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Eigenvalue condition numbers: Zero-structured versus traditional☆

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Abstract

We discuss questions of eigenvalue conditioning. We study in some depth relationships between the classical theory of conditioning and the theory of the zero-structured conditioning, and we derive from the existing theory formulae for the mathematical objects involved. Then an algorithm to compare the zero-structured individual condition numbers of a set of simple eigenvalues with the traditional ones is presented. Numerical tests are reported to highlight how the algorithm provides interesting information about eigenvalue sensitivity when the perturbations in the matrix have an arbitrarily assigned zero-structure. Patterned matrices (Toeplitz and Hankel) will be investigated in a forthcoming paper (Eigenvalue patterned condition numbers: Toeplitz and Hankel cases, Tech. Rep. 3, Mathematics Department, University of Rome ' La Sapienza', 2005.).

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1. Introduction

It is well known [3,4,19] that the worst perturbation which may affect a simple eigenvalue λ of a given matrix $A \in \mathbb{C}^{n\times n}$ arises under the action of the matrix

$$W_\lambda := yx^{\mathrm{H}}.$$

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Here x and y, respectively, are the right and the left eigenvector associated with λ and $\|x\|_2 = \|y\|_2 = 1$. To be more rigorous, one has (see e.g. [4]), for $\varepsilon > 0$ small enough,

$$(A + \varepsilon E)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon), \quad \|E\|_2 = 1,$$

where x and λ are differentiable functions such that $\lambda(0) = \lambda$, $x(0) = x$. The function λ can be represented as follows:

$$\lambda(\varepsilon) = \lambda + \left[\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}\right]_{\varepsilon=0} \varepsilon + \mathrm{O}(\varepsilon^2),$$

and it can be shown that

$$\left|\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left|\frac{y^{\mathrm{H}} E x}{y^{\mathrm{H}} x}\right| \leqslant \frac{\|y\|_2 \|E\|_2 \|x\|_2}{|y^{\mathrm{H}} x|} = \frac{1}{|y^{\mathrm{H}} x|}, \tag{1}$$

where the eigenvectors x and y are normalized as above. The upper-bound

$$\kappa(\lambda) := \frac{1}{|y^{\mathrm{H}} x|}$$

is attained if $E = W_\lambda$ and it is the individual condition number of λ. In the sequel, we shall refer to $\kappa(\lambda)$ as the *traditional* condition number of λ and to W_λ as the *Wilkinson perturbation* in A.

Even though the maximum rate of change in (1) is attained for infinitely many perturbations E [4, p. 586; 20, p. 250], not all perturbations E produce the effects of the Wilkinson one. (More on this matter can be found in [10,11].)

In addition, in most cases it does not make sense to consider an arbitrary norm-one perturbation E. In fact, only perturbations satisfying specific requirements should be considered. As an example, this happens when A is a structured matrix and the machine perturbations are considered. In such situations, the traditional condition number results very often in a pessimistic estimate since it takes into consideration all the unit norm matrices E, including W_λ, which is typically a full matrix since right and left eigenvectors are usually full vectors. On the contrary, the so-called structured condition number [7,14], which considers only the E's belonging to a suitable subspace, offers a more realistic evaluation of the conditioning of the problem.

Taking account of the above considerations, in this paper we consider matrix perturbations E that belong to a subspace $\mathscr{S}$ formed by the matrices having an assigned zero-structure and we allow the zero-structure, that is the subspace $\mathscr{S}$, to be arbitrarily chosen.

Remark 1.1. Note that $\mathscr{S}$ might not contain A. This allows the study of the effects on the eigenvalues of perturbations in A that are of some particular interest (not machine perturbations, for instance).

Remark 1.2. Up to now, we have used the 2-norm since this is the norm used to state the just outlined theory in most books and articles we cite. On the other hand, results from [14] that we need (see Eqs. (4), (5) below) and arguments that we are going to develop, require the Frobenius norm. This causes no complications since only formal changes occur in the above theory if the Frobenius norm replaces the 2-norm everywhere. Thus, we use the Frobenius norm in this paper.

We start with the following definition of the absolute and relative *zero-structured* condition numbers of λ, respectively denoted by $s\kappa(\lambda)$ and $rs\kappa(\lambda)$. To extend in the most direct way the procedure that leads to the traditional condition number, we put

$$s\kappa(\lambda) := \max\left\{\left|\frac{y^{\mathrm{H}}Ex}{y^{\mathrm{H}}x}\right|, \|E\|_{\mathrm{F}} = 1, \ E \in \mathscr{S}\right\}, \tag{2}$$

and, consequently,

$$rs\kappa(\lambda) := \frac{s\kappa(\lambda)\|A\|_{\mathrm{F}}}{|\lambda|}. \tag{3}$$

Then, using results from [14], we derive new expressions for $s\kappa(\lambda)$ and $rs\kappa(\lambda)$ that are advantageous both from a theoretical and a computational point of view. Subsequently, we find one of the unit norm matrices E that belongs to $\mathscr{S}$ and maximize the ratio $|y^{\mathrm{H}}Ex/y^{\mathrm{H}}x|$. Such a matrix will be referred to as the *$\mathscr{S}$-structured analogue* of the Wilkinson perturbation W_λ. In fact, it is defined in terms of the Wilkinson perturbation W_λ. Finally, we present an algorithm that uses the outlined theoretical results to compare the absolute and relative traditional condition numbers of selected simple eigenvalues with the zero-structured ones. The number v of the selected eigenvalues, the v eigenvalues, and the zero-structure, i.e. the subspace $\mathscr{S}$, can be chosen arbitrarily. Moreover, the user is allowed to modify the zero-structure as he desires and as many times as he wants. Each time, the analysis of the results obtained can be completed visualizing the moduli of both W_λ and of its $\mathscr{S}$-structured analogue. The MATLAB code is available upon request.

Remark 1.3. Note that obviously one has $s\kappa(\lambda) \leqslant \kappa(\lambda)$ and $rs\kappa(\lambda) \leqslant r\kappa(\lambda)$, $r\kappa(\lambda)$ denoting the relative traditional condition number of λ.

The outline of the paper is as follows.

In Section 2 we tell more about structured conditioning. Section 3 describes our algorithm. Finally, significant numerical tests can be found in Section 4. They have been run on an Intel Pentium 4 PC, using MATLAB 6.5 (R13).

2. More on the zero-structured condition number

We start with Eq. (4.2) in [14]. We apply it to the zero-structured case, assuming $\mathscr{S} = \mathscr{S}_A$. We regard the parameter α that appears therein as the Frobenius norm of A and, using our notation, we get

$$s\kappa(\lambda) = \frac{\|y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B\|_2}{|y^{\mathrm{H}}x|}, \tag{4}$$

$$rs\kappa(\lambda) = \frac{\|y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B\|_2\|A\|_{\mathrm{F}}}{|y^{\mathrm{H}}x||\lambda|}. \tag{5}$$

Here $\otimes$ denotes the Kronecker product and $B = B(\mathscr{S})$ is a suitable matrix [6,14].

A few details on B are opportune. It belongs to $\mathbb{C}^{n^2\times m}$, $m = m(\mathscr{S})$ being the number of the structure-positions in the matrices in $\mathscr{S}$ ($m = n^2$ if $\mathscr{S} = \mathbb{C}^{n\times n}$; $m = 3n - 2$ if $\mathscr{S}$ is the subspace of $\mathbb{C}^{n\times n}$ formed

by the tridiagonal matrices, and so on), and it is the unique matrix such that

$$\mathrm{vec}(E) = B\xi, \quad \text{for each } E \in \mathscr{S}. \tag{6}$$

In (6), $\xi = \xi(E)$ is an m-length column vector whose components are the m entries of E located in the above-mentioned m structure-positions in E, arranged by columns, and vec is the operator that stacks the columns of a matrix into one long column vector (see e.g. [8, Chapter 12, Section 1]). It is easy to see that B is a full rank matrix and that it depends only on $\mathscr{S}$ and not on E [on ξ]. Its structure can be described as follows. If the ith ($i = 1 : n^2$) component of $\mathrm{vec}(E)$ comes from an entry in E that is out of the structure of the matrices in $\mathscr{S}$, the ith row of B is a row of zeros. The remaining rows of B are those of the identity matrix I_m. If the zero-rows were omitted, the resulting submatrix of B would be I_m.

Remark 2.1. Note that, no matter what $\mathscr{S}$ is, and even if $\mathscr{S}$ does not contain A, it is always possible to construct the relevant matrix B.

Now, we propose an improvement of (4), (5) by representing the vector $y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B$ in a form which is theoretically more significant and cheaper from a computational point of view.

Proposition 2.1. *One has*

$$y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B = \left(\underset{\mathscr{S}}{\mathrm{vec}}(\overline{W_\lambda|_{\mathscr{S}}})\right)^{\mathrm{T}},$$

where $\mathrm{vec}_{\mathscr{S}}$ *is the restriction of the* vec *operator to the positions in the zero-structure of the matrices in* $\mathscr{S}$ *and* $\overline{W_\lambda|_{\mathscr{S}}}$ *denotes the restriction of* $\overline{W_\lambda}$ *to* $\mathscr{S}$.

Proof. We first observe that $y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)$ is a row vector of length n^2 whose components are the entries of $\overline{W_\lambda}$ arranged by columns:

$$(y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I))_{i+n(j-1)} = \overline{y}_i x_j, \quad i = 1 : n, \ \ j = 1 : n.$$

In other words,

$$y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I) = (\mathrm{vec}(\overline{W_\lambda}))^{\mathrm{T}}.$$

Then, we take the structure of B into account. This leads us to see that post-multiplying $y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)$ by B results in deleting the components $\overline{y}_i x_j$ of $y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)$ whose indices locate the positions out of the structure of the matrices in $\mathscr{S}$.

The proof easily follows. □

Corollary 2.2. *One has*

$$\|y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B\|_2 = \|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}.$$

Proof. Of course, we have

$$\left\|\underset{\mathscr{S}}{\mathrm{vec}}(\overline{W_\lambda|_{\mathscr{S}}})\right\|_2 = \|\mathrm{vec}(\overline{W_\lambda|_{\mathscr{S}}})\|_2,$$

and, by virtue of Proposition 2.1,

$$\|y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B\|_2 = \|\mathrm{vec}(\overline{W_\lambda|_{\mathscr{S}}})\|_2.$$

Using the Frobenius norm (see Remark 1.2) leads to the equalities

$$\|\mathrm{vec}(\overline{W_\lambda|_{\mathscr{S}}})\|_2 = \|\overline{W_\lambda|_{\mathscr{S}}}\|_{\mathrm{F}} = \|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}},$$

and this concludes the proof. □

Corollary 2.2 leads to our improved formulae, which are as follows:

$$s\kappa(\lambda) = \frac{\|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}}{|y^{\mathrm{H}}x|}, \tag{7}$$

$$rs\kappa(\lambda) = \frac{\|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}\|A\|_{\mathrm{F}}}{|y^{\mathrm{H}}x||\lambda|}. \tag{8}$$

The expressions in (7) and (8) are interesting from several points of view.

- *Theoretical*. They show how matrix W_λ affects the conditioning.
- *Computational*. The computation of $\|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}$ is much less expensive than that of $\|y^{\mathrm{H}}(x^{\mathrm{T}} \otimes I)B\|_{\mathrm{F}}$.
- *Storage requirement.* It is less than that required by the matrices $(x^{\mathrm{T}} \otimes I)$ and B which have dimensions $n \times n^2$ and $n^2 \times m$, respectively.
- *Predictability*. A significant information on $\|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}$ is promptly yielded by MATLAB's *imagesc* function applied to $|W_\lambda|$.

But the role played by $W_\lambda|_{\mathscr{S}}$ is fully highlighted by the following:

Proposition 2.3. *The $\mathscr{S}$-structured analogue of the Wilkinson perturbation W_λ is given by*

$$\frac{W_\lambda|_{\mathscr{S}}}{\|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}}. \tag{9}$$

Proof. We can write

$$y^{\mathrm{H}}W_\lambda|_{\mathscr{S}}x = \sum_{i=1}^{n} \overline{y}_i \sum_{j=1}^{n} (W_\lambda|_{\mathscr{S}})_{ij}x_j = \sum_{\mathscr{S}} \overline{y}_i y_i \overline{x}_j x_j = \sum_{\mathscr{S}} |y_i|^2|x_j|^2 = \sum_{\mathscr{S}} |y_i x_j|^2 = \|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}^2$$

and, consequently,

$$y^{\mathrm{H}} \frac{W_\lambda|_{\mathscr{S}}}{\|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}} x = \|W_\lambda|_{\mathscr{S}}\|_{\mathrm{F}}.$$

Dividing by $|y^{\mathrm{H}}x|$ leads to the structured condition number in (7) and, as a consequence, the matrix in (9) is one of the unit norm matrices that yield the maximum in (2). The proof is concluded. □

2.1. A particular case

Tisseur [15] brought to our attention an issue in [2] that concerns the traditional condition number for a complex eigenvalue of a real matrix under real perturbations. Such an issue can be of interest for

instance when machine perturbations are considered (as a matter of fact, W_λ is nonreal in such cases) or, in general, each time it is appropriate to consider only real perturbations E. In [2] it is proved that

$$\frac{\kappa(\lambda)}{\sqrt{2}} \leqslant \kappa_{\mathbb{R}}(\lambda) \leqslant \kappa(\lambda),$$

$\kappa_{\mathbb{R}}(\lambda)$ being the *traditional* condition number with respect to only real perturbations.

A question naturally arises. Let $s\kappa_{\mathbb{R}}(\lambda)$ be the *zero-structured* condition number with respect to only real perturbations E. Can similar inequalities be proved even in the case of the zero-structured conditioning?

The answer is positive.

Proposition 2.4. *One has*

$$\frac{s\kappa(\lambda)}{\sqrt{\min(2,m)}} \leqslant s\kappa_{\mathbb{R}}(\lambda) \leqslant s\kappa(\lambda).$$

Proof. We start with Eq. (2.5) in [2]. Using the definition of ξ — which implies $\|E\|_{\mathrm{F}} = \|\mathrm{vec}(E)\|_{\mathrm{F}} = \|\xi\|_2$ for each $E \in \mathscr{S}$ — allows us to write $\sup_{\xi\in\mathbb{R}^m, \|\xi\|_2=1}$ instead of $\sup_{E\in\mathbb{R}^{n\times n}, \|E\|_{\mathrm{F}}=1}$ and instead of $\sup_{E\in\mathbb{R}^{n\times n}, \|\mathrm{vec}(E)\|_{\mathrm{F}}=1}$. Then we take (6) into account and we rewrite the matrix in an equivalent form. Eq. (2.5) in [2] becomes

$$\begin{aligned} s\kappa_{\mathbb{R}}(\lambda) &= \frac{1}{|y^{\mathrm{H}}x|} \sup_{\xi\in\mathbb{R}^m, \|\xi\|_2=1} \left\| \begin{bmatrix} \mathrm{Re}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)) \\ \mathrm{Im}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)) \end{bmatrix} B\xi \right\|_2 \\ &= \frac{1}{|y^{\mathrm{H}}x|} \sup_{\xi\in\mathbb{R}^m, \|\xi\|_2=1} \left\| \begin{bmatrix} \mathrm{Re}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B) \\ \mathrm{Im}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B) \end{bmatrix} \xi \right\|_2 \\ &= \frac{1}{|y^{\mathrm{H}}x|} \left\| \begin{bmatrix} \mathrm{Re}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B) \\ \mathrm{Im}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B) \end{bmatrix} \right\|_2. \end{aligned}$$

Now, taking account of the equality

$$\left\| \begin{bmatrix} \mathrm{Re}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B) \\ \mathrm{Im}(y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B) \end{bmatrix} \right\|_{\mathrm{F}} = \|y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B\|_{\mathrm{F}},$$

of the definition in (4) of $s\kappa(\lambda)$, and of the well-known relations between the 2-norm and the Frobenius norm, it follows that

$$\frac{s\kappa(\lambda)}{\sqrt{\min(2,m)}} = \frac{\|y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B\|_{\mathrm{F}}}{\sqrt{\min(2,m)}|y^{\mathrm{H}}x|} \leqslant s\kappa_{\mathbb{R}}(\lambda) \leqslant \frac{\|y^{\mathrm{H}}(x^{\mathrm{T}}\otimes I)B\|_{\mathrm{F}}}{|y^{\mathrm{H}}x|} = s\kappa(\lambda),$$

and this concludes the proof. □

3. The algorithm

Our algorithm computes the traditional and zero-structured condition numbers — absolute and relative — of simple eigenvalues of a given matrix $A \in \mathbb{C}^{n\times n}$. The zero-structure of the perturbations E (the subspace $\mathscr{S}$) can be chosen arbitrarily.

Function *structures* (see Section 3.2) allows us to select one of the *basic zero-structures*, i.e. tridiagonal, tridiagonal with zero diagonal, upper bidiagonal, lower bidiagonal, upper Hessemberg, lower Hessemberg, pentadiagonal and full. Any other zero-structure can be chosen. In particular, the user can obtain sub-structures of each of the preceding ones by annihilating in it a properly selected sub-set of p, $0 < p < m$ entries (see Section 2 for the definition of m). An additional way to select any kind of zero-structure is shown in Section 3.1.2.

A particularly useful tool to forecast structures that will yield small [large] zero-structured condition numbers is the *imagesc* function. It permits the user to examine the weight distribution of Wilkinson perturbations W_λ related to the selected eigenvalues and to try, if possible, a zero-structure (i.e. a subspace $\mathscr{S}$) matching a light part [a heavy part] of the involved W_λ's. (Here and in the sequel a light part/entry [a heavy part/entry] of a matrix stands for a part/entry that affects little [much] the Frobenius norm of the matrix.) A similar idea can be derived from the analysis of the matrices $W_\lambda|_{\mathscr{S}}/\|W_\lambda|_{\mathscr{S}}\|_F$ too. In fact, as we saw at the end of Section 2, these matrices are the $\mathscr{S}$-structured analogues of the W_λ's and such an analysis might locate entries to be annihilated in order to select a sub-structure yielding better [worse] zero-structured condition numbers.

The MATLAB code consists of a script (*strcnd*) and of a function (*structures*). A brief description of both of them is given here below.

3.1. Script strcnd

We divide this section into two parts.

3.1.1. User's interactions

Script *strcnd* asks the user to

1. enter the matrix A,
2. select v, $1 \leqslant v \leqslant n$, eigenvalues of A,
3. choose whether to examine some [all] of the v Wilkinson perturbations W_λ or not,
4. make the following choices:

   4.1. specify the zero-structure of the perturbation matrices E,
   4.2. choose whether to examine some [all] of the $\mathscr{S}$-structured analogues (9) of the v Wilkinson perturbations W_λ or not,
5. decide whether to return to point 4 or not.

3.1.2. Detail

After point 1, MATLAB's *eig* function computes the eigenvalues as well as the right and left eigenvectors. We disabled the default MATLAB's *balance* function — which implements the balancing procedure in [12] — to avoid possible changes in the traditional and structured conditioning of the v eigenvalues. The full vector of the eigenvalues is listed by *strcnd*.

To select the v eigenvalues, the user is allowed to enter their relevant indices as they appear in the list displayed by *strcnd*.

After point 2, the traditional absolute and relative condition numbers and the Wilkinson perturbations W_λ related to the chosen eigenvalues are computed. The condition numbers are printed.

A first choice of the zero-structure of the matrices E, that is to say a first choice of the subspace $\mathscr{S}$, happens at point 4.1. To define that structure, the user enters one of the following strings: 'trid', 'trizd', 'ubid', 'lbid', 'uhess', 'lhess', 'penta', 'full' and 'others'. The first eight stand for the above-mentioned basic zero-structures: tridiagonal, tridiagonal with zero diagonal, upper bidiagonal, lower bidiagonal, upper Hessemberg, lower Hessemberg, pentadiagonal and full, respectively. The last one offers the above-mentioned additional feature to choose any kind of zero-structure. This can be done by entering the m relevant couples of indices.

Then, function *structures* is called and returns the current matrices $W_\lambda|_{\mathscr{S}}$.

It is possible to eliminate p of the m structure entries, entering their relevant p couples of indices. Of course, this opportunity is not given in the case of 'others'.

Now, the structure [the subspace $\mathscr{S}$] is settled and, in the case of $p > 0$, the final matrices $W_\lambda|_{\mathscr{S}}$ are constructed. Except for the case $\mathscr{S} = \mathbb{C}^{n\times n}$ (i.e. if the basic structure 'full' has been chosen, and $p = 0$), the zero-structured absolute and relative condition numbers in (7), (8) are computed. Then they are printed together with the traditional ones to make the comparison easier.

3.2. Function structures

It carries out the restriction of the v Wilkinson perturbations W_λ to the selected subspace $\mathscr{S}$. To save computational cost, if one of the four basic structures: tridiagonal, tridiagonal with zero diagonal, upper bidiagonal and lower bidiagonal is selected, function *tridiag* in MATLAB's toolbox "gallery — Higham test matrices" is used.

Along with the matrices $W_\lambda|_{\mathscr{S}}$, function *structures* also returns two flags (see the Output arguments below).

Input arguments

n	= dimension of A
$choice$	= index vector of the v eigenvalues
$wlambda$	= tridimensional matrix of the v matrices W_λ
$kind$	= string denoting the selected zero-structure
$entries$	= index vector of the m structure entries (if kind = 'others')

Output arguments

$wlambdas$	= tridimensional matrix of the v matrices $W_\lambda|_{\mathscr{S}}$
$strflag$	= flag equal to 0 if $kind$ = full', to 1 otherwise
$bsstflag$	= flag equal to 0 if kind = 'others', to 1 otherwise

4. Numerical tests

To better document how the chosen structure [the selected subspace $\mathscr{S}$] influences the conditioning of the eigenvalues, we have taken from the literature matrices that are renowned to have seriously ill-conditioned eigenvalues and we have always included the worst of them in the list of the v selected ones.

Such a strategy often implies restrictions on the choice of the structure since most entries in the matrices in the literature are usually machine numbers and, on the other hand, the main practical interest is of course of considering only the machine perturbation matrices E. However, also investigating the

Table 1
Lesp matrix of dimension 50. 'others' structure: machine perturbations

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_{33}	2.0774e + 13	1.4451e + 14	1.4304e + 1	9.9500e + 1
λ_{34}	2.8325e + 13	1.9148e + 14	2.0099e + 1	1.3588e + 2
λ_{35}	3.8126e + 13	2.5068e + 14	1.3433e + 1	8.8323e + 1
λ_{36}	4.8100e + 13	3.0783e + 14	2.2461e + 1	1.4374e + 2
λ_{37}	4.4881e + 13	2.7977e + 14	1.6873e + 1	1.0517e + 2
λ_{38}	6.0638e + 13	3.6841e + 14	1.4400e + 1	8.7487e + 1
λ_{39}	4.3754e + 13	2.5927e + 14	3.6849e + 1	2.1835e + 2
λ_{40}	5.4537e + 13	3.1538e + 14	8.2579e + 0	4.7754e + 1
λ_{41}	3.8445e + 13	2.1709e + 14	9.7468e + 1	5.5037e + 2
λ_{42}	1.6353e + 13	9.0219e + 13	1.1027e + 1	6.0837e + 1

effects of perturbations that are not due to the floating-point representation might answer to theoretical motivations and this argument leads to treat any kind of zero-structure.

In this section we follow both the ideas. We report examples which treat only the machine perturbation matrices E (Sections 4.1 and 4.2), as well as examples which treat even matrices whose entries are all machine numbers (Sections 4.3 and 4.4).

In the tables, the indices of the v selected eigenvalues are in accordance with the list displayed by *strcnd*. In the figure titles, W_i stands for W_{λ_i}.

4.1. Lesp matrix

The Lesp matrices are real tridiagonal matrices with real, negative, sensitive simple eigenvalues, smoothly distributed in the interval approximately $[-2n - 3.5, -4.5]$ [9,16]. Roughly speaking, the sensitivities of the eigenvalues increase as the eigenvalues grow more negative.

Here the Lesp matrix of dimension $n = 50$ is considered. The command to enter it was taken from MATLAB's toolbox 'gallery — Higham test matrices'.

We selected the eigenvalues whose indices range from 33 to 42 since they are the ten worst conditioned ones. Then we observed that, besides the zeros, most entries are machine numbers: the upper-diagonal ones, the diagonal ones, and five of the sub-diagonal entries (2^{-k}, $k = 1 : 5$). Thus, to know the actual condition numbers, we strictly considered only the machine perturbation matrices to simulate the errors introduced just from storing the matrix in the computer. To do so, we chose 'others' and then we entered the indices of the remaining sub-diagonal entries. Table 1 reports the results we got.

The gain in the conditioning is remarkable. This means that, actually, the eigenvalues can be regarded as quite well-conditioned.

Visualizing the involved W_λ's (see Section 3.1.1) yields interesting information. In fact, a quick look at their weight distributions points out that each of them is essentially confined in very few (six, at most) adjacent entries in the last row, that slightly shift to the right as the relevant eigenvalue grows more negative. This suggests choosing as a new structure the one formed by the whole set of these entries and to conjecture that the resulting zero-structured condition numbers do not differ too much from the traditional ones. The set is that formed by the entries in the columns of indices from 20 to 30 and the results we report in Table 2 confirm the conjecture.

Table 2
Lesp matrix of dimension 50. 'others' structure: heavy entries

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_{33}	2.0774e + 13	1.4451e + 14	1.4897e + 13	1.0363e + 14
λ_{34}	2.8325e + 13	1.9148e + 14	2.1527e + 13	1.4553e + 14
λ_{35}	3.8126e + 13	2.5068e + 14	3.0583e + 13	2.0108e + 14
λ_{36}	4.8100e + 13	3.0783e + 14	3.9895e + 13	2.5531e + 14
λ_{37}	4.4881e + 13	2.7977e + 14	3.8457e + 13	2.3972e + 14
λ_{38}	6.0638e + 13	3.6841e + 14	5.3479e + 13	3.2492e + 14
λ_{39}	4.3754e + 13	2.5927e + 14	3.8922e + 13	2.3064e + 14
λ_{40}	5.4537e + 13	3.1538e + 14	4.9679e + 13	2.8728e + 14
λ_{41}	3.8445e + 13	2.1709e + 14	3.4352e + 13	1.9398e + 14
λ_{42}	1.6353e + 13	9.0219e + 13	1.4527e + 13	8.0144e + 13

Table 3
Lesp matrix of dimension 50. 'uhess' structure (light entries)

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_{33}	2.0774e + 13	1.4451e + 14	1.4418e + 1	1.0029e + 2
λ_{34}	2.8325e + 13	1.9148e + 14	2.0116e + 1	1.3599e + 2
λ_{35}	3.8126e + 13	2.5068e + 14	1.3444e + 1	8.8392e + 1
λ_{36}	4.8100e + 13	3.0783e + 14	2.2477e + 1	1.4385e + 2
λ_{37}	4.4881e + 13	2.7977e + 14	1.6884e + 1	1.0525e + 2
λ_{38}	6.0638e + 13	3.6841e + 14	1.4410e + 1	8.7551e + 1
λ_{39}	4.3754e + 13	2.5927e + 14	3.6875e + 1	2.1851e + 2
λ_{40}	5.4537e + 13	3.1538e + 14	8.2675e + 0	4.7810e + 1
λ_{41}	3.8445e + 13	2.1709e + 14	9.7547e + 1	5.5082e + 2
λ_{42}	1.6353e + 13	9.0219e + 13	1.1034e + 1	6.0872e + 1

Note that if we had used 'uhess' we would have got Table 3, that is essentially the same as Table 1. The results in the three tables confirm that it is not so much how large m is (see Section 2 for the definition of m) that influences the value of the zero-structured condition numbers, but how much the structure matches the weight distribution of the Wilkinson perturbation W_λ.

Remark 4.1. The results obtained in the case 'uhess' are of interest even with reference to the backward error analysis applied to the QR algorithm (*eig*). In fact, just because 'uhess' has been used, they indicate that not only will the inherent error be favorably bounded, but also the algorithmic one. The user can easily check such an assertion by perturbing the Lesp matrix with the $\mathscr{S}$-structured analogues of the Wilkinson perturbations W_λ and observing the induced errors in the ten eigenvalues. A strictly similar argument applies to the case of 'trid' and of the HR algorithm. In fact, after a slight modification, the tridiagonal form of a real matrix is preserved by the HR algorithm [1, p. 156], and using 'trid' will yield zero-structured condition numbers less than or equal to the ones in Table 3.

Table 4
Bessel matrix of dimension 25. 'trizd' structure (machine perturbations)

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_{25}	3.9408e + 12	9.3612e + 13	2.0839e + 12	4.9659e + 13

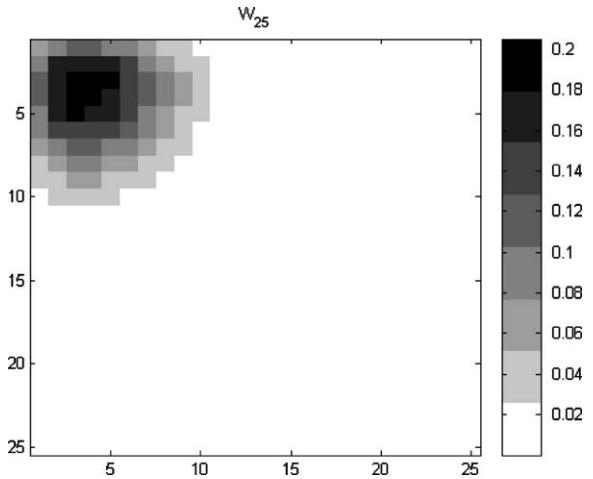

Fig. 1. Weight distribution of the Wilkinson perturbation $W_{\lambda_{25}}$ related to the Bessel matrix.

4.2. Bessel matrix

The Bessel matrices are real tridiagonal matrices associated with the Ordinary Bessel Polynomials (OBPs), in the sense that their eigenvalues are the zeros of the OBPs [5]. Even though they differ from skew-symmetric tridiagonal matrices only by the rank-one matrix $-e_1e_1^{\mathrm{T}}$, they have very badly conditioned eigenvalues [13]. Their spectra lie in the left half of the complex plane [5]. Here the Bessel matrix of dimension 25 is considered. It can be entered with the command $A =$ *full*(*gallery*('tridiag',1./*sqrt*(4*(1:24).^2-1),[-1 *zeros*(1,24)],-1./*sqrt*(4*(1:24).^2-1))).

We selected the unique real eigenvalue (25th), which is the worst conditioned one.

As before, we restricted ourselves to consider only the machine perturbations. To do so, we chose the zero-diagonal tridiagonal structure 'trizd'. This time we bring an example of a situation completely different from the previous one. In fact, the zero-structured condition numbers are just a bit more favorable than the traditional ones (see Table 4). Again, the reason for this can be found by looking at the weight distribution of $W_{\lambda_{25}}$ (see Fig. 1) and comparing it with the structure defined by 'trizd'.

Fig. 2 represents the $\mathscr{S}$-structured analogue of $W_{\lambda_{25}}$. It suggests trying the substructure of 'trizd' defined by the fourteen entries which are appreciable in the figure to verify if, as expected, the relevant zero-structured condition numbers do not differ essentially from the traditional ones. The results reported in Table 5 confirm the conjecture. On the contrary, if we try the substructure of 'trizd' defined by the fourteen

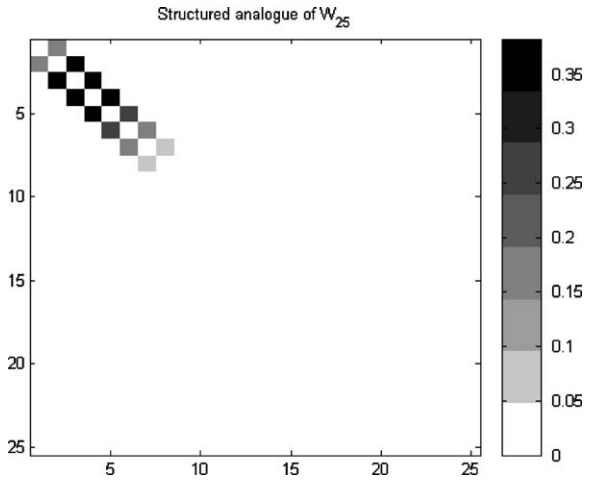

Fig. 2. Weight distribution of the $\mathscr{S}$-structured analogue of the $W_{\lambda_{25}}$ related to the Bessel matrix. 'trizd' structure.

Table 5
Bessel matrix of dimension 25. 'others' structure: heavy entries

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_{25}	3.9408e + 12	9.3612e + 13	2.0800e + 12	4.9567e + 13

Table 6
Bessel matrix of dimension 25. 'others' structure: light entries

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_{25}	3.9408e + 12	9.3612e + 13	1.5999e + 5	3.8127e + 6

entries counterdiagonally symmetric with respect to the above ones — whose weight is practically zero — we get the results in Table 6. The gain amounts to seven orders of magnitude.

4.3. Wilkinson matrix

This is one of the famous matrices introduced by Wilkinson for theoretical purposes [19, p. 90]. It is an upper bidiagonal machine matrix of dimension 20 with ill-conditioned simple eigenvalues. It can be entered with the command $A = diag(20{:}\text{-}1{:}1)+diag(20*ones(19,1),1)$.

In this case we selected all the eigenvalues. The reason for this choice is that all the W_{λ_i}, $i = 1 : 20$, share interesting properties. They have the heavy entries contained in their strictly triangular lower parts and, in all the cases, the $(20, 1)$ entry is by far the heaviest one. In fact, the unique nonzero entry in the matrix perturbation considered in examples in [19, p. 90; 17, p. 467], is the $(20, 1)$ entry. So did

Table 7
Wilkinson matrix of dimension 20. 'others' structure: the heaviest entry (20, 1)

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_1	8.4482e + 7	4.3222e + 8	4.3100e + 7	2.2051e + 8
λ_2	1.4550e + 9	7.8359e + 9	8.1889e + 8	4.4101e + 9
λ_3	1.2065e + 10	6.8587e + 10	7.3702e + 9	4.1897e + 10
λ_4	6.3888e + 10	3.8454e + 11	4.1761e + 10	2.5136e + 11
λ_5	2.4185e + 11	1.5467e + 12	1.6707e + 11	1.0685e + 12
λ_6	6.9407e + 11	4.7346e + 12	5.0112e + 11	3.4184e + 12
λ_7	1.5650e + 12	1.1438e + 13	1.1692e + 12	8.5456e + 12
λ_8	2.8368e + 12	2.2328e + 13	2.1729e + 12	1.7103e + 13
λ_9	4.1802e + 12	3.5644e + 13	3.2545e + 12	2.7751e + 13
λ_{10}	5.0770e + 12	4.7227e + 13	3.9852e + 12	3.7070e + 13
λ_{11}	5.0706e + 12	5.1884e + 13	3.9797e + 12	4.0722e + 13
λ_{12}	4.1830e + 12	4.7557e + 13	3.2570e + 12	3.7030e + 13
λ_{13}	2.8369e + 12	3.6285e + 13	2.1730e + 12	2.7793e + 13
λ_{14}	1.5643e + 12	2.2866e + 13	1.1687e + 12	1.7083e + 13
λ_{15}	6.9440e + 11	1.1842e + 13	5.0137e + 11	8.5502e + 12
λ_{16}	2.4178e + 11	4.9479e + 12	1.6702e + 11	3.4181e + 12
λ_{17}	6.3896e + 10	1.6345e + 12	4.1766e + 10	1.0684e + 12
λ_{18}	1.2065e + 10	4.1151e + 11	7.3700e + 9	2.5137e + 11
λ_{19}	1.4550e + 9	7.4442e + 10	8.1890e + 8	4.1896e + 10
λ_{20}	8.4482e + 7	8.6444e + 9	4.3100e + 7	4.4101e + 9

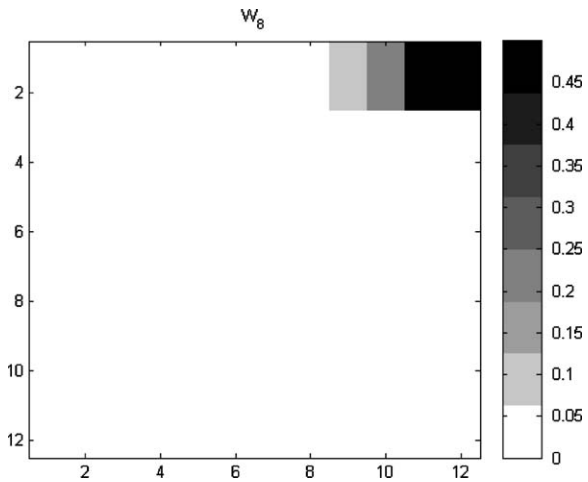

Fig. 3. Weight distribution of the Wilkinson perturbation W_{λ_8} related to the Frank matrix.

we, choosing 'others' and then entering [20, 1]. The results are reported in Table 7. As expected, the zero-structured condition numbers are practically the same as the traditional ones.

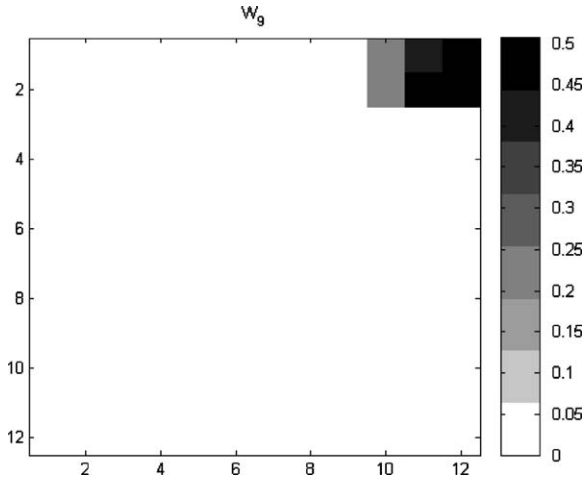

Fig. 4. Weight distribution of the Wilkinson perturbation W_{λ_9} related to the Frank matrix.

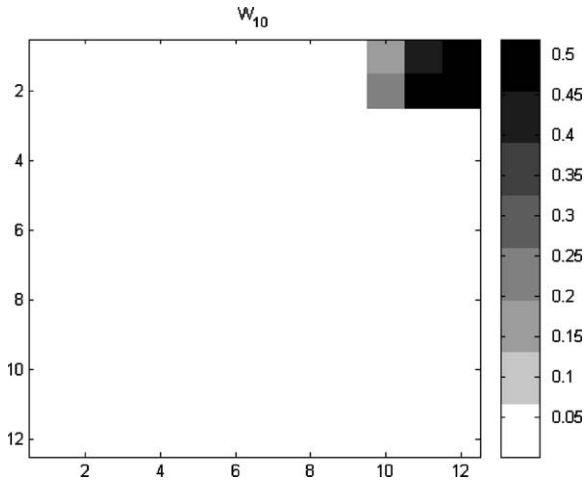

Fig. 5. Weight distribution of the Wilkinson perturbation $W_{\lambda_{10}}$ related to the Frank matrix.

4.4. *Frank matrix*

This is a famous upper Hessenberg matrix of dimension 12 [18; 4, Section 13; 20, Section 5]. The command to enter it was taken from MATLAB's toolbox "gallery — Higham test matrices". We selected the eigenvalues λ_i, $i = 8 : 11$, since they are the four worst conditioned ones. Also in this case all the entries are machine numbers. Taking the weight distributions of the W_{λ_i}, $i = 8 : 11$, into account (see Figs. 3–6), we chose 'uhess' and 'lhess'. Tables 8 ('uhess') and 9 ('lhess') report the results, which fully confirm the expectation.

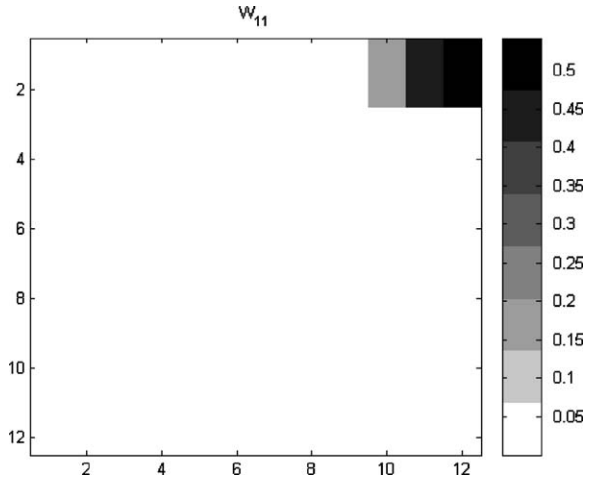

Fig. 6. Weight distribution of the Wilkinson perturbation $W_{\lambda_{11}}$ related to the Frank matrix.

Table 8
Frank matrix of dimension 12. 'uhess' structure

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_8	1.8283e + 7	3.1579e + 10	1.8283e + 7	3.1579e + 10
λ_9	3.8774e + 7	4.1972e + 10	3.8774e + 7	4.1972e + 10
λ_{10}	2.6646e + 7	1.7580e + 10	2.6646e + 7	1.7580e + 10
λ_{11}	6.7014e + 6	2.5001e + 9	6.7014e + 6	2.5001e + 9

Table 9
Frank matrix of dimension 12. 'lhess' structure

λ	$\kappa(\lambda)$	$r\kappa(\lambda)$	$s\kappa(\lambda)$	$rs\kappa(\lambda)$
λ_8	1.8283e + 7	3.1579e + 10	6.6138e + 0	1.1423e + 4
λ_9	3.8774e + 7	4.1972e + 10	3.6994e + 0	4.0045e + 3
λ_{10}	2.6646e + 7	1.7580e + 10	2.6031e + 0	1.7175e + 3
λ_{11}	6.7014e + 6	2.5001e + 9	2.2663e + 0	8.4550e + 2

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