Canonical Forms and Spectral Determination for a Class of Hyperbolic Distributed Parameter Control Systems*

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In this article we examine the effect of linear feedback control $u(t) = (u_1(t), u_2(t))$, in the hyperbolic distributed parameter control system

$$\frac{\partial}{\partial t} \begin{pmatrix} \omega \\ \psi \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \omega \\ \psi \end{pmatrix} - A(x) \begin{pmatrix} \omega \\ \psi \end{pmatrix} = g(x) u(t).$$

By means of a reduction to canonical form similar to the one already familiar for finite-dimensional systems we show this system to be equivalent to the controlled difference-delay system

$$\zeta(t + 2) = e^{\alpha_2} \zeta(t) + \int_0^2 p(2 - s) \zeta(t + s) \, ds + u(t).$$

The theory of nonharmonic Fourier series is then employed to investigate the placement of eigenvalues in the closed loop system. Boundary value control and canonical form for observed systems are also studied.

INTRODUCTION

The purpose of this paper is to study the problem of spectral determination via linear state feedback for a particular, simple class of hyperbolic control systems. These systems are two-dimensional, first order equations of the form

$$\frac{\partial}{\partial t} \begin{pmatrix} \omega \\ \psi \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \omega \\ \psi \end{pmatrix} - A(x) \begin{pmatrix} \omega \\ \psi \end{pmatrix} = g(x) u(t),$$  \tag{0.1}

where $A(x)$ is a continuous $2 \times 2$ matrix and $g(x) = (g_1(x), g_2(x))$ is a two-dimensional vector.

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vector function in $L^2[0, 2]$. The symbol $u$ denotes a scalar control function. We will be concerned with solutions of (0.1) which obey boundary conditions

\begin{align}
  a_0 w(0, t) + b_0 v(0, t) &= 0, \quad |a_0|^2 + |b_0|^2 \neq 0, \\
  a_1 w(1, t) + b_1 v(1, t) &= 0, \quad |a_1|^2 + |b_1|^2 \neq 0.
\end{align}

(0.2) (0.3)

We will also have occasion to consider the case wherein the term $g(x) u(t)$ in (0.1) is replaced by zero while the boundary condition (0.3) becomes

\begin{align}
  a_1 w(1, t) + b_1 v(1, t) &= u(t),
\end{align}

(0.4)

a case of "boundary value control."

With certain additional restrictions on $a_0, b_0, a_1, b_1$ in (0.2), (0.3), the equation (0.1), with $u(t) \equiv 0$, has solutions $(w(x, t), v(x, t))$ such that the operators $S(t): L^2([0, 1]; E^2) \to L^2([0, 1]; E^2)$

\[ S(t) \begin{pmatrix} w(\cdot, 0) \\ v(\cdot, 0) \end{pmatrix} = \begin{pmatrix} w(\cdot, t) \\ v(\cdot, t) \end{pmatrix} \]

constitute a group of bounded operators on $L^2([0, 1]; E^2)$ for $-\infty < t < \infty$ with generator

\[ L_0 \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w \\ v \end{pmatrix} + A(x) \begin{pmatrix} w \\ v \end{pmatrix} \]

defined on the domain

\[ D = \left\{ \begin{pmatrix} w \\ v \end{pmatrix} \in H^1([0, 1]; E^2) \mid a_0 w(0) + b_0 v(0) = a_1 w(1) + b_1 v(1) = 0 \right\}. \]

It will be seen in Section 3 that $L_0$ has spectrum

\[ \sigma(L_0) = \{ \sigma_j \mid \sigma_j = \alpha + 2j\pi i + o(1/j), j = 0, \pm 1, \pm 2, \ldots \} \]

for some complex $\alpha$.

Suppose now that the control $u(t)$ in (0.1) is determined by a linear feedback relation

\[ u(t) = \int_0^1 k(x)^* \begin{pmatrix} w(x, t) \\ v(x, t) \end{pmatrix} \, dx \]

(0.5)

for some two-dimensional vector function $k \in L^2([0, 1]; E^2)$. The result is a "closed loop" system

\[ \frac{\partial}{\partial t} \begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w \\ v \end{pmatrix} - A(x) \begin{pmatrix} w \\ v \end{pmatrix} - g(x) \int_0^1 k(\xi)^* \begin{pmatrix} w(\xi, t) \\ v(\xi, t) \end{pmatrix} \, d\xi = 0 \]

(0.6)
wherein the "dyadic" \(-g(x) \int_0^1 k(\xi) \ast (w(\xi, t)) \, d\xi\) has been added to the equation. Associated with (0.6) is the group of bounded operators \(S_k(t)\) with generator

\[
L_k \left( \begin{array}{c} w \\ v \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \frac{\partial}{\partial x} \left( \begin{array}{c} w \\ v \end{array} \right) + A(x) \left( \begin{array}{c} w \\ v \end{array} \right) + g(x) \int_0^1 k(\xi) \ast (w(\xi, t)) \, d\xi
\]

and spectrum \(\rho_j, j = 0, \pm 1, \pm 2, \ldots\). We ask: what values can we assign to the \(\xi\), by appropriate choice of the feedback vector function \(k\)?

We will show in Section 6 that the \(\rho_j\) can be assigned any distinct values whatsoever, provided that

\[
\sum_j \left| \frac{\rho_j - \alpha_j}{g_j} \right|^2 < \infty,
\]

where the numbers \(g_j\) represent the expansion coefficients of \(g\) with respect to the eigenfunctions of the operator \(L_0\). A similar (and, in fact, improved) result is obtained in the case of boundary value control (0.4).

Our work here is a natural extension of the eigenvalue assignment problem for finite-dimensional control systems

\[
\dot{w} = Aw + Gu \tag{0.7}
\]

where it is desired to specify the spectrum of the matrix \(A + GK\) in the closed loop system

\[
\dot{w} = (A + GK) w \tag{0.8}
\]

realized by setting \(u = Kw\) [12, 24]. The familiar result here is that this can be done at will proved that the pair \(A, G\) is controllable; analogous to the condition \(g_j \neq 0, j = 0, \pm 1, \pm 2, \ldots\) in our case. The present work also represents an improvement over the author's earlier attempts [18, 19] to relate controllability of hyperbolic systems to stabilization and the problem of spectral determination for the generator. Recent work in the area of stabilization for higher-dimensional hyperbolic systems [15, 20, 23], suggests that analogous results for these systems may eventually be forthcoming. Some remarks concerning possible extensions of our theory are provided in Section 7.

The problem of eigenvalue assignment, or spectral determination, for (0.7), (0.8) is resolved by reducing the system to a more easily analyzed canonical form—which in fact is the Frobenius form long familiar in matrix theory [6]. We carry out an analogous reduction to canonical form in our infinite-dimensional system and the reducing transformations are easily seen to be the counterparts of those employed in the finite-dimensional situation once we provide an appropriate interpretation for them. Because this interpretation is of some importance we will briefly review the finite-dimensional theory for the recursion
equation (whose control theory in many ways bears more resemblance to that of (0.1) than does the theory of the differential system (0.7))

\[ w_{k+1} = Aw_k + gu_k, \quad w_k \in E^n, \quad u_k \in E^1. \] (0.9)

We will also find it convenient to review canonical forms for observed systems

\[ y_{k+1} = Cy_k, \] (0.10)
\[ \omega = h^*y_k, \] (0.11)

and, later, to develop analogous forms for observed systems

\[ \frac{\partial}{\partial t} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} - C(x) \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = 0, \] (0.12)
\[ \omega(t) = \int_0^1 h(x)^* \begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} dx \] (0.13)

dual to the control systems (0.1).

1. Review of Finite-Dimensional Results

Before beginning the main body of our article it will be instructive to review known results concerning canonical forms and eigenvalue assignment for finite-dimensional control systems governed by linear differential or recursion equations. These results are, of course, very well known and may be found, e.g., in [12]. A particularly detailed treatment appears in [24] together with further references. In this paper, in order to avoid complexities which might make our presentation obscure, we will confine our attention to scalar observations and controls. We have indicated that the linear recursion model is most appropriate for comparison with (0.1), (0.2), (0.3) but the differential equation case can be treated essentially simultaneously.

We consider then the linear continuous observed system

\[ \dot{y} = Cy, \quad y \in E^n, \] (1.1)
\[ \omega = h^*y, \quad \omega \in E^1, \]

and the corresponding linear discrete observed system

\[ y_{k+1} = Cy_k, \quad y_k \in E^n, \] (1.2)
\[ \omega_k = h^*y_k, \quad \omega_k \in E^1. \]

Paired with these are the linear continuous control system

\[ \dot{w} = Aw + gu, \quad w \in E^n, \quad u \in E^1 \] (1.3)
and the linear discrete control system

$$w_{k+1} = Aw_k + g u_k, \quad w_k \in E^n, \quad u_k \in E^1. \quad (1.4)$$

We introduce the following assumptions.

**Observability Assumption.**

\[
\begin{pmatrix}
h*C_{n-1} \\
h*C_{n-2} \\
\vdots \\
h*C \\
h*
\end{pmatrix}
\]

\[
\text{rank} = n.
\]

**Controllability Assumption.**

\[
\text{rank}(A^{n-1}g, A^{n-2}g, \ldots, Ag, g) = n.
\]

When $C = A^*$, $g = h$ the systems (1.1) and (1.3) are dual to each other, as are (1.2) and (1.4). In this case the above assumptions are readily seen to be equivalent.

Let us begin our discussion with the system (1.2). We assume some $y_0 \in E^n$ is given, but not known to us a priori. The solution $\{y_k\}$ to (1.2) with the given initial vector is allowed to develop and we record the observations $\omega_k = h*y_k$, $k = 0, 1, 2, \ldots$. Since $y_k = C^k y_0$, we have

\[
\begin{pmatrix}
\omega_{n-1} \\
\omega_{n-2} \\
\vdots \\
\omega_1 \\
\omega_0
\end{pmatrix} = \begin{pmatrix}
h*C_{n-1} \\
h*C_{n-2} \\
\vdots \\
h*C \\
h*
\end{pmatrix} y_0 = Oy_0. \quad (1.5)
\]

The Observability Assumption implies that the "observation matrix" $O$ is nonsingular. Thus we may define the "reconstruction matrix"

$$R = O^{-1}.$$  

Multiplying (1.5) on the left by $R$ we have

$$y_0 = R \begin{pmatrix}
\omega_{n-1} \\
\omega_{n-2} \\
\vdots \\
\omega_1 \\
\omega_0
\end{pmatrix}.$$
The matrices $O$ and $R$ establish a linear isomorphism between states $y_0$ and observation sequences $\omega_{n-1}, \omega_{n-2}, \ldots, \omega_1, \omega_0$ of length $n$. Let us think of $y_0$ as lying in $E^n$ and the corresponding observation sequence as lying in $E^n$. The discrete dynamical system (1.2) is carried over, via $O$, into a transformed system in $\tilde{E}^n$, which is

\[
\tilde{\eta}_{k+1} = OCR\tilde{\eta}_k, \quad (1.6)
\]
\[
\omega_k = h^*R\tilde{\eta}_k. \quad (1.7)
\]

Now

\[
OC = \left(\begin{array}{c}
    h^*C^n \\
    h^*C^{-1} \\
    \vdots \\
    h^*C^2 \\
    h^*C
\end{array}\right).
\]

Since the last $n - 1$ rows of $OC$ are the same as the first $n - 1$ rows of $O = R^{-1}$, we clearly have

\[
OCR = \left(\begin{array}{c}
    h^*C^nR \\
    e^*_1 \\
    \vdots \\
    e^*_{n-2} \\
    e^*_n
\end{array}\right)
\]

where $e^*_j$ is the $j$th row of the $n \times n$ identity matrix. On the other hand, $h^*$ is the last row of $O = R^{-1}$, so

\[
h^*R = e^*_n.
\]

If we now write

\[
h^*C^nR = (-c_1, -c_2, \ldots, -c_{n-1}, -c_n)
\]

the system (1.6), (1.7) in $\tilde{E}^n$ becomes

\[
\tilde{\eta}_{k+1} = \tilde{C}\tilde{\eta}_k, \quad (1.8)
\]
\[
\omega_k = e^*n\tilde{\eta}_k = \tilde{\eta}_k^n, \quad \text{(last component of } \tilde{\eta}_k), \quad (1.9)
\]

where

\[
\tilde{C} = \begin{pmatrix}
-c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix} \quad (1.10)
\]
has characteristic polynomial

\[ c(\lambda) = \lambda^n + \sum_{k=1}^{n} c_k \lambda^{n-k}. \]

For want of a better term we shall follow Gantmacher [6, Vol. I, Chap. VII, Section 5] and refer to (1.8), (1.9) as the observation normal form. To obtain what is usually called the observation canonical form we introduce the matrix

\[
I + N = \begin{pmatrix}
1 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
0 & 1 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
0 & 0 & 1 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & c_1
\end{pmatrix}.
\]

Since \( N \) is nilpotent (\( N^n = 0 \)), we have

\[
(I + N)^{-1} = I - N + N^2 \cdots (-1)^{n-1} N^{n-1}.
\]

Letting

\[ \eta = (I + N) \bar{\eta} \]

(1.8), (1.9) becomes

\[
\eta_{k+1} = \hat{C} \eta_k, \quad \omega_k = e_n^*(I + N)^{-1} \eta_k = e_n^* \eta_k, \quad \text{(1.11)}
\]

where

\[
\hat{C} = \begin{pmatrix}
0 & 0 & \cdots & -c_n \\
1 & 0 & \cdots & 0 & -c_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -c_2 \\
0 & 0 & \cdots & 1 & -c_1
\end{pmatrix} \quad \text{(1.13)}
\]

There are numerous reasons why the canonical form (1.11), (1.12), (1.13) is important. One of the most significant arises from the fact that in filtering theory one frequently modifies the system (1.11) to

\[
\eta_{k+1} = \hat{C} \eta_k + \ell \omega_k \quad \text{(1.14)}
\]

with some appropriate vector \( \ell \in \mathbb{E}^n \). Using (1.12), (1.14) becomes

\[
\eta_{k+1} = (\hat{C} + \ell e_n^*) \eta_k. \quad \text{(1.15)}
\]
Now, letting the components of $e$ be $e_j, j = 1, 2, \ldots, n,$

\[
\mathcal{C} + \ell e^*_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & -c_n + \ell^1 \\
1 & 0 & \cdots & 0 & -c_{n-1} + \ell^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -c_2 + \ell^{n-1} \\
0 & 0 & \cdots & 1 & -c_1 + \ell^n 
\end{pmatrix}
\]

whose characteristic polynomial

\[
p(\lambda) = \lambda^n + \sum_{k=1}^{n} (c_k - \ell^k) \lambda^{n-k}
\]

can be assigned arbitrary complex zeros by choosing $\ell$ appropriately. Thus various types of dynamical behavior can be achieved for (1.13).

For (1.1) the reduction to normal form

\[
\dot{\eta} = \mathcal{C}\eta, \\
\omega = e^*_n \mathcal{C}\eta,
\]

is again accomplished via $y = \mathcal{R}\eta$, with $\mathcal{R} = O^{-1}, O$ as in (1.5). Then $\mathcal{R} = (I + N)^{-1} \eta$ produces

\[
\dot{\eta} = \mathcal{C}\eta, \\
\omega = e^*_n \eta.
\]

The matrices $\mathcal{C}, \mathcal{C}$ are the same as in (1.10) and (1.13) and the form of $\mathcal{C}$ is important for the same reason as cited above. Note in this case, however, that the map

\[
\mathcal{R} = O \eta
\]

takes the state vector $y$ into the vector $\mathcal{R} \in \mathcal{E}^n$ whose components are the successive derivatives of the observation $\omega = h^*y$, i.e.,

\[
\mathcal{R} = \begin{pmatrix}
\omega^{(n-1)} \\
\omega^{(n-2)} \\
\vdots \\
\omega' \\
\omega
\end{pmatrix} = \begin{pmatrix}
h^* C^{n-1} \\
h^* C^{n-2} \\
\vdots \\
h^* C' \\
h^*
\end{pmatrix} y.
\]

Now we pass to control systems. Beginning with (1.4), we let $u_0, u_1, \ldots, u_{n-1}$
be a sequence of scalar control values. Starting with \( w_0 = 0 \), this control sequence produces the state

\[
W_n = (A^{n-1}g, A^{n-2}g, \ldots, A^2g, Ag, g) \equiv Uw.
\]

Our controllability assumption makes \( U \) nonsingular, so we may define

\[ S = U^{-1}. \]

We may view \( S \) as the "steering matrix" for the system since, given \( w_n \in \mathbb{E}^n \),

\[
\begin{pmatrix}
U^0 \\
U^1 \\
\vdots \\
U^{n-2} \\
U^{n-1}
\end{pmatrix} = u = Sw_n
\]

provides the unique control sequence steering (1.4) from 0 to \( w_n \) in \( n \) steps. Again, the linear isomorphism \( w_j = U\xi_j \) carries the control system (1.4) in \( \mathbb{E}^n \) into a transformal system in \( \mathbb{E}^m \) (the space of control sequences now)

\[
\xi_{j+1} = SAU\xi_j + Sg\xi_j.
\]

Now \( S = U^{-1} \) and, since

\[
AU = (A^{n}g, A^{n-1}g, \ldots, A^2g, Ag)
\]

shows that the last \( n - 1 \) columns of \( AU \) are the first \( n - 1 \) columns of \( U \), we have

\[
SAU = (SA^n g, e_1, e_2, \ldots, e_{n-1}).
\]

On the other hand \( g \) is the \( n \)th column of \( U \) so

\[ Sg = e_n. \]

Finally we let

\[
SA^n g = \begin{pmatrix}
-a^1 \\
-a^2 \\
\vdots \\
-a^{n-1} \\
-a^{n-2}
\end{pmatrix}
\]
and we see that (1.17), the transformed control system in the space of control sequences, has the form

$$\xi_{j+1} = \bar{A}\xi_j + e_n u_j, \quad (1.18)$$

with

$$\bar{A} = \begin{pmatrix} -a^1 & 1 & 0 & \cdots & 0 \\ a^2 & 0 & 1 & \cdots & 0 \\ -a^3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a^{n-1} & 0 & 0 & \cdots & 1 \\ -a^n & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (1.19)$$

We shall refer to (1.18) as the control normal form for (1.4). When $A = C^*$ the matrix here is the adjoint of the matrix $\hat{C}$ occurring in the observation canonical form. Passage to the control canonical form is effected by setting

$$\xi_j = (I + M)^{-1} \xi_j = (I + M)^{-1} \xi_j, \quad (1.20)$$

Again, $M$ is nilpotent and

$$(I + M)^{-1} = I - M + M^2 \cdots (-1)^{n-1} M^{n-1}.$$ The control canonical form is then

$$\xi_{j+1} = (I + M)^{-1} \bar{A}(I + M) \xi_j + (I + M)^{-1} e_n u_j = \bar{A}\xi_j + e_n u_j, \quad (1.21)$$

$$\bar{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ a^1 & 1 & \cdots & 0 & 0 \\ a^2 & a^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{n-2} & a^{n-3} & \cdots & 1 & 0 \\ a^{n-1} & a^{n-2} & \cdots & a^1 & 1 \end{pmatrix}. \quad (1.22)$$

The significance of this form lies in the fact that it enables us to see with facility the effect of a linear feedback control

$$u_j = k^* \xi_j = (k_1, k_2, \ldots, k_n) \xi_j.$$
For with this specification of $u$ the closed loop system becomes

$$\xi_{j+1} = (A + e_n k^*) \xi_j$$

and

$$\tilde{A} + e_n k^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a^n + k_1 & -a^{n-1} + k_2 & -a^{n-2} + k_2 & \cdots & -a^1 + k_n \end{pmatrix}$$

whose characteristic polynomial

$$\lambda^n + \sum_{j=1}^n (a^j - k_{n+1-j}) \lambda^{n-j}$$

can be assigned any desired roots by appropriate choice of the feedback functional $k^*$.

The continuous control system (1.3) is transformed in exactly the same way to

$$\xi = \tilde{A} \xi + e_n u.$$

The point which we wish to stress in connection with the reduction to canonical form of both (1.2) and (1.4) is the nature of the preliminary transformations

$$\tilde{\eta} = Oy,$$

$$\tilde{\xi} = Sw,$$

respectively, which carry the system state space into the space of observation sequences in the first instance and the space of control sequences in the second instance, so that the transformed systems act within these latter spaces. This interpretation of the reduction to observation or control normal form, respectively, provides the key to generalization of this process to more general systems. The hyperbolic system (0.1) will be treated in this manner in the sequel.

2. EIGENVALUES, BIORTHOGONAL SEQUENCES, CONTROLLABILITY

If we look for solutions of

$$\frac{\partial}{\partial t} \begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w \\ v \end{pmatrix} - A(x) \begin{pmatrix} w \\ v \end{pmatrix} = 0$$

(2.1)
which obey (1.2), (1.3) and take the form

\[
\begin{pmatrix}
\psi(x, t) \\
\phi(x, t)
\end{pmatrix} = e^{ot} \begin{pmatrix}
\phi_1(x) \\
\phi_2(x)
\end{pmatrix}
\]

we arrive at the eigenvalue problem for the operator \( L_0 \):

\[
\begin{align*}
\sigma \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} &- \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} - A(x) \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} = 0, \\
\alpha_\phi \phi_1(0) + b_\phi \phi_2(0) &= 0, \quad a_\phi \phi_1(1) + b_\phi \phi_2(1) = 0.
\end{align*}
\] (2.2, 2.3)

It is instructive to consider first of all the case where \( A(x) = 0 \). We then have

\[
\phi_1(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x},
\phi_2(x) = c_1 e^{\alpha x} - c_2 e^{-\alpha x}
\]

for some scalar constants \( c_1, c_2 \). Substituting these expressions in (2.3) we find

\[
\begin{pmatrix}
\alpha_\phi + b_\phi \\
\alpha_\phi - b_\phi
\end{pmatrix} \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = 0.
\]

The condition for a nonzero solution \( c_1, c_2 \) is that the determinant should vanish, which reduces to the condition

\[
e^{2\alpha} = \frac{(\alpha_\phi + b_\phi)(\alpha_\phi - b_\phi)}{(\alpha_\phi - b_\phi)(\alpha_\phi + b_\phi)} = \gamma.
\] (2.4)

We will assume that the right-hand side, \( \gamma \), of (2.4) is neither zero nor infinite. Then we have an infinite collection of eigenvalues \( \hat{\sigma}_k, k = 0, \pm 1, \pm 2, \ldots \), given by

\[
\hat{\sigma}_k = \frac{1}{2} \log \gamma + k\pi i = \alpha + k\pi i,
\] (2.5)

where \( \log \gamma \) refers to the principal value of the logarithm.

Using methods of the type found in [1, 2, 8, 9] and many other places, one can show that for arbitrary differentiable \( A(x) \) the eigenvalue problem (2.2), (2.3) for the operator \( L_0 \) has eigenvalues \( \sigma_k, k = 0, \pm 1, \pm 2, \ldots \) obeying the asymptotic relation

\[
\sigma_k = \hat{\sigma}_k + O(1/k) = \alpha + k\pi i + O(1/k).
\] (2.6)

We will assume in this paper that the \( \sigma_k \) all have single multiplicity. Sufficient
conditions for this to be true are given in [1]. Corresponding to these eigenvalues are two-dimensional eigenfunctions

\[ \phi_k(x) = \begin{pmatrix} \phi_k^1(x) \\ \phi_k^2(x) \end{pmatrix}, \quad k = 0, \pm 1, \pm 2, \ldots \]

forming a Riesz basis for the space \( L^2([0, 1]; E^2) \) (a Riesz basis is the image of an orthonormal basis under a bounded and boundedly invertible linear transformation). Every element \((w, v) \in L^2([0, 1]; E^2)\) has an expansion

\[ \begin{pmatrix} w \\ v \end{pmatrix} = \sum_{k=\pm\infty} c_k \begin{pmatrix} \phi_k^1 \\ \phi_k^2 \end{pmatrix} \quad (2.7) \]

convergent in \( L^2([0, 1]; E^2) \) and there are positive numbers \( d, D \) such that

\[ d^2 \left\| \begin{pmatrix} w \\ v \end{pmatrix} \right\|^2 _{L^2([0,1];E^2)} \leq \sum_{k=\pm\infty} |c_k|^2 \leq D^2 \left\| \begin{pmatrix} w \\ v \end{pmatrix} \right\|^2 _{L^2([0,1];E^2)} . \]

If we introduce the adjoint operator \( L^*_0 \) defined by

\[ L^*_0 \begin{pmatrix} y \\ z \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} y \\ z \end{pmatrix} + A(x)^* \begin{pmatrix} y \\ z \end{pmatrix} \quad (2.8) \]

on the domain \( D^* \subseteq L^2([0, 1]; E^2) \) consisting of function pairs \((\psi)_y \in H^1([0, 1]; E^2)\) which obey boundary conditions

\[ \bar{a}_0 y(0) - \bar{b}_0 z(0) = 0, \quad \bar{a}_1 y(1) - \bar{b}_1 z(1) = 0 \quad (2.9) \]

its eigenvalues are \( \bar{\sigma}_k, \quad k = 0, \pm 1, \pm 2, \ldots \) with corresponding eigenfunctions

\[ \psi_k(x) = \begin{pmatrix} \psi_k^1(x) \\ \psi_k^2(x) \end{pmatrix}, \quad k = 0, \pm 1, \pm 2, \ldots \]

Correctly scaled, the \( \psi_k \) form the dual basis for \( L^2([0, 1]; E^2) \), satisfying the biorthogonality relations

\[ (\phi_k, \psi_\ell)_{L^2([0,1];E^2)} = \delta_k^\ell = 1, \quad k = \ell \]

\[ = 0, \quad k \neq \ell \]

and the coefficients \( c_k \) in the expansion (2.7) are given by

\[ c_k = \left( \begin{pmatrix} w \\ v \end{pmatrix}, \begin{pmatrix} \psi_k^1 \\ \psi_k^2 \end{pmatrix} \right)_{L^2([0,1];E^2)} \]
We are ready now to consider the controllability problem for (0.1). Let the initial state
\[
\begin{pmatrix}
v(x, 0) \\
v(x, 0)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\] (2.10)
be prescribed and let it be required to find \( u \in L^2[0, 2] \) (we know from earlier work [16, 17, 19] that a time interval of length 2 is minimal and sufficient for controllability) such that the solution \( (\psi) \) of (0.1), (0.2), (0.3) satisfies
\[
\begin{pmatrix}
v(x, 2) \\
v(x, 2)
\end{pmatrix} = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \sum_{k=-\infty}^{\infty} c_k \varphi_k(x).
\] (2.11)

Letting \( (\psi) \) be a solution of the adjoint system
\[
\frac{\partial}{\partial t} \begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} y \\ z \end{pmatrix} + A(z)^* \begin{pmatrix} y \\ z \end{pmatrix} = 0
\] (2.12)
satisfying boundary conditions of the form (2.9) we find (see [19] for details) that, taking inner products in \( L^2([0, 1]; E^2) \),
\[
((v(\cdot, 2), (\psi(\cdot, 2))) - ((v(\cdot, 0), (\psi(\cdot, 0)))) = \int_0^1 ((g^1(s), (\psi(\cdot, t))) u(t) dt,
\] (2.13)
where \( g(x) = (g^1(x), g^2(x)) \) is the control distribution function in (0.1). Since (2.12) has the form (cf. (2.8))
\[
\frac{\partial}{\partial t} \begin{pmatrix} y \\ z \end{pmatrix} + L^* \begin{pmatrix} y \\ z \end{pmatrix} = 0
\]
it has solutions of the form
\[
\begin{pmatrix} y(x, t) \\ z(x, t) \end{pmatrix} = e^{S(x-t)} \begin{pmatrix} \psi^1_k(x) \\ \psi^2_k(x) \end{pmatrix}.
\]

Employing these solutions in (2.13) and assuming for \( g(x) \) the expansion
\[
g(x) = \begin{pmatrix} g^1(x) \\ g^2(x) \end{pmatrix} = \sum_{k=-\infty}^{\infty} g_k \begin{pmatrix} \varphi^1_k(x) \\ \varphi^2_k(x) \end{pmatrix}
\] (2.14)
we find that (2.10) and (2.11) can be satisfied if and only if
\[
c_k = g_k \int_0^1 e^{\sigma_{k-1} a - b} u(t) dt, \quad k = 0, \pm 1, \pm 2, \ldots
\] (2.15)
If we take \( g(x) \equiv 0 \) but employ boundary value control (0.4) this equation is replaced by

\[
\begin{align*}
  c_k &= \hat{g}_k \int_0^1 e^{\sigma_k (t-\tau)} u(t) \, dt, \quad k = 0, \pm 1, \pm 2, \ldots \tag{2.16}
\end{align*}
\]

where

\[
\begin{align*}
  \hat{g}_k &= a_1^{-1} \phi_k^2(1), \quad a_1 \neq 0, \\
  &= b_1^{-1} \phi_k^1(1), \quad b_1 \neq 0.
\end{align*}
\]

Because the \( \sigma_k \) have the form (2.6) we can use the substantial body of theory on nonharmonic Fourier series \([13, 14, 17, 21]\) to see that the functions \( e^{\sigma_k (t-\tau)} \), \( k = 0, \pm 1, \pm 2, \ldots \) form a Riesz basis for \( L^2[0, 2] \). The dual basis consists of the unique sequence \( \{ p_k \} \) of functions in \( L^2[0, 2] \) which satisfy the biorthogonality relations

\[
\begin{align*}
  (e^{\sigma_k (t-\tau)}, p_r)_{L^2[0, 2]} &= \int_0^2 e^{\sigma_k (t-\tau)} \overline{p_r(t)} \, dt = \delta_{rk}. \tag{2.18}
\end{align*}
\]

Setting

\[
\begin{align*}
  u(t) &= \sum_{k=-\infty}^{\infty} \frac{c_k}{\hat{g}_k} \overline{p_k(t)}, \quad u(t) = \sum_{k=-\infty}^{\infty} \frac{c_k}{\hat{g}_k} \overline{p_k(t)}
\end{align*}
\]

in (2.15), (2.16), respectively, those moment problems are solved. The condition that \( u \in L^2[0, 2] \) is, respectively,

\[
\begin{align*}
  \sum_{k=-\infty}^{\infty} \left| \frac{c_k}{\hat{g}_k} \right|^2 < \infty, \quad \sum_{k=-\infty}^{\infty} \left| \frac{c_k}{\hat{g}_k} \right|^2 < \infty. \tag{2.19}
\end{align*}
\]

With a very simple argument it can be shown that the \( \hat{g}_k \) are bounded and bounded away from zero. Thus the second inequality in (2.9) is equivalent to

\[
\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty,
\]

i.e., all final states (2.11) in \( L^2([0, 1]; E^2) \) can be achieved with a boundary control \( u \in L^2[0, 2] \). The first inequality in (2.19) is satisfied if and only if the \( c_k \) have the form

\[
\begin{align*}
  c_k &= d_k g_k, \quad k = 0, \pm 1, \pm 2, \ldots
\end{align*}
\]

with \( \sum_{k=-\infty}^{\infty} |d_k|^2 < \infty \). In particular \( c_k = 0 \) when \( g_k = 0 \). A dense set of
terminal states (2.11) can be reached with controls \( u \in L^2[0, 2] \) just in case we have the "approximate controllability" condition

\[ g_k \neq 0, \quad k = 0, \pm 1, \pm 2, \ldots \]

This condition is analogous to the controllability assumption of Section 1.

The results of this section can also be interpreted in terms of observability in accord with the general duality theory of observation and control [5]. We will carry this out in Section 4.

3. A Difference-Delay Equation

Let us consider a collection of complex numbers \( \rho_k \) having a form similar to that given for the \( \sigma_k \) in Section 2:

\[ \rho_k = \beta + kni + \nu_k, \quad k = 0, \pm 1, \pm 2, \ldots \]  

(3.1)

with \( \sum_{k=\pm\infty} |\nu_k|^2 < \infty \). We form the exponential functions \( e^{\rho_k t} \) and we let \( r(t) \) denote a linear combination of these:

\[ r(t) = \sum_{k=-\infty}^{\infty} r_k e^{\rho_k t}. \]

If the \( r_k \) satisfy the condition \( \sum_{k=\pm\infty} |kr_k|^2 < \infty \) (so that \( r(t) \in L^2_{loc} \)) then \( r(t) \) has a well-defined value for each \( t \) and we can consider differences

\[ r(t + 2) - e^{2\beta} r(t) = \sum_{k=\pm\infty} r_k (e^{2\beta} - e^{\rho_k}) e^{\rho_k t}. \]  

(3.2)

In this paper we assume the \( \rho_k \) to be distinct. Then as in Section 2, the exponentials \( e^{\rho_k(t+2-t)} \) form a Riesz basis for \( L^2[0, 2] \) and there is a unique dual Riesz basis consisting of functions \( q_k \in L^2[0, 2] \) satisfying the biorthogonality relations

\[ (e^{\rho_k(t+2-t)}, q_{\ell})_{L^2[0, 2]} = \int_0^2 e^{\rho_k(t+2-t)} \overline{q_{\ell}(t)} \, dt = \delta_{\ell-k}. \]

(3.3)

Since the \( \rho_k \) satisfy (3.1) one verifies easily that

\[ \sum_{k=\pm\infty} |e^{2\beta_k} - e^{2\beta_k}|^2 < \infty \]

and we may define a function \( q(t) \in L^2[0, 2] \) by

\[ \overline{q(t)} = \sum_{k=\pm\infty} (e^{2\beta_k} - e^{2\beta_k}) \overline{q_k(t)}. \]  

(3.4)
Then

$$\int_0^2 q(2 - s) r(s + t) \, ds = \int_0^2 r(2 - s + t) \frac{q(s)}{q(2 - s)} \, ds$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} r_k e^{\ell \gamma} (e^{2\gamma t} - e^{2\ell \gamma}) \int_0^2 e^{\ell \gamma (2 - s)} \frac{q(s)}{q(2 - s)} \, ds$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} r_k e^{\ell \gamma} (e^{2\gamma t} - e^{2\ell \gamma}) \delta_k = \sum_{k=-\infty}^{\infty} r_k (e^{2\gamma k} - e^{2\ell \gamma}) e^{\ell \gamma t}$$

and (3.2) reduces to the difference-delay equation

$$r(t + 2) - e^{2\gamma} r(t) = \int_0^2 q(2 - s) r(t + s) \, ds.$$  \hspace{1cm} (3.5)$$

If we put $r(s, t) = r(t + s)$ then $r(s, t)$ satisfies (in an appropriately generalized sense) the partial differential equation

$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial s}.$$  \hspace{1cm} (3.6)$$

and (3.5) becomes a "Stieltjes type" (cf. [3, 10]) boundary condition

$$r(t, 2) - e^{2\gamma} r(t, 0) = \int_0^2 q(2 - s) r(t, s) \, ds.$$  \hspace{1cm} (3.7)$$

Now compute

$$\frac{\partial}{\partial t} \int_0^2 e^{-2|\Re(\beta)|s} | r(t, s) |^2 \, ds - 2|\Re(\beta)| \int_0^2 | e^{-\beta r(t, s)} |^2 \, ds$$

$$= \int_0^2 \left\{ e^{-2|\Re(\beta)|s} \left( \frac{\partial r(t, s)}{\partial s} + \frac{\partial r(t, s)}{\partial s} r(t, s) \right) + \frac{\partial e^{-2|\Re(\beta)|s}}{\partial s} | r(t, s) |^2 \right\} \, ds$$

$$= e^{-2|\Re(\beta)|} | r(t, 2) |^2 - | r(t, 0) |^2$$

$$= | r(t, 0) + e^{-2\beta} \int_0^2 q(2 - s) r(t, s) \, ds |^2 - | r(t, 0) |^2$$

$$= | e^{-2\beta} \int_0^2 q(2 - s) r(t, s) \, ds |^2 + \int_0^2 \overline{r(t, 0)} e^{-2\beta} \int_0^2 q(2 - s) r(t, s) \, ds$$

$$+ e^{-2\beta} \int_0^2 q(2 - s) r(t, s) \, ds \overline{r(t, 0)}$$
from which we conclude via the Schwartz inequality that for some $B_1 > 0$

\[
\frac{\partial}{\partial t} \int_0^2 e^{-2\text{Re}(\beta)s} |r(t, s)|^2 \, ds \leq B_1 \int_0^2 e^{-2\text{Re}(\beta)s} |r(t, s)|^2 \, ds + 2 \left| \int_0^2 \frac{q(2-s)}{r(t,0)} r(t, s) \, ds \right| \leq B_1 \int_0^2 e^{-2\text{Re}(\beta)s} |r(t, s)|^2 \, ds + |r(t,0)|^2 + \left| e^{-2\beta} \int_0^2 \frac{q(2-s)}{r(t, s)} r(t, s) \, ds \right|^2.
\]

(3.8)

Noting that (3.6) implies that, for $t \geq \tau$,

\[ r(t,0) = r(\tau, t - \tau), \]

(3.8) may be integrated to give

\[
\int_0^2 e^{-2\text{Re}(\beta)s} |r(\tau + 2, s)|^2 \, ds \leq K_1 \int_0^2 e^{-2\text{Re}(\beta)s} |r(\tau, s)|^2 \, ds \leq B_2 \int_\tau^{\tau+2} \int_0^2 e^{-2\text{Re}(\beta)s} |r(\tau + \sigma, s)|^2 \, ds \, d\sigma
\]

and from this it follows readily that

\[
\int_0^2 |r(t, s)|^2 \, ds \leq M e^{\mu t} \int_0^2 |r(0, s)|^2 \, ds, \quad t \geq 0,
\]

for some $M > 0, \mu$ real. This bound together with the density in $H^1[0, 2]$ of functions satisfying the boundary condition (3.7) enables us to see that (3.6), with the given domain, generates a bounded semigroup (in fact we can easily see that it is a group), $G(t)$, on $L^2[0, 2]$. We may then speak of "solutions" of (3.6), (3.7) for arbitrary $r(0, s)$ in $L^2[0, 2]$ or solutions of (3.5) for arbitrary initial $r(s)$, $0 \leq s \leq 2$, which lie in $L^2[0, 2]$. Let the operator $d\mathcal{R}/ds$ with domain consisting of $H^1[0, 2]$ functions which satisfy (3.7) be denoted by $T$. Consider then the operator

\[
T^* \mathcal{R} = - (d\mathcal{R}/ds) + e^{-2\beta}q(2-s) \mathcal{R}(0)
\]

(3.9)

defined on the domain consisting of those functions $\mathcal{R} \in H^1[0, 2]$ which satisfy the boundary condition

\[
\mathcal{R}(2) = e^{-2\beta}\mathcal{R}(0).
\]

(3.10)
We compute
\[ (Tr, \hat{r}) = \int_0^2 \frac{dr}{ds} \hat{f}(s) \, ds \]
\[ = - \int_0^2 r(s) \frac{d\hat{f}}{ds} \, ds + r(2) \hat{f}(2) - r(0) \hat{f}(0) \]
\[ = (r, T^* \hat{r}) - e^{-2g} \int_0^2 r(s) q(2 - s) \, ds \hat{f}(0) + (e^{-2g}r(2) - r(0)) \hat{f}(0) \]
\[ = (r, T^* \hat{r}) - e^{-2g} \int_0^2 r(s) q(2 - s) \, ds \hat{f}(0) + e^{-2g} \left( \int_0^2 q(2 - s) r(s) \, ds \right) \hat{f}(0) \]
\[ = (r, T^* \hat{r}). \]

Since both \( T \) and \( T^* \) are readily shown to have dense domain and to be closed operators, we conclude \( T^* \) is the adjoint of \( T \), as the notation anticipates. Since \( T \) generates a group \( G(t) \) in \( L^2[0, 2] \), \( T^* \) generates the adjoint group \( G(t)^* \) and, for \( \hat{f}_0(s) \) arbitrary in \( L^2[0, 2] \) we may identify \( G(t) \hat{f}_0(s) \) with the generalized solution of
\[
\begin{align*}
\hat{\partial}\hat{f}/\partial t &= -(\hat{\partial}\hat{f}/\partial s) + e^{-2g}q(2 - s) \hat{f}(0), \quad (3.11) \\
\hat{f}(t, 2) &= e^{-2g}\hat{f}(t, 0), \quad (3.12) \\
\hat{f}(0, s) &= \hat{f}_0(s). \quad (3.13)
\end{align*}
\]

Finally, for this section, we note the following fact. Let \( q_\ell, \ell = 0, \pm 1, \pm 2, \) denote the biorthogonal functions defined in (3.3). For any finite sum with \( K \geq |\ell| \):
\[
r(s) = \sum_{k=-\infty}^K c_k e^{\rho_k (2-s)}
\]
we have
\[
(Tr, q_\ell) = \int_0^2 \frac{\partial r}{\partial s} q_\ell(2 - s) \, ds = \int_0^2 \sum_{k=-\infty}^K \rho_k e^{\rho_k s} q_\ell(2 - s) \, ds
\]
\[ = \rho_\ell c_\ell = (r, \tilde{\rho}_\ell, q_\ell(2 - \cdot)) = \rho_\ell (r, q_\ell(2 - \cdot)). \]

We conclude then that \( (Tr, q_\ell(2 - \cdot)) \) extends to a continuous linear functional defined for all \( r \in L^2[0, 2] \). Hence \( q_\ell(2 - \cdot) \) belongs to the domain of \( T^* \) and is, in fact, the eigenfunction of \( T^* \) which corresponds to the eigenvalue \( \rho_\ell \) of \( T^* \), just as \( e^{\rho_k s} \) is the eigenfunctions of \( T \) corresponding to the eigenvalue \( \rho_k \), as one readily verifies. In particular \( q_\ell(2 - \cdot) \) satisfies (3.10), i.e.,
\[
q_\ell(0) = e^{-2g}q_\ell(2) \quad \text{or} \quad q_\ell(2) = e^{2g}q_\ell(0).
\]
It may be verified readily that the results of this section apply equally well to repeated eigenvalues $\rho_k$ with corresponding generalized eigenfunctions $e^{\rho_k z}$, $se^{\rho_k z}, \ldots, s^{k-1}e^{\rho_k z}$, where $\mu_k$ denotes the multiplicity of $\rho_k$, provided (3.1) still holds (which implies, of course, that only finitely many $\rho_k$ are multiple eigenvalues and the multiplicity is finite in each case). An indication of the manner in which the biorthogonal functions are constructed in such a case appears in [17]. It should be emphasized, however, that in such an eventuality the indexing of the $\rho_k$ in (3.1) must take the multiplicity of $\rho_k$ into account, i.e., if, say, $\rho_3$ has multiplicity 3 then $\rho_3 = \rho_4 = \rho_5$ and the next, presumably distinct, eigenvalue will be designated $\rho_6$, etc.

4. THE CANONICAL FORM FOR THE OBSERVED HYPERBOLIC SYSTEM

We begin with the development of the normal and canonical forms associated with a certain observed linear system. While the theory for observed systems is equivalent, via duality, to that for control systems, its formulation is a little more straightforward in that no inhomogeneous term appears in the partial differential equation. So we treat this subject first. We consider the partial differential system

$$\frac{\partial}{\partial t} \left( \begin{array}{l} y \\ z \end{array} \right) + \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \frac{\partial}{\partial x} \left( \begin{array}{l} y \\ z \end{array} \right) - C(x) \left( \begin{array}{l} y \\ z \end{array} \right) = 0$$

(4.1)

with boundary conditions

$$c_0 y(0, t) + d_0 z(0, t) = c_1 y(1, t) + d_1 z(1, t) = 0$$

(4.2)

and (cf. (2.4))

$$\frac{(c_1 - d_1)(c_0 + d_0)}{(c_1 + d_1)(c_0 - d_0)} = \gamma$$

with $\gamma$ finite and nonzero. Associated with this system is the linear observation functional

$$\omega(t) = \int_0^1 h(x) \ast \left( \begin{array}{l} y(x, t) \\ z(x, t) \end{array} \right) dx = \left( \begin{array}{l} y' \\ z' \end{array} \right) \left( \begin{array}{l} h' \\ z' \end{array} \right)_{L^2([0, 1]; E^2)}$$

(4.3)

where $h$ is an element of the space $L^2([0, 1]; E^2)$.

As in Section 2, (4.1) has solutions

$$e^{\rho_k t} \psi_k(x), \quad k = 0, \pm 1, \pm 2, \ldots$$

with

$$\rho_k = \beta + \kappa t + O(1/k), \quad \beta = \frac{1}{2} \log \gamma,$$
and these $\rho_k$ are assumed distinct. The vector functions

$$\psi_k(x) = \begin{pmatrix} \psi_{k,1}(x) \\ \psi_{k,2}(x) \end{pmatrix}$$

are normalized so as to form a Riesz basis for the space $L^2([0, 1]; E^2)$. We then have

$$h = \sum_{k=-\infty}^{\infty} h_k \psi_k$$

with

$$d \| h \|^2 \leq \sum_{k=-\infty}^{\infty} | h_k |^2 \leq D \| h \|^2$$

for some positive numbers $d, D$.

Each generalized solution $(y^{(x,t)}_{\alpha(x,t)})$ of (4.1), (4.2) corresponding to an initial state $(\tilde{y})$ in $L^2([0, 1]; E^2)$:

$$\begin{pmatrix} \tilde{y} \\ z \end{pmatrix} = \sum_{k=-\infty}^{\infty} \tilde{y}_k \psi_k, \quad \sum_{k=-\infty}^{\infty} | y_k |^2 < \infty,$$

has the expansion

$$\begin{pmatrix} y(\cdot, t) \\ z(\cdot, t) \end{pmatrix} = \sum_{k=-\infty}^{\infty} y_k(t) \psi_k, \quad y_k(t) = e^{\rho_k t} \tilde{y}_k.$$

This enables us to establish equivalence between such solutions of (4.1), (4.2) and solutions of the infinite set of linear scalar initial value problems

$$dy_k/dt = \rho_k y_k, \quad y_k(0) = \tilde{y}_k, \quad k = 0, \pm 1, \pm 2, \ldots,$$

whose solution $\{y_k(t)\} \in \ell^2$ for each $t$. The observation (4.3) becomes

$$\omega(t) = \sum_{k=-\infty}^{\infty} \tilde{h}_k y_k(t).$$

We will assume the observability condition

$$h_k \neq 0, \quad k = 0, \pm 1, \pm 2, \ldots.$$

If instead of taking the form (4.3), the observation $\omega$ is given by

$$\omega(t) = \xi_1 y(1, t) + \xi_2 z(1, t), \quad \det \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \neq 0,$$
then in (4.6) we obtain coefficients $\tilde{h}_k$ which, rather than being square summable as indicated by (4.4), have the property of being bounded and bounded away from zero, uniformly for all $k$. Then

$$\omega(t) = \sum_{k=-\infty}^{\infty} \tilde{h}_ky_k(t)$$

is not in general defined for each $t$ but (4.5) together with the properties of the exponential functions $e^{\sigma t}$ discussed in Section 3 shows $\omega(t)$ to be square integrable on each finite interval.

To actually construct the observation normal form we use the state to observation map analogous to the matrix $O$ defined in (1.5) of Section 1. Equations (4.5) and (4.6) show that for $\sigma \geq 0$ we have

$$\omega(t + \sigma) = \sum_{k=-\infty}^{\infty} \tilde{h}_k e^{\sigma t} y_k(t),$$

thereby associating with the state $\{y_k(t)\} \in \ell^2$ the observation function $\omega(t + \sigma)$. We define (cf. (4.5), (4.6))

$$\tilde{\eta}(t, \sigma) = \sum_{k=-\infty}^{\infty} \tilde{h}_k e^{\sigma t} y_k(t), \quad 0 \leq \sigma \leq 2, \quad t \geq 0, \quad (4.9)$$

which we denote by

$$\tilde{\eta}(t, \cdot) = O(\{y_k(t)\}).$$

As described in Section 3, we let the functions $q_\sigma(s)$ in $L^2[0, 2]$ form the biorthogonal sequence in that space relative to the functions $e^{\sigma z_{2-n}}$, i.e.,

$$\int_{0}^{2} e^{\sigma z_{2-n}} q_\sigma(s) \, ds = \delta_{\sigma}. \quad (4.10)$$

Then the map inverse to $O$ is

$$\{y_k(t)\} = R\tilde{\eta}(t, \cdot)$$

defined by

$$y_k(t) = h_k^{-1} \int_{0}^{2} \frac{q_\sigma(2 - \sigma)}{q_\sigma(\sigma)} \tilde{\eta}(t, \sigma) \, d\sigma. \quad (4.10)$$

The map $O$ takes $\ell^2$ onto the dense subspace $H$ of $L^2[0, 2]$ consisting of sequences $\sum_{k=-\infty}^{\infty} \eta_k e^{\sigma z_{2-n}}$ for which $\sum_{k=-\infty}^{\infty} |\eta_k/h_k|^2 < \infty$. $R$ maps $H$ back onto $\ell^2$. In the case of boundary observation (4.8), where the square summable $h_k$ are replaced
by the $h_k$ which are bounded away from zero and infinity, $O$ is a Hilbert space isomorphism from $L^2$ onto $L^2[0, 2]$.

Now compute, using (4.5), (4.9),

$$\frac{\partial \eta(t, \sigma)}{\partial t} = \sum_{k=-\infty}^{\infty} h_k e^{s \sigma} \frac{dy_k}{dt}$$

$$= \sum_{k=-\infty}^{\infty} h_k \rho_k e^{s \sigma} y_k(t) = \sum_{k=-\infty}^{\infty} h_k \frac{de^{s \sigma}}{d\sigma} y_k(t) = \frac{\partial \eta(t, \sigma)}{\partial \sigma}.$$

This identity holds in the classical sense if

$$\sum_{k=-\infty}^{\infty} |h_k \rho_k \dot{y}_k| < \infty;$$

otherwise it must be interpreted in the context of the theory of distributions.

We adjoin to this partial differential equation, i.e., to

$$\frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial \sigma},$$

the boundary condition

$$\eta(t, 2) = e^{2b} \eta(t, 0) + \int_0^2 q(2-\sigma) \eta(t, \sigma) d\sigma$$

(4.12)

where

$$q(\sigma) = \sum_{k=-\infty}^{\infty} (e^{2s \sigma} - e^{2b} q_k(\sigma)),$$

(4.13)

as discussed in Section 3. The partial differential equation (4.11) and boundary condition (4.12) constitute the observation normal form for the linear system (4.1), (4.2). The observation relation (4.3) becomes

$$\omega(t) = \sum_{k=-\infty}^{\infty} h_k y_k(t) = \sum_{k=-\infty}^{\infty} h_k e^{s \sigma} y_k(t) = \eta(t, 0).$$

(4.14)

It should be noted that (4.11), (4.12), (4.14) are independent of the particular coefficients $h_k$ in (4.6) and continue to have the same form for boundary observation (4.8). However, for a given solution of (4.1), (4.2) (or, equivalently, of (4.5)) the degree of regularity of the solution $\eta(t, \sigma)$ of (4.11), (4.12) varies, improving as the $h_k$ decay more rapidly.

To pass to the observation canonical form we now let (cf. (4.13))

$$\eta(t, \sigma) = \eta(t, \sigma) - \int_0^\sigma \frac{q(\sigma - \tau)}{q(\sigma)} \eta(t, \tau) d\tau.$$

(4.15)
Then

$$\eta(t, 2) = \tilde{\eta}(t, 2) - \int_0^2 \frac{q(2 - \tau)}{q(2 - \tau)} \tilde{\eta}(t, \tau) \, d\tau = e^{2\beta} \tilde{\eta}(t, 0) = e^{2\beta} \eta(t, 0)$$

so the boundary condition (4.12) is replaced by

$$\eta(t, 2) = e^{2\beta} \eta(t, 0). \quad (4.16)$$

The computation

$$\frac{\partial \eta(t, \sigma)}{\partial t} - \frac{\partial \eta(t, \sigma)}{\partial \sigma}$$

$$= \frac{\partial}{\partial t} \left( \tilde{\eta}(t, \sigma) - \int_0^\sigma \frac{q(\sigma - \tau)}{q(\sigma - \tau)} \tilde{\eta}(t, \tau) \, d\tau \right) - \frac{\partial}{\partial \sigma} \left( \tilde{\eta}(t, \sigma) - \int_0^\sigma \frac{q(\sigma - \tau)}{q(\sigma - \tau)} \tilde{\eta}(t, \tau) \, d\tau \right)$$

$$= \frac{\partial \tilde{\eta}(t, \sigma)}{\partial t} - \frac{\partial \tilde{\eta}(t, \sigma)}{\partial \sigma} - \int_0^\sigma \frac{q(\sigma - \tau)}{q(\sigma - \tau)} \frac{\partial \tilde{\eta}(t, \tau)}{\partial t} \, d\tau$$

$$+ \tilde{q}(0) \tilde{\eta}(t, \sigma) - \int_0^\sigma \frac{\partial}{\partial \tau} \frac{q(\sigma - \tau)}{q(\sigma - \tau)} \tilde{\eta}(t, \tau) \, d\tau$$

$$= (\text{using (4.11) and integration by parts})$$

$$- \int_0^\sigma \frac{\partial \tilde{\eta}(t, \tau)}{\partial \tau} \frac{q(\sigma - \tau)}{q(\sigma - \tau)} \, d\tau - \left[ \frac{q(\sigma - \tau)}{q(\sigma - \tau)} \tilde{\eta}(t, \tau) \right]_{\tau=0}^{\tau=\sigma} + \tilde{q}(0) \tilde{\eta}(t, \sigma)$$

$$= \tilde{q}(\sigma) \tilde{\eta}(t, 0) = \tilde{q}(\sigma) \eta(t, 0)$$

shows that (4.11) is replaced now by

$$\frac{\partial \eta(t, \sigma)}{\partial t} - \frac{\partial \eta(t, \sigma)}{\partial \sigma} = \tilde{q}(\sigma) \eta(t, 0). \quad (4.17)$$

Equations (4.17) and (4.16) constitute the observation canonical form for (4.1), (4.2) and should be compared with (1.13) of Section 1. From (4.15) we see that the observation relation remains

$$\eta(t) = \eta(t, 0). \quad (4.18)$$

It is quite possible to discuss spectral modification in the context of the present section, in the same vein as developed in (1.14)ff of Section 1. We choose, however, to carry out this program in the control context of the next section.
5. The Canonical Form for the Controlled Hyperbolic System

Let \((u(x, t))\) be a generalized solution of (0.1), (0.2), (0.3). If we develop in eigenfunction series (cf. (2.7))

\[
\begin{bmatrix}
u(t, x)
\end{bmatrix} = \sum_{k=-\infty}^{\infty} w_k(t) \varphi_k(x)
\]

and let \(g(x)\) have the development (2.14) then we find that the \(w_k(t), k = 0, \pm 1, \pm 2,...\) satisfy

\[
dw_k/dt = \sigma_k w_k + g_k(t).
\]

These \(g_k\) are, of course, replaced by the \(\hat{g}_k\) of (2.17) in the case of boundary control. Assuming \(w_k(0) = 0, k = 0, \pm 1, \pm 2,...\), the control to state map at time \(t = 2\) is (cf. (1.16))

\[
w_k(2) = g_k \int_0^2 e^{\sigma_k(t-s)} \, ds, \quad k = 0, \pm 1, \pm 2,...
\]

The first step in derivation of the control normal form is to use this map to define a new dependent variable \(\xi\):

\[
w_k(t) = g_k \int_0^t e^{\sigma_k(t-s)} \varphi(t, s) \, ds, \quad t \geq 0, \quad k = 0, \pm 1, \pm 2,...
\]

We abbreviate this relationship by

\[
\{w_k(t)\} = U(\xi(t, \cdot)), U: L^2[0, 2] \rightarrow \ell^2.
\]

Letting \(p_k(t), k = 0, \pm 1, \pm 2,...\) denote the biorthogonal functions defined in (2.18) and assuming the controllability condition

\[
g_k \neq 0, \quad k = 0, \pm 1, \pm 2,...
\]

the inverse map for \(U\) is

\[
\xi(t, s) = \sum_{k=-\infty}^{\infty} \frac{w_k(t)}{g_k} p_k(t)
\]

which we designate by

\[
\xi(t, \cdot) = S(\{w_k(t)\}).
\]
The linear transformation $U$ carries $L^2[0, 2]$ into a dense subspace $G \subseteq \ell^2$, consisting of sequences $\{w_k\}$ such that

$$\sum_{k=-\infty}^{\infty} \left| \frac{w_k}{g_k} \right|^2 < \infty$$

and $S$ carries $G$ onto $L^2[0, 2]$. In the case of boundary control (0.4), (2.17) we have $G = \ell^2$ since the $\dot{g}_k$ are bounded away from $O$. The differential equation satisfied by $\tilde{\zeta}(t, s)$, at least in the generalized sense, is obtained formally as follows:

$$\frac{\partial \tilde{\zeta}(t, s)}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{d w_k(t)}{dt} \frac{p_k(s)}{g_k} = \sum_{k=-\infty}^{\infty} \left( \alpha_k w_k(t) + g_k(t) \right) \frac{p_k(s)}{g_k}$$

$$= \sum_{k=-\infty}^{\infty} \left( \sigma_k \int_0^2 e^{s_k(2-s)} \tilde{\zeta}(t, s) \, ds \right) \frac{p_k(s)}{g_k} + u(t) \sum_{k=-\infty}^{\infty} \frac{p_k(s)}{g_k}$$

$$= \sum_{k=-\infty}^{\infty} \left[ -\tilde{\zeta}(t, 2) + e^{2s_k} \tilde{\zeta}(t, 0) + (t) \right] \frac{p_k(s)}{g_k}$$

$$+ \sum_{k=-\infty}^{\infty} \left( \int_0^2 e^{s_k(2-s)} \frac{\partial \tilde{\zeta}(t, s)}{\partial s} \, ds \right) \frac{p_k(s)}{g_k}.$$

Now we proceed as in Section 3, noting that

$$\sum_{k=-\infty}^{\infty} (e^{2s_k} - e^{2s}) \frac{p_k(s)}{g_k} = \sum_{k=-\infty}^{\infty} \varnothing \left( \frac{1}{k} \right) \frac{p_k(s)}{g_k} = \varnothing(s)$$

for some $\varnothing \in L^2[0, 2]$, since the $p_k$ form a Riesz basis for that space. Thus

$$\frac{\partial \tilde{\zeta}(t, s)}{\partial t} = \sum_{k=-\infty}^{\infty} \left( \int_0^2 e^{s_k(2-s)} \frac{\partial \tilde{\zeta}(t, s)}{\partial s} \, ds \right) \frac{p_k(s)}{g_k}$$

$$+ \varnothing(s) \tilde{\zeta}(t, 0) + \sum_{k=-\infty}^{\infty} \left[ e^{2s} \tilde{\zeta}(t, 0) - \tilde{\zeta}(t, 2) + u(t) \right] \frac{p_k(s)}{g_k}.$$
we obtain
\[
\frac{\partial \xi(t, s)}{\partial t} = \frac{\partial \xi(t, s)}{\partial s} + \overline{p(s)} \xi(t, 0). \tag{5.7}
\]

Equation (5.7) together with the boundary condition (5.6) constitute the control normal form for (5.2) or equivalently, for (0.1), (0.2), (0.3). The above formal development will be justified later in this section.

To pass to the control canonical form we set (cf. (1.20))
\[
\xi(t, s) = \zeta(t, s) - \int_0^s \overline{p(s - \tau)} \zeta(t, \tau) \, d\tau. \tag{5.8}
\]

Then (5.6) together with \(\zeta(t, 0) = \xi(t, 0)\) gives
\[
\begin{align*}
\zeta(t, 2) &= \xi(t, 2) + \int_0^2 \overline{p(2 - \tau)} \zeta(t, \tau) \, d\tau \\
&= e^{2\alpha} \zeta(t, 0) + \left[ \int_0^2 \overline{p(2 - \tau)} \zeta(t, \tau) \, d\tau \right]
\end{align*}
\tag{5.9}
\]

which supplies the boundary condition for \(\zeta(t, s)\). Equation (5.7) now becomes
\[
0 = \frac{\partial}{\partial t} \left( \zeta(t, s) - \int_0^s \overline{p(s - \tau)} \zeta(t, \tau) \, d\tau \right) - \frac{\partial \zeta(t, s)}{\partial s} - \int_0^s \overline{p(s - \tau)} \left( \frac{\partial \zeta(t, \tau)}{\partial t} - \frac{\partial \zeta(t, t)}{\partial \tau} \right) \, d\tau
\]
\[
= \left( \text{integrating by parts in the second integral} \right)
\]
\[
= \frac{\partial \xi(t, s)}{\partial t} - \frac{\partial \xi(t, s)}{\partial s} - \int_0^s \overline{p(s - \tau)} \left( \frac{\partial \xi(t, \tau)}{\partial t} - \frac{\partial \xi(t, \tau)}{\partial \tau} \right) \, d\tau
\]
\[
+ \overline{p(0)} \xi(t, s) - \overline{p(0)} \zeta(t, s) + \overline{p(s)} \zeta(t, 0) - \overline{p(s)} \zeta(t, 0)
\]
\[
= \frac{\partial \zeta(t, s)}{\partial t} - \frac{\partial \zeta(t, s)}{\partial s} - \int_0^s \overline{p(s - \tau)} \left( \frac{\partial \zeta(t, \tau)}{\partial t} - \frac{\partial \zeta(t, \tau)}{\partial \tau} \right) \, d\tau.
\]
Thus \(\partial \xi(t, s)/\partial t - \partial \xi(t, s)/\partial s\) satisfies the integral equation
\[
\frac{\partial \xi(t, s)}{\partial t} - \frac{\partial \xi(t, s)}{\partial s} - \int_0^s \overline{p(s - \tau)} \left( \frac{\partial \xi(t, \tau)}{\partial t} - \frac{\partial \xi(t, \tau)}{\partial \tau} \right) \, d\tau = 0
\]
whose only solution is
\[
\frac{\partial \xi(t, s)}{\partial t} - \frac{\partial \xi(t, s)}{\partial s} = 0. \tag{5.10}
\]
Equation (5.10) and the boundary condition (5.9) constitute the control canonical
form for (5.2), or, equivalently, for (0.1), (0.2), (0.3). Up to this point we can only claim that it has been obtained formally. Certainly the weakest link in our argument occurs in (5.6), since that condition has not been shown to be a logical consequence of the preceding arguments. We will use the rest of this section to justify the above development and its conclusions.

We begin this process by analyzing the nature of the solutions of (5.7), (5.6).

**Lemma 5.1.** Consider the equations

\[
\begin{align*}
\frac{\partial \xi(t, s)}{\partial t} & = \frac{\partial \xi(t, s)}{\partial s} + p(s) \xi(t, 0), \\
\xi(t, 2) & = e^{\omega t} \xi(t, 0) + u(t), \\
\xi(0, s) & = \xi_0(s),
\end{align*}
\]

wherein we assume \( p, \xi \in L^2[0, 2] \) and \( u \in L^2[0, T] \) for every \( T > 0 \). These equations have a unique generalized solution \( \xi(t, s) \), \( t \geq 0, \ 0 \leq s \leq 2 \) with \( \xi \in C([0, T], L^2[0, 2]) \) on each interval \([0, T]\) and

\[
\xi(T, \cdot) = \xi_1(T) \xi_0 + \xi_2(T) u
\]

where \( \xi_1(T) \) is a strongly continuous group of bounded operators on \( L^2[0, 2] \) and \( \xi_2(t): L^2[0, T] \rightarrow L^2[0, 2] \) is a bounded linear operator for each \( T \).

**Proof.** Equation (5.11) may be reinterpreted in terms of integration along characteristic lines \( t + s = \text{constant} \) (cf. [4, Chap. V]):

\[
\xi(t, s) = \xi_0(0, s + t) + \int_0^t p(s + t - \tau) \xi(\tau, 0) \, d\tau, \quad t \geq 0.
\]

In particular, for \( s = t, \ 0 \leq t \leq 2 \) we have

\[
\xi(t, 0) = \xi_0(t) + \int_0^t p(t - \tau) \xi(\tau, 0) \, d\tau
\]

which we immediately recognize as a Fredholm equation with Volterra kernel for \( \xi(t, 0), \ t \in [0, 2] \). Thus \( \xi(t, 0) \) is determined by \( \xi(t) \) and it is well known that the map \( \xi_0 \rightarrow \xi(\cdot, 0) \) is a bounded linear operator on \( L^2[0, 2] \). Once \( \xi(t, 0) \) has been thus determined, \( \xi(t, s) \) can be obtained by integration, using (5.11) and (5.12) in the region \( 0 \leq t \leq 2, \ 0 \leq s \leq 2 \). Indeed, we have (5.14) for \( t \geq 0, \ 0 \leq s \leq 2, \ t + s < 2 \), and

\[
\begin{align*}
\xi(t, s) & = e^{2\omega t} \xi(t - (2 - s), 2) + (t - (2 - s)) \\
& \quad + \int_{t-(2-s)}^t p(s + t - \tau) \xi(\tau, 0) \, d\tau, \quad t \geq 0, \ 0 \leq s \leq 2, \ t + s \geq 2.
\end{align*}
\]
The formulas (5.14) and (5.15) provide an explicit rendering of $G_1(t)$ and $G_2(t)$ for $0 < t < 2$ from which the stated properties of $\xi$ are easily obtained. The solution can be continued to arbitrarily large $t$ by repeating the above analysis in intervals $[2, 4], [4, 6], [6, 8], \ldots$.

Consider now the observed system

\begin{equation}
\frac{\partial}{\partial t} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} y \\ z \end{bmatrix} - A(x)^* \begin{bmatrix} y \\ z \end{bmatrix} = 0,
\end{equation}

\begin{equation}
\omega(t) = \int_0^2 g(x)^* \begin{bmatrix} y(x, t) \\ z(x, t) \end{bmatrix} \, dx,
\end{equation}

\begin{equation}
\bar{a}_0 y(0, t) - \bar{b}_0 z(0, t) = 0, \quad \bar{a}_1 y(1, t) - \bar{b}_1 z(1, t) = 0
\end{equation}

obtained by setting $C(x) = A(x)^*$ in (4.1), $h(x) = y(x)$ in (4.3), and taking $c_0 = \bar{a}_0$, $c_1 = \bar{a}_1$, $d_0 = -\bar{b}_0$, $d_1 = -\bar{b}_1$ in (4.2). When we reduce this observed linear system to the form (4.5), (4.6) we have

\begin{equation}
\rho_k = \bar{\sigma}_k, \quad h_k = g_k.
\end{equation}

Denoting a solution of (4.5), as just noted, by $\{y_k(t)\}$ and a solution of (5.2) by $\{w_k(t)\}$ an easy computation (see [19]) shows

\begin{equation}
\omega(t) = \int_0^T u(t) \omega(T - t) \, dt,
\end{equation}

$\omega(t)$ being given by (4.6).

Consider now the maps

\begin{equation}
\eta(t, \cdot) = O\{y_k(t)\},
\end{equation}

\begin{equation}
\{w_k(t)\} = U\xi(t, \cdot)
\end{equation}

defined by (4.9), (5.3), respectively. For any function $\xi(s) \in L^2[0, 2]$ let

\begin{equation}
(\Sigma\xi)(s) = \xi(2 - s).
\end{equation}

A brief computation shows that the adjoint of $O$, $O^*: L^2[0, 2] \rightarrow \ell^2$, is given by

\begin{equation}
O^* = U\Sigma,
\end{equation}

and, since $\Sigma$ is unitary we also have

\begin{equation}
U^* = \Sigma O.
\end{equation}
Making use of these adjoint relationships in (5.19) we find
\[ \int_0^T u(t) \overline{\omega(T - t)} \, dt = (\{w_k(T)\}, \{y_k(0)\})_{L^2} - (\{w_k(0)\}, \{y_k(T)\})_{L^2} \]
\[ = (U\xi(T, \cdot), \{y_k(0)\})_{L^2} - (U\xi(0, \cdot), \{y_k(T)\})_{L^2} \]
\[ = (\xi(T, \cdot), \Sigma O\{y_k(0)\})_{L^2[0,1]} - (\xi(0, \cdot), \Sigma O\{y_k(T)\})_{L^2[0,2]} \]
\[ = (\xi(T, \cdot), \eta(0, 2 - \cdot))_{L^2[0,2]} - (\xi(0, \cdot), \eta(T, 2 - \cdot))_{L^2[0,2]} \]  
(5.21)

Here \( \eta(t, s) \) is obtained from \( \{y_k(t)\} \), which is, in turn, obtained from (4.1), (4.2), following the procedure of Section 4. We know, therefore, that \( \eta(t, s) \) satisfies
\[ \partial \eta(t, s)/\partial t = \partial \eta(t, s)/\partial s, \]  
(5.22)
\[ \eta(t, 2) = e^{2\overline{\alpha}} \eta(t, 0) + \int_0^2 q(2 - s) \eta(t, s) \, ds. \]  
(5.23)

Comparing (4.13) and (5.5) and noting that in our present case \( \rho_k = \overline{\sigma}_k \), \( \beta = \overline{\alpha} \), we see that \( q(2 - s) = p(2 - s) \) and hence (5.23) becomes
\[ \eta(t, 2) = e^{2\overline{\alpha}} \eta(t, 0) + \int_0^2 p(2 - s) \eta(t, s) \, ds. \]  
(5.24)

The observation is (cf. (4.14))
\[ \omega(t) = \eta(t, 0). \]  
(5.25)

Now let \( \xi(t, s) \) be the solution of (5.11), (5.12), (5.13) with
\[ \xi_{\omega}(s) = \xi(0, s), \]
where \( \xi(0, s) = U^{-1}\{w_k(0)\} = S\{w_k(0)\} \) is the function appearing in (5.21). The following computations may be justified by appealing to the theory of distributions or by approximating \( \xi \) and \( \eta \) by smooth solutions.
\[ \frac{d}{dt} \int_0^2 \xi(t, s) \overline{\eta(T - t, 2 - s)} \, ds \]
\[ = \int_0^2 \partial \xi(t, s) \overline{\eta(T - t, 2 - s)} \, ds + \int_0^2 \xi(t, s) \overline{\partial \eta(T - t, 2 - s)} \, ds \]
\[ = (\text{cf. (5.22)}) \]
\[
\begin{align*}
&= \int_0^2 \frac{\partial \xi(t, s)}{\partial t} \eta(T - t, 2 - s) \, ds + \int_0^2 \frac{\partial \eta(T - t, 2 - s)}{\partial s} \, ds \\
&= \int_0^2 \left( \frac{\partial \xi(t, s)}{\partial t} - \frac{\partial \xi(t, s)}{\partial s} \right) \eta(T - t, 2 - s) \, ds \\
&\quad + \xi(t, 2) \eta(T - t, 0) - \xi(t, 0) \eta(T - t, 2) \\
&= \left( \text{cf. (5.11), (5.12)} \right) = \int_0^2 p(s) \eta(T - t, 2 - s) \, ds \\
&\quad + \xi(t, 0) [e^{2s} \eta(T - t, 0) - \eta(T - t, 2)] + u(t) \eta(T - t, 0) \\
&= \xi(t, 0) \int_0^2 \left[ p(s) \eta(T - t, 2 - s) - p(2 - s) \eta(T - t, 2) \right] \, ds \\
&\quad + u(t) \eta(T - t, 0) = u(t) \eta(T - t, 0).
\end{align*}
\]

Integrating with respect to \( t \) and using (5.13) and (5.25) we have

\[
\langle \xi(T, \cdot), \eta(0, 2 - \cdot) \rangle_{L^2[0, 2]} - \langle \xi(0, \cdot), \eta(T, 2 - \cdot) \rangle_{L^2[0, 2]} = \int_0^T u(t) \omega(T - t) \, dt. \tag{5.26}
\]

Subtracting (5.26) from (5.21) we have

\[
\langle \xi(T, \cdot) - \xi(T, \cdot), \eta(0, 2 - \cdot) \rangle_{L^2[0, 2]} = 0. \tag{5.27}
\]

Since we can take \( \eta(0, 2 - \cdot) \) in (5.26) to be an arbitrary element of the dense subspace \( H \) of \( L^2[0, 2] \), we conclude

\[
\xi(T, \cdot) = \xi(T, \cdot),
\]

thus establishing rigorously that \( \xi(t, s) \) does indeed satisfy (5.6), (5.7), as our earlier formal calculations have anticipated.

The passage from (5.6), (5.7) to (5.9), (5.10) via (5.8) may be justified by approximating \( \xi(t, s) \) by smooth solutions of (5.6), (5.7) so the canonical form (5.9), (5.10) is now established.

6. SPECTRAL DETERMINATION VIA LINEAR FEEDBACK

We return now to the system (0.1) of Section 1 with the boundary conditions (0.2) and (0.3) (or (0.4) if boundary control is used). With linear feedback

\[
u(t) = \int_0^1 k(x)^* \left( \frac{\partial w(x, t)}{\partial x} - \left( \frac{\partial w(\cdot, t)}{\partial x}, \frac{\partial v(\cdot, t)}{\partial x} \right) \right) \, dx - \left( \frac{\partial w(\cdot, t)}{\partial x}, \frac{\partial v(\cdot, t)}{\partial x} \right) \right)_{L^2([0,1];E^n)} \tag{6.1}
\]

we obtain the closed loop system (0.6). Referring to the material of Sections 2 and 5, we have
\[ u(t) = \sum_{j=-\infty}^{\infty} \tilde{k}_j w_j(t), \] (6.2)
where the \( w_j(t) \) have been defined in (5.1) and the \( \tilde{k}_j \) are the expansion coefficients of \( k \in L^2([0, 1]; E^2) \) with respect to the basis \( \{\varphi_j\} \) defined in Section 2.

When we pass to the system (5.6), (5.7) with state \( \tilde{z}(t, s) \) the feedback relation (6.2) (equivalently (6.1)) becomes (cf. (5.3))
\[ u(t) = \int_0^1 \tilde{k}_s(2 - s) \tilde{z}(t, s) \, ds \] (6.3)
\[ = \int_0^1 \tilde{k}_s(2 - s) \tilde{z}(t, s) \, ds = (\tilde{k}_s(2 - \cdot), \tilde{z})_{L^2[0, 2]} \]
where
\[ \tilde{k}_s(2 - s) = \sum_{j=-\infty}^{\infty} k_j \varphi_j e^{\sigma_j(2 - s)} \] (6.4)
is an element of \( L^2[0, 2] \). In fact, if \( \{k_j\}, \{g_j\} \) are both square summable we have \( \tilde{k}_s(2 - s) \) continuous on \([0, 2]\). In the case of boundary control (0.4), (2.16), (2.17) the \( g_j \) are replaced by the \( \eta_j \) which are bounded but not square summable and in that case we can only conclude, using the square summability of the \( k_j \), that \( \tilde{k}_s(2 - s) \in L^2[0, 2] \). Finally, denoting the linear transformation (5.8) by \((I + M)\), (6.3) now becomes
\[ u(t) = \int_0^1 \tilde{k}_s(2 - s) ((I + M) \tilde{z})(t, s) \, ds = ((I + M) \tilde{z}(t, \cdot), \tilde{k}_s(2 - \cdot))_{L^2[0, 2]} \]
\[ = (\tilde{z}(t, \cdot), (I + M)^* \tilde{k}_s(2 - \cdot))_{L^2[0, 2]} \] (6.5)

**Lemma 6.1.**
\[ (I + M)^* e^{\sigma_j(2 - \cdot)} = p_j(2 - \cdot), \quad j = 0, \pm 1, \pm 2, \ldots \] (6.6)
or, equivalently,
\[ (I + M) e^{\sigma_j s} = p_j, \quad j = 0, \pm 1, \pm 2, \ldots \] (6.7)

**Proof.** As in the work of Section 3 (see paragraph preceding (3.9)), let \( T \) denote the operator \((T \xi) = \partial \xi / \partial s\).

\(^1\) See Note added in proof.
defined on the domain in $L^2[0, 2]$ consisting of functions $\zeta \in H^1[0, 2]$ which satisfy the boundary condition

$$\zeta(2) = e^{2\alpha} \zeta(0) + \int_0^2 \rho(2 - s) \zeta(s) \, ds.$$  

We have seen in Section 3 that the adjoint of $T$ is the operator

$$T^* \zeta = - (\partial \zeta / \partial s) + e^{-2\alpha} \rho(2 - s) \zeta(0)$$

defined on the domain consisting of functions $\zeta \in H^1[0, 2]$ which satisfy the boundary condition

$$\zeta(2) = e^{-2\alpha} \zeta(0).$$  

If we let $u(t) \equiv 0$, $\zeta(t, s) = \overline{\zeta(t, 2 - s)}$ in (5.11), (5.12) we have

$$\frac{\partial \zeta(t, s)}{\partial s} = - \frac{\partial \zeta(t, s)}{\partial s} + e^{-2\alpha} \rho(2 - s) \zeta(t, 0),$$

(6.8)

$$\zeta(t, 2) = e^{-2\alpha} \zeta(t, 0).$$  

(6.9)

Clearly the semigroup associated with (6.8), (6.9) has generator $T^*$. Letting $\Sigma_1$ be defined by $(\Sigma_1 \zeta)(s) = \overline{\zeta(2 - s)}$ we conclude that the semigroup associated with (5.11), (5.12) has generator $\Sigma_1 T^* \Sigma_1$. On the other hand, taking the transformation (5.8), i.e., $I + M$, into account the generators $\tilde{T}$ and $T$ associated with (5.6), (5.7) and (5.9), (5.10) (with $u \equiv 0$), respectively, must be related by

$$T = (I + M)^{-1} \tilde{T}(I + M)$$

and we conclude, therefore, that

$$T = (I + M)^{-1} \Sigma_1 T^* \Sigma_1 (I + M).$$

(Note that this is not a similarity transformation because $\Sigma_1$ is antilinear rather than linear.) Since the eigenfunctions of $T$ are the functions $e^{\sigma \cdot}$, the eigenfunctions of $T^*$ are the functions $\rho_\sigma(2 - \cdot)$ (with associated eigenvalue $\sigma$). We conclude therefore that

$$\Sigma_1 (I + M) e^{\sigma \cdot} = \rho_\sigma(2 - \cdot)$$

or

$$(I + M) e^{\sigma \cdot} = \overline{\rho_\sigma(\cdot)},$$

which is (6.7). Thus

$$((I + M) e^{\sigma \cdot}, e^{\delta_\tau(\cdot)} )_{L^2[0, 2]} = \delta_{\tau, \sigma}.$$
so that

\[(e^{\sigma t}, (I + M)^* \ e^{\rho_1/(2 - s)}) L^2[0, 2] = \delta_j \]

from which (6.6) follows.

Using this lemma with (6.4) we conclude

\[(I + M)^* \tilde{k}_q(2 - \cdot) = (I + M)^* \sum_{j = -\infty}^\infty k_j g_j e^{\rho_j/(2 - s)} \]

(6.10)

\[= \sum_{j = -\infty}^\infty k_j g_j \rho_j(2 - \cdot) \equiv k_0(2 - \cdot). \]

Then (6.5) becomes

\[u(t) = (\zeta, k(2 - \cdot)) L^2[0, 2] = \int_0^2 \tilde{k}_q(2 - s) \zeta(t, s) \, ds. \]

(6.11)

We now return to the system (5.9), (5.10) and substitute (6.11) for \(u(t)\) in (5.9), thereby obtaining the closed loop system (equivalent to (0.6), (0.2), (0.3))

\[\frac{d\zeta(t, s)}{dt} = \frac{d\zeta(t, s)}{ds}, \]

(6.12)

\[\zeta(t, 2) = e^{2\alpha} \zeta(t, 0) + \int_0^2 (p(2 - \tau) + \tilde{k}_q(2 - \tau)) \zeta(t, \tau) \, d\tau, \]

(6.13)

which should be compared with (1.23). Moreover, we have the formula (cf. (5.5), (6.10))

\[p(2 - \tau) + \tilde{k}_q(2 - \tau) = \sum_{j = -\infty}^\infty (e^{2\alpha} - e^{2\alpha_j} + k_j g_j) \rho_j(2 - \tau). \]

We are now in a position to prove the main result of this paper.

**Theorem 6.2.** Let \(\{\rho_j\}\) be a sequence of distinct complex numbers satisfying

\[
\sum_{j = -\infty}^\infty \left| \frac{\rho_j - \sigma_j}{g_j} \right|^2 < \infty.
\]

(6.14)

Then there exists a function \(k \in L^2([0, 1], E^2)\) (cf. 6.1)) such that the eigenvalues of the closed loop system (6.12), (6.13) are \(\rho_j, j = 0, \pm 1, \pm 2, \ldots\).

**Proof.** Following the material of Section 3, linear combinations of the form

\[r(t) = \sum_{j = -\infty}^\infty c_j e^{\sigma_j t}, \]

where \(c_j \in \mathbb{C}\) and \(|c_j| < \infty\), are solutions of the closed loop system.
with \( \rho_j \) satisfying the inequality (6.14), satisfy the equation

\[
r(t + 2) = e^{2\sigma_j}(t) + \int_0^3 q(2 - s) r(t + s) \, ds
\]

where

\[
q(2 - s) = \sum_{\ell = -\infty}^{\infty} (e^{2\sigma_\ell} - e^{2\sigma_j}) q(2 - s), \tag{6.15}
\]

\[
\int_0^2 q(2 - s) e^{\sigma_j} \, ds = \delta_j. \tag{6.16}
\]

To realize the \( \rho_j \) as eigenvalues of the system (6.12), (6.13), therefore, it is necessary and sufficient that we should have

\[
k_j(2 - s) = q(2 - s) - \rho(2 - s). \tag{6.17}
\]

Let us estimate the coefficients of the right-hand side of (6.17) when expanded in terms of the biorthogonal functions \( p_j(2 - s) \). Since we already know this precisely for \( \rho(2 - s) \) it is enough to look at \( q(2 - s) \). If

\[
qu(2 - s) = \sum_{j = -\infty}^{\infty} d_j p_j(2 - s)
\]

then, clearly

\[
d_j = \int_0^2 e^{\sigma_j} q(2 - s) \, ds = \int_0^2 e^{\sigma_j} q(2 - s) \, ds + \int_0^2 (e^{\sigma_j} - e^{\sigma_j}) q(2 - s) \, ds
\]

\[
= (\text{cf. (6.15), (6.16)}) (e^{2\sigma_j} - e^{2\sigma_j}) \int_0^2 \left( \int_{\rho_j}^{\sigma_j} \frac{d}{d\tau} e^{\tau} \, d\tau \right) q(2 - s) \, ds
\]

\[
= (e^{2\sigma_j} - e^{2\sigma_j}) + \int_0^2 \left( \int_{\rho_j}^{\sigma_j} e^{\sigma_j} \, ds \right) q(2 - s) \, ds.
\]

The integral here clearly has absolute value

\[
d_j \leq \int_0^2 \left( \int_{\rho_j}^{\sigma_j} e^{\sigma_j} \, d\tau \right) q(2 - s) \, ds \leq \int_0^2 \int_{\rho_j}^{\sigma_j} e^{\sigma_j} \, d\tau \cdot |q(2 - s)| \, ds
\]

\[
\leq 2^{1/2} |\sigma_j - \rho_j| \sup |e^{\sigma_j}||q(2 - s)|_{L^2(0,2)},
\]

where the sup is taken over the straight line segment joining \( \sigma_j \) and \( \rho_j \) in the complex plane. From the asymptotic form (2.6) for the \( \sigma_j \) together with (6.14) we infer that this sup is uniformly bounded for all \( j \). Therefore we have

\[
d_j = e^{2\sigma_j} - e^{2\sigma_j} + O(|\sigma_j - \rho_j|), \quad |j| \to \infty.
\]
Then using (5.5) we conclude from (6.17) that
\[ \frac{k_j(2 - s)}{2} = \sum_{j=-\infty}^{\infty} k_j \rho_j (2 - s) \]
where
\[ |k_j| = O(|\sigma_j - \rho_j|), \quad |j| \to \infty. \]

Referring to (6.10) we see that $k_j$ must be equal to $k_j \rho_j$ so that
\[ |k_j \rho_j| = O(|\sigma_j - \rho_j|), \quad |j| \to \infty \]
from which it follows that the $k_j$ are square summable (and hence $k$ in (6.1) yields a continuous functional of the state $(w)$ just in case (6.14) is true. Since
\[ \lim_{|j| \to \infty} \left| \frac{\int_{\rho_j}^{\sigma_j} e^{\pi j \sigma} d\tau}{|\sigma_j - \rho_j|} \right| = \tilde{s} \left( \lim_{|j| \to \infty} |e^{\pi j \sigma}| \right) = se^{Re(\sigma)s} \]
it does not appear easy to show that there is any positive $d$ with $d_j - (e^{2\alpha_j} - e^{2\sigma}) \geq d_j - |\sigma_j - \rho_j|$. Hence (6.14) has only been shown to be a sufficient condition here, rather than a necessary condition.

We remark that while the above theorem has been established for distinct $\sigma_j$ and distinct $\rho_j$, it is quite easy (but cumbersome of description) to extend the result to include finitely many $\sigma_j$ having finite multiplicities $\mu_j$ and finitely many $\rho_j$ having finite multiplicities $\nu_j$, provided the criteria spelled out at the end of Section 3 are met.

The case of boundary control deserves special mention. Assuming it takes the form
\[ a_1 w(1, t) + b_1 v(1, t) = u(t), \quad (6.18) \]
we shall suppose $u(t)$ to be synthesized by a linear feedback law
\[ u(t) = k_1 w(1, t) + k_2 v(1, t) + \int_{0}^{1} k(x) \left( \frac{w(x, t)}{v(x, t)} \right) dx, \]
where $k \in L^2([0, 1]; E^2)$ and
\[ \det \begin{pmatrix} a_1 & b_1 \\ k_1 & k_2 \end{pmatrix} \neq 0 \quad (6.19) \]
unless $k_1 = k_2 = 0$. The boundary condition (6.18) then becomes
\[ (a_1 - k_1) w(1, t) + (b_1 - k_2) v(1, t) = \int_{0}^{1} k(x) \left( \frac{w(x, t)}{v(x, t)} \right) dx. \quad (6.20) \]
What we might call the "base point" for the eigenvalues is the complex number

\[ \alpha(k_1, k_2) = \frac{1}{2} \log \gamma(k_1, k_2) \]

where (cf. (2.4))

\[ \gamma(k_1, k_2) = \frac{(a_1 - b_1 - (k_1 - k_2))(a_0 + b_0)}{(a_1 + b_1 - (k_1 + k_2))(a_0 - b_0)}. \]

From this we see that \( \alpha(k_1, k_2) \) can be determined as any desired complex number by appropriate choice of \( k_1 \) and \( k_2 \). Replacing \( a_1 \) by \( a_1 - k_1 \), \( b_1 \) by \( b_1 - k_2 \) we have returned to the case where the system takes the form (0.1), (0.2), (0.3) with \( u(t) \) determined by

\[ u(t) = \int_0^1 k(x)^* \left( \frac{w(x, t)}{w(x, t)} \right) dx. \]

As noted in Section 2, this case can be subsumed under the one already considered, replacing the square integrable \( \xi \) by coefficients \( \xi \) which are bounded away from 0 and \( \infty \). The condition (6.14) then becomes

\[ \sum_{j=-\infty}^{\infty} | \rho_j - \sigma_j |^2 < \infty \]

and we have the following result.

**Theorem 6.3.** Consider the boundary control system (0.1), (0.2), (0.4). By appropriate choice of \( k_1, k_2 \) satisfying (6.19) and \( k \in L^2([0, 1], E^p) \) the closed loop system (0.1), (0.2), (6.20) can be assigned eigenvalues

\[ \rho_j = \alpha + j\sigma_j + \alpha_j, \quad j = 0, \pm 1, \pm 2, \ldots, \]

where \( \alpha \) is any complex number and

\[ \sum_{j=-\infty}^{\infty} | \alpha_j |^2 < \infty. \]

It is shown in [19] that if \( k_1 \) and \( k_2 \) are selected so that

\[ a_1 - k_1 = b_1 - k_2 \neq 0 \]

then \( k(x) \in L^2([0, 1]; E^p) \) can be found so that all solutions of (0.1), (0.2), (6.20)
vanish identically for \( t \geq 2 \). This yields a "deadbeat" control law which has no counterpart in finite-dimensional control systems

\[
\dot{x} = Ax + bu
\]

but is a familiar and well studied phenomenon in the case of the discrete system

\[
x_{k+1} = Ax_k + bu_k.
\]

(See [11, 22] for further details.)

7. Concluding Remarks

It should be clear that the study carried out in the foregoing sections of this article does not exhaust the possibilities inherent in the method. By "the method" we mean the use of the control to state operator, called \( U \) in this paper, followed by a convolution transformation of the type \( I + M \), to reduce a controllable distributed parameter system to a canonical form wherein certain properties of the system will be more amenable to analysis. Certainly one goal must be to carry out this procedure for a general system without specific reference to eigenfunctions, etc. Here, however, a certain caveat is in order. It must be regarded exceptional that there should exist a one to one linear control to state map \( U \) from a space \( L^2([0, T], E^n) \) to the state space of the control process. In most cases \( U \) will have to be defined on a subspace \( \tilde{L} \) of \( L^2([0, T], E^n) \) in order to make it a one-to-one map. This will always be true for parabolic control systems, for example, where no minimal control interval exists and it will be true for hyperbolic systems involving several different wave speeds [16]. The normal form obtained with the transformation \( U \) will have \( \tilde{L} \) as an invariant subspace when \( u(t) = 0 \). The resulting canonical forms will necessarily have somewhat greater complexity than is the case in the present article. Even if we take the scalar hyperbolic system

\[
\left( \frac{\partial w}{\partial t} \right) - \left( \frac{\partial^2 w}{\partial x^2} \right) = g(x) u(t), \quad 0 \leq x \leq 1, \quad t \geq 0, \\
w(0, t) = w(1, t) = 0
\]

the domain of \( U \) becomes \( \{ u \in L^2[0, 2] \mid \int_0^2 u(t) \, dt = 0 \} \). The canonical systems in this and other hyperbolic cases where \( \tilde{L} \) is a proper subspace of \( L^2([0, 1], E^n) \) can be expected to correspond to groups whose generators are operators of the form studied in [3, 10], e.g., wherein the domain is described by means of "Stieltjes" boundary conditions together with certain linear side conditions.

We remark also that the case of multiple eigenvalues, in the original uncontrolled \( (u(t) = 0) \) system or in the closed loop system remains inadequately
explored for the hyperbolic systems studied in this paper. As remarked in Sections 3 and 6, however, we expect this work to be a fairly straightforward extension of what we have carried out here.

*Note added in proof.* Lemma 6.1 should be corrected so that (6.6), (6.7) read

\[(I + M)^* \Phi_j^{(2-\cdot)} = b_j \Phi_j^{(2-\cdot)}, j = 0, \pm 1, \pm 2, \ldots\]

\[(I + M) e^{\alpha j} = b_j e^{\alpha j}, j = 0, \pm 1, \pm 2, \ldots,\]

where, for some positive numbers \(b, B,\)

\[b < |b_j| < B, j = 0, \pm 1, \pm 2, \ldots.\]

The existence of \(b, B\) follows from the boundedness and bounded invertibility of \(I + M.\) No essential change in the subsequent analysis results.

Additional references: (Canonical forms for ordinary differential systems.)


**REFERENCES**


