z, x) \leqslant c_1|x|^p + c_2|x|^{p^*}$. So we can say that $pF(z, x) \leqslant (\theta(z) + \varepsilon)|x|^p + \gamma|x|^{p*}$ for almost all $z \in Z$ and all $x \in R$ with $\gamma \geqslant (c_1 - \theta(z) - \varepsilon)\delta^{p-p*} + c_2$.

Then we obtain

$$\|Dx_n\|_p^p \leqslant \int_Z \big(\theta(z)+\varepsilon\big)\big|x_n(z)\big|^p + \gamma\big|x_n(z)\big|^{p^*}\,dz. \tag{8}$$

Dividing inequality (8) with $\|x_n\|_{1,p}^p$, we have

$$\begin{aligned}\|Dy_n\|^p &\leqslant \int_Z \big(\theta(z)+\varepsilon\big)\big|y_n(z)\big|^p\,dz + \gamma\frac{\int_Z |x_n(z)|^{p^*}\,dz}{\|x_n\|_{1,p}^p}\\ &\leqslant (\lambda_1+\varepsilon)\|y_n\|_p^p + \gamma_1\|x_n\|_{1,p}^{p^*-p};\end{aligned}$$

here we have used the fact that $W_o^{1,p}(Z)$ is continuously embedded on $L^{p^*}(Z)$.

Using the variational characterization of the first eigenvalue we have that

$$\lambda_1\|y_n\|_p^p \leqslant \|Dy_n\|_p^p \leqslant (\lambda_1+\varepsilon)\|y_n\|_p^p + \gamma_1\|x_n\|_{1,p}^{p^*-p}.$$

Recall that $\|y_n\|=1$ so $y_n\to y$ weakly in $W_o^{1,p}(Z)$, $y_n(z)\to y(z)$ a.e. on Z. Thus, from the last inequality we have that $\|Dy_n\|\to\lambda_1\|y\|_p$. Also, from the weak lower semicontinuity of the norm we have $\|Dy\|\leqslant\liminf\|Dy_n\|\to\lambda_1\|y\|_p$. Using the Rayleigh quotient we have that $\|Dy\|=\lambda_1\|y\|_p$. Recall that $y_n\to y$ weakly in $W_o^{1,p}(Z)$ and $\|Dy_n\|\to\|Dy\|$. So, from a well-known argument we obtain $y_n\to y$ in $W_o^{1,p}(Z)$ and since $\|y_n\|=1$ we have that $\|y\|=1$. That is, $y\neq 0$ and from the equality $\|Dy\|=\lambda_1\|y\|_p$ we have that $y(z)=\pm u_1(z)$. Suppose that $y(z)=u_1(z)>0$.

Dividing now inequality (8) with $\|x_n\|_{1,p}^p$ we have

$$\begin{aligned}&\lambda_1\|y_n\|_p^p \leqslant \|Dy_n\|_p^p\\ &\qquad\leqslant \int_Z \theta(z)\frac{|x_n(z)|^p}{\|x_n\|_{1,p}^p}\,dz + \varepsilon\int_Z \frac{|x_n(z)|^p}{\|x_n\|_{1,p}^p}\,dz + \gamma\int_Z \frac{|x_n(z)|^{p^*}}{\|x_n\|_{1,p}^p}\,dz\\ &\Rightarrow\quad \int_Z \big(\lambda_1-\theta(z)\big)\big|y_n(z)\big|^p\,dz \leqslant \varepsilon\|y_n\|_p^p + \gamma_1\|x_n\|_{1,p}^{p^*-p}.\end{aligned}$$

So in the limit we have

$$\begin{aligned}&(\lambda_1-\mu)\int_A u_1^p(z)\,dz \leqslant \int_Z \big(\lambda_1-\theta(z)\big)u_1^p(z)\,dz \leqslant \varepsilon\|u_1\|_p^p \quad \text{for every } \varepsilon>0,\\ &\Rightarrow\quad (\lambda_1-\mu)\leqslant \varepsilon\frac{\|u_1\|_p^p}{\int_A u_1^p(z)\,dz} \quad \text{for every } \varepsilon>0.\end{aligned}$$

Recall that we have $\theta(z)\leqslant\mu<\lambda_1$ on $A\subseteq Z$ with $|A|>0$.

Thus we have a contradiction. So, there exists $\rho>0$ such that $R(x)\geqslant\eta>0$ for all $x\in W_o^{1,p}(Z)$ with $\|x\|=\rho$.

Also, from the hypothesis $H(f)_1$(ii), for almost all $z \in Z$ and all $x \in R$ we have

$$F(z,x) \geqslant c|x|^{\theta} - c_1, \quad \text{for some } c, c_1 > 0. \tag{9}$$

Then for all $\xi > 0$, we have

$$\begin{aligned} R(\xi u_1) &= \frac{\xi^p}{p}\|Du_1\|_p^p - \int_Z F\big(z, \xi u_1(z)\big)\,dz \\ &\leqslant \frac{\xi^p}{p}\|Du_1\|_p^p - c_2\xi^{\theta}\|u_1\|_{\theta}^{\theta} \quad (\text{for some } c_2 > 0) \\ &\leqslant \xi^p\big(c_1 - c_2\xi^{\theta-p}\big). \end{aligned}$$

By virtue of hypothesis for ξ big enough we have that $R(\xi u_1) \leqslant 0$. So we can apply Theorem 2.1 and have that $R(\cdot)$ has a critical point $x \in W_o^{1,p}(Z)$. So $0 \in \partial(\psi(x) + \Phi(x))$. Let $\psi_1(x) = \|Dx\|^p/p$. Then let $\hat{\psi}_1 : L^p(Z) \to \mathbf{R}$ the extension of ψ_1 in $L^p(Z)$. Then $\partial\psi_1(x) \subseteq \partial\hat{\psi}_1(x)$ (see Chang [3]). It is easy to prove that the nonlinear operator $\hat{A} : D(A) \subseteq L^p(Z) \to L^q(Z)$ such that

$$\langle \hat{A}x, y\rangle = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z)\big)\,dz \quad \text{for all } y \in W^{1,p}(Z),$$

with $D(A) = \{x \in W_o^{1,p}(Z):\ \hat{A}x \in L^q(Z)\}$, satisfies $\hat{A} = \partial\hat{\psi}_1$. Indeed, first we show that $\hat{A} \subseteq \partial\hat{\psi}$ and then it suffices to show that $\hat{A}$ is maximal monotone:

$$\begin{aligned} \langle \hat{A}x, y - x\rangle &= \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z) - Dx(z)\big)_{R^N}\,dz \\ &= \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z)\big)_{R^N}\,dz - \int_Z \|Dx(z)\|^p\,dz \\ &\leqslant \int_Z \left(\frac{\|Dx(z)\|^{q(p-2)}\|Dx(z)\|^q}{q} + \frac{\|Dy(z)\|^p}{p}\right)dz - \|Dx\|_p^p \\ &= \frac{\|Dx\|_p^p}{q} - \|Dx\|^p + \frac{\|Dy\|_p^p}{p} = \hat{\psi}_1(y) - \hat{\psi}_1(x). \end{aligned}$$

The monotonicity part is obvious using inequality (4). The maximality needs more work. Let $J : L^p(Z) \to L^q(Z)$ be defined as $J(x) = |x(\cdot)|^{p-2}x(\cdot)$. We will show that $R(\hat{A} + J) = L^q(Z)$. Assume for the moment that this holds. Then let $v \in L^p(Z)$, $v^* \in L^q(Z)$ such that

$$\big(\hat{A}(x) - v^*, x - v\big)_{pq} \geqslant 0$$

for all $x \in D(\hat{A})$. Therefore there exists $x \in D(\hat{A})$ such that $\hat{A}(x) + J(x) = v^* + J(v)$ (recall that we assumed that $R(\hat{A} + J) = L^q(Z)$). Using this in the above inequality we have that

$$\big(J(v) - J(x), x - v\big)_{pq} \geqslant 0.$$

But J is strongly monotone. Thus we have that $v = x$ and $\hat{A}(x) = v^*$. Therefore $\hat{A}$ is maximal monotone. It remains to show that $R(\hat{A} + J) = L^q(Z)$. But $\hat{J} = J|_{W^{1,p}(Z)} : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is maximal monotone, because is demicontinuous and monotone. So $A + \hat{J}$ is maximal monotone. But it is easy to see that the sum is coercive. So is surjective. Therefore, $R(A + \hat{J}) = W^{1,p}(Z)^*$. Then for every $g \in L^q(Z)$, we can find $x \in W^{1,p}(Z)$ such that

$$A + \hat{J}(x) = g \quad \Rightarrow \quad A(x) = g - \hat{J}(x) \in L^q(Z) \quad \Rightarrow \quad A(x) = \hat{A}(x).$$

Thus, $R(\hat{A} + J) = L^q(Z)$.

So, we can say that

$$\int_Z w(z)y(z)\,dz = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z)\big)\,dz \tag{10}$$

for some $w \in W^{-1,q}(Z)$ and in fact $w \in L^{q^*}(Z)$ such that $w(z) \in [f_1(z, x(z)), f_2(z, x(z))]$ (note that $\partial(-\Phi)(x) \subseteq [f_1(z, x(z)), f_2(z, x(z))]$, see Chang [3]) for every $y \in W_o^{1,p}(Z)$. Let $y = \phi \in C_o^\infty(Z)$. Then we have

$$\int_Z w(z)\phi(z)\,dz = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), D\phi(z)\big)\,dz.$$

But

$$\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \in W^{-1,q}(Z);$$

then we have that

$$\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \in L^{q^*}(Z)$$

because $w \in L^{q^*}(Z)$. So x is of type I. $\square$

Remark 3.2. Notice that we have used an extend nonresonace hypotheses at zero from that of Ambrosseti–Rabinowitz used (see De Figueiredo [5, p. 53]).

The question whenever problem (1) has a solution of type II remains open.

4. Neumann problems

As before we introduce two types of solutions for problem (2). Solution of type I and of type II. The first result concerns solutions of type I.

Let us state the hypotheses for the function f and j of problem (2).

$H(f)_3$ $f : Z \times \mathbf{R} \to \mathbf{R}$ is a function such that

(i) for almost all $z \in Z$ is N-measurable (i.e., if $x(\cdot) \in W^{1,p}(Z)$ is measurable so is $f_1(z, x(z)), f_2(z, x(z))$);

(ii) there exists $h : \mathbf{R} \to \mathbf{R}$ such that $h(x) \to \infty$ as $n \to \infty$ and there exists $M > 0$ such that for almost all $z \in Z$ $-F(z,x) \geqslant h(|x|)$ for $|x| \geqslant M$ with $F(z,x) = \int_o^x f(z,r)\,dr$;

(iii) for almost all $z \in Z$ and all $x \in \mathbf{R}$ $|f(z,x)| \leqslant a(z) + c|x|^{\mu-1}$, $\mu < p$ with $a \in L^q(Z)$.

$H(j)$ $j : Z \times \mathbf{R} \to \mathbf{R}$ such that $z \to j(z,x)$ is measurable and $x \to j(z,x)$ locally Lipschitz. Also $j(z,\cdot) \geqslant 0$ for almost all $z \in Z$ and finally $|w(z)| \leqslant a_1(z) + c|x|^{\theta-1}$ with $\theta < p^* = Np/(N-p)$ for every $w(z) \in \partial j(z,x)$.

Remark 4.1. If hypothesis $H(j)$ holds, then Theorem 2.7.5 of Clarke [4] is satisfied.

Proposition 4.1. *If hypotheses $H(f)_3, H(j)$ hold, then problem* (2) *have a solution of type* I.

Proof. Let

$$\Phi(x) = -\int_Z F\big(z, x(z)\big)\,dz$$

and

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j\big(z, \tau\big(x(z)\big)\big)\,d\sigma.$$

Then the energy functional is $R(x) = \Phi(x) + \psi(x)$. It is well known that R is locally Lipschitz.

Claim 1. $R(\cdot)$ satisfies the (PS)-condition of Chang [3].

Indeed, let $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}$ such that $R(x_n) \to c$ as $n \to \infty$. We shall prove that this sequence is bounded in $W^{1,p}(Z)$. Suppose not. Then $\|x_n\| \to \infty$. Let $y_n(z) = x_n(z)/\|x_n\|$. Then clearly we have $y_n \xrightarrow{w} y$ in $W^{1,p}(Z)$. From the choice of the sequence we have

$$\Phi(x_n) + \frac{1}{p}\|Dx_n\|_p^p \leqslant M \tag{11}$$

(recall that $j(z,\cdot) \geqslant 0$). Dividing with $\|x_n\|$ the last inequality, we have

$$-\int_Z \frac{F(z,x(z))}{\|x_n\|^p}\,dz + \frac{1}{p}\|Dy_n\|_p^p \leqslant \frac{M}{\|x_n\|^p}.$$

By virtue of hypothesis $H(f)_3$(iii) we have that $F(z, x(z))/\|x_n\|_p^p \to 0$. So $\limsup \|Dy_n\|_p^p \to 0$. Thus, $\|Dy\| = 0$. So it arises that $y = c \in R$. But $\|y_n\| = 1$, so $c \neq 0$. So we have that $|x_n(z)| \to \infty$. From hypotheses $H(f)_3$(ii) we have that there exists some $a \in L^q(Z)$ such that for all $x \in R$ and for almost all $z \in Z$ we have $-F(z, x) \geqslant h(|x|) - a(z)$. Going back to (11) and using this fact we have a contradiction. So $\|x_n\|$ is bounded, i.e., $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$. It remains to show that $x_n \to x$ in $W^{1,p}(Z)$. From the properties of the subdifferential of Clarke, we have

$$\begin{aligned}\partial R(x_n) &\subseteq \partial \Phi(x_n) + \partial \psi(x_n) \\ &\subseteq \partial \Phi(x_n) + \partial\left(\frac{1}{p}\|Dx_n\|_p^p\right) + \int_\Gamma \partial j\bigl(z, \tau\bigl(x_n(z)\bigr)\bigr)\, d\sigma\end{aligned}$$

(see Clarke [4, p. 83]).

So we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle + \langle r_n, y\rangle_\Gamma - \int_Z v_n(z)y(z)\, dz$$

with $r_n(z) \in \partial j(z, x_n(z))$, $v_n(z) \in [f_1(z, x_n(z)), f_2(z, x_n(z))]$ and w_n the element with minimal norm of the subdifferential of R, and $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is such that

$$\langle Ax, y\rangle = \int_Z \bigl\|Dx(z)\bigr\|^{p-2}\bigl(Dx(z), Dy(z)\bigr)_{R^N}\, dz.$$

But $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W^{1,p}(Z) \subseteq L^p(Z)$. Thus, r_n is bounded in $L^q(Z)$ (see Chang [3, p. 104, Proposition 2]), i.e., $r_n \xrightarrow{w} r$ in $L^{\theta'}(Z)$. Choose $y = x_n - x$. Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$ (note that v_n is bounded). By virtue of the inequality (4) we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W^{1,p}(Z)$. The claim is proved.

Claim 2. $R(\cdot)$ is bounded from below.

Indeed, suppose not. Then there exists some sequence $\{x_n\}_{n\geqslant 1}$ such that $R(x_n) \leqslant -n$. Then we have

$$\Phi(x_n) + \psi(x_n) \leqslant -n$$

(recall that $j(z, \cdot) \geqslant 0$). By virtue of the continuity of $\Phi + \psi$ we have that $\|x_n\| \to \infty$ (because if $\|x_n\|$ was bounded then $\Phi(x_n) + \psi(x_n)$ shall was bounded). Dividing with $\|x_n\|^p$ and letting $n \to \infty$ we have as before a contradiction (by virtue of hypothesis $H(f)_3$(ii)). Therefore $R(\cdot)$ is bounded from below.

So by Theorem 2.2 we have that there exists $x \in W^{1,p}(Z)$ such that $0 \in \partial R(x)$. That is, $0 \in \partial \Phi(x) + \partial \psi(x)$. Let $\psi_1(x) = \|Dx\|^p/p$ and $\psi_2(x) = \int_\Gamma j(z, \tau(x)(z))\, d\sigma$. Then let $\hat{\psi}_1 : L^p(Z) \to R$ be the extension of ψ_1 in $L^p(Z)$. Then $\partial \psi_1(x) \subseteq \partial \hat{\psi}_1(x)$ (see Chang [3]). From Theorem 3.1 we know that the nonlinear operator $\hat{A} : D(A) \subseteq L^p(Z) \to L^q(Z)$ such that

$$\langle \hat{A}x, y \rangle = \int_Z \|Dx(Z)\|^{p-2}\big(Dx(z), Dy(z)\big)\, dz \quad \text{for all } y \in W^{1,p}(Z),$$

with $D(A) = \{x \in W^{1,p}(Z)\colon\ \hat{A}x \in L^q(Z)\}$, satisfies $\hat{A} = \partial \hat{\psi}_1$.

So, we can say that

$$\int_Z w(z)y(z) = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), Dy(z)\big)\, dz + \int_\Gamma v(z)y(z)\, d\sigma, \qquad (12)$$

with $w(z) \in [f_1(z, x(z)), f_2(z, x(z))]$ and $v(z) \in \partial j(z, \tau(x(z)))$, for every $y \in W^{1,p}(Z)$. Let $y = \phi \in C_o^\infty(Z)$. Then we have

$$\int_Z w(z)\phi(z)\, dz = \int_Z \|Dx(z)\|^{p-2}\big(Dx(z), D\phi(z)\big)\, dz.$$

But

$$\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \in W^{-1,q}(Z);$$

then we have that

$$\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \in L^q(Z)$$

because $w(Z) \in L^q(Z)$.

Thus $-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in [f_1(z, x(z)), f_2(z, x(z))]$ a.e. on Z. Going back to (12) and letting $y = C^\infty(Z)$, and finally using the Green formula (1.6) of Kenmochi [9], we have that $-\partial x/\partial n_p \in \partial j(z, \tau(x)(z))$. So $x \in W^{1,p}(Z)$ is of type I. □

Let now state the following condition on f.

$H(f)_4$ f satisfies $H(f)_3$ but is independent of z and is nondecreasing.

Theorem 4.1. *If the hypotheses* $H(f)_4, H(j)$ *holds, then problem* (2) *has a solution of type* II.

Proof. If

$$\Phi(x) = -\int_Z F\big(x(z)\big)\, dz, \qquad \psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j\big(z, x(z)\big)\, dz$$

then the energy functional now is $R = \Phi + \psi$.

From Proposition 4.1 we know that there exists $x \in W^{1,p}(Z)$ such that minimizes R. So $0 \leqslant R(y) - R(x)$ for all $y \in W^{1,p}(Z)$. Thus $0 \leqslant \Phi(y) - \Phi(x) + \psi(y) - \psi(x)$ for all $y \in W^{1,p}(Z)$. Then $(-\Phi)(y) - (-\Phi)(x) \leqslant \psi(y) - \psi(x)$ for all $y \in W^{1,p}(Z)$. Choose now $y = x + tv$ with $v \in W^{1,p}(Z)$ and divide with $t > 0$. Then in the limit we have (note that $-\Phi$ is convex)

$$(-\Phi)'(x; v) \leqslant \psi'(x; v) \leqslant \psi^o(x; v).$$

So we infer that $\langle w, y\rangle = \langle Ax, y\rangle + \langle v, y\rangle_\Gamma$ for all $w \in \partial(-\Phi)(x)$ for some $v \in \partial(\int_\Gamma j(z, x(z))\,dz$ and all $y \in W^{1,p}(Z)$.

We will show that $\lambda\{z \in Z\colon x(z) \in D(f)\} = 0$ with $D(f) = \{x \in R\colon f(x^+) > f(x^-)\}$, that is the set of upward discontinuities.

So let $w \in \partial(-\Phi(x))$ and for any $t \in D(f)$, set

$$\rho^\pm(z) = \bigl[1 - \chi_t\bigl(x(z)\bigr)\bigr]w(z) + \chi_t\bigl(x(z)\bigr)\bigl[f\bigl(x(z)^\pm\bigr)\bigr], \tag{13}$$

where

$$\chi_t(s) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then $\rho^\pm \in L^p(Z)$ and $\rho^\pm \in \partial(-\Phi)(x)$. So

$$\int_Z \rho^\pm(z) y(z)\,dz = \int_Z \|Dx(z)\|^{p-2}\bigl(Dx(z), Dy(z)\bigr)_{R^N}\,dz + \int_\Gamma v(z)y(z)\,d\sigma$$

for all $y \in W^{1,p}(Z)$.

So for $y = \phi \in C_o^\infty(Z)$ we have

$$\int_Z \rho^\pm(z)\phi(z)\,dz = \int_Z \|Dx(z)\|^{p-2}\bigl(Dx(z), D\phi(z)\bigr)_{R^N}\,dz.$$

Thus, $\rho^+ = \rho^-$ for almost all $z \in Z$. From this it follows that $\chi_t(x(z)) = 0$ for almost all $z \in Z$. Since $D(f)$ is countable and

$$\chi\bigl(x(z)\bigr) = \sum_{t \in D(f)} \chi_t\bigl(x(z)\bigr),$$

it follows that $\chi(x(z)) = 0$ almost everywhere (with $\chi(t) = 1$ if $t \in D(f)$ and $\chi(t) = 0$ otherwise).

Now it is clear that x is a solution of type II. $\square$

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