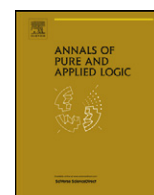


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Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Integration in algebraically closed valued fields with sections

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ARTICLE INFO

Article history:

Received 3 January 2012

Received in revised form 28 March 2012

Accepted 18 May 2012

Available online 17 September 2012

Communicated by T. Scanlon

MSC:

03C48

18C35

Keywords:

Motivic integration

Algebraically closed valued field

Cross-section

ABSTRACT

We construct Hrushovski–Kazhdan style motivic integration in certain expansions of ACVF. Such an expansion is typically obtained by adding a full section or a cross-section from the RV-sort into the VF-sort and some (arbitrary) extra structure in the RV-sort. The construction of integration, that is, the inverse of the lifting map \mathbb{L} , is rather straightforward. What is a bit surprising is that the kernel of \mathbb{L} is still generated by one element, exactly as in the case of integration in ACVF. The overall construction is more or less parallel to the main construction of Hrushovski and Kazhdan (2006) [10], as presented in Yin (2010) [19] and Yin (2011) [20]. As an application, we show uniform rationality of Igusa zeta functions for non-archimedean local fields with unbounded ramification degrees.

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1. Introduction

We have presented the main construction of the Hrushovski–Kazhdan integration theory [10] in [19,20]. The integration constructed there is “unrefined” in the sense that, although the kernel of the lifting map \mathbb{L} , that is, the congruence relation I_{sp} , is surprisingly simple, being generated by a single element, and the whole theory is structurally sound, satisfying, among other things, a Fubini-type theorem and a change of variables formula, computation of most integrals appear to be too complicated or utterly intractable. This is so even without volume forms and when only simple geometrical objects are involved, such as an open ball with one closed hole and a closed ball with two open holes, computing the standard contractions of which, according to [19, Proposition 6.18], would tell us whether there is a definable bijection of the two in ACVF. Refinement may proceed in several directions, for example, see [10, §10] and [11], all of which involve manipulations of the Grothendieck (semi)rings that provide values for motivic integrals, such as groupifying, coarsening (usually by way of introducing external algebraic structures), and decomposing into tensor product. This last manipulation makes computation of certain integrals much more transparent, especially when integrating functions with one variable, such as the one mentioned above.

In this paper we shall first construct “unrefined” motivic integration maps in certain expansions of algebraically closed valued fields and then refine the target semirings of these maps by decomposing them into tensor products in a canonical way. Such an expansion of algebraically closed valued fields is typically obtained in two independent steps: adding a full section (an RV-section) or a cross-section from the RV-sort into the VF-sort and then adding arbitrary relations and functions in the RV-sort. Expansions with extra structure in the RV-sort has been considered in [10, §12], where a homomorphism

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¹ I would like to thank Udi Hrushovski and François Loeser for their guidance. I would also like to thank the anonymous referee whose thorough reports have led to vast improvements of the paper. The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007–2013)/ERC Grant Agreement 246903 NMNAG.

between Grothendieck semirings is obtained more or less along the line of the main construction, in particular, the congruence relation I_{sp} retains the same degree of simplicity. Expansions with a section from the residue field into the valued field (a \bar{K} -section) has been considered in [12]. This is in the context of adelic structures over curves, where an integration in the style of [10] is not needed and hence is not developed.

Our motivation for extending the Hrushovski–Kazhdan theory to such expansions is twofold. Firstly, this is to prepare the ground for a plausible theory of motivic characters, especially multiplicative ones, which is something we should have if we are to further the (already far-reaching) application of the theory of motivic integration to, say, geometry and representation theory, as demonstrated, for example, in [1–3,12,13]. The use of characters in constructing representations in function spaces is beautifully expounded in the (perhaps a bit old-fashioned but still tremendously insightful) work [9]. Secondly, motivic integration in real closed fields is alluded to in the introduction of [10] as a hope. We shall realize this hope in a future paper [21]. The framework for doing so calls for a cross-section and its technical aspects closely resemble those of this paper.

The construction in this paper is entirely modeled on and heavily relies on the (auxiliary) results of the construction presented in [19,20]. In particular, we still adhere to the three-step procedure as laid out in the introduction of [20]. For clarity, let us repeat it once again. Let \mathbb{T} be an expansion of ACVF, which includes an RV-section $sn : RV \rightarrow VF$ or a cross-section $csn : \Gamma \rightarrow VF$ or both. Let VF_* and $RV[*]$ be two suitable categories of definable sets that are respectively associated with the VF-sort and the RV-sort. To construct a canonical homomorphism from the Grothendieck semigroup $\mathbf{K}_+ VF_*$ to the Grothendieck semigroup $\mathbf{K}_+ RV[*]/I_{sp}$, where I_{sp} is a suitable semigroup congruence relation, we proceed as follows:

- *Step 1.* There is a natural lifting map \mathbb{L} from the set of objects of $RV[*]$ into the set of objects of VF_* . We show that \mathbb{L} hits every isomorphism class of VF_* .
- *Step 2.* We show that \mathbb{L} induces a semigroup homomorphism from $\mathbf{K}_+ RV[*]$ into $\mathbf{K}_+ VF_*$, which is also denoted by \mathbb{L} .
- *Step 3.* In order to obtain a precise description of the semigroup congruence relation on $\mathbf{K}_+ RV[*]$ induced by \mathbb{L} , that is, the kernel of \mathbb{L} , we introduce two operations: special bijection in the VF-sort and blowup in the RV-sort. In a sense these two operations mirror each other. Using this correlation we show that, for any objects $\mathbf{U}_1, \mathbf{U}_2$ in $RV[*]$, there are isomorphic blowups $\mathbf{U}_1^\sharp, \mathbf{U}_2^\sharp$ of $\mathbf{U}_1, \mathbf{U}_2$ if and only if $\mathbb{L}(\mathbf{U}_1), \mathbb{L}(\mathbf{U}_2)$ are isomorphic.

Through certain standard algebraic manipulations, the inverse of \mathbb{L} gives rise to various ring homomorphisms and module homomorphisms. These are understood as generalized Euler characteristic or, if volume forms are present, integration. Note that, in principle, the construction is already completed in Step 2 (see Section 4). However, to facilitate computation in future applications, it seems much more satisfying to have a precise description of the semigroup congruence relation as obtained in Step 3 (see Section 5). Perhaps a bit surprisingly, this kernel of \mathbb{L} is still generated by one element, exactly as in the case of integration in ACVF.

There is really just one new (nontechnical) idea in this paper, which is very straightforward. For every \mathbb{T} -definable set A we seek a definable function $\pi : A \rightarrow RV^m$ such that each fiber $\pi^{-1}(\bar{t})$ is $sn(\bar{t})$ -definable in ACVF, similarly if the RV-section sn is replaced by the cross-section csn (we have to work with csn instead of sn in the situation with volume forms). Such a function is called an RV- or a Γ -partition of A . If it exists then we may assign a volume to A by first computing the volumes of the fibers, using the results for ACVF, and then sum them up more or less formally. In fact such a partition always exists for a definable set. Conceptually, the few foregoing sentences capture the gist of this paper so well that it is actually tempting to end the discussion right here. But that is probably not very convincing for someone who is not already familiar with the intricate working of the Hrushovski–Kazhdan theory, especially when highly nontrivial modifications of certain technical results are called for. So, we opt for spelling out more details in a few pages. Inevitably, the writing will repeat (variations of) some things that have already been said in [19,20].

In [18] we have compared expansions with RV-section and expansions with \bar{K} -section in terms of minimality conditions. It is not hard to see that our method here also works for expansions of ACVF with \bar{K} -section.

We now describe an application to local zeta functions. Let $f(\vec{X}) \in \mathbb{Q}_p[X_1, \dots, X_n]$, κ be a positive real number, and L be a finite extension of \mathbb{Q}_p . The norm of $a \in L$ is denoted by $|a|_L$ and the Haar measure on L is denoted by $|d\vec{X}|_L$. Suppose that $A \subseteq L^n$ is bounded and is \mathbb{Q}_p -definable in the language with a cross-section. Note that here the parameters used to define f and A are allowed to vary in a suitable way as p and L vary, for example, the ramification degree of L may be a defining parameter for A . Consider the Igusa local zeta function

$$\zeta(A, L, \kappa) = \int_A |f(\vec{X})|_L^\kappa |d\vec{X}|_L.$$

Following the specialization procedure in [10], we can show that $\zeta(A, L, \kappa)$ is uniformly rational for all p -adic fields (see Definition 6.4 for the precise meaning of uniformity). This can also be derived using the Denef–Pas method in [6,7,16,17].

The paper is organized as follows. In Section 2 we first introduce the class of expansions of ACVF that shall be considered. Obviously not much can be done without quantifier elimination, which is derived immediately. Other basic structural properties are also collected in this section, which shall be used throughout the rest of the paper. In Section 3 categories associated with the RV-sort are introduced and their Grothendieck semigroups are studied. Here the reader should notice that, by having a cross-section, the various target semirings of the Grothendieck homomorphisms actually become simpler

than those in [10]. The main result of this section is the expression of these semirings as certain tensor products. This essentially repeats some of the work in [10, §§9, 10]. However, as in [19,20], we give much simpler and more direct proofs. In Section 4 we begin with an investigation of dimension in the VF-sort and other related notions, such as the Jacobian. Then the categories associated with the VF-sort are introduced. This is parallel to the corresponding discussion in [19] and the modifications are all very natural for the current setting. The first two steps of the three-step procedure described above are completed in Section 4. In order to obtain a precise description of the kernel of the lifting map \mathbb{L} , we need an analog of [19, Theorem 5.4], which guarantees, after modification using only special bijections, contractibility of an arbitrary function. This is also done in Section 4, which is the most technical part of the construction and is needed for the application to local zeta functions. In Section 5 we study blowups in the RV-categories and then describe the kernel of \mathbb{L} . Subsequently various Grothendieck homomorphisms are constructed. These follow very closely the corresponding discussion in [20]. In the last section, we specialize some of the results to non-archimedean local fields, which is more or less automatic by compactness, and derive the uniform rationality of local zeta functions described above.

2. Preliminaries and some basic structural properties

The reader is referred to [18,20,19] for notation and terminology. For example, the various notational conventions concerning coordinate projection maps in [19, Notation 2.10] shall be used frequently:

Notation 2.1. Let $A \subseteq \text{VF}^n \times \text{RV}^m$. For any $n \in \mathbb{N}$, let $I_n = \{1, \dots, n\}$. Let $I = I_n \uplus I_m$, $E \subseteq I$, and $\bar{E} = I \setminus E$. If E is a singleton $\{i\}$ then we always write E as i and \bar{E} as \bar{i} . We write $\text{pr}_E(A)$ for the projection of A to the coordinates in E . For any $\bar{a} \in \text{pr}_{\bar{E}}(A)$, the fiber $\{\bar{b} : (\bar{b}, \bar{a}) \in A\}$ is denoted by $\text{fib}(A, \bar{a})$. Note that we shall often tacitly identify the two subsets $\text{fib}(A, \bar{a})$ and $\text{fib}(A, \bar{a}) \times \{\bar{a}\}$. Also, it is often more convenient to use simple descriptions as subscripts. For example, if $E = \{1, \dots, k\}$, etc., then we may write $\text{pr}_{\leq k}$, etc. If E contains exactly the VF-indices (respectively RV-indices) then pr_E is written as pvf (respectively prv). If E' is a subset of the coordinates of $\text{pr}_E(A)$ then the composition $\text{pr}_{E'} \circ \text{pr}_E$ is written as $\text{pr}_{E, E'}$. Naturally $\text{pr}_{E'} \circ \text{pvf}$ and $\text{pr}_{E'} \circ \text{prv}$ are written as $\text{pvf}_{E'}$ and $\text{prv}_{E'}$, respectively.

We shall work with certain expansions of the \mathcal{L}_{RV} -theory ACVF (see [19, Definitions 2.1, 2.2]). Recall that the RV-sort contains an element $\infty = \text{rv}(0)$. It also serves as the element 0 in the residue field \bar{K} . For psychological reasons, we shall write it as 0 when \bar{K} is concerned (also see Convention 2.5).

The expansions of ACVF that we shall consider are obtained in two steps: we first add a section of RV and a cross-section of Γ (see below), and then arbitrary relations and functions in the RV-sort.

Definition 2.2. A function $\text{sn} : \text{RV} \rightarrow \text{VF}$ is a *section* of RV if

- (1) $\text{sn} \upharpoonright \text{RV}^\times$ is a homomorphism of multiplicative groups and $\text{sn}(\infty) = 0$,
- (2) $\text{sn}(t) \in t$ for every $t \in \text{RV}$,
- (3) $\text{sn}(\bar{K}^\times) \cup \{0\}$ is a subfield of \mathcal{O} .

Similarly, sn is a *section* of \bar{K} if it is the restriction of a section of RV to \bar{K}^\times augmented by $\text{sn}(0) = 0$.

Remark 2.3. Any non-archimedean local field of positive characteristic carries a natural section. However, a non-archimedean local field of characteristic 0 is only equipped with a natural *weak section*, that is, a function $\text{sn} : \text{RV} \rightarrow \text{VF}$ that only satisfies the first two conditions in Definition 2.2, which is given by the nonzero Teichmüller representatives and a choice of uniformizer.

Definition 2.4. A *cross-section* of Γ is a group homomorphism $\text{csn} : \Gamma \rightarrow \text{VF}^\times$ such that $\text{val} \circ \text{csn} = \text{id}$. The *reduced cross-section* of Γ is the function $\overline{\text{csn}} = \text{rv} \circ \text{csn} : \Gamma \rightarrow \text{RV}^\times$. Set $\text{csn}(\infty) = 0$ and $\overline{\text{csn}}(\infty) = \infty$. The *twistback* function $\text{tbk} : \text{RV} \rightarrow \bar{K}$ is given by $u \mapsto u / \overline{\text{csn}}(\text{rv}(u))$, where we set $\infty / \infty = 0$.

The expansions of \mathcal{L}_{RV} with the function symbols sn , csn are respectively denoted by $\mathcal{L}_{\text{RV}}^1$, $\mathcal{L}_{\text{RV}}^2$. The expansion of $\mathcal{L}_{\text{RV}}^1$ with the function symbol $\overline{\text{csn}}$ is denoted by $\mathcal{L}_{\text{RV}}^3$. The theories ACVF¹ in $\mathcal{L}_{\text{RV}}^1$, ACVF² in $\mathcal{L}_{\text{RV}}^2$, and ACVF³ in $\mathcal{L}_{\text{RV}}^3$ state that, in addition to the axioms of ACVF, sn is a section of RV, csn is a cross-section of Γ , and $\overline{\text{csn}}$ is a reduced cross-section of Γ . If the characteristics are specified then we write ACVF¹(0, p), etc.

Convention 2.5. Let $\text{res} : \text{RV} \rightarrow \bar{K}$ be the function given by $\text{res} \upharpoonright \bar{K}^\times = \text{id}$ and $\text{res}(t) = 0$ for all $t \notin \bar{K}^\times$. Technically speaking, $+$: $\bar{K}^2 \rightarrow \bar{K}$ is a function symbol only in the imaginary sort \bar{K} , which, as in [10,18–20], is subsumed into the RV-sort. Terms that appear potentially ill-formed should be interpreted accordingly. For example, in the term $\text{sn}(\tau + \tau')$, the symbol sn should be understood as a section of \bar{K} and τ , τ' should be replaced by $\text{res}(\tau)$, $\text{res}(\tau')$.

Theorem 2.6. *The theories ACVF, ACVF¹, ACVF², and ACVF³ all admit quantifier elimination. Consequently, if the characteristics are specified then these theories are complete.*

Proof. For ACVF and ACVF¹ quantifier elimination is proved in [18, Theorem 3.10, Theorem 3.14]. It is easy to adapt the proof there for ACVF² and ACVF³ (also see Proposition 2.7 and Remark 2.8 below). Completeness is clear by inspecting the quantifier-free sentences. \square

Let \mathbb{T} be an expansion of ACVF¹ in a language $\mathcal{L}_{\mathbb{T}}$. We assume that the language $\mathcal{L}_{\mathbb{T}}$ contains additional relation and function symbols only in the RV-sort, for example, a cross-section, a Denef–Pas angular component map, or a subfield of the residue field. After Proposition 2.7 below we shall work exclusively with such expansions of ACVF³. But, before that, there is no need to require the presence of a cross-section. Note that if \mathbb{T} does expand ACVF³ then it makes sense to speak of the $\mathcal{L}_{\tilde{\mathbb{T}}}$ -reduct $\tilde{\mathbb{T}}$ of \mathbb{T} , where $\mathcal{L}_{\tilde{\mathbb{T}}}$ is the language obtained from $\mathcal{L}_{\mathbb{T}}$ by replacing the functions sn and $\overline{\text{csn}}$ with the function csn . Also note that since, for example, ACVF¹(0, p) is complete, every model of it embeds into a sufficiently saturated model of $\mathbb{T}(0, p)$. By adding more primitives, without changing the class of definable sets, we also assume that the reduct of \mathbb{T} to the RV-sort eliminates quantifiers.

Let $M_{\mathbb{T}} \models \mathbb{T}$ and $A \subseteq M_{\mathbb{T}}$. Let M_1, M be the $\mathcal{L}_{\text{RV}}^1$ -, \mathcal{L}_{RV} -reducts of $M_{\mathbb{T}}$, respectively. We shall write $\text{dcl}^{\mathbb{T}}(A)$, $\text{acl}^{\mathbb{T}}(A)$ for the definable and the (model-theoretic) algebraic closures of A in $M_{\mathbb{T}}$, $\text{dcl}^1(A)$, $\text{acl}^1(A)$ for those of A in M_1 , etc. Note that, in general, $\text{dcl}(A)$ is not closed under sn and hence cannot be expanded to a substructure of M_1 without changing the underlying set. If $\text{dcl}(A)$ is closed under sn then it may be identified with $\text{dcl}^1(A)$. In this case we shall write $\text{dcl}(A) = \text{dcl}^1(A)$. The same convention applies when the operators $\text{dcl}^{\mathbb{T}}$, acl^1 , etc., are involved. For example, we have

$$\text{acl}(\text{sn}(\text{RV}(M))) = \text{acl}^1(\text{sn}(\text{RV}(M))) = \text{acl}^1(\text{RV}(M)).$$

The proof of [18, Theorem 3.14] works more or less for the following proposition. For clarity, we give some details.

Proposition 2.7. *The theories \mathbb{T} and $\tilde{\mathbb{T}}$ eliminate all quantifiers.*

Proof. We shall only be concerned with \mathbb{T} , since the argument for $\tilde{\mathbb{T}}$ is similar. Let $M, N \models \mathbb{T}$ such that N is $\|M\|^+$ -saturated. Let S be a substructure of M and $f : S \rightarrow N$ a monomorphism. It is enough to extend f to a monomorphism $M \rightarrow N$.

Since \mathbb{T} eliminates quantifiers in the RV-sort, there is a monomorphism $g : \text{RV}(M) \rightarrow \text{RV}(N)$ extending $f \upharpoonright \text{RV}(S)$. Since the henselization of S is an immediate extension (in the sense of valuation theory), we may extend f accordingly and hence may just assume that S is henselian.

Let $\dot{f} : \dot{S} \rightarrow N$ be the $\mathcal{L}_{\text{RV}}^1$ -reduct of f . Let $t \in \bar{K}(M)$ be algebraic over $\bar{K}(\dot{S})$ in the field theoretic sense. We have

$$[\text{VF}(\dot{S})(\text{sn}(t)) : \text{VF}(\dot{S})] = [\bar{K}(\dot{S})(t) : \bar{K}(\dot{S})]$$

and hence $\Gamma(\text{dcl}^1(\dot{S} \cup \text{sn}(t))) = \Gamma(\dot{S})$. Since the fields $\bar{K}(\dot{S})(t)$ and $\bar{K}(f(\dot{S}))(g(t))$ are isomorphic via g , we may extend \dot{f} to an $\mathcal{L}_{\text{RV}}^1$ -monomorphism $\dot{f}_t : \text{dcl}^1(\dot{S} \cup \text{sn}(t)) \rightarrow N$ by $\text{sn}(t) \mapsto \text{sn}(g(t))$. Clearly \dot{f}_t is compatible with g . Repeating this procedure, we may assume that $\bar{K}(\dot{S})$ is algebraically closed. Next, let $t \in \text{RV}(M) \setminus \bar{K}(M)$ such that $t^n \in \text{RV}(\dot{S})$ for some $n > 0$ and n is minimal with respect to this condition. We have

$$[\Gamma(\text{dcl}^1(\dot{S} \cup \text{sn}(t))) : \Gamma(\dot{S})] = n, \quad \bar{K}(\text{dcl}^1(\dot{S} \cup \text{sn}(t))) = \bar{K}(\dot{S}).$$

Since $\text{sn}(t)^n = \text{sn}(t^n)$ and $\text{sn}(g(t))^n = \dot{f}(\text{sn}(t^n))$, as above, by setting $\text{sn}(t) \mapsto \text{sn}(g(t))$, we obtain an extension of \dot{f} that is compatible with g . So we may assume that $\Gamma(\dot{S})$ is divisible. Now, $\text{acl}^1(\dot{S})$ is a model of ACVF¹ and $\text{RV}(\text{acl}^1(\dot{S})) = \text{RV}(\dot{S})$, by Theorem 2.6, we may assume $\dot{S} = \text{acl}^1(\dot{S})$.

For any $t \in \text{RV}(M) \setminus \dot{S}$, the proof of [18, Lemma 3.13] goes through with the choice $\text{sn}(t) \mapsto \text{sn}(g(t))$, which yields an extension of \dot{f} that is compatible with g . Repeating the whole process thus far, we eventually obtain an extension \dot{f}_1 of \dot{f} that includes the $\mathcal{L}_{\text{RV}}^1$ -reduct of g . Hence $f_1 = \dot{f}_1 \cup g$ is an $\mathcal{L}_{\mathbb{T}}$ -monomorphism. At this point, any $\mathcal{L}_{\text{RV}}^1$ -extension of \dot{f}_1 induces an obvious $\mathcal{L}_{\mathbb{T}}$ -extension of f_1 , so we are done by Theorem 2.6. \square

Remark 2.8. If \mathbb{T} is the expansion ACVF³ of ACVF¹ or the one with a reduced angular component map $\bar{\text{ac}} : \text{RV}^{\times} \rightarrow \bar{K}^{\times}$ then \mathbb{T} eliminates quantifiers in the RV-sort and hence Proposition 2.7 holds for \mathbb{T} . The proofs are routine and are left to the reader. Since there is a section of RV, we can always define an angular component map from a cross-section and vice versa.

Quantifier elimination still holds if \mathbb{T} is an expansion of ACVF (in the RV-sort only). This follows from a simpler version of the above proof, or from standard syntactical manipulations that reduces it to the case of ACVF.

From now on we assume that \mathbb{T} expands ACVF³. We fix a sufficiently saturated model $\mathcal{C}^{\mathbb{T}} \models \mathbb{T}$ of pure characteristic 0. The (imaginary) sort of value group is denoted by Γ . The \mathcal{L}_{RV} -reduct (resp. $\mathcal{L}_{\text{RV}}^1$ -reduct, etc.) of $\mathcal{C}^{\mathbb{T}}$ is denoted by \mathcal{C} (resp. \mathcal{C}^1 , etc.).

Convention 2.9. Except in the last section, for convenience and without loss of generality, by a substructure we shall always mean a substructure that is equal to its definable closure. Let S be a small substructure of $\mathcal{C}^{\mathbb{T}}$. Note that any reduct of S is VF-generated. For simplicity, all the reducts of S shall simply be denoted by S if there is no danger of confusion. The corresponding expanded languages (with constants in S) are still referred to as \mathcal{L}_{RV} , \mathcal{L}_{RV}^1 , etc. Parameters from S are allowed and they will not be specified unless it is necessary. So in effect we shall be working with the complete theories $ACVF(S)$, $ACVF^1(S)$, etc., and by an \mathcal{L}_{RV} -definable (resp. \mathcal{L}_{RV}^1 -definable, etc.) subset we mean an S - \mathcal{L}_{RV} -definable (resp. S - \mathcal{L}_{RV}^1 -definable, etc.) subset. In general, by a definable subset we mean an $\mathcal{L}_{\mathbb{T}}$ -definable subset, unless indicated otherwise in context. Parameters from sources other than S will be specified in context.

Notation 2.10. If $A \subseteq VF$ then the field generated by A over $VF(S)$ is denoted as usual by $VF(S)(A)$ and the field-theoretic algebraic closure of $A \cup VF(S)$ is denoted by A^{ac} .

Lemma 2.11. For any $U \subseteq RV$, the \mathcal{L}_{RV} -reduct of $dcl^1(U)$ is $dcl(sn(U))$ and hence $RV(dcl^1(U))$ is equal to $RV(dcl(sn(U))) = RV(dcl(U))$.

Proof. Let $M = acl^1(U) \models ACVF^1(S)$ and $N = acl(U \cup sn(U)) \models ACVF(S)$. It is clear from the proof of [18, Theorem 3.14] that $VF(M) = VF(N) = sn(U)^{ac}$. Hence $\sigma \in Aut_{dcl^1(U)}(M)$ if and only if $\sigma \in Aut_{dcl(sn(U))}(N)$. The claim follows. \square

Since a VF-sort equality can be equivalently expressed as an RV-sort equality, we may and shall assume that an $\mathcal{L}_{\mathbb{T}}$ -formula contains no VF-sort equalities at all.

Definition 2.12. Let $M, N \subseteq \mathcal{C}$ be substructures and $\sigma : M \rightarrow N$ be an \mathcal{L}_{RV} -isomorphism. We say that σ is an *immediate isomorphism* if $\sigma(t) = t$ for all $t \in RV(M)$.

Lemma 2.13. Let $s_1, s_2 : RV \rightarrow VF^\times$ be two full sections. Then any immediate isomorphism $\sigma : M \rightarrow N$ such that $\sigma(s_1(t)) = s_2(t)$ for all $t \in RV(M)$ may be extended to an immediate automorphism $\bar{\sigma} \in Aut_S(\mathcal{C})$ such that $\sigma(s_1(t)) = s_2(t)$ for all $t \in RV$.

Proof. With extra bookkeeping, the proof of [18, Theorem 3.10] works. \square

Lemma 2.14. Let $U \subseteq RV$ and σ be an automorphism of \mathcal{C} over $dcl(U)$. Then there is an automorphism ρ of \mathcal{C}^1 over $dcl^1(U)$ and an immediate automorphism $\bar{\sigma} \in Aut_S(\mathcal{C})$ such that $\sigma = \bar{\sigma} \circ \rho$.

Proof. First note that $\sigma(sn(RV))$ induces a full section $sn^* : RV \rightarrow VF^\times$. By Lemma 2.11, $RV(dcl^1(U)) = RV(dcl(U))$ and hence the restriction of σ to the \mathcal{L}_{RV} -reduct of $dcl^1(U)$ is an immediate automorphism with $\sigma(sn(t)) = sn^*(t)$ for all $t \in RV(dcl(U))$. By Lemma 2.13, this restriction of σ may be extended to an immediate automorphism $\bar{\sigma}$ of \mathcal{C} with $\bar{\sigma}(sn(t)) = sn^*(t)$ for all $t \in RV$. Now set $\rho = \bar{\sigma}^{-1} \circ \sigma$. \square

Corollary 2.15. Let $\vec{t} \in RV$. If $A \subseteq RV^m$ is parametrically \mathcal{L}_{RV} -definable and is also \vec{t} - \mathcal{L}_{RV}^1 -definable then it is \vec{t} - \mathcal{L}_{RV} -definable.

Proof. We only need to show that any automorphism of \mathcal{C} over $dcl(\vec{t})$ fixes A setwise. This is immediate by Lemma 2.14, since A is trivially invariant under immediate automorphisms. \square

Definition 2.16. Let τ be an $\mathcal{L}_{\mathbb{T}}$ -term. For any variable X , the X -complexity $|\tau|_X \in \mathbb{N}$ of τ is defined inductively as follows.

- (1) If either X does not occur in τ or τ is an \mathcal{L}_{RV} -term then $|\tau|_X = 0$.
- (2) If X occurs in τ and τ is of the form $sn(\sigma)$ then $|\tau|_X = |\sigma|_X + 1$.
- (3) If τ is not of the form $sn(\sigma)$ then $|\tau|_X$ is the maximum of the X -complexities of the proper subterms of τ .

The *complexity* $|\tau|$ of τ is the maximum of all X -complexities of τ .

Let $\phi(\vec{X}, \vec{Y})$ be an $\mathcal{L}_{\mathbb{T}}$ -formula, where $\vec{X} = (X_1, \dots, X_n)$ are the occurring VF-sort variables and $\vec{Y} = (Y_1, \dots, Y_m)$ are the occurring RV-sort variables. The X_i -complexity $|\phi|_{X_i}$ of ϕ is the maximal X_i -complexity of the terms occurring in ϕ ; the Y_i -complexity $|\phi|_{Y_i}$ of ϕ is defined similarly. Let $|\phi|_{VF}$ be the maximum of the X_i -complexities of ϕ ; similarly for $|\phi|_{RV}$. Lastly set $|\phi| = \max\{|\phi|_{VF}, |\phi|_{RV}\}$.

Let ϕ be an $\mathcal{L}_{\mathbb{T}}$ -term or a quantifier-free $\mathcal{L}_{\mathbb{T}}$ -formula. If a term F occurs in ϕ in the form $rv(F)$ (respectively $sn(F)$) then F is said to be an *occurring VF-term* (respectively *occurring RV-term*) of ϕ . Note that if F is an occurring VF-term of ϕ with $|F| = 0$ then it is called an occurring polynomial of ϕ in [20,19]. We shall keep this terminology. Obviously if $|\phi| > 0$ then we have

$$|\phi| = \max\{|F| : F \text{ is an occurring VF-term of } \phi\} = \max\{|F| : F \text{ is an occurring RV-term of } \phi\} + 1.$$

If F is an occurring VF-term of ϕ that is not a subterm of an occurring VF-term of a higher complexity then F is a *top occurring VF-term* of ϕ ; similarly for a *top occurring RV-term* of ϕ .

Lemma 2.17. *If $A \subseteq RV$ is \mathcal{L}_{RV}^1 -definable then it is \mathcal{L}_{RV} -definable.*

Proof. Let $\phi(Y)$ be a quantifier-free formula that defines A , where Y is an RV-sort variables. We do induction on $|\phi|$. Since the base case $|\phi| = 0$ is tautological, we proceed to the inductive step directly.

Let $F_k(Y)$ enumerate the occurring VF-terms of $\phi(Y)$ of complexity 1. We may write each $F_k(Y)$ in the form $\sum_i a_i \text{sn}(Y^i)$, where $a_i \in \text{VF}(S)$. For each $t \in A$ and each k let

$$I_{k,t} = \{i: \text{vrv}(\text{rv}(a_i)t^i) \leq \text{vrv}(\text{rv}(a_j)t^j) \text{ for all } j\}.$$

Then set $e_i = \text{sn}(\text{rv}(a_i)) \in \text{VF}(S)$ and $E_{k,t}(Y) = \sum_{i \in I_{k,t}} e_i \text{sn}(Y^i)$. Across a disjunction we may assume that, for every k and all $t, s \in A$, $I_{k,t} = I_{k,s}$ and hence $E_{k,t}(Y) = E_{k,s}(Y)$. Then we may write I_k and $E_k(Y)$ instead. Note that the equality $E_k(Y) = 0$ is equivalent to an \mathcal{L}_{RV} -formula. Therefore we may further assume that, for every k , either $E_k(t) = 0$ for all $t \in A$ or $E_k(t) \neq 0$ for all $t \in A$.

If $E_k(t) = 0$ for some k and some $t \in A$ then A is finite and hence, by Corollary 2.15, A is \mathcal{L}_{RV} -definable. So we may assume that $E_k(t) \neq 0$ for all k and all $t \in A$. Then $\text{rv}(F_k(t)) = \text{rv}(E_k(t))$ for all k and all $t \in A$. Since, without loss of generality, $E_k(Y)$ is of the form $1 + \sum_i e_i \text{sn}(Y^i)$, we have $\text{rv}(E_k(t)) = 1 + \text{rv}(e_i)t^i$ for all $t \in A$. This means that $\phi(Y)$ is equivalent to a formula of complexity $< |\phi|$ and hence, by the inductive hypothesis, A is \mathcal{L}_{RV} -definable. \square

Lemma 2.18. *If $A \subseteq RV^m$ is \mathcal{L}_{RV}^1 -definable then it is \mathcal{L}_{RV} -definable.*

Proof. We do induction on m . The base case $m = 1$ is proved above. For the inductive step, by the inductive hypothesis, $\text{pr}_{<m}(A)$ is \mathcal{L}_{RV} -definable. On the other hand, for every $\vec{t} \in \text{pr}_{<m}(A)$, $\text{fib}(A, \vec{t})$ is both $\text{dcl}^1(\vec{t})$ - \mathcal{L}_{RV} -definable and \vec{t} - \mathcal{L}_{RV}^1 -definable and hence, by Corollary 2.15, it is \vec{t} - \mathcal{L}_{RV} -definable. For any \mathcal{L}_{RV} -formula $\phi(\vec{Y}, Z)$, let $B_\phi \subseteq \text{pr}_{<m}(A)$ be the \mathcal{L}_{RV}^1 -definable subset such that $\vec{t} \in B_\phi$ if and only if $\phi(\vec{t}, Z)$ defines $\text{fib}(A, \vec{t})$. By the inductive hypothesis again, B_ϕ is \mathcal{L}_{RV} -definable. Now the claim follows from compactness. \square

Corollary 2.19. *Any $\mathcal{L}_{\mathbb{T}}$ -definable subset $A \subseteq RV^m$ may be defined by an $\mathcal{L}_{\mathbb{T}}$ -formula that does not involve sn , that is, A is definable in the reduct of $\mathcal{C}^{\mathbb{T}}$ to the RV-sort.*

Proof. Let ϕ be a quantifier-free formula that defines A . We do induction on $|\phi|$. Let $F_k(\vec{Y})$ enumerate the top occurring VF-terms of ϕ . We may write each $F_k(\vec{Y})$ in the form $\sum_i a_i \text{sn}(\tau_{ki}(\vec{Y}))$, where $a_i \in \text{VF}(S)$. Let F_k^* be the VF-term obtained from $F_k(\vec{Y})$ by replacing each $\tau_{ki}(\vec{Y})$ with a new variable X_{ki} . Let ϕ^* be the formula obtained from ϕ by replacing each $\text{rv}(F_k(\vec{Y}))$ with a new variable Z_k . Let A^* be the subset defined by the formula

$$\phi^* \wedge \bigwedge_{k,i} (Z_k = \text{rv}(F_k^*) \wedge X_{ki} = \tau_{ki}(\vec{Y})).$$

Since $A = \text{pr}_{\leq m}(A^*)$, the claim follows from the inductive hypothesis and Lemma 2.18. \square

Therefore, as far as the RV-sort is concerned, \mathbb{T} and $\tilde{\mathbb{T}}$ are the same theory (in the sense that they have the same definable subsets) and there is no need to treat $\tilde{\mathbb{T}}$ separately. Consequently, if $\mathcal{C}^{\mathbb{T}}$ is a Γ -minimal expansion of \mathcal{C}^3 , that is, if any $\mathcal{L}_{\mathbb{T}}$ -definable subset $I \subseteq \Gamma^m$ is \mathcal{L}_{RV}^3 -definable, then we may unambiguously speak of definable subsets in the Γ -sort:

Definition 2.20. An *imaginary \bar{K} -term* is a term of the form $\sum_{i=1}^k \text{res}(\text{rv}(F_i(\vec{X})) \cdot r_i \cdot \vec{Y}^{\vec{n}_i})$, where \vec{X} are VF-sort variables, \vec{Y} are RV-sort variables, $\vec{n}_i \in \mathbb{N}$, $r_i \in RV$, and $F_i(\vec{X})$ is a polynomial with coefficients in VF. An *imaginary Γ -term* is a term of the same form with res replaced by vrv .

We should think of these as real terms if we work with the language $\mathcal{L}_{\bar{K}\Gamma}$ (resp. $\mathcal{L}_{\bar{K}\Gamma}^{\text{csn}}$) that corresponds to the three-sorted structure of the reduct of \mathcal{C} (resp. \mathcal{C}^2 or \mathcal{C}^3) to the RV-sort. The complexity of an $\mathcal{L}_{\bar{K}\Gamma}^{\text{csn}}$ -formula with respect to vrv and csn is defined as in Definition 2.16.

Lemma 2.21. *Let $\Gamma_\infty = \Gamma \cup \{\infty\}$. If $I \subseteq \Gamma_\infty^m$ is \mathcal{L}_{RV}^3 -definable then it is \mathcal{L}_{RV} -definable.*

Proof. By Corollary 2.19, we may work in the reduct of \mathcal{C}^3 to the RV-sort and hence with the language $\mathcal{L}_{\bar{K}\Gamma}^{\text{csn}}$, where we still have quantifier elimination. Let $\phi(\vec{Z})$ be a quantifier-free formula that defines I . Consider any term $\tau(\vec{Z})$ that occurs in $\phi(\vec{Z})$ in one of the following ways: $\text{vrv}(\tau(\vec{Z}))$, $\text{res}(\tau(\vec{Z}))$, and $\tau(\vec{Z}) \square 1$ or $\tau(\vec{Z}) \square \infty$, where \square is either $=$ or \neq in the RV-sort.

Then $\tau(\vec{Z})$ may be written as $t \text{csn}(F(\vec{Z}))$, where $t \in \text{RV}(S)$. If $\text{vrv}(\tau(\vec{Z}))$ occurs then it may be replaced by $\text{vrv}(t) + F(\vec{Z})$. If $\text{res}(\tau(\vec{Z}))$ occurs then it may be replaced by either 0 or $t \overline{\text{csn}}(\text{vrv}(t))^{-1}$. If $\tau(\vec{Z}) \square 1$ occurs then it may be replaced by $\text{vrv}(t) + F(\vec{Z}) \square 0$ (note that this is so because if $t \neq \overline{\text{csn}}(\text{vrv}(t))$ then $\forall \vec{Z} \tau(\vec{Z}) \neq 1$ is true); similarly for the case $\tau(\vec{Z}) \square \infty$. In all situations, across a disjunction, the complexity of the formula decreases. So the claim follows from a routine induction on complexity. \square

Remark 2.22. Recall that the (imaginary) Γ -sort is stably embedded in \mathfrak{C} ; that is, any parametrically \mathcal{L}_{RV} -definable subset in the Γ -sort can be parametrically defined in the reduct of \mathfrak{C} to the Γ -sort (see the discussion preceding [18, Lemma 4.17]). Therefore, all \mathcal{L}_{RV} -definable functions in the Γ -sort are piecewise \mathbb{Q} -linear. Here an R -linear map for any ring R is allowed to have a constant term, unless indicated otherwise. By Lemma 2.21, this is also true in $\mathfrak{C}^{\mathbb{T}}$ if it is a Γ -minimal expansion of \mathfrak{C}^3 .

There are two ways of treating an element $\gamma \in \Gamma_\infty$: as a point (when we study Γ as an independent structure) or a subset of $\mathfrak{C}^{\mathbb{T}}$ (when we need to remain in the realm of definable subsets of $\mathfrak{C}^{\mathbb{T}}$). The former perspective simplifies the notation but is of course dispensable. We shall write $\text{vrv}^{-1}(\gamma)$ when we want to emphasize that $\gamma \in \Gamma$ is a subset of $\mathfrak{C}^{\mathbb{T}}$.

In fact, Lemma 2.21 may be strengthened:

Corollary 2.23. *Let $\vec{t} \in \text{RV}$ and $I \subseteq \Gamma_\infty^m$ be a \vec{t} - $\mathcal{L}_{\text{RV}}^3$ -definable subset. Then I is \vec{t} - \mathcal{L}_{RV} -definable.*

Proof. By stable embeddedness, I is $\Gamma(\text{dcl}^3(\vec{t}))$ -definable in the reduct of \mathfrak{C}^3 (or \mathfrak{C}) to the Γ -sort. On the other hand, it is not hard to see that, by Corollary 2.19, $\Gamma(\text{dcl}^3(\vec{t})) = \Gamma(\text{dcl}(\vec{t}))$, that is, the subgroup of Γ generated by $\text{vrv}(\vec{t})$. So I is also \vec{t} - \mathcal{L}_{RV} -definable. \square

Notation 2.24. Given a function $f : A \rightarrow B$, we shall often write A_b for the fiber over $b \in B$ under f . In particular, given a definable subset A , we shall often write A_x for the fiber over x under a function of the form $\text{rv} \upharpoonright A$, $\text{val} \upharpoonright A$, $\text{vrv} \upharpoonright A$, etc. Of course which function is being considered should always be clear in context.

Definition 2.25. For any subset $U \subseteq \text{RV}^n$ and $\vec{\gamma} \in \Gamma_\infty^n$, the subset $\text{tbk}(U_{\vec{\gamma}}) \subseteq \bar{\mathbb{K}}^n$ is called the $\vec{\gamma}$ -twistback of U . The subset $\bigcup_{\vec{\gamma} \in \text{vrv}(U)} \{\vec{\gamma}\} \times \text{tbk}(U_{\vec{\gamma}}) \subseteq \Gamma_\infty^n \times \bar{\mathbb{K}}^n$ is denoted by $\Omega(U)$. Conversely, $U_{\vec{\gamma}}$ is called the $\vec{\gamma}$ -twist of $\text{tbk}(U_{\vec{\gamma}})$. If $\text{tbk}(U_{\vec{\gamma}}) = \text{tbk}(U_{\vec{\gamma}'})$ for all $\vec{\gamma}, \vec{\gamma}' \in \text{vrv}(U)$ then U is called a *twistoid*, in which case we simply write $\text{tbk}(U)$ for the unique twistback.

These notions of course depend on the choice of the cross-section csn . Note that for a subset $W \subseteq \bar{\mathbb{K}}^n$ and a $\vec{\gamma} \in \Gamma_\infty^n$, the $\vec{\gamma}$ -twist $W_{\vec{\gamma}}$ of W is defined only if the 0-coordinates in W match the ∞ -coordinates in $\vec{\gamma}$. For $D \subseteq \Gamma^n$ we write $\mathcal{E}(W, D)$ for $\bigcup_{\vec{\gamma} \in D} W_{\vec{\gamma}}$.

Lemma 2.26. *Let $U \subseteq \text{RV}^n$ be an \mathcal{L}_{RV} -definable subset and $\text{vrv}(U) = D$. Then there is a definable finite partition D_k of D such that each $U_k = U \cap \text{vrv}^{-1}(D_k)$ is a twistoid and the corresponding twistback is \mathcal{L}_{RV} -definable.*

Proof. We work in the reduct of \mathfrak{C}^3 to the RV -sort, considered as an $\mathcal{L}_{\bar{\mathbb{K}}\Gamma}^{\overline{\text{csn}}}$ -structure. Let $\phi(\vec{Z}, \vec{Y}) = \bigvee_i \phi_i(\vec{Z}, \vec{Y})$ be a quantifier-free $\mathcal{L}_{\bar{\mathbb{K}}\Gamma}^{\overline{\text{csn}}}$ -formula in disjunctive normal form that defines $\Omega(U) \subseteq \Gamma_\infty^n \times \bar{\mathbb{K}}^n$. Let $\text{res}(t \text{csn}(F(\vec{Z})))$ be a term that occurs in ϕ . If $\text{vrv}(t) + F(\vec{Z}) \neq 0$ then $\text{res}(t \text{csn}(F(\vec{Z})))$ may be replaced by 0, otherwise it may be replaced by $t \overline{\text{csn}}(\text{vrv}(t))^{-1}$. Therefore, without loss of generality, each $\phi_i(\vec{Z}, \vec{Y})$ may be written as a conjunction of $\theta_i(\vec{Z})$ and $\psi_i(\vec{Y})$, where the variables are displayed. Let $B_i \subseteq \Gamma_\infty^n$ be the subset defined by $\theta_i(\vec{Z})$ and $V_i \subseteq \bar{\mathbb{K}}^n$ the subset defined by $\psi_i(\vec{Y})$.

Now we may easily translate each $\psi_i(\vec{Y})$ back into an \mathcal{L}_{RV} -formula. On the other hand, by Lemma 2.21, each B_i is also \mathcal{L}_{RV} -definable. Let D_k be the \mathcal{L}_{RV} -definable finite partition of D induced by the subsets B_i . Clearly, for every $\vec{\gamma} \in D_k$, $\text{tbk}(U_{\vec{\gamma}}) = \bigcup_{\vec{\gamma} \in B_i} V_i$. \square

Definition 2.27. Let $f : A \rightarrow B$ be a function. If A, B only have VF - and RV -coordinates then f is *rv-contractible* if $(\text{rv} \circ f)(\mathfrak{p} \cap A)$ is a singleton for every rv -polydisc $\mathfrak{p} \subseteq \text{RVH}(A)$ (see [18, Definition 2.4, Definition 4.21]). If A, B only have RV -coordinates then f is *vrv-contractible* if $(\text{vrv} \circ f)(A_{\vec{\gamma}})$ is a singleton for every $\vec{\gamma} \in \text{vrv}(A)$. The contractions of f , that is, the induced functions $\text{rv}(A) \rightarrow \text{rv}(B)$, $\text{vrv}(A) \rightarrow \text{vrv}(B)$, are usually denoted by f_\downarrow .

In context, we shall often drop the prefixes and simply say that f is a contractible function.

Remark 2.28. Unlike in [10], the conclusion of this remark is not needed for the Γ -categories below (see Definition 3.20). We present it here for the sake of comparison (see [10, Lemma 3.28, Definition 9.1]).

Obviously the composition of two vrv -contractible functions is a vrv -contractible function. Let $f : A \rightarrow B$ be an \mathcal{L}_{RV} -definable vrv -contractible function. For all $\vec{t} \in \text{RV}$, since the underlying substructure S is VF -generated (see Convention 2.9),

it is clear that if $\gamma \in \Gamma(\text{dcl}(\vec{t}))$ then $\text{vrv}^{-1}(\gamma) \cap \text{RV}(\text{dcl}(\vec{t})) \neq \emptyset$. This implies that, if $f_{\downarrow} : \text{vrv}(A) \rightarrow \text{vrv}(B)$ is a bijection then there is a \vec{t} - \mathcal{L}_{RV} -definable $g(\vec{t}) \in \text{vrv}^{-1}(f_{\downarrow}^{-1}(\text{vrv}(\vec{t})))$ for every $\vec{t} \in B$ and hence, by compactness, we have an \mathcal{L}_{RV} -definable function g on B such that $g_{\downarrow} = f_{\downarrow}^{-1}$. Let G be a $\text{GL}(\mathbb{Z})$ -transformation on $\text{vrv}(A)$, that is, a bijection of the form $T\vec{x} + \vec{\gamma}$ with $T \in \text{GL}(\mathbb{Z})$ and $\vec{\gamma} \in \Gamma(S)$. Let $\vec{t} \in \text{RV}(S)$ with $\text{vrv}(\vec{t}) = \vec{\gamma}$ and G^{\uparrow} the $\text{GL}(\mathbb{Z})$ -transformation on A given by $T\vec{x} \cdot \vec{t}$. Then $f_{\downarrow} \circ G^{-1}$ is the contraction of $f \circ (G^{\uparrow})^{-1}$; similarly if G is a $\text{GL}(\mathbb{Z})$ -transformation on $\text{vrv}(B)$. Lastly, let pr_E be a coordinate projection on B and $\text{vrv}(B)$. It is straightforward to check that $\text{pr}_E \circ f_{\downarrow}$ is the contraction of $\text{pr}_E \circ f$.

We have just shown that the class \mathfrak{D} of contractions of \mathcal{L}_{RV} -definable vrv -contractible functions is closed under composition, inversion, composition with $\text{GL}(\mathbb{Z})$ -transformations, and composition with coordinate projections.

Now suppose that $A \subseteq \text{RV}^n$, $B \subseteq \text{RV}$, and $(\vec{\alpha}, \beta) \in \text{vrv}(f) \cap \Gamma^{n+1}$. Let $\phi(\vec{Y}, Z)$ be a quantifier-free $\mathcal{L}_{\overline{\text{RV}}}$ -formula with parameters $\vec{\alpha}, \beta$ that defines $f_{(\vec{\alpha}, \beta)} : A_{\vec{\alpha}} \rightarrow B_{\beta}$. Clearly we may assume that all RV -sort literals occurring in ϕ are of the form $t\vec{Y}^{\vec{n}}Z^m \square 1$, where $t \in \text{RV}(S)$, $\vec{n}, m \in \mathbb{Z}$, and \square is $=$ or \neq . Since $t\vec{Y}^{\vec{n}}Z^m = 1$ is equivalent to $\text{res}(t\vec{Y}^{\vec{n}}Z^m) = 1$ and $f_{(\vec{\alpha}, \beta)}$ is a function, we see that ϕ contains irredundant $\overline{\text{K}}$ -sort equalities between sums of terms of the form $\text{res}(t\vec{Y}^{\vec{n}}Z^m)$ with $\text{vrv}(t) + \sum_i n_i \alpha_i + m\beta = 0$. Observe that

$$\text{res}(t\vec{Y}^{\vec{n}}Z^m) = (t / \overline{\text{csn}}(\text{vrv}(t))) \cdot (\vec{Y} / \overline{\text{csn}}(\vec{\alpha}))^{\vec{n}} \cdot (Z / \overline{\text{csn}}(\beta))^m.$$

We may treat $\vec{Y} / \overline{\text{csn}}(\vec{\alpha})$, $Z / \overline{\text{csn}}(\beta)$ as variables in these equalities and consequently may assume $\vec{n}, m \in \mathbb{N}$. Applying the Euclidean algorithm, we see that, away from a $\overline{\text{csn}}(\vec{\alpha}, \beta)$ - \mathcal{L}_{RV} -definable subset of $\text{vrv}^{-1}(\vec{\alpha})$ of RV -dimension $< n$ (recall [19, Definition 4.9]), the twistback of $f_{(\vec{\alpha}, \beta)}$ is given by

$$Z / \overline{\text{csn}}(\beta) = \sum_i F_i(\vec{Y} / \overline{\text{csn}}(\vec{\alpha})) / \sum_j G_j(\vec{Y} / \overline{\text{csn}}(\vec{\alpha})),$$

where $F_i(\vec{Y} / \overline{\text{csn}}(\vec{\alpha}))$, $G_j(\vec{Y} / \overline{\text{csn}}(\vec{\alpha}))$ are monomials such that, for any i, j and any $\vec{t} \in \text{vrv}^{-1}(\vec{\alpha})$,

$$\beta = \text{vrv}(F_i(\vec{t})) - \text{vrv}(G_j(\vec{t})).$$

This means that there are integers $n_i \in \mathbb{Z}$ and a $\delta \in \Gamma(S)$ (note that $(\vec{\alpha}, \beta)$ is not needed to define this δ) such that $\beta = \delta + \sum_i n_i \alpha_i$.

In summary, by compactness, all functions in \mathfrak{D} are definably piecewise \mathbb{Z} -linear (with constant terms). Moreover, if $h : I \rightarrow J$ is a bijection in \mathfrak{D} and $I, J \subseteq \Gamma^n$ then h is definably a piecewise $\text{GL}_n(\mathbb{Z})$ -transformation. This follows from the next lemma, which holds in a more general setting.

Lemma 2.29. *Let R be an integral domain and M be a torsion-free R -module, viewed as the main sort of a first-order structure of some expansion of the usual R -module language. Let \mathfrak{D} be a class of definable functions in the sort M such that*

- (1) \mathfrak{D} contains all the identity functions and all functions in \mathfrak{D} are definably piecewise R -linear,
- (2) \mathfrak{D} is closed under composition, inversion, composition with $\text{GL}(R)$ -transformations, and composition with coordinate projections (in the sense described above).

If $g : D \rightarrow E$ is a bijection in \mathfrak{D} , where $D, E \subseteq M^n$, then h is definably a piecewise $\text{GL}_n(R)$ -transformation.

Proof. We proceed by induction on n . The base case $n = 1$ is clear. For the inductive step, without loss of generality, we may assume that both g and g^{-1} are R -linear, given respectively by $\vec{x} \mapsto A\vec{x} + \vec{a}$ and $\vec{x} \mapsto B\vec{x} + \vec{b}$. Observe that if there are distinct $\vec{x}_1, \vec{x}_2 \in D$ such that $\text{pr}_k(\vec{x}_1) = \text{pr}_k(\vec{x}_2)$ for some $1 \leq k \leq n$ then the k th column of $BA - I_n$ must be 0; similarly for E and $AB - I_n$. Therefore we are reduced to the situation where this fails for D, E with respect to some $1 \leq k, l \leq n$. After $\text{GL}_n(R)$ -transformations if necessary, we may assume that $k = l = n$. Then D is the graph of a function $\alpha : \text{pr}_{<n}(D) \rightarrow \text{pr}_n(D)$ and E is the graph of a function $\beta : \text{pr}_{<n}(E) \rightarrow \text{pr}_n(E)$. Note that $\alpha = (\text{pr}_n \upharpoonright D) \circ (\text{pr}_{<n} \upharpoonright D)^{-1}$ and $\beta = (\text{pr}_n \upharpoonright E) \circ (\text{pr}_{<n} \upharpoonright E)^{-1}$. Let $g^* : \text{pr}_{<n}(D) \rightarrow \text{pr}_{<n}(E)$ be the bijection $(\text{pr}_{<n} \upharpoonright E) \circ g \circ (\text{pr}_{<n} \upharpoonright D)^{-1}$. By the assumed two conditions, α, β , and g^* are all in \mathfrak{D} .

By compactness, we may further assume that α is given by $\vec{x} \mapsto \vec{r} \cdot \vec{x} + c$, where $\vec{r} \in R^{n-1}$, and, by the inductive hypothesis, g^* is given by $\vec{x} \mapsto T\vec{x} + \vec{d}$, where $T \in \text{GL}_{n-1}(R)$. Let (\vec{s}_n, s_n) be the last row of A and a_n the last entry of \vec{a} . Set

$$A^* = \begin{bmatrix} T & 0 \\ \vec{s}_n + s_n \vec{r} + \vec{r} & -1 \end{bmatrix} \quad \text{and} \quad \vec{a}^* = (\vec{d}, s_n c + c + a_n).$$

Then g is given by $\vec{x} \mapsto A^* \vec{x} + \vec{a}^*$, as required. \square

Lemma 2.30. *Let $A \subseteq \text{RV}^k \times \Gamma^l$ be an $\mathcal{L}_{\text{RV}}^3$ -definable (resp. \mathcal{L}_{RV} -definable) subset. Set $\text{pr}_{\leq k}(A) = U$ and suppose that $\text{vrv}(U)$ is finite. Then there is an $\mathcal{L}_{\text{RV}}^3$ -definable (resp. \mathcal{L}_{RV} -definable) finite partition U_i of U such that, for each i , $\text{fib}(A, \vec{t}) = \text{fib}(A, \vec{t}')$ for all $\vec{t}, \vec{t}' \in U_i$.*

Proof. Since the Γ -sort is stably embedded (see Remark 2.22), by Lemma 2.23, $\text{fib}(A, \vec{t})$ is $\text{vrv}(\vec{t})$ - \mathcal{L}_{RV} -definable for every $\vec{t} \in U$. Since $\text{vrv}(\vec{t})$ is definable, the lemma simply follows from compactness. \square

Remark 2.31. We clearly have $\text{acl}^1(RV) = \text{acl}^{\mathbb{T}}(RV)$, which is a model of $\mathbb{T}(S)$, and $\text{VF}(\text{acl}^1(RV)) = (\text{VF}(S) \cup \text{sn}(RV))^{\text{ac}}$. But $\text{acl}^1(RV)$, as a valued field, is not maximally complete. In fact the underlying valued field of \mathfrak{C} may be taken to be the unique maximal completion of $\text{acl}^1(RV)$, which is isomorphic to the field $\bar{K}((\Gamma))$ of generalized formal Laurent series. Each element $a \in \text{VF}$ may be written in the form $\sum_{i \in I} \text{sn}(t_i)$, where I is a well-ordered set and if $i < i'$ then $\text{vrv}(t_i) < \text{vrv}(t_{i'})$. We say that $\text{sn}(t_i)$ is the $\text{vrv}(t_i)$ -component of a and denote it by $(a)_{\text{vrv}(t_i)}$. Observe that if $a_1, \dots, a_n \in \text{VF}$ are of the same value γ then $\text{val}(\sum_i a_i) > \gamma$ if and only if $\sum_i (a_i)_{\gamma} = 0$.

For any consistent set $\Phi(\vec{X})$ of $\mathcal{L}_{\mathbb{T}}$ -formulas with parameters in $\text{acl}^1(RV)$, where \vec{X} are the free variables and are all of the VF-sort, if $\Phi(\vec{X})$ is realized in an immediate extension of $\text{acl}^1(RV)$ then it is realized in \mathfrak{C} , because any immediate extension of $\text{acl}^1(RV)$ may be embedded into \mathfrak{C} .

Convention 2.32. We reiterate [18, Convention 4.20] here, since this trivial looking convention is actually quite crucial for understanding the whole construction, especially the parts that involve special bijections. For a subset $A \subseteq \text{VF}^n \times \text{RV}^m$, let

$$\mathbf{c}(A) = \{(\vec{a}, \text{rv}(\vec{a}), \vec{t}) : (\vec{a}, \vec{t}) \in A\}.$$

This is called the *canonical image* of A and $\mathbf{c} : A \rightarrow \mathbf{c}(A)$ is called the *canonical bijection* on A . The convention is that we shall tacitly substitute $\mathbf{c}(A)$ for A in the discussion below if it is necessary or is just more convenient. Whether or not this substitution has been performed should be clear in context.

3. The Grothendieck semirings of RV

The main purpose of this section is to express the Grothendieck semirings of RV-categories as tensor products of the Grothendieck semirings of Γ -categories and RES-categories, which will be defined below. This works if certain conditions are met by \mathbb{T} , in particular, if $\mathbb{T} = \text{ACVF}^3(0, 0)$. On the other hand, it does not seem straightforward to work out these conditions and it does seem to be an unworthy distraction here to digress into that direction. It is perhaps better to deal with it on a case-by-case basis when it is called for in future applications. In Hypothesis 3.15 we describe what some of these conditions might be.

Of course, at the very least we can assume that $\mathfrak{C}^{\mathbb{T}}$ is an RV-minimal expansion of \mathfrak{C}^3 , that is, all definable RV-sort subsets in $\mathfrak{C}^{\mathbb{T}}$ are already definable in \mathfrak{C}^3 . However, for concreteness, we shall work in \mathfrak{C}^3 throughout this section. Hence, all definable subsets in this section are \mathcal{L}_{RV}^3 -definable, unless indicated otherwise.

Definition 3.1. Let $A \subseteq \text{RV}^n$ be a definable subset. A Γ -partition of A is a definable function $\pi : A \rightarrow \Gamma_{\infty}^l$ such that, for $\vec{\gamma} \in \Gamma_{\infty}^l$, $\pi^{-1}(\vec{\gamma})$ is contained in a $(\bar{K}^{\times})^n$ -coset and is $\overline{\text{csn}}(\vec{\gamma})$ - \mathcal{L}_{RV} -definable. If π is a Γ -partition of A then the RV-dimension of π , denoted by $\text{dim}_{RV}(\pi)$, is the number $\max\{\text{dim}_{RV}(\pi^{-1}(\vec{\gamma})) : \vec{\gamma} \in \Gamma_{\infty}^l\}$.

By Corollary 2.15, the existence of such a Γ -partition of A is easily verified by straightforward syntactical manipulation of any quantifier-free formula that defines A . This definition can be extended to definable subsets $A \subseteq \text{RV}^n \times \Gamma_{\infty}^m$ in the obvious way.

Lemma 3.2. For any two Γ -partitions π_1, π_2 of $A \subseteq \text{RV}^n \times \Gamma_{\infty}^m$, we have $\text{dim}_{RV}(\pi_1) = \text{dim}_{RV}(\pi_2)$.

Proof. We may assume that π_1 is constant. Recall [19, Lemma 4.10, Lemma 4.11], which essentially say that the RV-dimension of any \mathcal{L}_{RV} -definable subset in the RV-sort equals its algebraic dimension (Zariski dimension). Now observe that the algebraic dimension of A over $\text{dcl}(\overline{\text{csn}}(\Gamma))$ is still $\text{dim}_{RV}(A)$. This implies that $\text{dim}_{RV}(\pi_2^{-1}(\vec{\gamma})) = \text{dim}_{RV}(A)$ for some $\vec{\gamma} \in \text{ran}(\pi_2)$ and hence $\text{dim}_{RV}(A) = \text{dim}_{RV}(\pi_2)$. \square

Therefore the RV-dimension $\text{dim}_{RV}(A)$ of a definable subset $A \subseteq \text{RV}^n \times \Gamma_{\infty}^m$ may be defined as the RV-dimension of any Γ -partition of A . Note that the proof of the above lemma shows that $\text{dim}_{RV}(A)$ does not depend on parameters and if $f : A \rightarrow \text{RV}^k \times \Gamma_{\infty}^l$ is a definable function then $\text{dim}_{RV}(A) \geq \text{dim}_{RV}(f(A))$. Hence there is a definable finite-to-one function $f : A \rightarrow \text{RV}^k \times \Gamma_{\infty}^l$ if and only if there is a definable function $f : A \rightarrow \text{RV}^k$ such that all fibers are of RV-dimension 0 if and only if $\text{dim}_{RV}(A) \leq k$. We say that a property holds *almost everywhere* on A or *for almost every element* in A if it holds away from a definable subset $A' \subseteq A$ of a smaller RV-dimension. This terminology will also be used when other notions of dimension are involved.

Lemma 3.3. Let $U \subseteq \text{RV}^n$ be a definable subset and $\text{vrv}(U) = D$. Then there is a definable finite partition D_i of D such that each $U_i = U \cap \text{vrv}^{-1}(D_i)$ is a definable twistoid and the corresponding twistback is \mathcal{L}_{RV} -definable.

Proof. Let $\pi : U \rightarrow \Gamma_\infty^l$ be a Γ -partition. Without loss of generality, we may assume $\pi(U) \subseteq \Gamma^l$. By Corollary 2.23 and compactness, there is an \mathcal{L}_{RV} -definable subset $B \subseteq RV^{n+l}$ such that, for every $\vec{\gamma} \in \Gamma^l$, $\pi^{-1}(\vec{\gamma}) \times \{\overline{\text{csn}}(\vec{\gamma})\}$ is precisely the fiber of B over $\overline{\text{csn}}(\vec{\gamma})$. Applying Lemma 2.26 to B , we find an \mathcal{L}_{RV} -definable finite partition I_i of $\text{vrv}(B)$ such that each $B_i = B \cap \text{vrv}^{-1}(I_i)$ is a twistoid with an \mathcal{L}_{RV} -definable $V_i \subseteq \overline{K}^{n+l}$ as its twistback. For each $\vec{\gamma} \in D$ let

$$E_{\vec{\gamma}} = \{i : (\{\vec{\gamma}\} \times \pi(U_{\vec{\gamma}})) \cap I_i \neq \emptyset\}.$$

Let D_k be the definable finite partition of D determined by the condition that $\vec{\gamma}, \vec{\gamma}'$ are in the same piece if and only if $E_{\vec{\gamma}} = E_{\vec{\gamma}'}$. Write E_k for any $E_{\vec{\gamma}}$ with $\vec{\gamma} \in D_k$. Observe that, for any $\vec{\gamma} \in D_k$, $\text{tbk}(U_{\vec{\gamma}}) = \bigcup_{i \in E_k} \text{fib}(V_i, \vec{1})$, where $\vec{1} \in \overline{K}^l$. The lemma follows. \square

The conclusion of this lemma shall be referred to as the *twistoid condition*. This is a condition that should be imposed on a more general \mathbb{T} (see Hypothesis 3.15). This will not interfere with the possibility of adding more structure to the residue field that expands the theory of algebraically closed fields.

Corollary 3.4. *If $U \subseteq RV^n$ is a definable subset such that $\text{vrv}(U)$ is a singleton then U is \mathcal{L}_{RV} -definable.*

Therefore, for any $A \subseteq RV^n$, $\text{vrv} \upharpoonright A$ is a Γ -partition of A . This implies that $\dim_{RV}(A) = k$ if and only if $\dim_{RV}(\text{tbk}(A_{\vec{\gamma}})) = k$ for some $\vec{\gamma}$.

Corollary 3.5. *Let $\vec{\gamma} \in \Gamma^n$, $A \subseteq \text{vrv}^{-1}(\vec{\gamma})$, and $f : A \rightarrow \Gamma^m$ be a definable function. Then $f(A)$ is finite.*

Proof. This is immediate by applying Lemma 3.3 to the subset $\bigcup_{\vec{\alpha} \in \Gamma^m} f^{-1}(\vec{\alpha}) \times \{\overline{\text{csn}}(\vec{\alpha})\}$. \square

Corollary 3.6. *Let $A \subseteq RV^n$, $B \subseteq RV^m$, and $F \subseteq A \times B$ be a definable finite-to-finite correspondence. Then $\text{vrv}(F)$ is a finite-to-finite correspondence between $\text{vrv}(A)$ and $\text{vrv}(B)$.*

Definition 3.7. A nonempty definable subset $U \subseteq RV^n$ is Γ -regular if $\dim_{RV}(U_{\vec{\gamma}}) = n$ for all $\vec{\gamma} \in \text{vrv}(U)$.

Note that if $U \subseteq RV^n$ is Γ -regular then we actually have $U \subseteq (RV^\times)^n$. By convention, $U \subseteq \overline{K}^n$ is Γ -regular if $U \cap (\overline{K}^\times)^n$ is Γ -regular.

Lemma 3.8. *Let $U \subseteq RV^n$ be Γ -regular and $\text{vrv}(U) = D$. Then $\dim_{RV}(\text{vrv}^{-1}(D) \setminus U) < n$.*

Proof. This is immediate by Corollary 3.4 and [19, Lemma 4.10]. \square

Lemma 3.9. *Let $D, E \subseteq \Gamma_\infty^n$. Let $W \subseteq \overline{K}^k$ be Γ -regular and $f : W \times D \rightarrow W \times E$ be a definable bijection. Then there are definable subsets $A \subseteq W \times D$, $B \subseteq W \times E$ and a definable bijection $e : D \rightarrow E$ such that, for all $\vec{\alpha} \in D$, $f(\text{fib}(A, \vec{\alpha})) = \text{fib}(B, e(\vec{\alpha}))$ and $\text{fib}(A, \vec{\alpha}), \text{fib}(B, e(\vec{\alpha}))$ are Γ -regular.*

Proof. By Corollary 3.4, Corollary 3.5, and Lemma 3.8, for each $\vec{\alpha} \in D$ there is a unique $\vec{\alpha}$ -definable $e(\vec{\alpha}) \in E$ such that $A_{\vec{\alpha}} = (W \times \{\vec{\alpha}\}) \cap f^{-1}(W \times \{e(\vec{\alpha})\})$ is $\overline{\text{csn}}(\vec{\alpha})$ - \mathcal{L}_{RV} -definable and $\dim_{RV}((W \times \{\vec{\alpha}\}) \setminus A_{\vec{\alpha}}) < k$. Symmetrically this also holds for each $\vec{\beta} \in E$. Now the assertion simply follows from compactness. \square

Since the Γ -sort is o -minimal, we can use the dimension theory of o -minimal structures. We shall call it Γ -dimension and denote the operator by \dim_Γ .

Definition 3.10 ($\overline{\Gamma}$ -categories). The objects of the category $\overline{\Gamma}[k]$ are the definable subsets with coordinates in Γ_∞ of Γ -dimension k . Its morphisms are the definable bijections between the objects. Set $\overline{\Gamma}_* = \bigcup_k \overline{\Gamma}[k]$.

Definition 3.11 (\overline{RV} - and \overline{RES} -categories). The objects of the category $\overline{RV}[k]$ are the definable subsets with coordinates in RV of RV -dimension k . Its morphisms are the definable bijections between the objects.

The category $\overline{RES}[k]$ is the full subcategory of $\overline{RV}[k]$ such that $U \in \overline{RES}[k]$ if and only if all coordinates of U are in \overline{K} . Set $\overline{RV}_* = \bigcup_k \overline{RV}[k]$; similarly for \overline{RES}_* .

By the computation in [14], $\mathbb{Z}[X]/(X^2 + X) = \mathbf{K}\overline{\Gamma}_*$ via the map $X \mapsto [(0, \infty)]$, which is much simpler than $\mathbf{K}_+\overline{\Gamma}_*$. On the other hand, it is well known that $\mathbf{K}\overline{RES}_*$ is still quite complicated (see [15, Example 3.7]). Anyway, following the philosophy of [10], we shall work with Grothendieck semirings whenever possible.

We clearly have $\mathbf{K}_+\overline{\Gamma}[0] = \mathbb{N}$. By Corollary 3.4, $U \in \overline{RES}[0]$ if and only if U is finite and hence $\mathbf{K}_+\overline{RES}[0]$ contains \mathbb{N} as a proper sub-semiring.

Each $\mathbf{K}_+ \overline{\Gamma}[k]$ is identified canonically with a sub-semigroup of $\mathbf{K}_+ \overline{\Gamma}_*$. These sub-semigroups satisfy the conditions:

$$\begin{cases} \mathbf{K}_+ \overline{\Gamma}[k] \cap \mathbf{K}_+ \overline{\Gamma}[l] = \{0\}, & \text{if } k \neq l, \\ \mathbf{K}_+ \overline{\Gamma}[k] + (\mathbf{K}_+ \overline{\Gamma}[l] \setminus \{0\}) = \mathbf{K}_+ \overline{\Gamma}[l], & \text{if } k \leq l. \end{cases}$$

In this situation, we may think of $\mathbf{K}_+ \overline{\Gamma}_*$ as a disjoint union $\bigsqcup_k \mathbf{K}_+ \overline{\Gamma}[k]$; similarly $\mathbf{K}_+ \overline{\text{RES}}_* = \bigsqcup_k \mathbf{K}_+ \overline{\text{RES}}[k]$ and $\mathbf{K}_+ \overline{\text{RV}}_* = \bigsqcup_k \mathbf{K}_+ \overline{\text{RV}}[k]$. Note that $\overline{\Gamma}_*$ is equivalent via the reduced cross-section to a full subcategory of $\overline{\text{RV}}[0]$ and hence $\mathbf{K}_+ \overline{\Gamma}_*$ may be canonically identified with a sub-semiring of $\mathbf{K}_+ \overline{\text{RV}}[0]$.

For $U \in \overline{\text{RV}}[k]$ and $I \in \overline{\Gamma}_*$ we write $U \times_{\overline{\text{csn}}} I$ for the object $U \times \overline{\text{csn}}(I) \in \overline{\text{RV}}[k]$. The map from $\mathbf{K}_+ \overline{\text{RES}}_* \times \mathbf{K}_+ \overline{\Gamma}_*$ to $\mathbf{K}_+ \overline{\text{RV}}_*$ naturally determined by the assignment $([U], [I]) \mapsto [U \times_{\overline{\text{csn}}} I]$ is clearly \mathbb{N} -bilinear. Hence it induces a semiring homomorphism:

$$\overline{\mathbb{D}} : \mathbf{K}_+ \overline{\text{RES}}_* \otimes \mathbf{K}_+ \overline{\Gamma}_* \longrightarrow \mathbf{K}_+ \overline{\text{RV}}_*.$$

Lemma 3.12. $\overline{\mathbb{D}}$ is a semiring isomorphism.

Proof. Surjectivity of $\overline{\mathbb{D}}$ follows immediately from Lemma 3.3. For injectivity, let $U_i, V_j \in \overline{\text{RES}}_*$, $I_i, J_j \in \overline{\Gamma}_*$, and

$$f : A = \bigsqcup_i U_i \times_{\overline{\text{csn}}} I_i \longrightarrow B = \bigsqcup_j V_j \times_{\overline{\text{csn}}} J_j$$

be a definable bijection. We need to show that $\sum_i [U_i] \otimes [I_i] = \sum_j [V_j] \otimes [J_j]$. Set

$$C = \{(\vec{t}, \vec{s}, \vec{\alpha}, \vec{\beta}) : f(\vec{t}, \overline{\text{csn}}(\vec{\alpha})) = (\vec{s}, \overline{\text{csn}}(\vec{\beta}))\}$$

and $W = \text{prv}(C)$. By Lemma 2.30, there is a definable finite partition W_k of W such that

$$\text{fib}(C, (\vec{t}, \vec{s})) = \text{fib}(C, (\vec{t}', \vec{s}')) \quad \text{for all } (\vec{t}, \vec{s}), (\vec{t}', \vec{s}') \in W_k.$$

Since f is a bijection, clearly each $\text{fib}(C, (\vec{t}, \vec{s}))$ is the graph of a bijection and, in each W_k , $\vec{t} = \vec{t}'$ if and only if $\vec{s} = \vec{s}'$. So W_k is also the graph of a bijection. Actually, we can form the disjoint unions A, B in such a way (say, by tagging on both sides of the products) that $W_k \subseteq U_i \times V_j$ for some i, j . The desired equality follows easily from these conditions. \square

Corollary 3.13. For any $[U] \in \mathbf{K}_+ \overline{\text{RES}}_*$, if U is Γ -regular then the semigroup homomorphism

$$[U] \otimes - : \mathbf{K}_+ \overline{\Gamma}_* \longrightarrow \mathbf{K}_+ \overline{\text{RES}}_* \otimes \mathbf{K}_+ \overline{\Gamma}_*$$

is injective.

Proof. Suppose that $[U] \otimes [I] = [U] \otimes [J]$. By Lemma 3.12, $U \times I$ is definably bijective to $U \times J$. By Lemma 3.9, $[I] = [J]$. \square

Let $U \subseteq \text{RV}^n \times \Gamma^m$, $V \subseteq \text{RV}^{n'} \times \Gamma^{m'}$, and $C \subseteq U \times V$ be definable subsets. For all $((\vec{u}, \vec{\alpha}), (\vec{v}, \vec{\beta})) \in C$, the Γ -Jacobian of C at $((\vec{u}, \vec{\alpha}), (\vec{v}, \vec{\beta}))$, written as $\text{Jcb}_\Gamma C((\vec{u}, \vec{\alpha}), (\vec{v}, \vec{\beta}))$, is the element

$$-\Sigma(\text{vrv}(\vec{u}), \vec{\alpha}) + \Sigma(\text{vrv}(\vec{v}), \vec{\beta}) \in \Gamma,$$

where $\Sigma(\gamma_1, \dots, \gamma_n) = \gamma_1 + \dots + \gamma_n$. If $U, V \subseteq \overline{\mathbb{K}}^n$, $\dim_{\text{RV}}(U) = \dim_{\text{RV}}(V) = n$, and $C \subseteq U \times V$ is a finite-to-finite correspondence then, for almost all $(\vec{u}, \vec{v}) \in C$, the Jacobian at (\vec{u}, \vec{v}) may be defined in the natural way (see the discussion preceding [19, Definition 9.14]), which is a (\vec{u}, \vec{v}) -definable element in $\overline{\mathbb{K}}^\times$ and is denoted by $\text{Jcb}_{\overline{\mathbb{K}}} C(\vec{u}, \vec{v})$. More generally, if $U, V \subseteq (\text{RV}^\times)^n$, then, for any $\vec{\alpha}, \vec{\beta} \in \Gamma^n$, we may consider the $(\vec{\alpha}, \vec{\beta})$ -twistback $\text{tbk}(C_{\vec{\alpha}, \vec{\beta}})$ of C :

Definition 3.14. The Jacobian $\text{Jcb}_{\text{RV}} C(\vec{u}, \vec{v}) = (\text{Jcb}_{\overline{\mathbb{K}}} C(\vec{u}, \vec{v}), \text{Jcb}_\Gamma C(\vec{u}, \vec{v}))$ of C at (\vec{u}, \vec{v}) is a (\vec{u}, \vec{v}) -definable pair in $\overline{\mathbb{K}}^\times \times \Gamma^\times$, where $\text{Jcb}_{\overline{\mathbb{K}}} C(\vec{u}, \vec{v})$, if it exists, is given by

$$\text{Jcb}_{\overline{\mathbb{K}}} \text{tbk}(C_{\text{vrv}(\vec{u}), \text{vrv}(\vec{v})})(\text{tbk}(\vec{u}), \text{tbk}(\vec{v})).$$

It is routine to check that the Jacobian is defined for almost all $(\vec{u}, \vec{v}) \in C$.

Hypothesis 3.15. Here we can provide a bit more information than at the beginning of this section on what conditions a more general \mathbb{T} should satisfy in order to make the construction work. The twistoid condition should hold. The Γ -sort should be o -minimal. There should be a notion of RV-dimension that agrees with the Zariski dimension, that is, if $U \subseteq \overline{\mathbb{K}}^n$ is an $\mathcal{L}_\mathbb{T}$ -definable subset then its RV-dimension equals the Zariski dimension of its Zariski closure. Consequently, the Jacobian in the RV-sort may be defined as in Definition 3.14.

Definition 3.16 (Coarse RV-categories). An object of the category $\text{RV}[k, \cdot]$ is a definable pair (U, f) , where $U \in \overline{\text{RV}}_*$ and $f : U \rightarrow (\text{RV}^\times)^k$ is a function. Given two such objects (U, f) and (V, g) , any definable bijection $F : U \rightarrow V$ is a *morphism* of $\text{RV}[k, \cdot]$. Such a morphism F induces a correspondence between $f(U)$ and $g(V)$:

$$\{(\vec{t}, \vec{s}) \in f(U) \times g(V) : \exists \vec{u} \in U (f(\vec{u}) = \vec{t} \wedge (g \circ F)(\vec{u}) = \vec{s})\},$$

which is denoted by F^\rightleftharpoons .

An object of the category $\mu_\Gamma \text{RV}[k]$ is a definable triple (U, f, ω_Γ) , where $(U, f) \in \text{RV}[k, \cdot]$ and $\omega_\Gamma : U \rightarrow \Gamma$ is a function, which is understood as a Γ -volume form on U . A *morphism* $F : (U, f, \omega_\Gamma) \rightarrow (U', f', \omega'_\Gamma)$ of $\mu_\Gamma \text{RV}[k]$ is an $\text{RV}[k, \cdot]$ -morphism such that, for all $(f(\vec{u}), (f' \circ F)(\vec{u})) \in F^\rightleftharpoons$,

$$\omega_\Gamma(\vec{u}) = \omega'_\Gamma(F(\vec{u})) + \text{Jcb}_\Gamma F^\rightleftharpoons(f(\vec{u}), (f' \circ F)(\vec{u})).$$

Set $\text{RV}[\leq k, \cdot] = \coprod_{i \leq k} \text{RV}[i, \cdot]$ and $\text{RV}[* , \cdot] = \coprod_k \text{RV}[k, \cdot]$; similarly for $\mu_\Gamma \text{RV}[\leq k]$ and $\mu_\Gamma \text{RV}[*]$.

Remark 3.17. The categories $\text{RV}[k, \cdot]$ only play an auxiliary role in the construction and could have been defined in a simpler way, that is, the function f may be deleted from (U, f) without any real consequences. We have chosen to define them in this way so to make other definitions below more compact. In those definitions the “presentation” f of U is indeed essential.

Note that, in the above definition and other similar ones below, all morphisms are actually isomorphisms. Also, for the cases $k = 0$, the reader should interpret things such as $(\text{RV}^\times)^0$ and how they interact with other things in a natural way. For example, $(\text{RV}^\times)^0$ may be treated as the empty tuple. This results in the interpretation that the requirement above on Γ -volume forms for $k = 0$ is simply $\omega_\Gamma(\vec{u}) = \omega'_\Gamma(F(\vec{u}))$.

About the notation: $\overline{\Gamma}_*$, etc., suggests that the category is filtrated and the notation $\text{RV}[* , \cdot]$, etc., suggests that the category is actually graded.

Definition 3.18 (Fine RV-categories). The category $\text{RV}[k]$ is the full subcategory of $\text{RV}[k, \cdot]$ such that $(U, f) \in \text{RV}[k]$ if and only if $\dim_{\text{RV}}(f^{-1}(\vec{t})) = 0$ for all $\vec{t} \in (\text{RV}^\times)^k$. Note that, for any $\text{RV}[k]$ -morphism $F : (U, f) \rightarrow (V, g)$, by Corollary 3.4, the correspondence $F^\rightleftharpoons_{\vec{\gamma}}$ is finite-to-finite for all $\vec{\gamma} \in \Gamma^{2k}$.

An object of the category $\mu \text{RV}[k]$ is a definable triple (U, f, ω) , where $(U, f) \in \text{RV}[k]$ and $\omega : U \rightarrow \overline{\mathbb{K}}^\times \times \Gamma$ is a function, which is understood as a *volume form* on U . We also write ω as a pair $(\omega_{\overline{\mathbb{K}}}, \omega_\Gamma)$. A $\mu_\Gamma \text{RV}[k]$ -morphism $F : (U, f, \omega) \rightarrow (U', f', \omega')$ is a *pseudo-morphism* of $\mu \text{RV}[k]$. If, in addition, for all $(f(\vec{u}), (f' \circ F)(\vec{u})) \in F^\rightleftharpoons$ that are away from a subset of F^\rightleftharpoons of RV-dimension $< k$,

$$\omega_{\overline{\mathbb{K}}}(\vec{u}) = \omega'_{\overline{\mathbb{K}}}(F(\vec{u})) \text{Jcb}_{\overline{\mathbb{K}}} F^\rightleftharpoons(f(\vec{u}), (f' \circ F)(\vec{u}))$$

then F is a *morphism* of $\mu \text{RV}[k]$.

Set $\text{RV}[*] = \coprod_k \text{RV}[k]$; similarly for $\mu \text{RV}[*]$.

Observe that if $(U, f, \omega), (V, g, \sigma) \in \mu \text{RV}[k]$ and $\dim_{\text{RV}}(f(U)) < k$ (in particular, if $\dim_{\text{RV}}(U) < k$) then any $\mu \text{RV}[k]$ -pseudo-morphism between them is indeed a $\mu \text{RV}[k]$ -morphism.

Definition 3.19 (RES-categories). The category $\text{RES}[k, \cdot]$ is the full subcategory of $\text{RV}[k, \cdot]$ such that $(U, f) \in \text{RES}[k, \cdot]$ if and only if all coordinates of U and $f(U)$ are in $\overline{\mathbb{K}}$. Similarly, $\text{RES}[k]$ is such a full subcategory of $\text{RV}[k]$, which is also a full subcategory of $\text{RES}[k, \cdot]$. The category $\mu \text{RES}[k]$ is the full subcategory of $\mu \text{RV}[k]$ such that $(U, f, \omega) \in \mu \text{RES}[k]$ if and only if $(U, f) \in \text{RES}[k]$ and $\omega_\Gamma = 0$.

The category $\text{RES}^c[k]$ (resp. $\mu \text{RES}^c[k]$) is the smallest full subcategory of $\text{RES}[k]$ (resp. $\mu \text{RES}[k]$) that contains the isomorphism class of $\mathbf{T}^k = ((\overline{\mathbb{K}}^\times)^k, \text{id})$ (resp. $\mathbf{T}^k_\mu = ((\overline{\mathbb{K}}^\times)^k, \text{id}, (1, 0))$) and is closed under disjoint union.

Set $\text{RES}[* , \cdot] = \coprod_k \text{RES}[k, \cdot]$; similarly for $\text{RES}[*]$, $\mu \text{RES}[*]$, $\text{RES}^c[*]$, and $\mu \text{RES}^c[*]$.

We do not have RES-categories with Γ -volume forms because, in light of Corollary 3.5, there will be no need to. Also note that $\text{RES}[k, \cdot]$ is canonically isomorphic to a full subcategory of $\mu_\Gamma \text{RV}[k]$ via the map $(U, f) \mapsto (U, f, 0)$ and $\text{RES}^c[k]$ is a full subcategory of $\text{RES}[k, \cdot]$. Also, by Corollary 3.4, every $(U, f) \in \text{RES}[k, \cdot]$ is \mathcal{L}_{RV} -definable and hence $(U, f) \in \text{RES}[k]$ if and only if f is finite-to-one.

Definition 3.20 (Γ -categories). The objects of the category $\Gamma[k]$ are the definable pairs (I, f) , where $I \in \overline{\Gamma}_*$ and $f : I \rightarrow \Gamma^k$ is a function. Given $(I, f), (J, g) \in \Gamma[k]$, any definable bijection $F : I \rightarrow J$ is a *morphism* of $\Gamma[k]$.

An object of the category $\mu \Gamma[k]$ is a definable triple (I, f, ω) , where $(I, f) \in \Gamma[k]$ and $\omega : I \rightarrow \Gamma$ is a function, which is understood as a *volume form* on I . Let $\omega_f : I \rightarrow \Gamma$ be the function given by $\vec{\gamma} \mapsto \Sigma f(\vec{\gamma}) + \omega(\vec{\gamma})$. A *morphism* $F : (I, f, \omega) \rightarrow (I', f', \omega')$ of $\mu \Gamma[k]$ is a $\Gamma[k]$ -morphism such that $\omega_f(\vec{\gamma}) = \omega'_{f'}(F(\vec{\gamma}))$ for all $\vec{\gamma} \in I$.

The category $\Gamma^c[k]$ is the full subcategory of $\Gamma[k]$ such that $(I, f) \in \Gamma^c[k]$ if and only if I is finite. The category $\mu\Gamma^c[k]$ is the full subcategory of $\mu\Gamma[k]$ such that $(I, f, \omega) \in \Gamma^c[k]$ if and only if I is finite and $\omega_f(\vec{\gamma}) = 0$ for all $\vec{\gamma} \in I$.

Set $\Gamma[*] = \coprod_k \Gamma[k]$; similarly for $\mu\Gamma[*]$, $\Gamma^c[*]$, and $\mu\Gamma^c[*]$.

Obviously $\mathbf{K}_+ \text{RV}[*] = \bigoplus_k \mathbf{K}_+ \text{RV}[k]$; similarly for the other graded categories.

Note that the semigroups $\mathbf{K}_+ \text{RES}^c[k]$, $\mathbf{K}_+ \mu\text{RES}^c[k]$, $\mathbf{K}_+ \Gamma^c[k]$, and $\mathbf{K}_+ \mu\Gamma^c[k]$ may be identified with $(\mathbb{N}, +)$ and hence the semirings $\mathbf{K}_+ \text{RES}^c[*]$, $\mathbf{K}_+ \mu\text{RES}^c[*]$, $\mathbf{K}_+ \Gamma^c[*]$, and $\mathbf{K}_+ \mu\Gamma^c[*]$ may be identified with $\mathbb{N}[X]$, the semiring of polynomials with coefficients in \mathbb{N} . Let us abbreviate

$$\mathbf{K}_+ \text{RES}[k, \cdot] \otimes \mathbf{K}_+ \Gamma[k], \quad \mathbf{K}_+ \text{RES}[k, \cdot] \otimes \mathbf{K}_+ \mu\Gamma[k], \quad \mathbf{K}_+ \mu\text{RES}[k] \otimes \mathbf{K}_+ \mu\Gamma[k]$$

as $\mathbf{K}_+ \text{VTP}[k]$, $\mathbf{K}_+ \mu_\Gamma \text{VTP}[k]$, $\mathbf{K}_+ \mu\text{VTP}[k]$, respectively. Note that both $\mathbf{K}_+ \text{VTP}[k]$ and $\mathbf{K}_+ \mu_\Gamma \text{VTP}[k]$ use $\mathbf{K}_+ \text{RES}[k, \cdot]$ as the first factor. Set $\mathbf{K}_+ \text{VTP}[*] = \bigoplus_k \mathbf{K}_+ \text{VTP}[k]$, similarly for $\mathbf{K}_+ \mu_\Gamma \text{VTP}[*]$ and $\mathbf{K}_+ \mu\text{VTP}[*]$. These are graded semirings.

For $(U, f) \in \text{RES}[k, \cdot]$ and $(I, g) \in \Gamma[k]$, let $f \times_{\overline{\text{csn}}} g : U \times_{\overline{\text{csn}}} I \rightarrow (\text{RV}^\times)^k$ be the function given by

$$(\vec{t}, \overline{\text{csn}}(\vec{\gamma})) \mapsto (f(\vec{t})_i \overline{\text{csn}}(g(\vec{\gamma}))_i).$$

We write $(U, f) \times_{\overline{\text{csn}}} (I, g)$ for the object

$$(U \times_{\overline{\text{csn}}} I, f \times_{\overline{\text{csn}}} g) \in \text{RV}[k, \cdot].$$

Note that if $(U, f) \in \text{RES}[k]$ then $(U, f) \times_{\overline{\text{csn}}} (I, g) \in \text{RV}[k]$. For $(I, g, \sigma) \in \mu\Gamma[k]$, let $(U, f) \times_{\overline{\text{csn}}} (I, g, \sigma)$ be the object

$$((U, f) \times_{\overline{\text{csn}}} (I, g), \sigma_\Gamma) \in \mu_\Gamma \text{RV}[k],$$

where σ_Γ is the volume form on $U \times_{\overline{\text{csn}}} I$ given by $(\vec{t}, \overline{\text{csn}}(\vec{\gamma})) \mapsto \sigma(\vec{\gamma})$. Finally, for $(U, f, \omega) \in \mu\text{RES}[k]$, let $(U, f, \omega) \times_{\overline{\text{csn}}} (I, g, \sigma)$ be the object

$$((U, f) \times_{\overline{\text{csn}}} (I, g), \omega \times_{\overline{\text{csn}}} \sigma) \in \mu\text{RV}[k],$$

where $\omega \times_{\overline{\text{csn}}} \sigma$ is the volume form on $U \times_{\overline{\text{csn}}} I$ given by $(\vec{t}, \overline{\text{csn}}(\vec{\gamma})) \mapsto (\omega(\vec{t}), \sigma(\vec{\gamma}))$.

The assignment $([U], [I]) \mapsto [U \times_{\overline{\text{csn}}} I]$ naturally determines a map

$$\mathbf{K}_+ \text{RES}[k, \cdot] \times \mathbf{K}_+ \Gamma[k] \rightarrow \mathbf{K}_+ \text{RV}[k, \cdot],$$

which is clearly \mathbb{N} -bilinear. Similarly there are such maps

$$\mathbf{K}_+ \text{RES}[k, \cdot] \times \mathbf{K}_+ \mu\Gamma[k] \rightarrow \mathbf{K}_+ \mu_\Gamma \text{RV}[k] \quad \text{and} \quad \mathbf{K}_+ \mu\text{RES}[k] \times \mathbf{K}_+ \mu\Gamma[k] \rightarrow \mathbf{K}_+ \mu\text{RV}[k].$$

Hence we have three induced semigroup homomorphisms:

$$\begin{aligned} \mathbb{D}_k : \mathbf{K}_+ \text{VTP}[k] &\rightarrow \mathbf{K}_+ \text{RV}[k, \cdot], & \mu_\Gamma \mathbb{D}_k : \mathbf{K}_+ \mu_\Gamma \text{VTP}[k] &\rightarrow \mathbf{K}_+ \mu_\Gamma \text{RV}[k], \\ \mu \mathbb{D}_k : \mathbf{K}_+ \mu\text{VTP}[k] &\rightarrow \mathbf{K}_+ \mu\text{RV}[k]. \end{aligned}$$

Proposition 3.21. $\mathbb{D}_k, \mu_\Gamma \mathbb{D}_k,$ and $\mu \mathbb{D}_k$ are isomorphisms.

Proof. Since $\mathbb{D}_0 = \mu_\Gamma \mathbb{D}_0 = \overline{\mathbb{D}}$ and $\mu \mathbb{D}_0$ is a restriction of $\overline{\mathbb{D}}$, let us assume $k > 0$. We shall only be concerned with $\mu \mathbb{D}_k$, since for \mathbb{D}_k or $\mu_\Gamma \mathbb{D}_k$ the argument is similar and simpler. In fact, the proof is more or less the same as that of Lemma 3.12 and hence we shall be brief.

For any $(U, f, \omega) \in \mu\text{RV}[k]$, by Corollary 3.5, there is a definable finite partition U_i of U such that the restrictions $(\text{vrv} \circ f) \upharpoonright U_i, \omega_\Gamma \upharpoonright U_i$ factor through $\text{vrv} \upharpoonright U_i$. So, without loss of generality, we may assume that U has this property. By Lemma 3.3, we may further assume that (the graphs of) f and $\omega_{\overline{\kappa}}$ are twistoids. Then it is clear that (U, f, ω) is isomorphic to a product in the desired form.

For injectivity, in a similar notation to that in the proof of Lemma 3.12, we are reduced to showing that the bijections coded in W_k are indeed $\mu\text{RES}[k]$ - and $\mu\Gamma[k]$ -morphisms. It is straightforward to check this. \square

Corollary 3.22. $\mathbb{D} = \bigoplus_k \mathbb{D}_k, \mu_\Gamma \mathbb{D} = \bigoplus_k \mu_\Gamma \mathbb{D}_k,$ and $\mu \mathbb{D} = \bigoplus_k \mu \mathbb{D}_k$ are isomorphisms of graded semirings.

Corollary 3.23. For any $(U, f) \in \mathbf{K}_+ \text{RES}[k, \cdot]$, if U is Γ -regular then the semigroup homomorphism $(U, f) \otimes - : \mathbf{K}_+ \Gamma[k] \rightarrow \mathbf{K}_+ \text{VTP}[k]$ is injective; similarly for the other two cases.

Proof. Since we have Proposition 3.21, bijections as described in Lemma 3.9 may be obtained as in the proof of Corollary 3.13, which are indeed morphisms of the corresponding categories. \square

For $(I, f) \in \Gamma[k]$ and $(I, f, \omega) \in \mu\Gamma[k]$, we define their *canonical lifting* into the corresponding RV-categories:

$$\mathbb{L}_\Gamma(I, f) = \mathbf{T}^k \times_{\text{c\overline{sn}}} (I, f), \quad \mu_\Gamma \mathbb{L}_\Gamma(I, f, \omega) = \mathbf{T}^k \times_{\text{c\overline{sn}}} (I, f, \omega), \quad \mu \mathbb{L}_\Gamma(I, f, \omega) = \mathbf{T}_\mu^k \times_{\text{c\overline{sn}}} (I, f, \omega).$$

Corollary 3.24. *These lifting maps induce canonical embeddings of graded semirings:*

$$\mathbb{L}_\Gamma : \mathbf{K}_+ \Gamma[*] \longrightarrow \mathbf{K}_+ \text{VTP}[*], \quad \mu_\Gamma \mathbb{L}_\Gamma : \mathbf{K}_+ \mu\Gamma[*] \longrightarrow \mathbf{K}_+ \mu_\Gamma \text{VTP}[*], \quad \mu \mathbb{L}_\Gamma : \mathbf{K}_+ \mu\Gamma[*] \longrightarrow \mathbf{K}_+ \mu \text{VTP}[*],$$

which yield the canonical identifications:

$$(\mathbb{D} \circ \mathbb{L}_\Gamma)(\mathbf{K}_+ \Gamma^c[*]) = (\mu_\Gamma \mathbb{D} \circ \mu_\Gamma \mathbb{L}_\Gamma)(\mathbf{K}_+ \mu\Gamma^c[*]) = \mathbf{K}_+ \text{RES}^c[*] \quad \text{and} \\ (\mu \mathbb{D} \circ \mu \mathbb{L}_\Gamma)(\mathbf{K}_+ \mu\Gamma^c[*]) = \mathbf{K}_+ \mu \text{RES}^c[*].$$

There is an alternative description of the semiring $\mathbf{K}_+ \mu\Gamma[*]$. For that, we introduce the following notation:

Notation 3.25. Let P be a subset of additional parameters. If \mathcal{C} is a category of P -definable subsets then we shall emphasize this by writing \mathcal{C}_P . Let A, B be two subsets. We write $[A] =_P [B]$ if A, B are isomorphic objects in \mathcal{C}_P .

Definition 3.26. A function $f : \Gamma \longrightarrow \mathbf{K}_+ \Gamma[k]$ is *definable* if there is a definable subset $I \subseteq \Gamma \times \Gamma^{m+k}$ such that, for all $\alpha \in \text{pr}_1(I)$, $\text{fib}(I, \alpha)$ encodes naturally a representative of $f(\alpha) \in \mathbf{K}_+ \Gamma[k]_\alpha$. The subset I is considered as a *representative* of f . With pointwise addition, such definable functions form a semigroup $\text{FN}(\Gamma, \mathbf{K}_+ \Gamma[k])$. Given another definable function $g : \Gamma \longrightarrow \mathbf{K}_+ \Gamma[l]$ with a representative J , it is routine to check that their convolution product is well defined as follows:

$$(f * g)(\gamma) = \left[\bigcup_{\alpha+\beta=\gamma} \text{fib}(I, \alpha) \times \text{fib}(J, \beta) \right] \in \mathbf{K}_+ \Gamma[k+l]_\gamma.$$

This makes $\text{FN}(\Gamma, \mathbf{K}_+ \Gamma[*]) = \bigoplus_k \text{FN}(\Gamma, \mathbf{K}_+ \Gamma[k])$ a graded semiring.

Lemma 3.27. *Each $\text{FN}(\Gamma, \mathbf{K}_+ \Gamma[k])$ is canonically isomorphic to $\mathbf{K}_+ \mu\Gamma[k]$ and hence $\text{FN}(\Gamma, \mathbf{K}_+ \Gamma[*])$ is canonically isomorphic to $\mathbf{K}_+ \mu\Gamma[*]$.*

Proof. For $(I, f, \omega) \in \mu\Gamma[k]$, set $\gamma \mapsto [(\omega_f^{-1}(\gamma), f \upharpoonright \omega_f^{-1}(\gamma))]$ for $\gamma \in \omega_f(I)$, which is a definable function in $\text{FN}(\Gamma, \mathbf{K}_+ \Gamma[k])$. Conversely, for any definable function $f : \Gamma \longrightarrow \mathbf{K}_+ \Gamma[k]$ with a representative I , let $U_I = I$, $f_I : U_I \longrightarrow \Gamma^k$ be the projection to the last k coordinates, and $\omega_I : U_I \longrightarrow \Gamma$ be the function given by $(\gamma, \vec{\alpha}, \vec{\beta}) \mapsto \gamma - \Sigma\vec{\beta}$. Then $(U_I, f_I, \omega_I) \in \mu\Gamma[k]$. It is routine to check that these maps induce isomorphisms as desired. \square

There are two Euler characteristics χ_g, χ_b that can be associated to the Γ -sort (see [8, §4.2], [14], and also [10, §9]). They are distinguished by $\chi_g((0, \infty)) = -1$ and $\chi_b((0, \infty)) = 0$. We shall denote both of them by χ if no distinction is needed. Using these and the groupifications of the results above, we can obtain various retractions to the Grothendieck rings of the RES-categories.

Lemma 3.28. *There are two homomorphisms $\mathcal{E}_g, \mathcal{E}_b : \mathbf{K}\Gamma[*] \longrightarrow \mathbb{Z}[\tau]$ and two homomorphisms $\mu\mathcal{E}_g, \mu\mathcal{E}_b : \mathbf{K}\mu\Gamma[*] \longrightarrow \mathbb{Z}[\tau]$ of graded rings.*

Proof. For $(I, f) \in \Gamma[k]$ and $(I, f, \omega) \in \mu\Gamma[k]$ we simply set $\mathcal{E}_k(I, f) = \chi(I)$ and $\mu\mathcal{E}_k(I, f, \omega) = \chi(I)$. Clearly these maps induce graded ring homomorphisms $\mathcal{E} = \bigoplus_k \mathcal{E}_k$ and $\mu\mathcal{E} = \bigoplus_k \mu\mathcal{E}_k$. \square

Notation 3.29. Let $\text{RV}^{>1} = \text{rv}(\mathcal{M})$ and $(\text{RV}^\times)^{>1} = \text{rv}(\mathcal{M} \setminus \{0\})$. We introduce the following shorthand for some elements of the Grothendieck semigroups and their groupifications (and closely related constructions):

$$[1]_0 = [\{1\}] \in \mathbf{K}_+ \text{RES}[0], \quad [1]_1 = [\{1, \text{id}\}] \in \mathbf{K}_+ \text{RES}[1], \quad [1_\mu]_1 = [\{1, \text{id}, \text{id}\}] \in \mathbf{K}_+ \mu\text{RES}[1], \\ [0]_0 = [\{0\}] \in \mathbf{K}_+ \Gamma[0], \quad [0]_1 = [\{0, \text{id}\}] \in \mathbf{K}_+ \Gamma[1], \quad [0_\mu]_1 = [\{0, \text{id}, \text{id}\}] \in \mathbf{K}_+ \mu\Gamma[1], \\ [\mathbf{H}]_1 = [\{(0, \infty), \text{id}\}] \in \mathbf{K}_+ \Gamma[1], \quad [\mathbf{H}_\mu]_1 = [\{(0, \infty), \text{id}, 0\}] \in \mathbf{K}_+ \mu\Gamma[1], \\ \mathbf{j} = [((\text{RV}^\times)^{>1}, \text{id})] - [1]_1 \in \mathbf{K}\text{RV}[1], \quad \mathbf{j}_{\mu\Gamma} = [((\text{RV}^\times)^{>1}, \text{id}, 0)] - [1]_1 \in \mathbf{K}\mu\Gamma\text{RV}[1],$$

$$\mathbf{j}_\mu = [((RV^\times)^{>1}, \text{id}, (1, 0))] - [1_\mu]_1 \in \mathbf{K}\mu\text{RV}[1],$$

$$\mathbf{A} = [\mathbf{T}^1] + [1]_1 \in \mathbf{K}\text{RES}[1], \quad \mathbf{A}_\mu = [\mathbf{T}_\mu^1] + [1_\mu]_1 \in \mathbf{K}\mu\text{RES}[1].$$

As in [10], the elements $[1]_0 + \mathbf{j} \in \mathbf{K}\text{RV}[*], \cdot]$, $\mathbf{j}_{\mu\Gamma} \in \mathbf{K}\mu_\Gamma\text{RV}[*]$, and $\mathbf{j}_\mu \in \mathbf{K}\mu\text{RV}[*]$ are instrumental in the discussions below.

Proposition 3.30. *There are two ring homomorphisms*

$$\mathbb{E}_g : \mathbf{K}\text{RV}[*], \cdot] \longrightarrow \mathbf{K}\text{RES}[*], \cdot][\mathbf{A}^{-1}] \quad \text{and} \quad \mathbb{E}_b : \mathbf{K}\text{RV}[*], \cdot] \longrightarrow \mathbf{K}\text{RES}[*], \cdot][[1]_1^{-1}]$$

such that

- (1) the ranges of $\mathbb{E}_g, \mathbb{E}_b$ are precisely the zeroth graded pieces of the targets,
- (2) $\mathbb{E}_g([1]_0 + \mathbf{j}) = \mathbb{E}_b([1]_0 + \mathbf{j}) = 0$,
- (3) for $x \in \mathbf{K}\text{RES}[k, \cdot]$, $\mathbb{E}_g(x) = x\mathbf{A}^{-k}$ and $\mathbb{E}_b(x) = x[1]_1^{-k}$.

With volume forms, we have two pairs of homomorphisms of graded rings:

$$\mu_\Gamma \mathbb{E}_g : \mathbf{K}\mu_\Gamma\text{RV}[*] \longrightarrow \mathbf{K}\text{RES}[*], \cdot]/(\mathbf{A}) \quad \text{and} \quad \mu_\Gamma \mathbb{E}_b : \mathbf{K}\mu_\Gamma\text{RV}[*] \longrightarrow \mathbf{K}\text{RES}[*], \cdot]/([1]_1),$$

$$\mu \mathbb{E}_g : \mathbf{K}\mu\text{RV}[*] \longrightarrow \mathbf{K}\mu\text{RES}[*]/(\mathbf{A}_\mu) \quad \text{and} \quad \mu \mathbb{E}_b : \mathbf{K}\mu\text{RV}[*] \longrightarrow \mathbf{K}\mu\text{RES}[*]/([1_\mu]_1)$$

such that their restrictions to $\mathbf{K}\text{RES}[*], \cdot]$, $\mathbf{K}\mu\text{RES}[*]$ are the natural projections and

$$\mu_\Gamma \mathbb{E}_g(\mathbf{j}_{\mu\Gamma}) = \mu_\Gamma \mathbb{E}_b(\mathbf{j}_{\mu\Gamma}) = 0, \quad \mu \mathbb{E}_g(\mathbf{j}_\mu) = \mu \mathbb{E}_b(\mathbf{j}_\mu) = 0.$$

Proof. For each n , let $E_{g,n} : \bigoplus_{i \leq n} \mathbf{KVTP}[i] \longrightarrow \mathbf{K}\text{RES}[n, \cdot]$ and $E_{b,n} : \bigoplus_{i \leq n} \mathbf{KVTP}[i] \longrightarrow \mathbf{K}\text{RES}[n, \cdot]$ be the surjective group homomorphisms given respectively by

$$x \otimes y \longmapsto \mathcal{E}_{g,k}(y)x\mathbf{A}^{n-k} \quad \text{and} \quad x \otimes y \longmapsto \mathcal{E}_{b,k}(y)x[1]_1^{n-k},$$

where $x \in \mathbf{K}\text{RES}[k, \cdot]$, $y \in \mathbf{K}\Gamma[k]$, and $\mathcal{E}_{g,k}, \mathcal{E}_{b,k}$ are defined with respect to χ_g, χ_b as in Lemma 3.28. For each $n > 0$, we have

$$(E_{g,n} \circ \mathbb{D}^{-1})([1]_0 + \mathbf{j}) = E_{g,n}([1]_0 \otimes [0]_0 + [\mathbf{T}^1] \otimes [\mathbf{H}]_1 - [1]_1 \otimes [0]_1) = \mathbf{A}^n - [\mathbf{T}^1]\mathbf{A}^{n-1} - [1]_1\mathbf{A}^{n-1} = 0,$$

$$(E_{b,n} \circ \mathbb{D}^{-1})([1]_0 + \mathbf{j}) = E_{b,n}([1]_0 \otimes [0]_0 + [\mathbf{T}^1] \otimes [\mathbf{H}]_1 - [1]_1 \otimes [0]_1) = [1]_1^n + 0 - [1]_1[1]_1^{n-1} = 0.$$

The group homomorphisms $g_n, b_n : \mathbf{K}\text{RES}[n, \cdot] \longrightarrow \mathbf{K}\text{RES}[n+1, \cdot]$ given respectively by $x \longmapsto x\mathbf{A}$, $x \longmapsto x[1]_1$ determine two colimit systems and the group homomorphisms $\mathbb{E}_{g,n} = E_{g,n} \circ \mathbb{D}^{-1}$, $\mathbb{E}_{b,n} = E_{b,n} \circ \mathbb{D}^{-1}$ determine two homomorphisms of colimit systems. Hence we have two surjective ring homomorphisms:

$$\text{colim}_n \mathbb{E}_{g,n} : \mathbf{K}\text{RV}[*], \cdot] \longrightarrow \text{colim}_{g_n} \mathbf{K}\text{RES}[n, \cdot] \quad \text{and} \quad \text{colim}_n \mathbb{E}_{b,n} : \mathbf{K}\text{RV}[*], \cdot] \longrightarrow \text{colim}_{b_n} \mathbf{K}\text{RES}[n, \cdot].$$

These yield the desired homomorphisms since the two colimits are respectively isomorphic to the zeroth graded pieces of $\mathbf{K}\text{RES}[*], \cdot][\mathbf{A}^{-1}]$ and $\mathbf{K}\text{RES}[*], \cdot][[1]_1^{-1}]$.

The cases with volume forms are not very different and are left to the reader. \square

Note that these homomorphisms are slightly different from the ones constructed in [10, Theorem 10.5, Theorem 10.11].

4. The Grothendieck semirings of VF and special bijections

Let $A \subseteq \text{VF}^n \times \text{RV}^m$ be a definable subset. Recall that if A equals its RV-hull $\text{RVH}(A)$ (see [18, Definition 4.21]) then A is an RV-pullback. An rv-polydisc $\mathfrak{p} \subseteq \text{VF}^n \times \text{RV}^m$ is degenerate if $\dim_{\text{VF}}(\mathfrak{p}) < n$. This happens if and only if some VF-coordinate of \mathfrak{p} is 0, if there is one at all. An RV-pullback is degenerate if it contains a degenerate rv-polydisc and is strictly degenerate if it only contains degenerate rv-polydiscs.

Let ψ be a quantifier-free formula that defines A . By inspection of the complexity of the occurring VF-terms of ψ , we see that there is a definable function $\pi : A \longrightarrow \text{RV}^l$ and an \mathcal{L}_{RV} -formula ϕ such that each $\pi^{-1}(\vec{t})$ is contained in an rv-polydisc and is defined by the formula $\phi(\text{sn}(\vec{t}))$. Hence the following definition is not empty.

Definition 4.1. An RV-partition of A is a definable function $\pi : A \longrightarrow \text{RV}^l$ such that, for every $\vec{t} \in \text{ran}(\pi)$, the fiber $\pi^{-1}(\vec{t})$ is $\text{sn}(\vec{t})$ - \mathcal{L}_{RV} -definable. We do not explicitly require that $\pi^{-1}(\vec{t})$ is contained in an rv-polydisc, but this can always be achieved if needed.

Similarly, by syntactical inspection, the twistoid condition (see Hypothesis 3.15), and compactness, the following specialization of the above definition is not empty either.

Definition 4.2. If A is $\mathcal{L}_{\tilde{\Gamma}}$ -definable then a Γ -partition of A is an $\mathcal{L}_{\tilde{\Gamma}}$ -definable function $\pi : A \rightarrow \Gamma_{\infty}^I$ such that each fiber $\pi^{-1}(\vec{\gamma})$ is contained in a coset of $(\text{rv}^{-1}(\vec{K}^{\times}))^n \times (\vec{K}^{\times})^m$ and is $\text{csn}(\vec{\gamma})$ - \mathcal{L}_{RV} -definable.

Note that if A has no VF-coordinates or is an RV-pullback then it is $\mathcal{L}_{\tilde{\Gamma}}$ -definable and hence admits a Γ -partition.

If A is \mathcal{L}_{RV} -definable then we may speak of the VF-dimension of A (see [19, Definition 4.1]). We may extend this notion of dimension to RV-partitions, which is parallel to how RV-dimension is extended to Γ -partitions above:

Definition 4.3. Let π be an RV-partition of A . The VF-dimension of π , denoted by $\dim_{\text{VF}}(\pi)$, is the number $\max\{\dim_{\text{VF}}(\pi^{-1}(\vec{t})) : \vec{t} \in \text{ran}(\pi)\}$.

Let $B \subseteq \text{VF}$ be an arbitrary subset. For any $(\vec{a}, \vec{t}) \in A$ let $\text{trdeg}_B(\vec{a}, \vec{t})$ be the transcendental degree of $B^{\text{ac}}(\vec{a})$ over B^{ac} (see Notation 2.10). Let $\text{trdeg}_B(A) = \max\{\text{trdeg}_B(\vec{a}, \vec{t}) : (\vec{a}, \vec{t}) \in A\}$. If $B = \emptyset$ then we omit it from the expression.

Lemma 4.4. Suppose that A is \mathcal{L}_{RV} -definable. Then $\text{trdeg}(A) = \text{trdeg}_{\text{sn}(\text{RV})}(A)$ and consequently $\dim_{\text{VF}}(A) = \dim_{\text{VF}}(\pi)$ for any RV-partition π of A .

Proof. By [19, Corollary 5.6], A is \mathcal{L}_{RV} -definably bijective to an RV-pullback A' . By [19, Lemma 3.3], we have $\text{trdeg}(A) = \text{trdeg}(A')$ and $\text{trdeg}_{\text{sn}(\text{RV})}(A) = \text{trdeg}_{\text{sn}(\text{RV})}(A')$. So, for the first equality, it is enough to show that $\text{trdeg}(A') = \text{trdeg}_{\text{sn}(\text{RV})}(A')$. By [19, Lemma 4.4, Lemma 4.6], A' contains an rv-polydisc $\{(0, \dots, 0)\} \times \text{rv}^{-1}(\vec{t}) \times \{\vec{s}\}$, where $\vec{t} \in (\text{RV}^{\times})^k$ and $\text{trdeg}(A') = k$. By Remark 2.31, it is easy to see that there is an $\vec{a} \in \text{rv}^{-1}(\vec{t})$ that is algebraically independent over $\text{VF}(\text{acl}^1(\text{RV}))$. Hence $\text{trdeg}_{\text{sn}(\text{RV})}(A') = k$.

Now, let π be an RV-partition of A . We have $\text{trdeg}_{\text{sn}(\vec{t})}(\pi^{-1}(\vec{t})) \leq \text{trdeg}(A)$ for every $\vec{t} \in \text{ran}(\pi)$ and, by the first equality, $\text{trdeg}_{\text{sn}(\vec{t})}(\pi^{-1}(\vec{t})) = \text{trdeg}(A)$ for some $\vec{t} \in \text{ran}(\pi)$. Hence the second equality follows from [19, Lemma 4.4]. \square

Lemma 4.5. For any two RV-partitions π_1, π_2 of A , we have $\dim_{\text{VF}}(\pi_1) = \dim_{\text{VF}}(\pi_2)$.

Proof. Let π be the RV-partition of A given by $\vec{x} \mapsto (\pi_1(\vec{x}), \pi_2(\vec{x}))$. For every $\vec{t} \in \text{ran}(\pi_1)$, since $\pi_1^{-1}(\vec{t})$ is $\text{sn}(\vec{t})$ - \mathcal{L}_{RV} -definable, by Lemma 4.4, $\dim_{\text{VF}}(\pi_1^{-1}(\vec{t})) = \dim_{\text{VF}}(\pi \upharpoonright \pi_1^{-1}(\vec{t}))$ and hence $\dim_{\text{VF}}(\pi_1) = \dim_{\text{VF}}(\pi)$. Since this also holds symmetrically for π_2 , the lemma follows. \square

Of course, [19, Definition 4.1] still makes sense in the current context:

Definition 4.6. The VF-dimension of A , denoted by $\dim_{\text{VF}}(A)$, is the smallest number k such that there is a definable finite-to-one function $f : A \rightarrow \text{VF}^k \times \text{RV}^l$ or, equivalently, there is a definable injection $f : A \rightarrow \text{VF}^k \times \text{RV}^l$ (see [19, Lemma 4.2]).

However, the VF-dimension of A itself and the VF-dimension of the RV-partitions of A are really the same thing:

Lemma 4.7. Let π be an RV-partition of A , $\dim_{\text{VF}}(A) = k$, and $A_{\pi} = \bigcup\{\pi^{-1}(\vec{t}) : \dim_{\text{VF}}(\pi^{-1}(\vec{t})) = k\}$. Then $\dim_{\text{VF}}(\pi) = k$ and $\dim_{\text{VF}}(A \setminus A_{\pi}) < k$.

Proof. By compactness, obviously $k \leq \dim_{\text{VF}}(\pi)$. For the other direction, suppose that $f : A \rightarrow \text{VF}^k \times \text{RV}^l$ is a witness to $\dim_{\text{VF}}(A) = k$. Let ρ be an RV-partition of (the graph of) f , which obviously also carries an RV-partition π' of A such that $\text{ran}(\pi') = \text{ran}(\rho)$ and $\pi'^{-1}(\vec{t}) = \text{dom}(\rho^{-1}(\vec{t}))$ for every $\vec{t} \in \text{ran}(\pi')$. By Lemma 4.5, $\dim_{\text{VF}}(\pi) = \dim_{\text{VF}}(\pi') \leq k$.

The second item is a corollary of the first. Note that it makes sense since, by [19, Lemma 4.6], A_{π} is definable. \square

For any definable function $f : \text{VF}^n \rightarrow \text{VF}^m$, the derivative and the partial derivatives of f at a point are defined exactly as in [19, Definition 9.6]. Standard properties of differentiation such as the product rule and the chain rule only depend on the valuation and hence hold regardless of the presence of a section sn and additional structure in the RV-sort.

Lemma 4.8. Let $f : \text{VF}^n \rightarrow \text{VF}^m$ be a definable function. Then each partial derivative $\partial^{ij} f$ is defined almost everywhere.

Proof. Let ρ be an RV-partition of f . For each $\vec{t} \in \text{ran}(\rho)$, $f_{\vec{t}} = \pi^{-1}(\vec{t}) \subseteq f$ is an $\text{sn}(\vec{t})$ - \mathcal{L}_{RV} -definable function and hence, by [19, Lemma 9.8], there is an $\text{sn}(\vec{t})$ - \mathcal{L}_{RV} -definable subset $A_{\vec{t}} \subseteq \text{dom}(f_{\vec{t}})$ with $\dim_{\text{VF}}(A_{\vec{t}}) < n$ such that every partial derivative $\partial^{ij} f_{\vec{t}}$ is defined everywhere in $\text{dom}(f_{\vec{t}}) \setminus A_{\vec{t}}$. By compactness, we may take $A = \bigcup_{\vec{t}} A_{\vec{t}} \subseteq \text{VF}^n$ to be definable,

and there is an RV-partition π of A such that $\text{ran}(\rho) = \text{ran}(\pi)$ and $\pi^{-1}(\vec{t}) = A_{\vec{t}}$ for every $\vec{t} \in \text{ran}(\pi)$. By Lemma 4.7, $\dim_{\text{VF}}(A) = \dim_{\text{VF}}(\pi) < n$. \square

We would like to differentiate functions between definable subsets with RV-coordinates. The procedure for this is the same, with or without a section sn (or a cross-section csn) and additional structure in the RV-sort, as described after [19, Corollary 9.9]. It follows from Lemma 4.8 and compactness that every partial derivative of f is defined almost everywhere. The *Jacobian of f at a point (\vec{a}, \vec{t})* is defined in the usual way, that is, the determinant of the Jacobian matrix, and is denoted by $\text{Jcb}_{\text{VF}} f(\vec{a}, \vec{t})$. By the chain rule, we have:

Lemma 4.9. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be definable functions. Then for any $\vec{x} \in A$,*

$$\text{Jcb}_{\text{VF}}(g \circ f)(\vec{x}) = \text{Jcb}_{\text{VF}} g(f(\vec{x})) \cdot \text{Jcb}_{\text{VF}} f(\vec{x}),$$

if both sides are defined.

Definition 4.10 (Coarse VF-categories). The objects of the category $\text{VF}[k, \cdot]$ are the definable subsets of VF-dimension $\leq k$. Its morphisms are the definable bijections between the objects. Set $\text{VF}_*[\cdot] = \bigcup_k \text{VF}[k, \cdot]$.

An object of the category $\mu_{\Gamma}\text{VF}[k]$ is a definable pair (A, ω_{Γ}) , where $\text{pvf}(A) \subseteq \text{VF}^k$ and $\omega_{\Gamma} : A \rightarrow \Gamma$ is a function, which is understood as a Γ -volume form on A . A *morphism* between two objects $(A, \omega_{\Gamma}), (B, \sigma_{\Gamma})$ is a definable *essential bijection* $F : A \rightarrow B$, that is, a bijection that is defined outside of definable subsets of A, B of VF-dimension $< k$, such that, for every $\vec{x} \in \text{dom}(F)$,

$$\omega_{\Gamma}(F(\vec{x})) = \sigma_{\Gamma}(F(\vec{x})) + \text{val}(\text{Jcb}_{\text{VF}} F(\vec{x})).$$

We also say that such an F is a Γ -measure-preserving map. Set $\mu_{\Gamma}\text{VF}[*] = \bigsqcup_k \mu_{\Gamma}\text{VF}[k]$.

Recall from [19, Remark 10.2] that conceptually $\mu_{\Gamma}\text{VF}[k]$ -morphisms (and $\mu\text{VF}[k]$ -morphisms below) should be treated as equivalence classes so that each of them is actually an isomorphism and the Grothendieck semigroup may be constructed in the traditional way. However, this viewpoint is not essential for our purpose and, as usual, it is less cumbersome to work with representatives.

In order to avoid verbosity, below we shall more or less ignore the coarse VF- and RV-categories with Γ -volume forms, since the results may be modified in the obvious way to hold for them.

For any $\mathbf{U} \in \text{RV}[k, \cdot]$, the *lift* $\mathbb{L}\mathbf{U} \in \text{VF}[k, \cdot]$ of \mathbf{U} is defined following [19, Definition 4.18]. For any $\text{RV}[k, \cdot]$ -morphism $F : \mathbf{U} \rightarrow \mathbf{V}$, a *lift* $F^{\uparrow} : \mathbb{L}\mathbf{U} \rightarrow \mathbb{L}\mathbf{V}$ of F is defined following [19, Definition 7.3]. With the presence of sn , such an F can always be lifted.

Proposition 4.11. *The lifting map $\mathbb{L} : \text{Ob RV}[\leq k, \cdot] \rightarrow \text{Ob VF}[k, \cdot]$ induces a surjective homomorphism, also denoted by \mathbb{L} , between the Grothendieck semigroups $\mathbb{L} : \mathbf{K}_+ \text{RV}[\leq k, \cdot] \rightarrow \mathbf{K}_+ \text{VF}[k, \cdot]$.*

Proof. Applying [19, Corollary 5.6] piecewise over RV-partition, it is clear that \mathbb{L} hits every isomorphism class of $\text{VF}[k, \cdot]$ (see also the discussion after [20, Proposition 6.18]). Due to the presence of a section sn and its immediate consequence that each rv-ball has a prescribed center, the work in [10, §6] (as well as [19, §7]) is not needed here, although it is needed below, and it is almost trivial that every isomorphism class of $\bigsqcup_{i \leq k} \text{RV}[i, \cdot]$ is mapped into an isomorphism class of $\text{VF}[k, \cdot]$. \square

Definition 4.12. For a definable subset $A \subseteq \text{VF}^n \times \text{RV}^m$, the *RV-fiber dimension* of A , written as $\dim_{\text{RV}}^{\text{fb}}(A)$, is the number $\max\{\dim_{\text{RV}}(\text{fib}(A, \vec{a})) : \vec{a} \in \text{pvf}(A)\}$.

Definition 4.13 (Fine VF-categories). The objects of the category $\text{VF}[k]$ are the $\mathcal{L}_{\vec{\pi}}$ -definable subsets of VF-dimension $\leq k$ and RV-fiber dimension 0. Its morphisms are the $\mathcal{L}_{\vec{\pi}}$ -definable bijections between the objects. Set $\text{VF}_* = \bigcup_k \text{VF}[k]$.

An object of the category $\mu\text{VF}[k]$ is a $\mathcal{L}_{\vec{\pi}}$ -definable pair (A, ω) , where $\text{pvf}(A) \subseteq \text{VF}^k$, $A \in \text{VF}[k]$, and $\omega : A \rightarrow \vec{\mathbb{K}}^{\times} \times \Gamma$ is a function, which is understood as a *volume form* on A . We also write ω as a pair $(\omega_{\vec{\mathbb{K}}}, \omega_{\Gamma})$. A $\mu_{\Gamma}\text{VF}[k]$ -morphism $F : (A, \omega_{\Gamma}) \rightarrow (A', \omega'_{\Gamma})$ is a *pseudo-morphism* of $\mu\text{VF}[k]$. If, in addition, F is $\mathcal{L}_{\vec{\pi}}$ -definable and, for every $\vec{x} \in \text{dom}(F)$,

$$\omega_{\vec{\mathbb{K}}}(\vec{x}) = \sigma_{\vec{\mathbb{K}}}(F(\vec{x})) \cdot (\text{tbk} \circ \text{rv})(\text{Jcb}_{\text{VF}} F(\vec{x}))$$

then F is a *morphism* of $\mu\text{VF}[k]$. We also say that such an F is a *measure-preserving map*. Set $\mu\text{VF}[*] = \bigsqcup_k \mu\text{VF}[k]$.

For $\mathbf{A} = (A, \omega) \in \mu\text{VF}[k]$, we shall sometimes write $\dim_{\text{VF}}(\mathbf{A})$ for $\dim_{\text{VF}}(A)$. Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be a $\mu\text{VF}[k]$ -morphism. For any Γ -partition π of F and every $\vec{\gamma} \in \text{ran}(\pi)$, $\pi^{-1}(\vec{\gamma}) = F_{\vec{\gamma}}$ is a $\text{csn}(\vec{\gamma})$ - \mathcal{L}_{RV} -definable $\mu\text{VF}[k]$ -morphism between the objects $\text{dom}(F_{\vec{\gamma}}) \subseteq \mathbf{A}$ and $\text{ran}(F_{\vec{\gamma}}) \subseteq \mathbf{B}$, which is also a morphism in the sense of [19, Definition 10.1]. If F is not a trivial morphism, that is, if $\dim_{\text{VF}}(\mathbf{A}) = k$, then, by Lemma 4.7, some $F_{\vec{\gamma}}$ is a nontrivial morphism.

Remark 4.14. For any $(\mathbf{U}, \omega) \in \mu\text{RV}[k]$, the lift $\mathbb{L}(\mathbf{U}, \omega) = (\mathbb{L}\mathbf{U}, \mathbb{L}\omega) \in \mu\text{VF}[k]$ of (\mathbf{U}, ω) is defined in the obvious way. Let $F : \mathbf{U} \rightarrow \mathbf{V}$ be an $\text{RV}[k]$ -morphism and F^\uparrow a lift of F . If F^\uparrow is a $\mu\text{VF}[k]$ -morphism between $(\mathbb{L}\mathbf{U}, \mathbb{L}\omega)$ and $(\mathbb{L}\mathbf{V}, \mathbb{L}\sigma)$ then, under Hypothesis 3.15, we see that the proof of [19, Lemma 9.15] may be easily adapted to show that F is indeed a $\mu\text{RV}[k]$ -morphism between (\mathbf{U}, ω) and (\mathbf{V}, σ) .

Proposition 4.15. Every $\mathbf{A} \in \mu\text{VF}[k]$ is isomorphic to another object $\mathbb{L}\mathbf{U}$ of $\mu\text{VF}[k]$, where $\mathbf{U} \in \mu\text{RV}[k]$.

Proof. Let π be a Γ -partition of \mathbf{A} . By [19, Theorem 10.4], there is an $\text{csn}(\tilde{\gamma})\text{-}\mathcal{L}_{\text{RV}}$ -definable $\mu\text{VF}[k]$ -isomorphism between $\pi^{-1}(\tilde{\gamma})$ and an object $\mathbb{L}\mathbf{U}_{\tilde{\gamma}} \in \mu\text{VF}[k]$, where $\mathbf{U}_{\tilde{\gamma}} \in \mu\text{RV}[k]$. By Lemma 4.7 and compactness, these isomorphisms may be glued together to form one isomorphism in $\mu\text{VF}[k]$. \square

Proposition 4.16. Let $F : (\mathbf{U}, \omega) \rightarrow (\mathbf{U}', \omega')$ be a $\mu\text{RV}[k]$ -isomorphism. Then there exists a measure-preserving lift $F^\uparrow : \mathbb{L}(\mathbf{U}, \omega) \rightarrow \mathbb{L}(\mathbf{U}', \omega')$ of F .

Proof. Let π be a Γ -partition of F . Every $\pi^{-1}(\tilde{\gamma}) = F_{\tilde{\gamma}}$ may be treated as a morphism as defined in [19, Definition 10.3]. So the assertion follows from [19, Theorem 10.5] and compactness. \square

Corollary 4.17. The lifting map $\mathbb{L} : \text{Ob } \mu\text{RV}[k] \rightarrow \text{Ob } \mu\text{VF}[k]$ induces a surjective homomorphism, also denoted by \mathbb{L} , between the Grothendieck semigroups $\mathbb{L} : \mathbf{K}_+ \mu\text{RV}[k] \rightarrow \mathbf{K}_+ \mu\text{VF}[k]$.

The inverse of \mathbb{L} , denoted by $\int_+ : \mathbf{K}_+ \mu\text{VF}[k] \rightarrow \mathbf{K}_+ \mu\text{RV}[k] / \ker(\mathbb{L})$, where $\ker(\mathbb{L})$ is the kernel of \mathbb{L} , is an isomorphism of semigroups and is in effect the integration we are after. However, to understand the isomorphism \int_+ better and to apply it effectively in the future, we need a concrete description of $\ker(\mathbb{L})$. To obtain that, as in [10,20], the notion of special bijections in VF-categories plays a key role.

Below we shall refer to a special bijection as defined in [19, Definition 5.1] as an \mathcal{L}_{RV} -definable special bijection.

Definition 4.18. Let $A \subseteq \text{VF}^n \times \text{RV}^m$ be a definable subset. A bijection $T : A \rightarrow A^\sharp$ is a special bijection on A of length 1 if for each $\text{rv-polydisc } p \subseteq \text{RVH}(A)$ there is an $\text{sn}(\text{rv}(p))\text{-}\mathcal{L}_{\text{RV}}$ -definable special bijection T_p on p of length at most 1 such that $T \upharpoonright (p \cap A) = T_p \upharpoonright (p \cap A)$ (all such T_p of length 1 target the same VF-coordinate). The subset $C \subseteq \text{RVH}(A)$ that contains exactly those $\text{rv-polydiscs } p$ such that T_p is of length 1 is called the locus of T . For each $\text{rv-polydisc } p \subseteq C$ let λ_p be the focus map of $T_p \upharpoonright (p \cap A)$. The function $\lambda = \bigcup_p \lambda_p$ is called the focus map of T .

Naturally a special bijection T on A of length n , denoted by $\text{lh}(T) = n$, is a composition of n special bijections T_i of length 1. Each T_i is a component of T .

These notions may be formulated in the same way if we work in $\mathcal{C}^{\tilde{\mathbb{T}}}$. Of course, in that case, the section sn is replaced by the cross-section csn and everything is $\mathcal{L}_{\tilde{\mathbb{T}}}$ -definable.

Remark 4.19. Let $A \subseteq \text{VF}^n \times \text{RV}^m$ and $T : A \rightarrow A^\sharp$ be a special bijection with components T_i . Clearly if A is an RV-pullback then A^\sharp is an RV-pullback . By definition, each T_i is a restriction of a special bijection R_i on $\text{RVH}(\text{dom}(T_i))$ and hence their composition R is a special bijection on $\text{RVH}(A)$. For any $\text{rv-polydisc } p \subseteq \text{RVH}(A^\sharp)$ and any $\text{sn}(\text{rv}(p))\text{-}\mathcal{L}_{\text{RV}}$ -definable (resp. $\text{csn}(\text{rv}(p))\text{-}\mathcal{L}_{\text{RV}}$ -definable) subset $B \subseteq p$, the restriction $R \upharpoonright R^{-1}(B)$ is an $\text{sn}(\text{rv}(p))\text{-}\mathcal{L}_{\text{RV}}$ -definable (resp. $\text{csn}(\text{rv}(p))\text{-}\mathcal{L}_{\text{RV}}$ -definable) special bijection.

Lemma 4.20. For any special bijection $T : A \rightarrow A^\sharp$ of length 1, the Jacobians of T and T^{-1} are equal to 1 almost everywhere. If A is a nondegenerate RV-pullback then they are equal to 1 everywhere.

Proof. This is immediate by [19, Lemma 9.11]. \square

Remark 4.21. Many results below hold in both $\mathcal{C}^{\mathbb{T}}$ and $\mathcal{C}^{\tilde{\mathbb{T}}}$ and the proofs are essentially identical if the section sn and the cross-section csn are interchanged everywhere. We shall quote them in both versions. However, to avoid repetition, whenever this is the case we shall only present the version for $\tilde{\mathbb{T}}$ and leave the other one for the reader. In particular, we shall work in $\mathcal{C}^{\tilde{\mathbb{T}}}$ in the rest of this section.

We can easily generalize [19, Theorem 5.4] if the terms in question do not contain any RV-sort variables:

Lemma 4.22. Let $\tau(\vec{X}) : \text{VF}^n \rightarrow \text{VF}$ be an $\mathcal{L}_{\tilde{\mathbb{T}}}$ -term, $\vec{t} \in \text{RV}^n$, $R : \text{rv}^{-1}(\vec{t}) \rightarrow A$ a special bijection, and $f = \tau \circ R^{-1}$. Then there is a special bijection T on A such that the function $f \circ T^{-1}$ is contractible.

Proof. First observe that if the assertion holds for one such term τ then it holds simultaneously for any finite number of such terms. Let $F_{ki}(\vec{X})$ enumerate all the occurring VF-terms of τ such that $|F_{ki}(\vec{X})| = k$. By compactness, it is enough

to concentrate on one rv-polydisc $p_0 \subseteq A$. By Remark 4.19 and [19, Theorem 5.5], there is an $\text{rv}(p_0)$ - \mathcal{L}_{RV} -definable special bijection T_0 on p_0 such that, for every rv-polydisc $q \subseteq T_0(p_0)$,

$$(\text{rv} \circ F_{0i} \circ R^{-1} \circ T_0^{-1})(q)$$

is a singleton $\{s_{0i}^q\}$ for all i .

By compactness again, it is enough to concentrate on one rv-polydisc $p_1 \subseteq T_0(p_0)$. Let $F_{1i}^{p_1}(\vec{X})$ be the \mathcal{L}_{RV} -term obtained from $F_{1i}(\vec{X})$ by replacing each $\text{rv}(F_{0i}(\vec{X}))$ with $s_{0i}^{p_1}$. Each $F_{1i}^{p_1}(\vec{X})$ may be written as a polynomial $\sum_j a_j \vec{X}^j$ with $a_j \in \text{dcl}^{\mathbb{T}}(\text{rv}(p_1))$. By Remark 4.19 and [19, Theorem 5.5] again, there is a $\text{dcl}^{\mathbb{T}}(\text{rv}(p_1))$ -definable special bijection T_1 on p_1 such that, for every rv-polydisc $q \subseteq T_1(p_1)$,

$$(\text{rv} \circ F_{1i}^{p_1} \circ R^{-1} \circ T_0^{-1} \circ T_1^{-1})(q)$$

is a singleton $\{s_{1i}^q\}$ for all i .

Repeating this procedure for all $F_{ki}(\vec{X})$ of higher complexity, we see that there is a special bijection T on A as desired. \square

The following lemma should be viewed as a joint generalization of [18, Lemma 4.10, Lemma 4.12].

Lemma 4.23. *Let $B \subseteq \text{rv}^{-1}(\vec{t}) \times \text{RV}^m$ be a definable subset such that $\text{prv} \upharpoonright B$ is finite-to-one. Then there is a special bijection T on $\text{rv}^{-1}(\vec{t})$ such that $\text{pvf}(T(\text{pvf}(B))) = \{\vec{0}\}$, that is, $T(\text{pvf}(B))$ is a union of rv-polydiscs of the form $(\vec{0}, \vec{\infty}, \vec{s})$. Consequently, there is a definable injection $\text{pvf}(B) \rightarrow \text{RV}^l$ for some l .*

Proof. Let $\phi(\vec{X})$ be a quantifier-free formula that defines $\text{pvf}(B)$. Let $F_i(\vec{X})$ enumerate all the top occurring VF-terms of $\phi(\vec{X})$. By Lemma 4.22, there is a special bijection T on $\text{rv}^{-1}(\vec{t})$ such that every function $F_i \circ T^{-1}$ is contractible. Therefore, for every rv-polydisc $p \subseteq T(\text{rv}^{-1}(\vec{t}))$, either $T^{-1}(p) \subseteq \text{pvf}(B)$ or $T^{-1}(p) \cap \text{pvf}(B) = \emptyset$. Since $\text{acl}^{\mathbb{T}}(\text{RV}) \models \mathbb{T}(S)$, we see that if $T^{-1}(p) \subseteq \text{pvf}(B)$ then p must be a point, that is, p must be of the form $(\vec{0}, \vec{\infty}, \vec{s})$. \square

Lemma 4.24. *Let T, R be two special bijections on $\text{rv}^{-1}(t)$. Let R_i be the components of R , λ_i the focus map of R_i , and $A_i \subseteq \text{rv}^{-1}(t)$ the image of (the graph of) λ_i under \widehat{R}_i^{-1} , where $\widehat{R}_i = R_i \circ \dots \circ R_1$. If $\text{pvf}(T(A_i)) = \{0\}$ for all i then $R \circ T^{-1}$ is contractible.*

Proof. Suppose for contradiction that $R \circ T^{-1}$ is not contractible. Then there is an rv-polydisc $p \subseteq T(\text{rv}^{-1}(t))$ such that $(R \circ T^{-1})(p)$ is a union of more than one rv-polydiscs. It is clear that there is an i such that $(\widehat{R}_i \circ T^{-1})(p)$ is contained in one rv-polydisc and $\lambda_{i+1} \cap (\widehat{R}_i \circ T^{-1})(p) \neq \emptyset$. Then $\text{pvf}(T(A_{i+1})) \neq \{0\}$, contradiction. \square

We are now ready to state a better generalization of [19, Theorem 5.4]:

Theorem 4.25. *Let $\tau(\vec{X}, \vec{Y}) : \text{VF}^n \times \text{RV}^m \rightarrow \text{VF}$ be an $\mathcal{L}_{\mathbb{T}}$ -term. For each $\vec{s} \in \text{RV}^m$ let $\tau_{\vec{s}} = \tau(\vec{X}, \vec{s})$. Let $\vec{t} \in \text{RV}^n$, $R : \text{rv}^{-1}(\vec{t}) \rightarrow A$ a special bijection, and $f_{\vec{s}} = \tau_{\vec{s}} \circ R^{-1}$. Then there is a special bijection T on A such that every function $f_{\vec{s}} \circ T^{-1}$ is contractible.*

Proof. As Lemma 4.22, if the assertion holds for one such term τ then it holds simultaneously for any finite number of such terms. We do induction on n . In a way the proof here combines those of Lemma 4.22 and [19, Theorem 5.4]. The inductive step below is copied almost verbatim from the proof of [19, Theorem 5.4].

For the base case $n = 1$, we simply write X for \vec{X} . Let $F_{ji}(X, \vec{Y})$ enumerate the occurring VF-terms of τ such that $|F_{ji}(X, \vec{Y})| = j$. Let p_0, T_0, s_{0i}^q , and p_1 be as in the proof of Lemma 4.22. For each $\vec{s} \in \text{RV}^m$ let $F_{1i}^{p_1, \vec{s}}(X)$ be the \mathcal{L}_{RV} -term obtained from $F_{1i}(X, \vec{s})$ by replacing each $\text{rv}(F_{0i}(\vec{X}))$ with $s_{0i}^{p_1}$. Each $F_{1i}^{p_1, \vec{s}}(X)$ may be written as a polynomial $\sum_j a_j X^j$ with $a_j \in \text{dcl}^{\mathbb{T}}(\text{rv}(p_1), \vec{s})$. As in the proof of Lemma 4.22 again, there is a $\text{dcl}^{\mathbb{T}}(\text{rv}(p_1), \vec{s})$ - \mathcal{L}_{RV} -definable special bijection $T_{1, \vec{s}}$ on p_1 such that, for every rv-polydisc $q \subseteq T_{1, \vec{s}}(p_1)$,

$$(\text{rv} \circ F_{1i}^{p_1, \vec{s}} \circ R^{-1} \circ T_0^{-1} \circ T_{1, \vec{s}}^{-1})(q)$$

is a singleton for all i . Let $\lambda_{1, \vec{s}, k}$ be the focus maps of the components of $T_{1, \vec{s}}$ and $h_{\vec{s}} : \biguplus_k \text{dom}(\lambda_{1, \vec{s}, k}) \times \{\vec{s}\} \rightarrow p_1$ the injection induced by $T_{1, \vec{s}}$. By compactness, $\bigcup_{\vec{s}} h_{\vec{s}}$ is an $\text{rv}(p_1)$ -definable injection into p_1 . By Lemma 4.23, there is a special bijection T_1 on p_1 such that $\text{pvf}(T_1(\text{ran}(h))) = \{0\}$. By Lemma 4.24, every function $T_{1, \vec{s}} \circ T_1^{-1}$ is contractible. This means that for every rv-polydisc $\tau \subseteq T_1(p_1)$ and every $\vec{s} \in \text{RV}^m$ there is an rv-polydisc $q_{\tau} \subseteq T_{1, \vec{s}}(p_1)$ such that $T_1^{-1}(\tau) \subseteq T_{1, \vec{s}}^{-1}(q_{\tau})$ and hence

$$(\text{rv} \circ F_{1i}^{p_1, \vec{s}} \circ R^{-1} \circ T_0^{-1} \circ T_1^{-1})(\tau)$$

is a singleton $\{s_{i,\vec{s}}^c\}$ for all i . Repeating this procedure for all $F_{k_i}(X, \vec{Y})$ of higher complexity, we see that there is a special bijection T on A as desired. This completes the base case of the induction.

We now proceed to the inductive step. As above, we may concentrate on one rv-polydisc $p = \text{rv}^{-1}(\vec{u}) \times \{(\vec{u}, \vec{r})\} \subseteq A$. Let $\phi(\vec{X}, \vec{Y}, Z)$ be a quantifier-free formula such that $\phi(\vec{X}, \vec{s}, Z)$ defines the function $(\text{rv} \circ f_{\vec{s}}) \upharpoonright p$. Let $F_i(\vec{X}, \vec{Y}, Z)$ enumerate the top occurring VF-terms of ϕ . For every $a \in \text{rv}^{-1}(u_1)$ and every $\vec{s} \in \text{RV}^{m+1}$ let

$$F_{i,\vec{s}} = F_i(\vec{X}, \vec{s}), \quad F_{i,a,\vec{s}} = F_i(a, X_2, \dots, X_n, \vec{s}).$$

By the inductive hypothesis, there is a special bijection R_a on $\text{rv}^{-1}(u_2, \dots, u_n)$ such that every function $F_{i,a,\vec{s}} \circ R_a^{-1}$ is contractible. Let $U_{k,a}$ enumerate the loci of the components of R_a and $\lambda_{k,a}$ the corresponding focus maps. By compactness,

- (1) for each i there is a quantifier-free formula ψ_i such that $\psi_i(a, \vec{s})$ defines the contraction of $F_{i,a,\vec{s}} \circ R_a^{-1}$,
- (2) there is a quantifier-free formula θ such that $\theta(a)$ determines the sequence $\text{rv}(U_{k,a})$ and the VF-coordinates targeted by $\lambda_{k,a}$.

Let $H_j(X_1)$ enumerate the top occurring VF-terms of the formulas ψ_i, θ . For every tuple $\vec{t} \in \text{RV}$ of the right length, let $H_{j,\vec{t}} = H_j(X_1, \vec{t})$. Applying the inductive hypothesis again, we obtain a special bijection T_1 on $\text{rv}^{-1}(u_1)$ such that every function $H_{j,\vec{t}} \circ T_1^{-1}$ is contractible. This means that, for every rv-polydisc $q \subseteq T_1(\text{rv}^{-1}(u_1))$ and every $a_1, a_2 \in T_1^{-1}(q)$,

- (1) for every $\vec{s} \in \text{RV}^{m+1}$, the formulas $\psi_i(a_1, \vec{s}), \psi_i(a_2, \vec{s})$ define the same function,
- (2) the special bijections R_{a_1}, R_{a_2} may be naturally glued together to form one special bijection on $\{a_1, a_2\} \times \text{rv}^{-1}(u_2, \dots, u_n)$.

Consequently, T_1 and R_a naturally induce a special bijection T on p such that each function $F_{i,\vec{s}} \circ T^{-1}$ is contractible. This implies that each function $f_{\vec{s}} \circ T^{-1}$ is contractible and hence T is as required. \square

We immediately give a slightly more general version of Theorem 4.25, which is easier to use:

Theorem 4.26. *Let $A \subseteq \text{VF}^n$ and $f : A \rightarrow \text{RV}^m$ be a definable function. Then there is a special bijection T on A such that $T(A)$ is an RV-pullback and the function $f \circ T^{-1}$ is contractible.*

Proof. By compactness, we may assume that A is contained in an rv-polydisc p . Let ϕ be a quantifier-free formula that defines f . Let $F_i(\vec{X}, \vec{Y})$ enumerate the top occurring VF-terms of ϕ . For $\vec{s} \in \text{RV}^m$ let $F_{i,\vec{s}} = F_i(\vec{X}, \vec{s})$. By Theorem 4.25 there is a special bijection T on p such that each function $F_{i,\vec{s}} \circ T^{-1}$ is contractible. This means that, for each rv-polydisc $q \subseteq T(p)$,

- (1) either $T^{-1}(q) \subseteq A$ or $T^{-1}(q) \cap A = \emptyset$,
- (2) if $T^{-1}(q) \subseteq A$ then $(f \circ T^{-1})(q)$ is a singleton.

So $T \upharpoonright A$ is as required. \square

Recall that a subset A is called a *deformed RV-pullback* if there is a special bijection T such that $T(A)$ is an RV-pullback. By Theorem 4.26 and compactness, we have:

Corollary 4.27. *Every definable subset $A \subseteq \text{VF}^n \times \text{RV}^m$ is a deformed RV-pullback.*

Lemma 4.28. *Let $A \subseteq \text{VF}^{n_1} \times \text{RV}^{m_1}$, $B \subseteq \text{VF}^{n_2} \times \text{RV}^{m_2}$, and $f : A \rightarrow B$ be a definable function. Then there exists a special bijection T on A such that $T(A)$ is an RV-pullback and the function $f \circ T^{-1}$ is contractible.*

Proof. By compactness, we may assume that A is contained in an rv-polydisc. Then this is immediate by applying Theorem 4.26 to the function $\text{prv} \circ f$ (recall that $\mathbf{c}(B)$ is substituted for B). \square

Recall the definition of the open-to-open property (see [20, Proposition 3.19] for subsets of VF and [20, Definition 3.20] for the general case). It is obviously still true that for functions between subsets that have only one VF-coordinate, composing with special bijections on the right and inverses of special bijections on the left preserves the open-to-open property.

Lemma 4.29. *Let $A \subseteq \text{VF} \times \text{RV}^{m_1}$, $B \subseteq \text{VF} \times \text{RV}^{m_2}$, and $f : A \rightarrow B$ be a definable bijection. Then there exist special bijections $T_A : A \rightarrow A^\sharp$ and $T_B : B \rightarrow B^\sharp$ such that A^\sharp, B^\sharp are RV-pullbacks and, in the commutative diagram*

$$\begin{array}{ccccc}
 A & \xrightarrow{T_A} & A^\# & \xrightarrow{\text{rv}} & \text{rv}(A^\#) \\
 f \downarrow & & f^\# \downarrow & & f^\#_\downarrow \\
 B & \xrightarrow{T_B} & B^\# & \xrightarrow{\text{rv}} & \text{rv}(B^\#)
 \end{array}$$

$f^\#_\downarrow$ is bijective and hence $f^\#$ is a lift of it.

Proof. By Corollary 4.27 we may assume that A, B are RV-pullbacks. Let π be a Γ -partition of f . For each $\vec{\gamma} \in \text{ran}(\pi)$, set $\text{dom}(f_{\vec{\gamma}}) = A_{\vec{\gamma}}$ and $\text{ran}(f_{\vec{\gamma}}) = B_{\vec{\gamma}}$. By [20, Proposition 3.21], there is a finite partition of $A_{\vec{\gamma}}$ into definable subsets $A_{i,\vec{\gamma}}$ such that each $f_{\vec{\gamma}} \upharpoonright A_{i,\vec{\gamma}}$ has the open-to-open property. For $(a, \vec{t}) \in A_{i,\vec{\gamma}}$ let $h(a, \vec{t}) = (\overline{\text{csn}}(\vec{\gamma}), i)$. Applying Lemma 4.28 to the function h , we obtain a special bijection T on A such that each $A_{i,\vec{\gamma}}$ is an RV-pullback. Applying it again to $f \circ T$, we may assume that f is contractible and has the open-to-open property. In particular, for each rv-polydisc $p \subseteq A$, $f(p)$ is an open polydisc contained in an rv-polydisc.

By Lemma 4.28 again, there is a special bijection $T_B : B \rightarrow B^\#$ such that $(T_B \circ f)^{-1}$ is contractible. Let $T_B = T_{B,n} \circ \dots \circ T_{B,1}$. It is enough to construct a special bijection $T_A = T_{A,n} \circ \dots \circ T_{A,1}$ on A such that, for each i , both $\widehat{T}_{B,i} \circ f \circ (\widehat{T}_{A,i})^{-1}$ and $\widehat{T}_{A,i} \circ (T_B \circ f)^{-1}$ are contractible, where

$$\widehat{T}_{B,i} = T_{B,i} \circ \dots \circ T_{B,1}, \quad \widehat{T}_{A,i} = T_{A,i} \circ \dots \circ T_{A,1}.$$

Now we may simply use the construction in the proof of [20, Lemma 5.2], since it only depends on the contractibility and the open-to-open property of f . \square

Recall from [20, Definition 5.4] the notion of a (special) relatively unary bijection.

Lemma 4.30. Let $A \subseteq \text{VF}^n \times \text{RV}^{m_1}$, $B \subseteq \text{VF}^n \times \text{RV}^{m_2}$, and $f : A \rightarrow B$ a definable bijection. Then there is a finite partition of A into definable subsets A_i such that each $f \upharpoonright A_i$ is a composition of definable relatively unary bijections.

Proof. Since there are finitely many VF-coordinates to choose from, this is immediate by applying [20, Lemma 5.6] over a Γ -partition of f and then compactness. \square

Let $A \subseteq \text{VF}^n \times \text{RV}^m$ be a definable subset and σ a permutation of $I_n = \{1, \dots, n\}$. We define a standard contraction \widehat{T}_σ of A exactly as in [20, Definition 5.5]. By Corollary 4.27, there are abundant standard contractions of A in stock.

Lemma 4.31. Let $1_2, 2_1$ denote the permutations of I_2 . Let $A \subseteq \text{VF}^2 \times \text{RV}^m$ be a definable subset. Then there is a definable injection $f : A \rightarrow \text{VF}^2 \times \text{RV}^l$ such that

- (1) f is unary relative to both coordinates,
- (2) there are standard contractions $\widehat{T}_{1_2}, \widehat{R}_{2_1}$ of $f(A)$ such that $(\widehat{T}_{1_2}(f(A)), \text{pr}_{\leq 2}), (\widehat{R}_{2_1}(f(A)), \text{pr}_{\leq 2})$ are $\text{RV}[2, \cdot]$ -isomorphic and, if $\dim_{\text{RV}}^{\text{fb}}(A) = 0$, then they are $\text{RV}[2]$ -isomorphic.

Proof. Let π be a Γ -partition of A . Since the bijection on A given by $\vec{x} \mapsto (\vec{x}, \overline{\text{csn}}(\pi(\vec{x})))$ is obviously unary relative to both coordinates, it is easily seen that the assertion simply follows from [20, Corollary 5.8] and compactness. \square

5. The kernel of \mathbb{L} and integration

To understand the kernels of the semigroup homomorphisms \mathbb{L} constructed above, we shall produce analogues of [20, Proposition 6.17, Proposition 7.8]. The key notion is still that of a blowup. This is defined in almost exactly the same way as in [20, Definition 6.1, Definition 7.1].

We shall first work in \mathfrak{e}^Γ and discuss the coarse VF- and RV-categories. However, as mentioned above, we shall concentrate on the categories without Γ -volume forms and the auxiliary results will only be stated for them. For the categories with Γ -volume forms the proofs are very similar and the extra computational work involving Γ -volume forms is always straightforward.

Definition 5.1. Suppose $k > 0$ and let $\mathbf{U} = (U, f) \in \text{RV}[k, \cdot]$. An elementary blowup of \mathbf{U} is an object $\mathbf{U}^\# = (U^\#, f^\#) \in \text{RV}[\leq k, \cdot]$ such that $U^\# = U \times \text{RV}^{>1}$ and, for some $1 \leq j \leq k$ and any $(\vec{t}, s) \in U^\#$,

$$f_i^\#(\vec{t}, s) = f_i(\vec{t}) \quad \text{for } i \neq j, \quad f_j^\#(\vec{t}, s) = sf_j(\vec{t}).$$

Note that $\mathbf{U}^\#$ is an object in $\text{RV}[\leq k, \cdot]$ (actually in $\text{RV}[k-1, \cdot] \sqcup \text{RV}[k, \cdot]$) but in general not an object in $\text{RV}[k, \cdot]$ because $f_j^\#(\vec{t}, \infty) = \infty$.

Let $\mathbf{V} = (V, g) \in \text{RV}[k, \cdot]$, $C \subseteq V$, and $\mathbf{C} = (C, g \upharpoonright C) \in \text{RV}[k, \cdot]$. Let $F : \mathbf{U} \rightarrow \mathbf{C}$ be an $\text{RV}[k, \cdot]$ -morphism. Then $\mathbf{U}^\sharp \uplus (V \setminus C, g \upharpoonright (V \setminus C))$ is a *blowup* of (V, g) via F , written as \mathbf{V}_F^\sharp . The subscript F may be dropped in context if there is no danger of confusion. The object \mathbf{C} (or the subset C) is called the *locus* of the blowup \mathbf{V}_F^\sharp . A *blowup of length n* is a composition of n blowups.

Lemma 5.2. *Let $\mathbf{U}, \mathbf{V} \in \text{RV}[\leq k, \cdot]$ and $\mathbf{U}^\sharp, \mathbf{V}^\sharp$ be two blowups. In $\mathbf{K}_+ \text{RV}[\leq k, \cdot]$, if $[\mathbf{U}] = [\mathbf{V}]$ then there are blowups $\mathbf{U}^{\sharp\sharp}, \mathbf{V}^{\sharp\sharp}$ of $\mathbf{U}^\sharp, \mathbf{V}^\sharp$ such that $[\mathbf{U}^{\sharp\sharp}] = [\mathbf{V}^{\sharp\sharp}]$. Therefore, if $[\mathbf{U}] = [\mathbf{U}']$, $[\mathbf{V}] = [\mathbf{V}']$ and there are isomorphic blowups of \mathbf{U}, \mathbf{V} then there are isomorphic blowups of \mathbf{U}', \mathbf{V}' .*

Proof. For the first assertion, the proof of [20, Lemma 6.5] works. The second assertion is a corollary. \square

Definition 5.3. Let $\text{I}_{\text{sp}}[k, \cdot]$ be the subclass of $\text{Ob RV}[\leq k, \cdot] \times \text{Ob RV}[\leq k, \cdot]$ of pairs (\mathbf{U}, \mathbf{V}) such that there exist isomorphic blowups $\mathbf{U}^\sharp, \mathbf{V}^\sharp$. Let $\text{I}_{\text{sp}}[* , \cdot] = \bigcup_k \text{I}_{\text{sp}}[k, \cdot]$.

We will just write I_{sp} for all these classes if there is no danger of confusion. By Lemma 5.2, I_{sp} may be regarded as a binary relation on isomorphism classes.

Lemma 5.4. *$\text{I}_{\text{sp}}[k, \cdot]$ is a semigroup congruence relation and $\text{I}_{\text{sp}}[* , \cdot]$ is a semiring congruence relation.*

Proof. The proof of [20, Lemma 6.8] works. \square

Let $\mathbf{U}_i = (U_i, f_i) \in \text{RV}[i, \cdot]$, $\mathbf{U} = \prod_{i \leq k} \mathbf{U}_i \in \text{RV}[\leq k, \cdot]$, and T a special bijection on $\mathbb{L}\mathbf{U}$. We write $U_{i,T}$ for the subset $(\text{prv} \circ T)(\mathbb{L}\mathbf{U}_i)$, $\mathbf{U}_{i,T}$ for the object $(U_{i,T}, \text{pr}_{\leq i}) \in \text{RV}[i, \cdot]$, and \mathbf{U}_T for the object $\prod_{i \leq k} \mathbf{U}_{i,T} \in \text{RV}[\leq k, \cdot]$. Recall from [20, Notation 2.37] the shorthand $[U_{i,T}]_{\leq i}$ for $[(U_{i,T}, \text{pr}_{\leq i})] \in \mathbf{K}_+ \text{RV}[i, \cdot]$.

Lemma 5.5. *The object \mathbf{U}_T is isomorphic to a blowup of \mathbf{U} of the same length as T .*

Proof. By induction on the length $\text{lh } T$ of T and Lemma 5.2, this is immediately reduced to the case $\text{lh } T = 1$. Then we may use the isomorphism constructed in the proof of [20, Lemma 6.9]. \square

Lemma 5.6. *Suppose that $[A] = [B]$ in $\text{VF}[1, \cdot]$ and $\mathbf{U}, \mathbf{V} \in \text{RV}[\leq 1, \cdot]$ are two standard contractions of A, B . Then $([\mathbf{U}], [\mathbf{V}]) \in \text{I}_{\text{sp}}$.*

Proof. By Lemma 4.29, there are special bijections T, R on $\mathbb{L}\mathbf{U}, \mathbb{L}\mathbf{V}$ such that $\mathbf{U}_T, \mathbf{V}_R$ are isomorphic. So the assertion follows from Lemma 5.5. \square

Lemma 5.7. *Let \mathbf{U}^\sharp be a blowup of $\mathbf{U} \in \text{RV}[\leq k, \cdot]$ of length l . Then $\mathbb{L}\mathbf{U}^\sharp$ and $\mathbb{L}\mathbf{U}$ are isomorphic.*

Proof. By induction this is immediately reduced to the case $l = 1$. Observe that, using the section sn , the special bijection T on $\mathbb{L}\mathbf{U}$ as described in the proof of [20, Lemma 6.12] can be (quite trivially) constructed. \square

Lemma 5.8. *Let $A_1, A_2 \in \text{VF}[k, \cdot]$ such that $\text{pvf}(A_1) = \text{pvf}(A_2) = A$. Suppose that there is a common subset E of the indices of the RV-coordinates of A_1, A_2 such that, for every $\bar{a} \in A$,*

$$([\text{fib}(A_1, \bar{a})]_E, [\text{fib}(A_2, \bar{a})]_E) \in \text{I}_{\text{sp}}.$$

Let $\widehat{T}_\sigma, \widehat{R}_\sigma$ be two standard contractions of A_1, A_2 . Set $E' = E \cup I_k$. Then

$$([\widehat{T}_\sigma(A_1)]_{E'}, [\widehat{R}_\sigma(A_2)]_{E'}) \in \text{I}_{\text{sp}}.$$

Proof. In the proof of [20, Lemma 6.14], the special bijection T_A is achieved by applying [19, Theorem 5.5] to the occurring polynomials of a suitable quantifier-free formula, as in [20, Lemma 5.1]. This procedure may be reproduced here by applying Theorem 4.26 to the top occurring VF-terms of a suitable quantifier-free formula, as in Lemma 4.28. For the rest of the proof, we may simply follow the proof of [20, Lemma 6.14]. \square

Corollary 5.9. *Let $A_1, A_2 \in \text{VF}[k, \cdot]$ and $f : A_1 \rightarrow A_2$ a unary bijection relative to the coordinate $i \in I_k$. Then for any permutation σ of I_k with $\sigma(1) = i$ and any standard contractions $\widehat{T}_\sigma, \widehat{R}_\sigma$ of A_1, A_2 ,*

$$([\widehat{T}_\sigma(A_1)]_{\leq k}, [\widehat{R}_\sigma(A_2)]_{\leq k}) \in \text{I}_{\text{sp}}.$$

Proof. This is immediate by Lemmas 5.6 and 5.8. \square

Lemma 5.10. Let $A \in \text{VF}[k, \cdot]$. Let $i, j \in I_k$ be distinct and σ_1, σ_2 two permutations of I_k such that

$$\sigma_1(1) = \sigma_2(2) = i, \quad \sigma_1(2) = \sigma_2(1) = j, \quad \sigma_1 \upharpoonright \{3, \dots, k\} = \sigma_2 \upharpoonright \{3, \dots, k\}.$$

Then, for any standard contractions $\widehat{T}_{\sigma_1}, \widehat{T}_{\sigma_2}$ of A ,

$$([\widehat{T}_{\sigma_1}(A)]_{\leq k}, [\widehat{T}_{\sigma_2}(A)]_{\leq k}) \in \text{I}_{\text{sp}}.$$

Proof. We have developed analogues of the results used in the proof of [20, Lemma 6.16]. Therefore its proof may be quoted here with virtually no changes. \square

Now we have reproduced for $\text{VF}[k, \cdot], \text{RV}[\leq k, \cdot]$ all the results that the proof of [20, Proposition 6.17] formally depends on, so the following crucial description of the kernel of $\mathbb{L} : \mathbf{K}_+ \text{RV}[\leq k, \cdot] \rightarrow \mathbf{K}_+ \text{VF}[k, \cdot]$ may be obtained by more or less the same proof, which is reproduced in its entirety below.²

Proposition 5.11. For $\mathbf{U}, \mathbf{V} \in \text{RV}[\leq k, \cdot], [\mathbb{L}\mathbf{U}] = [\mathbb{L}\mathbf{V}]$ if and only if $([\mathbf{U}], [\mathbf{V}]) \in \text{I}_{\text{sp}}$.

Proof. The “if” direction simply follows from Lemma 5.7 and Proposition 4.11.

For the “only if” direction, we show a stronger claim: if $[A] = [B]$ in $\text{VF}[k, \cdot]$ and $\mathbf{U}, \mathbf{V} \in \text{RV}[\leq k, \cdot]$ are two standard contractions of A, B then $([\mathbf{U}], [\mathbf{V}]) \in \text{I}_{\text{sp}}$. We do induction on k . The base case $k = 1$ is of course Lemma 5.6. For the inductive step, suppose that $F : \mathbb{L}\mathbf{U} \rightarrow \mathbb{L}\mathbf{V}$ is a definable bijection. By Lemma 4.30, there is a partition of $\mathbb{L}\mathbf{U}$ into definable subsets A_1, \dots, A_n such that each $F_i = F \upharpoonright A_i$ is a composition of relatively unary bijections. Applying Theorem 4.26 as in Lemma 4.28, we obtain special bijections T, R on $\mathbb{L}\mathbf{U}, \mathbb{L}\mathbf{V}$ such that $T(A_i), (R \circ F)(A_i)$ are RV-pullbacks for each i . By Lemma 5.5, it is enough to show that there are standard contractions $\widehat{T}_\sigma, \widehat{R}_\tau$ of $T(A_i), (R \circ F)(A_i)$ for each i such that

$$([\widehat{T}_\sigma \circ T(A_i)]_{\leq k}, [\widehat{R}_\tau \circ R \circ F(A_i)]_{\leq k}) \in \text{I}_{\text{sp}}.$$

To that end, first note that each $(R \circ F \circ T^{-1}) \upharpoonright T(A_i)$ is a composition of relatively unary bijections, say

$$T(A_i) = B_1 \xrightarrow{G_1} B_2 \cdots B_l \xrightarrow{G_l} B_{l+1} = (R \circ F)(A_i).$$

For each $j \leq l - 2$ we may choose five contractions $[U_j]_{\leq k}, [U_{j+1}]_{\leq k}, [U'_{j+1}]_{\leq k}, [U''_{j+1}]_{\leq k}$, and $[U_{j+2}]_{\leq k}$ with the permutations $\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma''_{j+1}$, and σ_{j+2} of I_k such that

- (1) $\sigma_{j+1}(1)$ and $\sigma_{j+1}(2)$ are the VF-coordinates targeted by G_j and G_{j+1} , respectively,
- (2) $\sigma''_{j+1}(1)$ and $\sigma''_{j+1}(2)$ are the VF-coordinates targeted by G_{j+1} and G_{j+2} , respectively,
- (3) $\sigma_j = \sigma_{j+1}, \sigma''_{j+1} = \sigma_{j+2}$, and $\sigma'_{j+1}(1) = \sigma''_{j+1}(1)$,
- (4) the relation between σ_{j+1} and σ'_{j+1} is as described in Lemma 5.10.

Then, by Corollary 5.9 and Lemma 5.10, all the adjacent pairs of these contractions are I_{sp} -congruent, except $([U'_{j+1}]_{\leq k}, [U''_{j+1}]_{\leq k})$. Since, without loss of generality, we may assume that $[U'_{j+1}]_{\leq k}$ and $[U''_{j+1}]_{\leq k}$ start with the same contraction on the first targeted VF-coordinate of B_{j+1} , the resulting objects in $\text{VF}[k - 1, \cdot]$ are the same. So, by the inductive hypothesis, this last pair is also I_{sp} -congruent. This completes the “only if” direction. \square

We now move on to work in $\mathfrak{C}^{\widehat{\Gamma}}$ and discuss the fine VF- and RV-categories. The definition of a blowup needs to be slightly modified and the results in [20, §7] are needed. However, applying Γ -partitions and compactness, analogues of the results above may be obtained by essentially the same proofs.

Definition 5.12. Suppose $k > 0$. Let $1 \leq j \leq k$ and $\mathbf{U} = (U, f) \in \text{RV}[k]$. Suppose that there is a Γ -partition π of (the graph of) f such that

$$\pi^{-1}(\vec{\gamma})_j(\vec{t}) \in \text{acl}(\pi^{-1}(\vec{\gamma})_{\neq j}(\vec{t}), \overline{\text{csn}}(\vec{\gamma}))$$

for all $\vec{\gamma} \in \text{ran}(\pi)$ and all $\vec{t} \in \text{dom}(\pi^{-1}(\vec{\gamma}))$. An elementary blowup of \mathbf{U} is an object $\mathbf{U}^\sharp = (U^\sharp, f^\sharp) \in \text{RV}[k]$ such that $U^\sharp = U \times (\text{RV}^\times)^{>1}$ and, for any $(\vec{t}, s) \in U^\sharp$,

$$f_i^\sharp(\vec{t}, s) = f_i(\vec{t}) \quad \text{for } i \neq j, \quad f_j^\sharp(\vec{t}, s) = sf_j(\vec{t}).$$

² In fact the wording of the proof of [20, Proposition 6.17] is somewhat terse and hence confusing. We take this opportunity to improve it.

Let ω be a volume form on U . An elementary blowup of (\mathbf{U}, ω) is an object $(\mathbf{U}^\sharp, \omega^\sharp) \in \mu\text{RV}[k]$, where \mathbf{U}^\sharp is an elementary blowup of \mathbf{U} and ω^\sharp is the volume form on U^\sharp given by $\omega^\sharp(\vec{t}, s) = \omega(\vec{t})$.

Other related notions are defined as in Definition 5.1.

Lemma 5.13. Let $\mathbf{U}, \mathbf{V} \in \mu\text{RV}[k]$ and $\mathbf{U}^\sharp, \mathbf{V}^\sharp$ be two elementary blowups. If $[\mathbf{U}] = [\mathbf{V}]$ then $[\mathbf{U}^\sharp] = [\mathbf{V}^\sharp]$.

Proof. This is immediate by applying [20, Lemma 7.2] over a Γ -partition of an isomorphism between \mathbf{U} and \mathbf{V} . \square

Definition 5.14. Let $\mu\text{I}_{\text{sp}}[k]$ be the subclass of $\text{Ob } \mu\text{RV}[k] \times \text{Ob } \mu\text{RV}[k]$ of pairs (\mathbf{U}, \mathbf{V}) such that there exist isomorphic blowups $\mathbf{U}^\sharp, \mathbf{V}^\sharp$. Let $\mu\text{I}_{\text{sp}}[*] = \coprod_k \mu\text{I}_{\text{sp}}[k]$.

Lemma 5.15. As a binary relation on isomorphism classes, $\mu\text{I}_{\text{sp}}[k]$ is a semigroup congruence relation and $\mu\text{I}_{\text{sp}}[*]$ is a semiring congruence relation.

Let $(A, \omega) \in \mu\text{VF}[k]$ and T a special bijection on A . Set $\omega_T = \omega \circ T^{-1}$. Then T is also understood as a special bijection from (A, ω) to $T(A, \omega) = (T(A), \omega_T)$. By Lemma 4.20, T is indeed a $\mu\text{VF}[k]$ -isomorphism. Set $A_\omega = \{(\vec{x}, \omega(\vec{x})) : \vec{x} \in A\}$. For simplicity the volume form on A_ω that is naturally induced by ω is still denoted by ω . Clearly (A, ω) and (A_ω, ω) are isomorphic. If \widehat{T}_σ is a standard contraction of A_ω then ω naturally induces a volume form $\omega_{\widehat{T}_\sigma}$ on $(\widehat{T}_\sigma(A_\omega), \text{pr}_{\leq k})$. The function \widehat{T}_σ (or the object $(\widehat{T}_\sigma(A_\omega), \text{pr}_{\leq k}, \omega_{\widehat{T}_\sigma}) \in \mu\text{RV}[k]$, which is completely determined by \widehat{T}_σ) is understood as a standard contraction of (A, ω) .

For $(\mathbf{U}, \omega) = (U, f, \omega) \in \mu\text{RV}[k]$ and a special bijection T on $\mathbb{L}(\mathbf{U}, \omega)$, we write ω_T for the volume form on U_T that is naturally induced by $(\mathbb{L}\omega)_T$ and $(\mathbf{U}, \omega)_T$ for the object $(\mathbf{U}_T, \omega_T) \in \mu\text{RV}[k]$.

It is straightforward to state and prove the analogues of the results from Lemma 5.5 to Lemma 5.7 (note that Remark 4.14 is needed for the analogues of Lemma 5.5 and Lemma 5.6). For the analogues of Lemma 5.8 and Lemma 5.10, the proofs of [20, Lemma 7.5, Lemma 7.7] can be easily adapted. From these we can deduce:

Proposition 5.16. For $\mathbf{U}, \mathbf{V} \in \mu\text{RV}[k]$, $[\mathbb{L}\mathbf{U}] = [\mathbb{L}\mathbf{V}]$ if and only if $([\mathbf{U}], [\mathbf{V}]) \in \mu\text{I}_{\text{sp}}$.

The dependence on [20, Proposition 6.17, Proposition 7.8] of the results concerning Grothendieck homomorphisms in [20, §§6, 7] is of a formal nature. Therefore, using the results above, their analogues may be derived in more or less the same way. The (obvious) proofs are again omitted.

We emphasize here that the statements below concern two situations: in $\mathfrak{C}^\mathbb{T}$ without volumes (or with Γ -volume forms) and in $\mathfrak{C}^\mathbb{T}$ with volume forms. It is a matter of restriction to transfer results from $\mathfrak{C}^\mathbb{T}$ to $\mathfrak{C}^\mathbb{T}$. But it is not clear if we can successfully incorporate volume forms in the categories associated to $\mathfrak{C}^\mathbb{T}$. The difficulty is that if we simply work with an analogue of Definition 4.13 then special bijections are not guaranteed to be morphisms, in particular, the inductive step of Theorem 4.25 seems to fail without an easy remedy.

Theorem 5.17. For each $k \geq 0$ there are canonical isomorphisms of Grothendieck semigroups

$$\int_+ : \mathbf{K}_+ \text{VF}[k, \cdot] \longrightarrow \mathbf{K}_+ \text{RV}[\leq k, \cdot] / \text{I}_{\text{sp}} \quad \text{and} \quad \int_+ \mathbf{K}_+ \mu\text{VF}[k] \longrightarrow \mathbf{K}_+ \mu\text{RV}[k] / \mu\text{I}_{\text{sp}}$$

such that

$$\int_+ [\mathbf{A}] = [\mathbf{U}] / \text{I}_{\text{sp}} \quad \text{if and only if} \quad [\mathbf{A}] = [\mathbb{L}\mathbf{U}] \quad \text{and} \quad \int_+ [\mathbf{A}] = [\mathbf{U}] / \mu\text{I}_{\text{sp}} \quad \text{if and only if} \quad [\mathbf{A}] = [\mathbb{L}\mathbf{U}].$$

Putting these together, we obtain canonical isomorphisms of Grothendieck semirings

$$\int_+ : \mathbf{K}_+ \text{VF}_*[\cdot] \longrightarrow \mathbf{K}_+ \text{RV}[*] / \text{I}_{\text{sp}} \quad \text{and} \quad \int_+ \mathbf{K}_+ \mu\text{VF}[*] \longrightarrow \mathbf{K}_+ \mu\text{RV}[*] / \mu\text{I}_{\text{sp}}.$$

Recall [19, Notation 3.16]. Let $A \subseteq \text{VF}^n$ and $f : A \longrightarrow \mathcal{P}(\text{RV}^m)$ be a definable function such that every $f(\vec{a})$ codes an object in $\text{RV}[\leq k, \cdot]_{\vec{a}}$ (note that, by compactness, k is bounded). We think of f , or rather the graph of f , as a representative of an equivalence class of definable functions induced by I_{sp} and the equivalence class as a definable function $A \longrightarrow \mathbf{K}_+ \text{RV}[*] / \text{I}_{\text{sp}}$, which, for simplicity, is also denoted by f . The set of all such functions, as in Definition 3.26, is denoted by

$$\text{FN}(A, \mathbf{K}_+ \text{RV}[*] / \text{I}_{\text{sp}}) = \bigoplus_i \text{FN}(A, \mathbf{K}_+ \text{RV}[i, \cdot] / \text{I}_{\text{sp}}),$$

which is a semimodule. Using Notation 3.25, f and g represent the same function if $[f(\vec{a})] =_{I_{sp}, \vec{a}} [g(\vec{a})]$ for every $\vec{a} \in A$. Let $\mathbb{L}f = \bigcup_{\vec{a} \in A} \{\vec{a}\} \times \mathbb{L}(f(\vec{a}))$. Set

$$\int_{+A} f = \int_{+} [f] = \int_{+} [\mathbb{L}f],$$

which, by Proposition 5.11 and compactness, does not depend on the representative f . Consequently we have a homomorphism of semimodules:

$$\int_{+A} : \text{FN}(A, \mathbf{K}_+ \text{RV}[*], \cdot / I_{sp}) \longrightarrow \mathbf{K}_+ \text{RV}[*], \cdot / I_{sp}.$$

Similarly, if each $f(\vec{a})$ codes an object in $\mu\text{RV}[*]$ then f represents a definable function, sometimes denoted by (f, ω) , in the semimodule $\text{FN}(A, \mathbf{K}_+ \mu\text{RV}[*] / \mu I_{sp})$, where ω is the volume form on the graph of f , that is, $\omega \upharpoonright f(\vec{a})$ is the volume form carried in $f(\vec{a})$. Let $\mathbb{L}\omega$ be the volume form on $\mathbb{L}f$ naturally induced by ω . Setting

$$\int_{+A} (f, \omega) = \int_{+} [(f, \omega)] = \int_{+} [(\mathbb{L}f, \mathbb{L}\omega)],$$

we obtain a homomorphism of semimodules:

$$\int_{+A} : \text{FN}(A, \mathbf{K}_+ \mu\text{RV}[*] / \mu I_{sp}) \longrightarrow \mathbf{K}_+ \mu\text{RV}[*] / \mu I_{sp}.$$

Proposition 5.18. For any nonempty subsets $E_1, E_2 \subseteq I_n = \{1, \dots, n\}$,

$$\int_{+\vec{a} \in \text{pr}_{E_1}(A) + \text{fib}(A, \vec{a})} f = \int_{+\vec{a} \in \text{pr}_{E_2}(A) + \text{fib}(A, \vec{a})} f.$$

Similarly for the case with volume forms.

Proof. This is immediate by Proposition 5.11, Proposition 5.16, and the definition of iterated integrals. \square

Let $B \subseteq \text{VF}^n$ and assume that the VF-dimensions of A, B are n . For any definable bijection $\phi : A \longrightarrow B$, we can define the *Jacobian transformation*

$$\phi^{\text{Jcb}} : \text{FN}(A, \mathbf{K}_+ \mu\text{RV}[*] / \mu I_{sp}) \longrightarrow \text{FN}(B, \mathbf{K}_+ \mu\text{RV}[*] / \mu I_{sp})$$

exactly as in [20, Section 7] and, as [20, Proposition 7.12], obtain

Proposition 5.19. $\int_{+A} (f, \omega) = \int_{+B} \phi^{\text{Jcb}}(f, \omega)$.

Let $\mathbf{I}, \mu\mathbf{I}$ be the ideals of the groupifications of $\mathbf{K}_+ \text{RV}[*], \cdot / I_{sp}, \mathbf{K}_+ \mu\text{RV}[*] / \mu I_{sp}$. By the same calculations as in [20, §§6, 7], we see that, the ideal \mathbf{I} is generated by $[1]_0 + \mathbf{j}$ and the ideal $\mu\mathbf{I}$ is generated by \mathbf{j}_μ (see Notation 3.29). Note that \mathbf{j} is equal to $-[1]_0 = -1$ in $\mathbf{KRV}[*], \cdot / \mathbf{I}$ and hence is not a zero-divisor in $\mathbf{KRV}[*], \cdot$ (for otherwise $[1]_0 + \mathbf{j}$ would be a zero-divisor in $\mathbf{KRV}[*], \cdot$, which is clearly impossible).

Theorem 5.20. The Grothendieck semiring isomorphism \int_{+} induces canonically an injective homomorphism

$$\int : \mathbf{KVF}_*[\cdot] \longrightarrow \mathbf{KRV}[*], \cdot [\mathbf{j}^{-1}],$$

whose range is the entire zeroth graded piece, and two homomorphisms

$$\int^g : \mathbf{KVF}_*[\cdot] \longrightarrow \mathbf{KRES}[*], \cdot [\mathbf{A}^{-1}] \quad \text{and} \quad \int^b : \mathbf{KVF}_*[\cdot] \longrightarrow \mathbf{KRES}[*], \cdot [[1]_1^{-1}].$$

Proof. The first homomorphism is similar to the one in [20, Theorem 6.22]. The other two come from Proposition 3.30. \square

Similarly, since $\mu\mathbf{I}$ is a homogeneous ideal, setting $\mu\mathbf{I}_k = \mathbf{K} \mu\text{RV}[k] \cap \mu\mathbf{I}$, we have:

Theorem 5.21. *The Grothendieck semiring isomorphism \int_+ induces canonically a graded ring isomorphism*

$$\int \mathbf{K}\mu\text{VF}[*] \longrightarrow \mathbf{K}\mu\text{RV}[*]/\mu\mathbf{I} = \bigoplus_k \mathbf{K}\mu\text{RV}[k]/\mu\mathbf{I}_k$$

and two graded ring homomorphisms

$$\int^g : \mathbf{K}\mu\text{VF}[*] \longrightarrow \mathbf{K}\mu\text{RES}[*]/(\mathbf{A}_\mu) \quad \text{and} \quad \int^b : \mathbf{K}\mu\text{VF}[*] \longrightarrow \mathbf{K}\mu\text{RES}[*]/([1_\mu]_1).$$

In future applications, we will often need to modify the target Grothendieck (semi)rings of the integration maps through some standard algebraic manipulations. The following procedure is an example.

Let us abbreviate $\mathbf{K}_+ \mu\text{RV}[k]/\mu\mathbf{I}_{\text{sp}}$, $\mathbf{K}_+ \mu\text{RV}[*]/\mu\mathbf{I}_{\text{sp}}$ as $\mathbf{K}_+ \text{RV}_k$, $\mathbf{K}_+ \text{RV}$, respectively. Their groupifications are abbreviated accordingly. Clearly the groupification of the completion $\widehat{\mathbf{K}_+ \text{RV}}$ of $\mathbf{K}_+ \text{RV}$ is the completion $\widehat{\mathbf{K}\text{RV}}$ of $\mathbf{K}\text{RV}$. So there is a canonical semiring homomorphism from the former into the latter. Recall [19, Notation 2.9]. For any $\vec{\gamma} \in \Gamma$, we write $\mathbf{o}_{\vec{\gamma}}$ and $\mathbf{c}_{\vec{\gamma}}$ for the canonical images of the elements $\int[(\mathbf{o}(0, \vec{\gamma}), (1, 0))]$ and $\int[(\mathbf{c}(0, \vec{\gamma}), (1, 0))]$ in $\widehat{\mathbf{K}\text{RV}}$. We localize $\widehat{\mathbf{K}\text{RV}}$ at \mathbf{o}_0 , \mathbf{c}_0 and obtain the ring $\widehat{\mathbf{K}\text{RV}}[\mathbf{o}_0^{-1}, \mathbf{c}_0^{-1}]$, which is abbreviated as $\mathbf{K}\mathbb{R}$.

An (ind-)definable function $A \longrightarrow \widehat{\mathbf{K}_+ \text{RV}}$ is a sequence $\mathbf{f} = (f_i)_{i \in \mathbb{N}}$, where the i th component f_i is a function in $\text{FN}(A, \mathbf{K}_+ \text{RV}_i)$. In other words, the set of all such functions is given by

$$\text{FN}(A, \widehat{\mathbf{K}_+ \text{RV}}) = \sum_i \text{FN}(A, \mathbf{K}_+ \text{RV}_i).$$

For $\mathbf{r} = (r_i)_{i \in \mathbb{N}} \in \widehat{\mathbf{K}_+ \text{RV}}$, let $\mathbf{r} \cdot \mathbf{f}$ be the function in $\text{FN}(A, \widehat{\mathbf{K}_+ \text{RV}})$ such that its k th component $(\mathbf{r} \cdot \mathbf{f})_k \in \text{FN}(A, \mathbf{K}_+ \text{RV}_k)$ is given by $(\mathbf{r} \cdot \mathbf{f})_k(\vec{a}) = \sum_{i+j=k} r_i \cdot f_j(\vec{a})$. This operation turns $\text{FN}(A, \widehat{\mathbf{K}_+ \text{RV}})$ into a natural $\widehat{\mathbf{K}_+ \text{RV}}$ -semimodule. Now we may integrate \mathbf{f} componentwise: $\int_{+A} \mathbf{f} = \sum_i \int_{+A} f_i$, where, since $\int_{+A} f_i \in \mathbf{K}_+ \text{RV}_{i+n}$, the first n terms of the right-hand side are 0. A simple computation shows that

$$\int_{+A} : \text{FN}(A, \widehat{\mathbf{K}_+ \text{RV}}) \longrightarrow \widehat{\mathbf{K}_+ \text{RV}}$$

is indeed a homomorphism of semimodules. Using the canonical semiring homomorphism

$$\widehat{\mathbf{K}_+ \text{RV}} \longrightarrow \widehat{\mathbf{K}\text{RV}} \longrightarrow \mathbf{K}\mathbb{R},$$

we define the set of (ind-)definable functions $A \longrightarrow \mathbf{K}\mathbb{R}$ by

$$\text{FN}(A, \mathbf{K}\mathbb{R}) = \text{FN}(A, \widehat{\mathbf{K}_+ \text{RV}}) \otimes_{\widehat{\mathbf{K}_+ \text{RV}}} \mathbf{K}\mathbb{R}.$$

Then \int_{+A} induces a canonical homomorphism of $\mathbf{K}\mathbb{R}$ -modules

$$\int_A : \text{FN}(A, \mathbf{K}\mathbb{R}) \longrightarrow \mathbf{K}\mathbb{R}.$$

For any definable bijection $\phi : A \longrightarrow B \subseteq \text{VF}^n$ we also have the (componentwise) Jacobian transformation:

$$\phi^{\text{Jcb}} : \text{FN}(A, \mathbf{K}\mathbb{R}) \longrightarrow \text{FN}(B, \mathbf{K}\mathbb{R}).$$

By Proposition 5.18 and Proposition 5.19, for any $\mathbf{f} \in \text{FN}(A, \mathbf{K}\mathbb{R})$, we obtain the following theorems:

Theorem 5.22 (Fubini theorem). *For any nonempty subsets $E_1, E_2 \subseteq I_n = \{1, \dots, n\}$,*

$$\int_{\vec{a} \in \text{pr}_{E_1}(A)} \int_{\text{fib}(A, \vec{a})} \mathbf{f} = \int_{\vec{a} \in \text{pr}_{E_2}(A)} \int_{\text{fib}(A, \vec{a})} \mathbf{f}.$$

Theorem 5.23 (Change of variables). $\int_A \mathbf{f} = \int_B \phi^{\text{Jcb}}(\mathbf{f})$.

We remind the reader that similar results are available for the coarse categories with Γ -volume forms (in $\mathfrak{C}^{\mathbb{T}}$).

6. The uniform rationality of certain Igusa local zeta functions

The integration theory developed so far is quite effective in showing uniform rationality of Igusa local zeta functions. In this last section we shall discuss such an application. The general idea is to specialize from the sufficiently saturated model to non-archimedean local fields. This is usually done in two steps: descent to an arbitrary henselian substructure and then specialization to all non-archimedean local fields of sufficiently large residue characteristic.

Lemma 6.1. *Let $M \supseteq S$ be a henselian substructure of \mathcal{C} . This means that $(VF(M), \mathcal{O}(M))$ is a nontrivially valued field and is henselian. If M is a substructure of \mathcal{C}^1 then $M = \text{dcl}^1(M)$. If M is a VF-generated substructure of \mathcal{C}^2 (resp. \mathcal{C}^3) and $\Gamma(M)$ is divisible then $M = \text{dcl}^2(M)$ (resp. $M = \text{dcl}^3(M)$).*

Proof. From the proof of [18, Theorem 3.14] it is clear that any valued field automorphism of \mathcal{C}^1 over M is an \mathcal{L}_{RV}^1 -automorphism of \mathcal{C} over M and hence the proof of [19, Lemma 6.2] may be easily adapted here. The other cases are similar. \square

Proposition 6.2. *In all the three cases above, M admits elimination of VF-quantifiers.*

Proof. It suffices to reproduce [19, Lemma 6.3] for the current setting. The key of its proof is to apply [19, Theorem 5.5] to the occurring polynomials in question and then apply [19, Lemma 6.2]. To imitate this argument, we may obviously apply Theorem 4.26 to the top occurring VF-terms in question and then apply Lemma 6.1. The details are left to the reader. \square

From now on we shall work in \mathcal{C}^2 and assume that the substructure $S = \text{dcl}^2(S)$ is generated by a “universal uniformizer” $\varpi \in \text{csn}(\Gamma)$, where $\Gamma(S)$ is identified with (the additive group of) \mathbb{Q} . A subset in the Γ -sort is a *rational polyhedron* if it is defined by a finite system of linear inequalities.

We shall only work with non-archimedean local fields. Let L range over all local fields and denote its residue characteristic, residue degree, and ramification degree by ϵ_L, δ_L , and ρ_L , respectively. Set $q_L = \epsilon_L^{\delta_L}$. We consider L as an \mathcal{L}_{RV}^2 -structure with $\varpi = p$ if L is an extension of \mathbb{Q}_p and $\varpi = t$ if L is an extension of $\mathbb{F}_p((t))$. This is why we do not normalize the value group of L to be \mathbb{Z} if $\text{char}(L) > 0$. The Haar measure $|\text{d}\vec{X}|_L$ on L is normalized so that the maximal ideal has measure 1. For any subset A in the VF-sort of L , the volume $\int_A |\text{d}\vec{X}|_L$ of A , if defined, is denoted by $\text{vol}(A)$. For example, $\text{vol}(\mathcal{O}(L)) = q_L$, $\text{vol}(\text{val}^{-1}(\gamma)) = (q_L - 1)q_L^{-\rho_L \gamma}$ for $\gamma \in \Gamma$, and, for $t \in \text{RV}(L)$ with $\text{vrv}(t) = \gamma$, $\text{vol}(\text{rv}^{-1}(t)) = q_L^{-\rho_L \gamma}$. We adopt the convention $q_L^{-\infty} = 0$. For any $a \in \text{VF}(L)$, the norm of a , denoted by $|a|_L$, is by definition the number $q_L^{-\rho_L \text{val}(a)}$.

Let F be either \mathbb{Q}_p or $\mathbb{F}_p((t))$ and $b_F \in \text{VF}(F)$ such that $\text{val}(b_F) \in \mathbb{Z}$ is constant as F varies. Let (L_i) be an infinite sequence of local fields such that (ϵ_{L_i}) is unbounded. Any ultraproduct M of (L_i) can be regarded as a substructure of \mathcal{C}^2 and the \mathcal{L}_{RV} -reduct of M as a substructure of \mathcal{C} . Let $b \in \text{VF}(M)$ be the element that corresponds to the sequence (b_F) . Suppose that $\Gamma(\text{dcl}^2(b)) = \mathbb{Q}$. Let $A \subseteq \text{VF}^n$ be a b -definable subset such that $\text{val}(A(M))$ is bounded from below in $\Gamma(M)^n$. Let $\vec{f} = (f_1, \dots, f_k)$ be a sequence of b -definable functions $A \rightarrow \Gamma$ such that $\vec{f}(A(M)) \subseteq \Gamma(M)^k$ is a Presburger subset that is independent of b , that is, $\vec{f}(A(M))$ is definable in the Γ -sort of M in the Presburger language without parameters. For each $\vec{\gamma} \in \Gamma^k$ let $\pi_{\vec{\gamma}}$ be a Γ -partition of the subset $\{\vec{a} \in A : \vec{f}(\vec{a}) = \vec{\gamma}\}$. By Lemma 2.21 and Remark 2.22, the image of $\pi_{\vec{\gamma}}$ is $\vec{\gamma}$ - \mathcal{L}_{RV} -definable (independent of b). Let π be the function on A given by $\vec{a} \mapsto (\vec{f}(\vec{a}), \pi_{\vec{\gamma}(\vec{a})}(\vec{a}))$ and $A_{\vec{\gamma}} = \pi^{-1}(\vec{\gamma})$. We write J for $\pi(A(M))$. It is easy to see that π may be chosen so that $J \subseteq \pi(A)(M)$, where the inclusion may be proper. Consequently, J is also a Presburger subset that is independent of b . Now, for every $A_{\vec{\gamma}}$, by Corollary 4.27 (in fact we only need the special case [19, Corollary 5.6]), there is an RV-pullback $A_{\vec{\gamma}}^{\sharp} \subseteq \text{VF}^n \times \text{RV}^m$ and a $\text{csn}(\vec{\gamma})$ - \mathcal{L}_{RV} -definable special bijection $T_{\vec{\gamma}}$ between $A_{\vec{\gamma}}$ and $A_{\vec{\gamma}}^{\sharp}$. By Lemma 6.1, $T_{\vec{\gamma}}(M)$ is a bijection between $A(M)$ and $A_{\vec{\gamma}}^{\sharp}(M)$. In particular, this implies that if $\vec{\gamma} \in \pi(A)(M) \setminus J$ then $A_{\vec{\gamma}}^{\sharp}(M) = \emptyset$.

Let $\vec{\kappa} = (\kappa_1, \dots, \kappa_k)$ be a sequence of positive real numbers. Consider the (generalized) Igusa local zeta function

$$\zeta(A, L, \vec{\kappa}) = \int_A q_L^{-\rho_L \sum_i \kappa_i f_i(\vec{X})} |\text{d}\vec{X}|_L.$$

The evaluation of $\zeta(A, L, \vec{\kappa})$ may be reduced to computing the volume of each $A_{\vec{\gamma}}$, which is $\text{csn}(\vec{\gamma})$ - \mathcal{L}_{RV} -definable. In [16, 17], it is shown that, if $\text{char}(L_i) = 0$ for all i and, among the three sequences (ϵ_{L_i}) , (δ_{L_i}) , and (ρ_{L_i}) , if the first or the second is the only unbounded one (here and below “unbounded” means “going to infinity”) then $\zeta(A, L_i, \vec{\kappa})$ is uniformly rational (see [5,4] for a motivic interpretation of these results). In fact, it is easy to see that the results in [16,17] imply that $\zeta(A, L_i, \vec{\kappa})$ is uniformly rational as long as (ρ_{L_i}) is bounded. We shall generalize these results such that local fields of positive characteristic are included and all three sequences are unbounded.

By Lemma 6.1, we may simultaneously work in all but finitely many L_i as far as \mathcal{L}_{RV} -formulas are concerned. In other words, from now on, by an \mathcal{L}_{RV} -definable subset A we mean a uniformly \mathcal{L}_{RV} -definable subset in (L_i) , which in turn means a sequence of subsets $(A_i)_{i \geq k}$ for some sufficiently large k such that every A_i is a subset of the \mathcal{L}_{RV} -reduct of L_i defined by

a fixed quantifier-free \mathcal{L}_{RV} -formula ϕ . To reason about such an A , or rather about all A_i for $i \geq k$, we can (and shall tacitly) work with the subset defined by ϕ in \mathfrak{C} and then state the results with respect to each L_i uniformly. The reader should note that in this process the number k may increase. With this understanding, for example, we may talk about the size of a definable subset in the \bar{K} -sort and infinite summation over a definable subset in the Γ -sort (since it may be identified with a union of rational polyhedrons). For simplicity, we shall drop “ L_i ” from some of the notations below. For example, $\text{VF}(L_i)$, ρ_{L_i} , etc., will simply be written as VF , ρ , etc.

Proposition 6.3. *The integral $\zeta(A, \vec{\kappa})$ may be written as a finite sum of μ terms of the form $e \sum_{d=1}^{\rho^\nu} \tau_d(q)$, where e, ν are natural numbers,*

$$\tau_d(q) = \sum_{(\vec{n}, \vec{m}, \vec{l}) \in \Delta_d} q^{-\vec{\kappa} \cdot \vec{l} - \Sigma \vec{n}},$$

and $\Delta_1, \dots, \Delta_{\rho^\nu}$ are pairwise disjoint Presburger subsets. This expression of $\zeta(A, \vec{\kappa})$ is uniform for the sequence (L_i) in the following sense:

- (1) μ and ν do not depend on L_i ,
- (2) if $\rho_{L_i} = c \rho_{L_j}$ then every Δ_d that occurs in $\tau_d(q_{L_j})$ also occurs in $\tau_d(q_{L_i})$ as $c \Delta_d$.

Proof. To compute $\zeta(A, \vec{\kappa})$, we may assume that, for all $\vec{\gamma} \in J$, the $\text{csn}(\vec{\gamma})$ - \mathcal{L}_{RV} -definable RV-pullback $A_{\vec{\gamma}}^\sharp$ is nondegenerate. Let $B_{\vec{\gamma}} = \text{prv}(A_{\vec{\gamma}}^\sharp)$. Then we have

$$\text{vol}(A_{\vec{\gamma}}) = \sum_{(\vec{t}, \vec{s}) \in B_{\vec{\gamma}}} q^{-\rho \Sigma \text{vrv}(\vec{t})}.$$

By Lemma 3.3, without loss of generality, we may assume that $B_{\vec{\gamma}}$ is a twistoid. Let $U_{\vec{\gamma}}$ be the twistback of $B_{\vec{\gamma}}$ and $I_{\vec{\gamma}} = \text{vrv}(B_{\vec{\gamma}})$. Note that $I_{\vec{\gamma}}$ is $\vec{\gamma}$ - \mathcal{L}_{RV} -definable (independent of b). Let $\sharp(U_{\vec{\gamma}})$ be the size of $U_{\vec{\gamma}}$. Then

$$\text{vol}(A_{\vec{\gamma}}) = \sharp(U_{\vec{\gamma}}) \sum_{(\vec{\alpha}, \vec{\beta}) \in I_{\vec{\gamma}}} q^{-\rho \Sigma \vec{\alpha}}.$$

We may assume that $U_{\vec{\gamma}}$ is the same and $\sharp(U_{\vec{\gamma}}) = e$ for all $\vec{\gamma} \in J$. Let $\Delta = \bigcup_{\vec{\gamma} \in J} I_{\vec{\gamma}} \times \{\vec{\gamma}\} \subseteq \Gamma^\nu$, which is of course a Presburger subset that is independent of b . Then

$$\zeta(A, \kappa) = \sum_{\vec{\gamma} \in J} q^{-\rho \vec{\kappa} \cdot \vec{\gamma}} \text{vol}(A_{\vec{\gamma}}) = e \sum_{(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) \in \Delta} q^{-\rho \vec{\kappa} \cdot \vec{\gamma} - \rho \Sigma \vec{\alpha}}.$$

For the desired uniformity, it remains to dispose of ρ in this expression. To that end, we identify Γ with \mathbb{Z} via the canonical homomorphism $\gamma \mapsto \rho\gamma$. Then Δ is identified with a Presburger subset $\Delta_* \subseteq \mathbb{Z}^\nu$. Obviously Δ_* may be decomposed into ρ^ν Presburger subsets Δ_d for $1 \leq d \leq \rho^\nu$ such that the i th coordinate of Δ_d equals d_i modulo ρ for some $1 \leq d_i \leq \rho$. Note that if $\text{gcd}(d_1, \dots, d_\nu) = c$ then Δ_d may also be defined using $\rho/c, d_1/c, \dots, d_\nu/c$. Now the uniformity condition is clear since $\Gamma(L_j) = (1/\rho_{L_j})\mathbb{Z}$ is a subgroup of $\Gamma(L_i) = (1/\rho_{L_i})\mathbb{Z}$. \square

Definition 6.4. Let \mathfrak{L} be a set of local fields. We say that $\zeta(A, \vec{\kappa})$ is *uniformly rational* for \mathfrak{L} if there is a finite set of rational functions $R_{m,i}(T_0, T_1, \dots, T_k)$ for each $m \in \mathbb{N}$ such that, for every $L \in \mathfrak{L}$,

$$\zeta(A, L, \vec{\kappa}) = \sum_{m|\rho_L} \sum_i e_{m,i,L} R_{m,i}(q_L^{\rho_L/m}, q_L^{-\rho_L \kappa_1/m}, \dots, q_L^{-\rho_L \kappa_k/m}),$$

where the coefficients $e_{m,i,L} \in \mathbb{N}$ only depend on q_L .

Note that our notion of uniformity is different from but implies that in [16,17]. It is clear from the proofs of the main results of [16,17] that they may be reformulated using our notion.

We can always make $\zeta(A, \vec{\kappa})$ uniformly rational for (L_i) by deleting finitely many entries from it. If local fields of positive characteristic are included then this cannot be improved at the moment, since rationality of Igusa zeta function, with or without cross-section, is not known in general for local fields of small positive characteristic. On the other hand, if we concentrate on p -adic fields then, using results in [6,7,16,17], we can deduce very general results about uniform rationality.

Theorem 6.5. *If each L_i is a local field of characteristic 0 then $\zeta(A, \vec{\kappa})$ is uniformly rational for (L_i) .*

Proof. This follows immediately from [7, Theorem 4.3], (the proof of) Proposition 6.3, and [5, Theorem 4.4.1]. There are (potentially) infinitely many rational functions because (ρ_{L_i}) may be unbounded, which gives rise to infinitely many Presburger subsets. \square

For each $n \in \mathbb{N}$ let \mathfrak{L}_n be the set of all local fields L of characteristic 0 such that $\rho_L \leq n$.

Theorem 6.6. *For all local fields of characteristic 0, $\zeta(A, \vec{\kappa})$ is uniformly rational.*

Proof. By [16, Theorem 7.1] and [17, Theorem 5.1], we only need to work with a finite subset $\mathfrak{L}'_n \subseteq \mathfrak{L}_n$ for each n . Then the assertion follows from Theorem 6.5. \square

Remark 6.7. There are variations of these theorems. For example, they hold if we work in \mathfrak{C} instead of \mathfrak{C}^2 (see [10, Theorem 1.3]). There are also analogues if we work in \mathfrak{C}^3 . However, in that case local fields of characteristic 0 have to be excluded (see Remark 2.3) and hence there always are finitely many exceptions about which we can say nothing at the moment.

Suppose that (ϵ_{L_i}) , (δ_{L_i}) are bounded and (ρ_{L_i}) is unbounded in (L_i) . What can one say about these infinitely many rational functions? Is it true that $\zeta(A, \vec{\kappa})$ is uniform for (L_i) with respect to finitely many rational functions? These are difficult and deep questions, and are related to the asymptotic behavior of the poles of $\zeta(A, \vec{\kappa})$ and its rationality in a local field of small positive characteristic. We hope that future development of the present theory will be able to offer some clues.

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