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Analysis of an optimal control problem for the three-dimensional coupled modified Navier–Stokes and Maxwell equations

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Abstract

The mathematical formulation and analysis of an optimal control problem associated with a viscous, incompressible, electrically conducting fluid in a bounded three-dimensional domain with fixed perfectly conducting boundaries is considered. The objective of control is the matching of the velocity and magnetic fields to given target fields; control is effected through distributed mechanical force and current controls. The existence of optimal solutions is shown, the Gâteaux differentiability for the magnetohydrodynamic system with respect to controls is proved, and the optimality system is obtained.

Keywords: Optimal control; Magnetohydrodynamics; Modified Navier–Stokes equations; Maxwell equations; Gâteaux derivatives

1. Introduction

In this paper, we study an optimal control problem for a viscous, incompressible, electrically conducting fluid. The controls applied are a distributed force and current and the object of control

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is to match the velocity and magnetic fields to given fields. The controls and states are constrained to satisfy a coupled system of partial differential equations consisting of a modified Navier–Stokes system and Maxwell's equations. The need to use a modification of the Navier–Stokes system is motivated by our interest in treating three-dimensional problems for which the global uniqueness of weak solutions of the Navier–Stokes system is not known. In two dimensions, this result is known and one can simply use the Navier–Stokes system; see, e.g., [13].

The particular form of the modified Navier–Stokes system that forms one part of the coupled MHD system is due to Ladyzhenskaya. The well-known Smagorinski turbulence model is a special case. The global uniqueness of solutions of the coupled modified Navier–Stokes/Maxwell equation model was proven in [6].

In the past decade, substantial attention has been devoted to optimal control problem for the two-dimensional MHD system, see, e.g., [7,8,12,14], only scant attention has been paid to the analysis of optimal control problems for the three-dimensional MHD system; [1,2,5] all treat the steady-state case.

The mathematical description of the control problem we study proceeds as follows. Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega \subset C^2$. Let $\mathbf{v}$ denote the velocity, p the pressure, and $\mathbf{h}$ the magnetic field. Denote by $\mathbf{f}$ an applied distributed force control and by $\mathbf{j}$ an applied current control. For given $T > 0$, the cost functional is defined by

$$J(\mathbf{v}, \mathbf{h}, \mathbf{f}, \operatorname{curl}\mathbf{j}) = \int_0^T \int_\Omega \left(\frac{\alpha_1}{2}|\mathbf{v} - \mathbf{v}_d|^2 + \frac{\alpha_2}{2}|\mathbf{h} - \mathbf{h}_d|^2 + \frac{\beta_1}{2}|\mathbf{f}|^2 + \frac{\beta_2}{2}|\operatorname{curl}\mathbf{j}|^2 \right) dx\, dt, \tag{1.1}$$

where $\mathbf{v}_d$ and $\mathbf{h}_d$ denote some desired velocity and magnetic fields, respectively, and α_1, α_2, β_1, and β_2 are nonnegative constants. The first two terms in (1.1) are the object of control, i.e., to match, in an $L^2(\Omega)$ sense, the velocity and magnetic fields to the given fields $\mathbf{v}_d$ and $\mathbf{h}_d$, respectively. If $\alpha_1 > 0$ and $\alpha_2 = 0$, then the object of control is to just match the velocity fields while if $\alpha_1 = 0$ and $\alpha_2 > 0$, then the object is to just match the magnetic field. If both $\alpha_1 > 0$ and $\alpha_2 > 0$, then the object is to match both the velocity and magnetic fields. The last two terms in (1.1) are penalization terms that serve to limit the size of the controls $\mathbf{f}$ and $\operatorname{curl}\mathbf{j}$. One sets $\beta_1 = 0$ or $\beta_2 = 0$ whenever only a current or distributed force control is used, respectively. If both are used, then $\beta_1 > 0$ and $\beta_2 > 0$. The relative sizes of the α_i's and β_i's are determined by the competing objectives of achieving a good match for the velocity and magnetic fields (in which case one wants relatively large α_i's) and of limiting the cost of control (in which case one wants relatively large β_i's.)

We wish to minimize (1.1) subject to the constraints which are the modified Navier–Stokes equations (see [9]) coupled with the Maxwell equations:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla\mathbf{v} - \operatorname{div} \Xi(\mathbf{v}) + \mu\mathbf{h} \times \operatorname{curl}\mathbf{h} + \nabla p = \mathbf{f}, \tag{1.2}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{1.3}$$

$$\mu\mathbf{h}_t + \frac{1}{\sigma}\operatorname{curl}(\operatorname{curl}\mathbf{h}) + \mu(\mathbf{v} \cdot \nabla\mathbf{h} - \mathbf{h} \cdot \nabla\mathbf{v}) = \frac{1}{\sigma}\operatorname{curl}\mathbf{j}, \tag{1.4}$$

$$\operatorname{div}\mathbf{h} = 0, \tag{1.5}$$

with $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{h} = (h_1, h_2, h_3)$, and

$$\Xi(\mathbf{v}) = \left.\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v})},$$

where

$$\varepsilon(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad v_{i,j} \equiv \frac{\partial v_i}{\partial x_j},$$

supplemented by the initial data

$$\mathbf{v}|_{t=0} = \mathbf{v}^0 \quad \text{and} \quad \mathbf{h}|_{t=0} = \mathbf{h}^0 \quad \text{in } \Omega, \tag{1.6}$$

and one of the following sets of boundary conditions: either

$$\mathbf{v}|_{S_T} = \mathbf{0}, \quad \mathbf{h}\cdot\mathbf{n}|_{S_T} = 0, \quad \text{and} \quad (\operatorname{curl}\mathbf{h})_\tau|_{S_T} = \mathbf{0}, \tag{1.7}$$

where $S_T = \partial\Omega \times [0, T]$, or

$$\mathbf{v},\ \mathbf{h},\ \text{and } p \text{ are periodic with respect to } x_k,\ k = 1, 2, 3. \tag{1.8}$$

Here, $\mathbf{n}$ is the outer normal to $\partial\Omega$ and $\mathbf{u}_\tau$ is the projection of the vector $\mathbf{u}$ onto the tangent plane to $\partial\Omega$. In (1.2) and (1.4), $\mu > 0$ denotes the constant magnetic permeability and $\sigma > 0$ the constant electric conductivity. We consider (1.2)–(1.8) in $Q_T = \Omega \times (0, T)$ with a fixed $T \in (0, \infty)$.

The potential $\mathcal{D}(\cdot)$ is a smooth function having the following properties:

(i) $\mathcal{D} : \mathbb{M}^{3\times3}_{\text{sym}} \to \mathbb{R}^1_+ = [0, \infty)$ and $\mathcal{D} \in C^3(\mathbb{M}^{3\times3}_{\text{sym}})$;
(ii) $\nu_1 m(\varepsilon) \leqslant \mathcal{D}(\varepsilon) \leqslant \nu_2 m(\varepsilon)$, where $m(\varepsilon) = |\varepsilon|^2 + |\varepsilon|^{2+2\delta}$;
(iii) $\nu_3 m(\varepsilon) \leqslant \frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon_{ij}}\varepsilon_{ij} \leqslant \nu_4 m(\varepsilon)$;
(iv) $\nu_5(1 + |\varepsilon|^{2\delta})|\kappa|^2 \leqslant \frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\kappa_{ij}\kappa_{kl} \leqslant \nu_6(1 + |\varepsilon|^{2\delta})|\kappa|^2$;
(v) $\frac{\partial^3\mathcal{D}(\varepsilon)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}\partial\varepsilon_{mn}}\kappa_{ij}\ell_{kl}\pi_{mn} \leqslant \nu_7|\varepsilon|^{2\delta-1}|\kappa||\ell||\pi|$

with $\nu_k > 0$, $k = 1, 2, \ldots, 7$, constants and κ, ℓ, π arbitrary elements in $\mathbb{M}^{3\times3}_{\text{sym}}$. For the Navier–Stokes equations, we have $\mathcal{D}(\varepsilon) = \nu|\varepsilon|^2$ and $\operatorname{div} \Xi(\mathbf{v}) = \nu\Delta\mathbf{v}$.

The global unique solvability of problems (1.2)–(1.8) was proved in [6] for the three-dimensional case, under the assumption $\delta \in [1/4, 2]$. For two-dimensional domains Ω, the parameter δ can be any nonnegative number.

The plan of the rest of the paper is as follows. In Section 2, we formulate the optimal control problem. In Section 3, we prove the existence of an optimal solution. Finally, in Section 4, we show that the magnetohydrodynamic system is Gâteaux differentiable with respect to controls and obtain the optimality system from which optimal states and controls may be determined.

2. Notations and formulation of the optimal control problem

We use the standard notations for the Lebesgue spaces $L^m(\Omega)$ with norms

$$\|\phi\|_{m,\Omega} = \left(\int_\Omega |\phi|^m \, d\mathbf{x}\right)^{1/m} \quad \text{for } m \in [1, \infty) \quad \text{and} \quad \|\phi\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |\phi|.$$

The inner product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$, i.e., $(\phi, \psi) = \int_\Omega \phi\psi \, d\mathbf{x}$. Sobolev spaces are denoted by $W^k_m(\Omega)$ with associated norms

$$\|\phi\|_{W_m^k(\Omega)} = \sum_{|i|=0}^{k} \left\| \frac{\partial^{|i|}\phi}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_\ell^{i_\ell}} \right\|_{m,\Omega},$$

where $i_1,\dots,i_\ell$ are nonnegative integers and $|i| = \sum_{j=1}^{\ell} i_j$.

We will use the same notation for spaces of vector-valued functions and their associated norms. For example, $\mathbf{u} = (u_1,\dots,u_\ell) \in L^m(\Omega)$ implies that each component $u_j \in L^m(\Omega)$.

The set of all infinitely differentiable functions with compact support with respect to Ω is denoted by $\mathcal{D}(\Omega)$. We then introduce the set

$$\mathcal{J}^\infty(\Omega) = \left\{ \mathbf{v} \in \mathcal{D}(\Omega) \;\middle|\; \operatorname{div}\mathbf{v} = \sum_{i=1}^{\ell} v_{i,i} = 0 \right\}$$

and the subspace of $L^2(\Omega)$,

$$\mathcal{J}(\Omega) = \left\{ \mathbf{v} \in L^2(\Omega) \;\middle|\; \operatorname{div}\mathbf{v} = 0 \right\},$$

where $\operatorname{div}\mathbf{v} = 0$ is understood in the sense of distributions, i.e.,

$$\int_\Omega \mathbf{v}\cdot\nabla\phi\,d\mathbf{x} = 0 \quad \forall\phi \in \mathcal{D}(\Omega).$$

Then, $\mathring{\mathcal{J}}(\Omega)$ is defined to be the closure of $\mathcal{J}^\infty(\Omega)$ in the norm of $L^2(\Omega)$. Thus,

$$\mathring{\mathcal{J}}(\Omega) \subset \mathcal{J}(\Omega) \subset L^2(\Omega).$$

We also define

$$\mathcal{J}_m^k(\Omega) = W_m^k(\Omega) \cap \mathcal{J}(\Omega)$$

and

$$\mathring{\mathcal{J}}_m^1(\Omega),\ \text{the closure of } \mathcal{J}^\infty(\Omega) \text{ in the norm of } W_m^1(\Omega).$$

The following subspaces of $\mathcal{J}_2^2(\Omega)$ and $\mathcal{J}_2^1(\Omega)$ will be needed:

$$\widetilde{\mathcal{J}}_2^2(\Omega) = \left\{ \mathbf{v} \in \mathcal{J}_2^2(\Omega) \;\middle|\; (\mathbf{v}\cdot\mathbf{n})|_{\partial\Omega} = 0,\ (\operatorname{curl}\mathbf{v})_\tau|_{\partial\Omega} = \mathbf{0} \right\}$$

and

$$\widetilde{\mathcal{J}}_2^1(\Omega),\ \text{the closure of } \widetilde{\mathcal{J}}_2^2(\Omega) \text{ in the norm of } W_2^1(\Omega).$$

Finally, C will denote several constants whose value changes with context.

Instead of Eqs. (1.2) and (1.4), we will use the integral identities

$$(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}, \boldsymbol{\eta}) + \left(\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=\varepsilon(\mathbf{v})}, \varepsilon(\boldsymbol{\eta}) \right) - (\mu\mathbf{h}\cdot\nabla\mathbf{h}, \boldsymbol{\eta}) = (\mathbf{f}, \boldsymbol{\eta}) \tag{2.1}$$

for any $\boldsymbol{\eta} \in \mathring{\mathcal{J}}_{2+2\delta}^1(\Omega)$ and

$$(\mu\mathbf{h}_t, \boldsymbol{\zeta}) - \frac{1}{\sigma}(\Delta\mathbf{h}, \boldsymbol{\zeta}) + \mu(\mathbf{v}\cdot\nabla\mathbf{h} - \mathbf{h}\cdot\nabla\mathbf{v}, \boldsymbol{\zeta}) = \frac{1}{\sigma}(\operatorname{curl}\mathbf{j}, \boldsymbol{\zeta}) \tag{2.2}$$

for any $\boldsymbol{\zeta} \in L^2(\Omega)$. It is easy to see that (2.1) follows from the inner product in $L^2(\Omega)$ of (1.2) and $\boldsymbol{\eta} \in \mathring{\mathcal{J}}_{2+2\delta}^1(\Omega)$ and that (2.2) follows from the inner product of (1.4) and $\boldsymbol{\zeta} \in L^2(\Omega)$, if we take into account (1.3), (1.5), and (1.7) or (1.8) and also the identity $\mathbf{h} \times \operatorname{curl} \mathbf{h} = -\mathbf{h} \cdot \nabla \mathbf{h} + \frac{1}{2}\nabla|\mathbf{h}|^2$.

We recall the following existence result from [6].

Theorem 1. *Suppose that Ω is a bounded domain in $\mathbb{R}^3$ with $\partial\Omega \subset C^2$ and let $Q_T = \Omega \times (0,T)$ and $S_T = \partial\Omega \times [0,T]$. Suppose that $\mathbf{f}, \operatorname{curl}\mathbf{j} \in L^2(Q_T)$, $\operatorname{div}\mathbf{j} = \mathbf{0}$, $\mathbf{v}^0 \in W_2^2(\Omega) \cap \mathring{\mathcal{J}}_2^1(\Omega)$, and $\mathbf{h}^0 \in \widetilde{\mathcal{J}}_2^1(\Omega)$. Then, the problem* (1.2)–(1.6) *along with either* (1.7) *or* (1.8) *with $\delta \in [\frac{1}{4}, 2]$ has a unique generalized solution* $\mathbf{v}$, $\mathbf{h}$. *Moreover, the generalized solution has the properties*

$$\|\mathbf{v}_t\|_{2,Q_T}, \max_{t\in[0,T]} \big\|\mathbf{v}_x(t)\big\|_{2+2\delta,\Omega} < \infty \tag{2.3}$$

and

$$\max_{t\in[0,t]} \big\|\mathbf{h}_x(t)\big\|_{2,\Omega}, \|\mathbf{h}_t\|_{2,Q_T}, \|\mathbf{h}_{xx}\|_{2,Q_T} < \infty. \tag{2.4}$$

We use the following notations:

$$|\boldsymbol{\psi}_x| = \left(\sum_{i,j=1}^{3} |\psi_{i,j}|^2\right)^{1/2}, \qquad |\boldsymbol{\psi}_{xx}| = \left(\sum_{i,j,k=1}^{3} |\psi_{i,jk}|^2\right)^{1/2}.$$

Given Ω, T, $\mathbf{v}^0 \in W_2^2(\Omega) \cap \mathring{\mathcal{J}}_2^1(\Omega)$, $\mathbf{h}^0 \in \widetilde{\mathcal{J}}_2^1(\Omega)$, and $\mathbf{v}_d, \mathbf{h}_d \in L^2(Q_T)$, the set of all admissible solutions is defined by

$$\mathcal{A}_{\mathrm{ad}} = \big\{(\mathbf{v}, \mathbf{h}, \mathbf{f}, \operatorname{curl}\mathbf{j}) \in L^2(Q_T) \;\big|\; J(\mathbf{v}, \mathbf{h}, \mathbf{f}, \operatorname{curl}\mathbf{j}) < \infty \text{ and (2.1)–(2.2) are satisfied}\big\}.$$

With this notation, the formulation of the optimal control problem is given by

$$\begin{aligned} &\textit{given } \Omega, T, \mathbf{v}^0 \in W_2^2(\Omega) \cap \mathring{\mathcal{J}}_2^1(\Omega), \mathbf{h}^0 \in \widetilde{\mathcal{J}}_2^1(\Omega), \textit{and } \mathbf{v}_d, \mathbf{h}_d \in L^2(Q_T), \\ &\textit{find } (\hat{\mathbf{v}}, \hat{\mathbf{h}}, \hat{\mathbf{f}}, \operatorname{curl}\hat{\mathbf{j}}) \in \mathcal{A}_{\mathrm{ad}} \textit{ such that the functional } (1.1) \textit{ is minimized.} \end{aligned} \tag{2.5}$$

We recall that $(\mathbf{u} \cdot \nabla\mathbf{v}, \mathbf{w}) = 0$ for all divergence free $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W_2^1(\Omega)$ satisfying $\int_{\partial\Omega} u_k v_i \times w_i n_k \, dS = 0$, and that $(\operatorname{curl}\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \operatorname{curl}\mathbf{v})$ holds if $\mathbf{u}_\tau|_{\partial\Omega} = \mathbf{0}$. We will use the inequality

$$\nu_8\|\mathbf{h}_x\|_{2,\Omega}^2 - \nu_9\|\mathbf{h}\|_{2,\Omega}^2 \leqslant \|\operatorname{curl}\mathbf{h}\|_{2,\Omega}^2 \tag{2.6}$$

with $\nu_8 > 0$ which holds for any solenoidal $\mathbf{h}$ satisfying the boundary condition $\mathbf{h} \cdot \mathbf{n}|_{\partial\Omega} = 0$; see, e.g., [3]. Also according to the Korn inequalities, we have

$$C(q)\|\mathbf{v}_x\|_{q,\Omega} \leqslant \big\|\varepsilon(\mathbf{v})\big\|_{q,\Omega} \quad \forall q \in (1,\infty), \tag{2.7}$$

that holds for some $C(q) > 0$ and for any $\mathbf{v} \in \mathring{\mathcal{J}}_q^1(\Omega)$.

3. Existence of optimal solutions

In the following theorem, we prove the existence of solutions for the optimal control problem.

Theorem 2. *Given $T > 0$, $\mathbf{v}^0 \in W_2^2(\Omega) \cap \mathring{\mathcal{J}}_2^1(\Omega)$, $\mathbf{h}^0 \in \widetilde{\mathcal{J}}_2^1(\Omega)$, and $\mathbf{v}_d, \mathbf{h}_d \in L^2(Q_T)$, then there exists a solution $(\hat{\mathbf{v}}, \hat{\mathbf{h}}, \hat{\mathbf{f}}, \operatorname{curl}\hat{\mathbf{j}})$ to the optimal control problem* (2.5).

Proof. The admissible set $\mathcal{A}_{\rm ad}$ is bounded and nonempty, e.g., $(\mathbf{v}_\circ, \mathbf{h}_\circ, \mathbf{0}, \mathbf{0}) \in \mathcal{A}_{\rm ad}$, where $\mathbf{v}_\circ$, $\mathbf{h}_\circ$ is the solution to (2.1)–(2.2) with $\mathbf{f} = 0$, $\operatorname{curl}\mathbf{j} = 0$. Let $\{(\mathbf{f}^{(n)}, \operatorname{curl}\mathbf{j}^{(n)})\}$ be a minimizing sequence for the optimal control problem and denote by $(\mathbf{v}^{(n)}, \mathbf{h}^{(n)}) = (\mathbf{v}(\mathbf{f}^{(n)}, \operatorname{curl}\mathbf{j}^{(n)}), \mathbf{h}(\mathbf{f}^{(n)}, \operatorname{curl}\mathbf{j}^{(n)}))$ the corresponding solution to (2.1) and (2.2). From (1.1), we see that the sequence $\{(\mathbf{f}^{(n)}, \operatorname{curl}\mathbf{j}^{(n)})\}$ is bounded in $L^2(Q_T)$. To obtain bounds on $(\mathbf{v}^{(n)}, \mathbf{h}^{(n)})$ we will use some estimates from [6], which we will sketch for the reader's convenience. Thus, (2.1) with $\boldsymbol{\eta} = \mathbf{v}^{(n)}$ and (2.2) with $\boldsymbol{\zeta} = \mathbf{h}^{(n)}$ yield

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}^{(n)}\|_{2,\Omega}^2 + \nu_3\big(C_1\|\mathbf{v}_x^{(n)}\|_{2,\Omega}^2 + C_2\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{2+2\delta}\big) + \frac{\mu}{2}\frac{d}{dt}\|\mathbf{h}^{(n)}\|_{2,\Omega}^2 + \frac{\nu_8}{\sigma}\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2$$
$$\leqslant \frac{\nu_9}{\sigma}\|\mathbf{h}^{(n)}\|_{2,\Omega}^2 + \|\mathbf{f}^{(n)}\|_{2,\Omega}\|\mathbf{v}^{(n)}\|_{2,\Omega} + \frac{1}{\sigma}\|\operatorname{curl}\mathbf{j}^{(n)}\|_{2,\Omega}\|\mathbf{h}^{(n)}\|_{2,\Omega}$$

and by the Gronwall lemma we obtain

$$\max_{t\in[0,T]}\|\mathbf{v}^{(n)}\|_{2,\Omega}^2 + \|\mathbf{v}_x^{(n)}\|_{2,Q_T}^2 + \|\mathbf{v}_x^{(n)}\|_{2+2\delta,Q_T}^{2+2\delta} + \max_{t\in[0,T]}\|\mathbf{h}^{(n)}\|_{2,\Omega}^2 + \|\mathbf{h}_x^{(n)}\|_{2,Q_T}^2$$
$$\leqslant \Phi\big(T, \|\mathbf{v}^0\|_{2,\Omega}, \|\mathbf{h}^0\|_{2,\Omega}, \|\mathbf{f}^{(n)}\|_{2,Q_T}, \|\operatorname{curl}\mathbf{j}^{(n)}\|_{2,Q_T}\big), \tag{3.1}$$

where Φ is a continuous function of the indicated arguments.

Now we take $\boldsymbol{\zeta} = -\Delta\mathbf{h}^{(n)}$ in (2.2) and obtain

$$\frac{\mu}{2}\frac{d}{dt}\|\operatorname{curl}\mathbf{h}^{(n)}\|_{2,\Omega}^2 + \frac{1}{\sigma}\|\Delta\mathbf{h}^{(n)}\|_{2,\Omega}^2$$
$$= \mu\big(\mathbf{v}^{(n)}\cdot\nabla\mathbf{h}^{(n)} - \mathbf{h}^{(n)}\cdot\nabla\mathbf{v}^{(n)}, \Delta\mathbf{h}^{(n)}\big) + \frac{1}{\sigma}\big(\operatorname{curl}\mathbf{j}^{(n)}, \Delta\mathbf{h}^{(n)}\big)$$
$$\leqslant -\mu\big(v_{k,j}^{(n)}h_{i,k}^{(n)}, h_{i,j}^{(n)}\big) + \frac{1}{2\sigma}\|\Delta\mathbf{h}^{(n)}\|_{2,\Omega}^2 + C\big(\|\mathbf{h}^{(n)}\cdot\nabla\mathbf{v}^{(n)}\|_{2,\Omega}^2 + \|\operatorname{curl}\mathbf{j}^{(n)}\|_{2,\Omega}^2\big). \tag{3.2}$$

We will use now the Hölder inequality with powers $q = 2 + 2\delta$ and $q' = (2 + 2\delta)/(1 + 2\delta)$, the multiplicative inequality (see [11])

$$\|\mathbf{u}\|_{q,\Omega} \leqslant C(q)\|\mathbf{u}_x\|_{2,\Omega}^{\alpha}\|\mathbf{u}\|_{2,\Omega}^{1-\alpha} + C_1(q)\|\mathbf{u}\|_{2,\Omega} \tag{3.3}$$

with

$$\alpha = 3\left(\frac{1}{2} - \frac{1}{q}\right) \in [0, 1], \quad q \in [2, 6]$$

(here $C_1(q) = 0$ if $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ or $\int_\Omega \mathbf{u}\,d\mathbf{x} = \mathbf{0}$), and Young's inequality to obtain

$$\mu\big|\big(v_{k,j}^{(n)}h_{i,k}^{(n)}, h_{i,j}^{(n)}\big)\big|$$
$$\leqslant C\int_\Omega |\mathbf{v}_x^{(n)}||\mathbf{h}_x^{(n)}|^2\,d\mathbf{x} \leqslant C\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}\|\mathbf{h}_x^{(n)}\|_{\frac{4(1+\delta)}{1+2\delta},\Omega}^2$$
$$\leqslant C_1\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}\big(\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^{2(1-\alpha)}\|\mathbf{h}_{xx}^{(n)}\|_{2,\Omega}^{2\alpha} + \|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2\big)$$
$$\leqslant \epsilon\|\mathbf{h}_{xx}^{(n)}\|_{2,\Omega}^2 + C_\epsilon\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{\frac{1}{1-\alpha}}\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2 + C_1\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2 \tag{3.4}$$

for any $\epsilon \in (0, 1]$ and $\alpha = \frac{3}{4(1+\delta)}$; see, e.g., [6].

Now we will use Hölder inequality with exponents $q = 1 + \delta$ and $q' = (1 + \delta)/\delta$, the imbedding inequality (see, e.g., [6,11])

$$\|\mathbf{u}\|_{m,\Omega} \leqslant C(m, r)\|\mathbf{u}_x\|_{r,\Omega} + C_1(m, r)\|\mathbf{u}\|_{2,\Omega}, \tag{3.5}$$

where

$$m \leqslant \frac{3r}{3-r} \quad \text{for } r \in [1,3),$$

with $m = 2(1+\delta)/\delta$ and $r = 6(1+\delta)/(2+5\delta)$ (also $C_1(m,r) = 0$ if $\mathbf{u}|_{\partial\Omega} = 0$ or $\int_\Omega \mathbf{u}\,d\mathbf{x}$), the inequality (3.3) with $q = 6(1+\delta)/(2+5\delta)$, and Young's inequality to obtain

$$\begin{aligned}
\|\mathbf{h}^{(n)} \cdot \nabla \mathbf{v}^{(n)}\|_{2,\Omega}^2 &\leqslant \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2 \|\mathbf{h}^{(n)}\|_{\frac{2+2\delta}{\delta},\Omega}^2 \\
&\leqslant C\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2 \big(\|\mathbf{h}_x^{(n)}\|_{\frac{6+6\delta}{2+5\delta},\Omega}^2 + \|\mathbf{h}^{(n)}\|_{2,\Omega}^2\big) \\
&\leqslant C_1\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2 \|\mathbf{h}_x^{(n)}\|_{2,\Omega}^{2(1-\gamma)} \|\mathbf{h}_{xx}^{(n)}\|_{2,\Omega}^{2\gamma} + C_1\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2 \big(\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2 + \|\mathbf{h}^{(n)}\|_{2,\Omega}^2\big) \\
&\leqslant \epsilon_1 \|\mathbf{h}_{xx}^{(n)}\|_{2,\Omega}^2 + C_{\epsilon_1}\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{\frac{2}{1-\gamma}} \|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2 \\
&\quad + C_2\|\mathbf{v}_x^{(n)}\|_{2+2\delta}^2 \big(\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2 + \|\mathbf{h}^{(n)}\|_{2,\Omega}^2\big)
\end{aligned} \tag{3.6}$$

for any $\epsilon_1 \in (0,1]$ and $\gamma = \frac{1-2\delta}{2+2\delta}$.

We recall also the inequality

$$\|\mathbf{h}_{xx}\|_{2,\Omega} \leqslant C_1(\Omega)\|\Delta \mathbf{h}\|_{2,\Omega} + C_2(\Omega)\|\mathbf{h}\|_{2,\Omega}$$

which holds for any solenoidal vector field $\mathbf{h}$ satisfying the boundary conditions $\mathbf{h} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $(\operatorname{curl} \mathbf{h})_\tau|_{\partial\Omega} = \mathbf{0}$; see, e.g., [10]. Now if

$$\delta \geqslant \frac{1}{4}, \tag{3.7}$$

then $\frac{1}{1-\alpha} \leqslant 2 + 2\delta$ and $\frac{2}{1-\gamma} \leqslant 2 + 2\delta$.

Using (3.4), (3.6), and (3.7), we conclude from (3.2) with sufficiently small ϵ and ϵ_1 that

$$\begin{aligned}
&\frac{d}{dt}\|\operatorname{curl} \mathbf{h}^{(n)}\|_{2,\Omega}^2 + \|\mathbf{h}_{xx}^{(n)}\|_{2,\Omega}^2 \\
&\quad \leqslant C\big(\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{\frac{1}{1-\alpha}} + \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega} + \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{\frac{2}{1-\gamma}} + \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2\big)\|\mathbf{h}_x\|_{2,\Omega}^2 \\
&\qquad + C_1\|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2 \|\mathbf{h}^{(n)}\|_{2,\Omega}^2 + C_2\|\operatorname{curl} \mathbf{j}^{(n)}\|_{2,\Omega}^2.
\end{aligned}$$

Integrate now over $(0,t)$, use (2.6), the Gronwall lemma, and (3.1) to obtain for $\delta \geqslant 1/4$ that

$$\max_{t\in[0,T]} \|\mathbf{h}_x^{(n)}(t)\|_{2,\Omega} + \|\mathbf{h}_{xx}^{(n)}\|_{2,Q_T} \leqslant \Phi_1\big(T, \|\mathbf{h}_x^0\|_{2,\Omega}\big). \tag{3.8}$$

In a similar way, we obtain from (2.2) with $\boldsymbol{\zeta} = \mathbf{h}_t^{(n)}$ that

$$\|\mathbf{h}_t^{(n)}\|_{2,Q_T} \leqslant \Phi_2\big(T, \|\mathbf{h}_x^0\|_{2,\Omega}\big).$$

Now, we let $\boldsymbol{\eta} = \mathbf{v}_t^{(n)}$ in (2.1), integrate over $(0,t)$, and use (2.7) to obtain

$$\begin{aligned}
&\|\mathbf{v}_t^{(n)}\|_{2,Q_T}^2 + 2\nu_3\big(\|\mathbf{v}_x^{(n)}\|_{2,\Omega}^2 + \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{2+2\delta}\big) \\
&\quad \leqslant 2\nu_4\big(\|\mathbf{v}_x^0\|_{2,\Omega}^2 + \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}^{2+2\delta}\big) + C\big(\||\mathbf{v}^{(n)}||\mathbf{v}_x^{(n)}|\|_{2,Q_T}^2 + \||\mathbf{h}^{(n)}||\mathbf{h}_x^{(n)}|\|_{2,Q_T}^2\big) \\
&\qquad + 2\|\mathbf{f}^{(n)}\|_{2,Q_T}^2.
\end{aligned} \tag{3.9}$$

Using the Hölder inequality with exponents $q = 1+\delta$ and $q' = (1+\delta)/\delta$, and inequality (3.5) with $m = (2+2\delta)/\delta$ and $r = 2+2\delta$, we obtain

$$\big\| |\mathbf{v}^{(n)}| |\mathbf{v}_x^{(n)}| \big\|_{2,Q_T}^2 \leqslant \int_0^t \|\mathbf{v}^{(n)}\|_{\frac{2+2\delta}{\delta},\Omega}^2 \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^2 \, d\tau \leqslant C \int_0^t \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^4 \, d\tau, \tag{3.10}$$

if $\delta \geqslant \frac{1}{5}$. Using the Hölder inequality with exponents $q = 3$ and $q' = 3/2$, the inequality (3.5) with $m = 6$ and $r = 2$, and the multiplicative inequality (3.3) with $q = 3$, we have

$$\begin{aligned} \big\| |\mathbf{h}^{(n)}| |\mathbf{h}_x^{(n)}| \big\|_{2,Q_T}^2 &\leqslant \int_0^t \|\mathbf{h}^{(n)}\|_{6,\Omega}^2 \|\mathbf{h}_x^{(n)}\|_{3,\Omega}^2 \, d\tau \\ &\leqslant C \int_0^t \big(\|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2 + \|\mathbf{h}^{(n)}\|_{2,\Omega}^2\big)\big(\|\mathbf{h}_x^{(n)}\|_{2,\Omega} \|\mathbf{h}_{xx}^{(n)}\|_{2,\Omega} + \|\mathbf{h}_x^{(n)}\|_{2,\Omega}^2\big)\, d\tau. \end{aligned} \tag{3.11}$$

From (3.9), (3.10), (3.11), and (3.8), we obtain

$$\begin{aligned} &\|\mathbf{v}_t^{(n)}\|_{2,Q_T}^2 + 2\nu_3\big(\|\mathbf{v}_x^{(n)}\|_{2,\Omega}^2 + \|\mathbf{v}_x^{(n)}\|_{2+2\delta,\Omega}^{2+2\delta}\big) \\ &\quad \leqslant C\Bigg[\|\mathbf{v}_x^0\|_{2,\Omega}^2 + \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}^{2+2\delta} + \|\mathbf{f}^{(n)}\|_{2,Q_T}^2 + C_1 \int_0^t \|\mathbf{v}_x^{(n)}(\tau)\|_{2+2\delta,\Omega}^{2+2\delta}\big(\|\mathbf{v}_x^{(n)}(\tau)\|_{2,\Omega}^2 \\ &\qquad + \|\mathbf{v}_x^{(n)}(\tau)\|_{2+2\delta,\Omega}^{2+2\delta}\big)\, d\tau + \Phi_1\big(T, \|\mathbf{h}_x^0\|_{2,\Omega}\big)\Bigg] \end{aligned}$$

so that (3.1) and the Gronwall lemma yield

$$\|\mathbf{v}_t^{(n)}\|_{2,Q_T} + \max_{t\in[0,T]}\big(\|\mathbf{v}_x^{(n)}(t)\|_{2,\Omega} + \|\mathbf{v}_x^{(n)}(t)\|_{2,\Omega}\big) \leqslant \Phi_3\big(T, \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}, \|\mathbf{v}_x^0\|_{2,\Omega}\big)$$

with a continuous function Φ_3 which depends also the known functions $\mathbf{v}^0$, $\mathbf{h}^0$, $\mathbf{f}^{(n)}$, and $\operatorname{curl}\mathbf{j}^{(n)}$.

Selecting subsequences, if necessary, we have

$$\begin{aligned} &\mathbf{f}^{(n)}, \operatorname{curl}\mathbf{j}^{(n)} \to \hat{\mathbf{f}}, \operatorname{curl}\hat{\mathbf{j}} \quad \text{weakly in } L^2(Q_T), \\ &\big(\mathbf{v}_x^{(n)}, \mathbf{h}_x^{(n)}\big) \to (\hat{\mathbf{v}}_x, \hat{\mathbf{h}}_x) \quad \text{weak-}* \text{ in } L^\infty\big(0,T; L^{2+2\delta}(\Omega)\big), \\ &\big(\mathbf{v}_t^{(n)}, \mathbf{h}_t^{(n)}, \mathbf{h}_{xx}^{(n)}\big) \to (\hat{\mathbf{v}}_t, \hat{\mathbf{h}}_t, \hat{\mathbf{h}}_{xx}) \quad \text{weakly in } L^2(Q_T), \\ &\Xi\big(\mathbf{v}^{(n)}\big) \to \boldsymbol{\chi} \quad \text{weakly in } L^{2(1+2\delta)/(1+2\delta)}\big(0,T; \big(W^1_{2+2\delta}(\Omega)\big)'\big). \end{aligned}$$

If $\mathbf{v}^{(n)}$ converges to $\hat{\mathbf{v}}$ in $L^2(0,T; W^1_{2+2\delta}(\Omega))$ weakly and $L^2(0,T; L^2(\Omega))$ strongly and $\mathbf{h}^{(n)}$ converges to $\hat{\mathbf{h}}$ in $L^2(0,T; W^1_2(\Omega))$ weakly and $L^2(0,T; L^2(\Omega))$ strongly, then, for any $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathcal{J}^\infty(\Omega)$ we have

$$\lim_{n\to\infty} \int_0^T \big(\mathbf{v}^{(n)} \cdot \nabla \mathbf{v}^{(n)} - \mu \mathbf{h}^{(n)} \cdot \nabla \mathbf{h}^{(n)}, \boldsymbol{\eta}\big)\, dt = \int_0^T (\hat{\mathbf{v}} \cdot \nabla \hat{\mathbf{v}} - \mu \hat{\mathbf{h}} \cdot \nabla \hat{\mathbf{h}}, \boldsymbol{\eta})\, dt,$$

$$\lim_{n\to\infty} \int_0^T \big(\mathbf{v}^{(n)} \cdot \nabla \mathbf{h}^{(n)} - \mathbf{h}^{(n)} \cdot \nabla \mathbf{v}^{(n)}, \boldsymbol{\zeta}\big)\, dt = \int_0^T (\hat{\mathbf{v}} \cdot \nabla \hat{\mathbf{h}} - \hat{\mathbf{h}} \cdot \nabla \hat{\mathbf{v}}, \boldsymbol{\zeta})\, dt.$$

Since $\mathcal{J}^\infty(\Omega)$ is dense in $\mathring{\mathcal{J}}^1_{2+2\delta}(\Omega)$, then this is still true for any $\boldsymbol{\eta} \in \mathring{\mathcal{J}}^1_{2+2\delta}(\Omega)$, $\boldsymbol{\zeta} \in L^2(\Omega)$, by a continuity argument.

To deduce $\boldsymbol{\chi} = \Xi(\hat{\mathbf{v}})$, we use the monotonicity of Ξ. Indeed, for any $\mathbf{w} \in L^{2+2\delta}(0, T; \mathring{\mathcal{J}}^1_{2+2\delta}(\Omega))$, we have

$$\begin{aligned}
&\int_0^T \big(\boldsymbol{\chi} - \Xi(\mathbf{w}), \hat{\mathbf{v}} - \mathbf{w}\big)\, dt \\
&\quad = \lim_{n\to\infty} \int_0^T \big(\Xi\big(\mathbf{v}^{(n)}\big) - \Xi(\mathbf{w}), \mathbf{v}^{(n)} - \mathbf{w}\big)\, dt \\
&\quad = \lim_{n\to\infty} \int_0^T \left(\int_0^1 \frac{\partial^2 \mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\tau\varepsilon(\mathbf{v}^{(n)})+(1-\tau)\varepsilon(\mathbf{w})} d\tau : \varepsilon\big(\mathbf{v}^{(n)} - \mathbf{w}\big), \varepsilon\big(\mathbf{v}^{(n)} - \mathbf{w}\big)\right) dt \geqslant 0.
\end{aligned}$$

Passing to the limit as $n \to \infty$ we obtain that $\hat{\mathbf{f}}, \operatorname{curl}\hat{\mathbf{j}}, \hat{\mathbf{v}}$ and $\hat{\mathbf{h}}$ satisfy the integral identities (2.1) and (2.2).

By the weak lower semicontinuity of the norms we finally get that the cost functional J attains its infimum at $(\hat{\mathbf{f}}, \operatorname{curl}\hat{\mathbf{j}}, \hat{\mathbf{v}}, \hat{\mathbf{h}})$. This completes the proof. □

4. First-order necessary condition

We now show that the optimal solution must satisfy the first-order necessary condition associated with the optimal control problem. By studying the case in which the Gâteaux derivative of the cost functional vanishes, we get a possible candidate solution for the optimal control; see [15].

Theorem 3. *Let* $\mathbf{v}^0 \in W_2^2(\Omega) \cap \mathring{\mathcal{J}}^1_2(\Omega)$ *and* $\mathbf{h}^0 \in \widetilde{\mathcal{J}}^1_2(\Omega)$. *The mapping* $(\mathbf{f}, \operatorname{curl}\mathbf{j}) \mapsto (\mathbf{v}, \mathbf{h}) \times (\mathbf{f}, \operatorname{curl}\mathbf{j})$ *from* $L^2(Q_T)$ *to* $L^2(0, T; W_2^1(\Omega))$, *defined as the solution of* (2.1), (2.2) *and* (1.6), *has a Gâteaux derivative* $(D(\mathbf{v}, \mathbf{h})/D(\mathbf{f}, \operatorname{curl}\mathbf{j})) \cdot (\mathbf{g}, \operatorname{curl}\mathbf{k})$ *for every* $(\mathbf{g}, \operatorname{curl}\mathbf{k}) \in L^2(Q_T)$. *Moreover,* $(\check{\mathbf{v}}, \check{\mathbf{h}})(\mathbf{g}, \operatorname{curl}\mathbf{k}) = (D(\mathbf{v}, \mathbf{h})/D(\mathbf{f}, \operatorname{curl}\mathbf{j})) \cdot (\mathbf{g}, \operatorname{curl}\mathbf{k})$ *is the solution of the linear problem*

$$(\check{\mathbf{v}}_t + \mathbf{v}\cdot\nabla\check{\mathbf{v}} + \check{\mathbf{v}}\cdot\nabla\mathbf{v}, \boldsymbol{\eta}) + \left(\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon(\mathbf{v})} : \varepsilon(\check{\mathbf{v}}), \varepsilon(\boldsymbol{\eta})\right) - \mu(\check{\mathbf{h}}\cdot\nabla\mathbf{h} + \mathbf{h}\cdot\nabla\check{\mathbf{h}}, \boldsymbol{\eta}) = (\mathbf{g}, \boldsymbol{\eta}) \tag{4.1}$$

for any $\boldsymbol{\eta} \in \mathring{\mathcal{J}}^1_{2+2\delta}(\Omega)$ *and*

$$\mu(\check{\mathbf{h}}_t, \boldsymbol{\zeta}) + \frac{1}{\sigma}(\operatorname{curl}\check{\mathbf{h}}, \operatorname{curl}\boldsymbol{\zeta}) + \mu(\check{\mathbf{v}}\cdot\nabla\mathbf{h} + \mathbf{v}\cdot\nabla\check{\mathbf{h}} - \check{\mathbf{h}}\cdot\nabla\mathbf{v} - \mathbf{h}\cdot\nabla\check{\mathbf{v}}, \boldsymbol{\zeta}) = \frac{1}{\sigma}(\operatorname{curl}\mathbf{k}, \boldsymbol{\zeta}) \tag{4.2}$$

for any $\boldsymbol{\zeta} \in L^2(\Omega)$, *with the initial data*

$$\check{\mathbf{v}}|_{t=0} = \check{\mathbf{h}}|_{t=0} = \mathbf{0} \quad \textit{in } \Omega.$$

Proof. Let $(\mathbf{f}, \operatorname{curl}\mathbf{j})$ and $(\mathbf{g}, \operatorname{curl}\mathbf{k})$ be given in $L^2(Q_T)$ and let $(\mathbf{v}, \mathbf{h})$ and $(\mathbf{v}_\lambda, \mathbf{h}_\lambda)$ denote the solutions of (2.1)–(2.2) with the right-hand sides $(\mathbf{f}, \operatorname{curl}\mathbf{j})$ and $(\mathbf{f}, \operatorname{curl}\mathbf{j}) + \lambda(\mathbf{g}, \operatorname{curl}\mathbf{k})$, respectively. We need to prove that

$$\lim_{\lambda\to 0}\frac{\|(\mathbf{v}_\lambda,\mathbf{h}_\lambda)-(\mathbf{v},\mathbf{h})-\lambda(\check{\mathbf{v}},\check{\mathbf{h}})\|_{L^2(0,T;W_2^1(\Omega))}}{\lambda}=0.$$

For $\tilde{\mathbf{v}}=\mathbf{v}_\lambda-\mathbf{v}-\lambda\check{\mathbf{v}}$ and $\tilde{\mathbf{h}}=\mathbf{h}_\lambda-\mathbf{h}-\lambda\check{\mathbf{h}}$, we have the identities

$$\int_0^T\Big((\tilde{\mathbf{v}}_t+\mathbf{v}\cdot\nabla\tilde{\mathbf{v}}+\tilde{\mathbf{v}}\cdot\nabla\mathbf{v},\boldsymbol{\eta})-\mu(\tilde{\mathbf{h}}\cdot\nabla\mathbf{h}+\mathbf{h}\cdot\nabla\tilde{\mathbf{h}},\boldsymbol{\eta})+\big(A,\varepsilon(\boldsymbol{\eta})\big)$$
$$-\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{v}-\mathbf{v}_\lambda),\boldsymbol{\eta}\big)-\mu\big((\mathbf{h}_\lambda-\mathbf{h})\cdot\nabla(\mathbf{h}-\mathbf{h}_\lambda),\boldsymbol{\eta}\big)\Big)\,dt=0 \tag{4.3}$$

and

$$\int_0^T\bigg(\mu(\tilde{\mathbf{h}}_t,\boldsymbol{\zeta})+\frac{1}{\sigma}(\operatorname{curl}\tilde{\mathbf{h}},\operatorname{curl}\boldsymbol{\zeta})+\mu(\tilde{\mathbf{v}}\cdot\nabla\mathbf{h}+\mathbf{v}\cdot\nabla\tilde{\mathbf{h}}-\tilde{\mathbf{h}}\cdot\nabla\mathbf{v}-\mathbf{h}\cdot\nabla\tilde{\mathbf{v}},\boldsymbol{\zeta})$$
$$-\mu\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{h}-\mathbf{h}_\lambda)-(\mathbf{h}_\lambda-\mathbf{h})\cdot\nabla(\mathbf{v}-\mathbf{v}_\lambda),\boldsymbol{\zeta}\big)\bigg)\,dt=0 \tag{4.4}$$

which follow from (2.1), (2.2), (4.1), and (4.2). Here

$$\begin{aligned}
A&=\int_0^1\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_\tau}d\tau:\varepsilon(\mathbf{v}_\lambda-\mathbf{v})-\lambda\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon(\mathbf{v})}:\varepsilon(\check{\mathbf{v}})\\
&=\int_0^1\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_\tau}d\tau:\varepsilon(\tilde{\mathbf{v}})+\int_0^1\bigg(\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_\tau}-\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon(\mathbf{v})}\bigg)d\tau:\lambda\varepsilon(\check{\mathbf{v}})\\
&=A_1+\int_0^1\bigg[\int_0^1\frac{d}{d\theta}\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\theta\varepsilon_\tau+(1-\theta)\varepsilon(\mathbf{v})}d\theta\bigg]d\tau:\lambda\varepsilon(\check{\mathbf{v}})=A_1+A_2
\end{aligned}$$

with $\varepsilon_\tau=\tau\varepsilon(\mathbf{v}_\lambda)+(1-\tau)\varepsilon(\mathbf{v})$ and

$$A_1=\int_0^1\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_\tau}d\tau:\varepsilon(\tilde{\mathbf{v}}),$$

$$A_2=\int_0^1\bigg[\int_0^1\frac{\partial^3\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_{\tau\theta}}d\theta:\tau\varepsilon(\mathbf{v}_\lambda-\mathbf{v})\bigg]d\tau:\lambda\varepsilon(\check{\mathbf{v}}).$$

By virtue of our hypothesis on $\mathcal{D}$, we have

$$\begin{aligned}
\big(A_1,\varepsilon(\tilde{\mathbf{v}})\big)&\geqslant\nu_5\int_\Omega\bigg(1+\int_0^1|\varepsilon_\tau|^{2\delta}\,d\tau\bigg)\big|\varepsilon(\tilde{\mathbf{v}})\big|^2\,d\mathbf{x}\\
&\geqslant\tilde{\nu}\|\tilde{\mathbf{v}}_x\|_{2,\Omega}^2+\nu_5\int_\Omega\int_0^1|\varepsilon_\tau|^{2\delta}\,d\tau\big|\varepsilon(\tilde{\mathbf{v}})\big|^2\,d\mathbf{x},
\end{aligned}\tag{4.5}$$

$\tilde{\nu}>0$, and

$$\begin{aligned}\big(A_2,\varepsilon(\tilde{\mathbf{v}})\big)&=\lambda\int_\Omega\left\{\left[\int_0^1\left(\int_0^1\frac{\partial^3\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon_{\tau\theta}}\tau\,d\theta\right)d\tau\right]:\varepsilon(\mathbf{v}_\lambda-\mathbf{v}):\varepsilon(\check{\mathbf{v}})\right\}\varepsilon(\tilde{\mathbf{v}})\,d\mathbf{x}\\&=\lambda\int_\Omega\left\{\left[\int_0^1\left(\int_0^\tau\frac{\partial^3\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon_\rho}:\varepsilon(\mathbf{v}_\lambda-\mathbf{v}):\varepsilon(\check{\mathbf{v}})\,d\rho\right)d\tau\right]\right\}\varepsilon(\tilde{\mathbf{v}})\,d\mathbf{x}\\&\leqslant\nu_7\lambda\int_\Omega\int_0^1\int_0^\tau|\varepsilon_\rho|^{2\delta-1}\big|\varepsilon(\mathbf{v}_\lambda-\mathbf{v})\big|\big|\varepsilon(\check{\mathbf{v}})\big|\big|\varepsilon(\tilde{\mathbf{v}})\big|\,d\rho\,d\tau\,d\mathbf{x}\\&\leqslant\nu_7\lambda\int_\Omega\int_0^1\big|\varepsilon(\tilde{\mathbf{v}})\big||\varepsilon_\rho|^\delta\big|\varepsilon(\mathbf{v}_\lambda-\mathbf{v})\big|\big|\varepsilon(\check{\mathbf{v}})\big||\varepsilon_\rho|^{\delta-1}\,d\rho\,d\mathbf{x}.\end{aligned}\tag{4.6}$$

Now, in (4.3), we set

$$\boldsymbol{\eta}(\mathbf{x},t)=\begin{cases}\tilde{\mathbf{v}}(\mathbf{x},s)&\text{for } s\leqslant t,\\\mathbf{0}&\text{for } s>t,\end{cases}$$

and in (4.4), we set

$$\boldsymbol{\zeta}(\mathbf{x},t)=\begin{cases}\tilde{\mathbf{h}}(\mathbf{x},s)&\text{for } s\leqslant t,\\\mathbf{0}&\text{for } s>t,\end{cases}$$

for arbitrary $t\leqslant T$ and transform them into the relations

$$\begin{aligned}&\frac{1}{2}\big\|\tilde{\mathbf{v}}(t)\big\|_{2,\Omega}^2+\int_0^t\big(A_1,\varepsilon(\tilde{\mathbf{v}})\big)\,ds\\&\quad=\int_0^t\Big((\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}},\mathbf{v})-\mu(\tilde{\mathbf{h}}\cdot\nabla\tilde{\mathbf{v}},\mathbf{h})+\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{v}-\mathbf{v}_\lambda),\tilde{\mathbf{v}}\big)\\&\qquad+\big((\mathbf{h}_\lambda-\mathbf{h})\cdot\nabla(\mathbf{h}-\mathbf{h}_\lambda),\tilde{\mathbf{v}}\big)-\big(A_2,\varepsilon(\tilde{\mathbf{v}})\big)\Big)\,ds\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}&\frac{1}{2}\big\|\tilde{\mathbf{h}}(t)\big\|_{2,\Omega}^2+\frac{1}{\sigma}\int_0^t\|\operatorname{curl}\tilde{\mathbf{h}}\|_{2,\Omega}^2\,ds\\&\quad=\int_0^t\Big(-\mu\big(\tilde{\mathbf{v}}\cdot\nabla\mathbf{h}-\tilde{\mathbf{h}}\cdot\nabla\mathbf{v}+\mu\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{h}-\mathbf{h}_\lambda)-(\mathbf{h}_\lambda-\mathbf{h})\cdot\nabla(\mathbf{v}-\mathbf{v}_\lambda),\tilde{\mathbf{h}}\big)\big)\Big)\,ds,\end{aligned}\tag{4.8}$$

respectively. We majorize the right-hand sides of (4.7) and (4.8) using our hypotheses on the potential $\mathcal{D}$. In detail, we apply the Hölder inequality with powers $p=2+2\delta$ and $q=(2+2\delta)/(1+2\delta)$, the multiplicative inequality (3.3) with $\alpha=\frac{3}{4(1+\delta)}$, and Young's inequality to get

$$(\tilde{\mathbf{v}}\cdot\nabla\tilde{\mathbf{v}},\mathbf{v}) \leqslant \int_\Omega |\mathbf{v}_x||\tilde{\mathbf{v}}|^2\,d\mathbf{x} \leqslant \|\mathbf{v}_x\|_{2+2\delta,\Omega}\|\tilde{\mathbf{v}}\|^2_{\frac{4(1+\delta)}{1+2\delta},\Omega} \leqslant C\|\mathbf{v}_x\|_{2+2\delta,\Omega}\|\tilde{\mathbf{v}}_x\|^{2\alpha}_{2,\Omega}\|\tilde{\mathbf{v}}\|^{2(1-\alpha)}_{2,\Omega}$$
$$\leqslant \epsilon\|\tilde{\mathbf{v}}_x\|^2_{2,\Omega} + C_\epsilon\|\mathbf{v}_x\|^{\frac{1}{1-\alpha}}_{2+2\delta,\Omega}\|\tilde{\mathbf{v}}\|^2_{2,\Omega}. \tag{4.9}$$

Using the Sobolev imbedding, (2.4), and Young's inequality, we have

$$(\tilde{\mathbf{h}}\cdot\nabla\tilde{\mathbf{v}},\mathbf{h}) \leqslant \|\tilde{\mathbf{v}}_x\|_{2,\Omega}\|\mathbf{h}\|_{\infty,\Omega}\|\tilde{\mathbf{h}}\|_{2,\Omega} \leqslant \epsilon\|\tilde{\mathbf{v}}_x\|^2_{2,\Omega} + C_\epsilon\|\mathbf{h}_{xx}\|^2_{2,\Omega}\|\tilde{\mathbf{h}}\|^2_{2,\Omega} \tag{4.10}$$

and

$$(\tilde{\mathbf{v}}\cdot\nabla\mathbf{h},\tilde{\mathbf{h}}) \leqslant \|\tilde{\mathbf{v}}\|_{2,\Omega}\|\tilde{\mathbf{h}}_x\|_{2,\Omega}\|\mathbf{h}\|_{\infty,\Omega} \leqslant \epsilon\|\tilde{\mathbf{h}}_x\|^2_{2,\Omega} + C_\epsilon\|\mathbf{h}_{xx}\|^2_{2,\Omega}\|\tilde{\mathbf{v}}\|^2_{2,\Omega}. \tag{4.11}$$

For $\delta > 1/2$, we have $W^1_{2+2\delta}(\Omega)\subset L^\infty(\Omega)$ since $\frac{1-2\delta}{6(1+\delta)} < 0$; therefore

$$\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{v}-\mathbf{v}_\lambda),\tilde{\mathbf{v}}\big) \leqslant \int_\Omega |\tilde{\mathbf{v}}_x||\mathbf{v}_\lambda-\mathbf{v}|^2\,d\mathbf{x} \leqslant \|\mathbf{v}_\lambda-\mathbf{v}\|_{\infty,\Omega}\|\tilde{\mathbf{v}}_x\|_{2,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}$$
$$\leqslant \big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|_{2+2\delta,\Omega}\|\tilde{\mathbf{v}}_x\|_{2,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}$$
$$\leqslant \epsilon\|\tilde{\mathbf{v}}_x\|^2_{2,\Omega} + C_\epsilon\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|^2_{2+2\delta,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|^2_{2,\Omega}.$$

For $\delta \leqslant 1/2$ we have $W^1_{2+2\delta}(\Omega)\subset L^{\frac{6(1+\delta)}{1-2\delta}}(\Omega)$ which yields

$$\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{v}-\mathbf{v}_\lambda),\tilde{\mathbf{v}}\big)$$
$$\leqslant \|\tilde{\mathbf{v}}_x\|_{2,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|_{\frac{6(1+\delta)}{1-2\delta},\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|_{\frac{6(1+\delta)}{2+5\delta},\Omega}$$
$$\leqslant C_1\|\tilde{\mathbf{v}}_x\|_{2,\Omega}\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|_{2+2\delta,\Omega}\Big(\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|^{\frac{1-2\delta}{2(1+\delta)}}_{2,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|^{\frac{1+4\delta}{2(1+\delta)}}_{2,\Omega} + \|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}\Big)$$
$$\leqslant \epsilon\|\tilde{\mathbf{v}}_x\|^2_{2,\Omega} + \bar{C}_\epsilon\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|^{\frac{3}{1+\delta}}_{2+2\delta,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|^{\frac{1+4\delta}{1+\delta}}_{2,\Omega}$$
$$+ C_\epsilon\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|^2_{2+2\delta,\Omega}\|\mathbf{v}_\lambda-\mathbf{v}\|^2_{2,\Omega}. \tag{4.12}$$

Here we used (3.3) with $\alpha = \frac{1-2\delta}{2(1+\delta)}$, since $\frac{6(1+\delta)}{2+5\delta}\in[2,6]$ for $\delta\leqslant 1/2$. Also note that $\frac{3}{1+\delta} < 2+2\delta$.

Using Young's inequality and the Sobolev imbedding we obtain

$$\big((\mathbf{h}_\lambda-\mathbf{h})\cdot\nabla(\mathbf{h}-\mathbf{h}_\lambda),\tilde{\mathbf{v}}\big)$$
$$\leqslant \int_\Omega |\tilde{\mathbf{v}}_x||\mathbf{h}_\lambda-\mathbf{h}|^2\,d\mathbf{x} \leqslant \|\tilde{\mathbf{v}}_x\|_{2,\Omega}\|\mathbf{h}_\lambda-\mathbf{h}\|_{\infty,\Omega}\|\mathbf{h}_\lambda-\mathbf{h}\|_{2,\Omega}$$
$$\leqslant \epsilon\|\tilde{\mathbf{v}}_x\|^2_{2,\Omega} + C_\epsilon\|\mathbf{h}_\lambda-\mathbf{h}\|^2_{W^2_2(\Omega)}\|\mathbf{h}_\lambda-\mathbf{h}\|^2_{2,\Omega}. \tag{4.13}$$

Similarly to (4.9) we have

$$(\tilde{\mathbf{h}}\cdot\nabla\mathbf{v},\tilde{\mathbf{h}}) \leqslant \int_\Omega |\mathbf{v}_x||\tilde{\mathbf{h}}|^2\,d\mathbf{x} \leqslant C\|\mathbf{v}_x\|_{2+2\delta,\Omega}\|\tilde{\mathbf{h}}\|^2_{\frac{4(1+\delta)}{1+2\delta}}$$
$$\leqslant C_1\|\mathbf{v}_x\|_{2+2\delta,\Omega}\|\tilde{\mathbf{h}}_x\|^{2\alpha}_{2,\Omega}\|\tilde{\mathbf{h}}\|^{2(1-\alpha)}_{2,\Omega}$$
$$\leqslant \epsilon\|\tilde{\mathbf{h}}_x\|^2_{2,\Omega} + C_\epsilon\|\mathbf{v}_x\|^{\frac{4(1+\delta)}{1+4\delta}}_{2+2\delta,\Omega}\|\tilde{\mathbf{h}}\|^2_{2,\Omega}. \tag{4.14}$$

Using Young's inequality and Sobolev imbedding we obtain

$$\begin{aligned}
&\big((\mathbf{v}_\lambda-\mathbf{v})\cdot\nabla(\mathbf{h}-\mathbf{h}_\lambda),\tilde{\mathbf{h}}\big)\\
&\quad\leqslant\int_\Omega|\mathbf{v}_\lambda-\mathbf{v}||\mathbf{h}_\lambda-\mathbf{h}||\tilde{\mathbf{h}}_x|\,d\mathbf{x}\leqslant\epsilon\|\tilde{\mathbf{h}}_x\|_{2,\Omega}^2+C_\epsilon\big\||\mathbf{h}_\lambda-\mathbf{h}||\mathbf{v}_\lambda-\mathbf{v}|\big\|_{2,\Omega}^2\\
&\quad\leqslant\epsilon\|\tilde{\mathbf{h}}_x\|_{2,\Omega}^2+C_\epsilon\|\mathbf{h}_\lambda-\mathbf{h}\|_{W_2^2(\Omega)}^2\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^2.
\end{aligned}\tag{4.15}$$

Adding (4.7) and (4.8), by the use of (2.4), (2.6), and (4.9)–(4.15) we obtain that

$$\begin{aligned}
&\big\|\tilde{\mathbf{v}}(t)\big\|_{2,\Omega}^2+\big\|\tilde{\mathbf{h}}(t)\big\|_{2,\Omega}^2+\int_0^t\left(\|\tilde{\mathbf{v}}_x\|_{2,\Omega}^2+\|\tilde{\mathbf{h}}_x\|_{2,\Omega}^2+\int_0^1\int_\Omega|\varepsilon_\tau|^{2\delta}\big|\varepsilon(\tilde{\mathbf{v}})\big|^2\,d\mathbf{x}\,d\tau\right)ds\\
&\quad\leqslant C\int_0^t\big(\|\tilde{\mathbf{v}}\|_{2,\Omega}^2+\|\tilde{\mathbf{h}}\|_{2,\Omega}^2\big)\Big(1+\|\mathbf{v}_x\|_{2+2\delta,\Omega}^{\frac{4(1+\delta)}{1+4\delta}}+\|\mathbf{h}_{xx}\|_{2,\Omega}^2\Big)\,ds\\
&\qquad+\int_0^t\Big[\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|_{2+2\delta,\Omega}^{\frac{3}{1+\delta}}\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^{\frac{1+4\delta}{1+\delta}}+\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|_{2+2\delta,\Omega}^2\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^2\\
&\qquad+\|\mathbf{h}_\lambda-\mathbf{h}\|_{W_2^2(\Omega)}^2\big(\|\mathbf{h}_\lambda-\mathbf{h}\|_{2,\Omega}^2+\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^2\big)\Big]\,ds\\
&\qquad+\lambda^2\int_0^t\int_\Omega\int_0^1\big|\varepsilon(\mathbf{v}_\lambda-\mathbf{v})\big|^2\big|\varepsilon(\check{\mathbf{v}})\big|^2|\varepsilon_\tau|^{2(\delta-1)}\,d\tau\,d\mathbf{x}\,ds\\
&\quad\leqslant C\int_0^t\big(\|\tilde{\mathbf{v}}\|_{2,\Omega}^2+\|\tilde{\mathbf{h}}\|_{2,\Omega}^2\big)\Big(1+\|\mathbf{v}_x\|_{2+2\delta,\Omega}^{\frac{4(1+\delta)}{1+4\delta}}+\|\mathbf{h}_{xx}\|_{2,\Omega}^2\Big)\,ds\\
&\qquad+\int_0^t\Big(\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|_{2+2\delta,\Omega}^{\frac{3}{1+\delta}}\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^{\frac{1+4\delta}{1+\delta}}+\big\|(\mathbf{v}_\lambda-\mathbf{v})_x\big\|_{2+2\delta,\Omega}^2\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^2\Big)\,ds\\
&\qquad+\int_0^t\|\mathbf{h}_\lambda-\mathbf{h}\|_{W_2^2(\Omega)}^2\big(\|\mathbf{h}_\lambda-\mathbf{h}\|_{2,\Omega}^2+\|\mathbf{v}_\lambda-\mathbf{v}\|_{2,\Omega}^2\big)\,ds\\
&\qquad+\lambda^2\int_0^t\int_0^1\big\|\varepsilon(\mathbf{v}_\lambda-\mathbf{v})\big\|_{2(\delta+1),\Omega}^2\big\|\varepsilon(\check{\mathbf{v}})\big\|_{2(\delta+1),\Omega}^2\|\varepsilon_\tau\|_{2(\delta+1),\Omega}^{2(\delta-1)}\,d\tau\,ds.
\end{aligned}\tag{4.16}$$

Now we set $\mathbf{u}=\mathbf{v}_\lambda-\mathbf{v}$, $\mathbf{B}=\mathbf{h}_\lambda-\mathbf{h}$ so that

$$\begin{aligned}
&(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{u},\boldsymbol{\eta})+\left(\int_0^1\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_\tau}d\tau:\varepsilon(\mathbf{u}),\varepsilon(\boldsymbol{\eta})\right)\\
&\quad-\mu(\mathbf{B}\cdot\nabla\mathbf{B}+\mathbf{B}\cdot\nabla\mathbf{h}+\mathbf{h}\cdot\nabla\mathbf{B},\boldsymbol{\eta})=\lambda(\mathbf{g},\boldsymbol{\eta})
\end{aligned}\tag{4.17}$$

and

$$\mu(\mathbf{B}_t,\boldsymbol{\zeta})+\frac{1}{\sigma}(\operatorname{curl}\mathbf{B},\operatorname{curl}\boldsymbol{\zeta})+\mu(\mathbf{u}\cdot\nabla\mathbf{B}+\mathbf{u}\cdot\nabla\mathbf{h}+\mathbf{v}\cdot\nabla\mathbf{B}$$
$$-\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}\cdot\nabla\mathbf{v}-\mathbf{h}\cdot\nabla\mathbf{u},\boldsymbol{\zeta})=\frac{\lambda}{\sigma}(\operatorname{curl}\mathbf{k},\boldsymbol{\zeta}). \tag{4.18}$$

After some calculation we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{2,\Omega}^2+\nu_5\big(\|\mathbf{u}_x\|_{2,\Omega}^2+\|\mathbf{u}_x\|_{2+2\delta,\Omega}^{2+2\delta}\big)$$
$$\leqslant-(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{u})+\mu(\mathbf{B}\cdot\nabla\mathbf{B}+\mathbf{B}\cdot\nabla\mathbf{h}+\mathbf{h}\cdot\nabla\mathbf{B},\mathbf{u})+\lambda(\mathbf{g},\mathbf{u})$$

and by (2.6)

$$\frac{\mu}{2}\frac{d}{dt}\|\mathbf{B}\|_{2,\Omega}^2+\frac{\nu_8}{\sigma}\|\mathbf{B}_x\|_{2,\Omega}^2$$
$$\leqslant\frac{\nu_9}{\sigma}\|\mathbf{B}\|_{2,\Omega}^2-\mu(\mathbf{u}\cdot\nabla\mathbf{h}-\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}\cdot\nabla\mathbf{v}-\mathbf{h}\cdot\nabla\mathbf{u},\mathbf{B})+\frac{\lambda}{\sigma}(\operatorname{curl}\mathbf{k},\mathbf{B}).$$

Integrating over t and using for the right-hand side the following estimates:

$$(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{u})\leqslant\int_\Omega|\mathbf{u}|^2|\mathbf{v}_x|\,d\mathbf{x}\leqslant\|\mathbf{u}\|^2_{\frac{4(1+\delta)}{1+2\delta},\Omega}\|\mathbf{v}_x\|_{2+2\delta,\Omega}\leqslant C\|\mathbf{u}_x\|_{2,\Omega}^{2\alpha}\|\mathbf{u}\|_{2,\Omega}^{2(1-\alpha)}\|\mathbf{v}_x\|_{2+2\delta,\Omega}$$
$$\leqslant\epsilon\|\mathbf{u}_x\|_{2,\Omega}^2+C_\epsilon\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{v}\|_{2+2\delta,\Omega}^{\frac{4(1+\delta)}{1+4\delta}},$$
$$(\mathbf{B}\cdot\nabla\mathbf{h},\mathbf{u})\leqslant\int_\Omega|\mathbf{B}||\mathbf{h}||\mathbf{u}_x|\,d\mathbf{x}\leqslant\|\mathbf{u}_x\|_{2,\Omega}\|\mathbf{B}\|_{2,\Omega}\|\mathbf{h}\|_{\infty,\Omega}$$
$$\leqslant\epsilon\|\mathbf{u}_x\|_{2+2\delta,\Omega}^2+C_\epsilon\|\mathbf{B}\|_{2,\Omega}^2\|\mathbf{h}_{xx}\|_{2,\Omega}^2,$$
$$(\mathbf{B}\cdot\nabla\mathbf{v},\mathbf{B})\leqslant\int_\Omega|\mathbf{u}|^2|\mathbf{v}_x|\,d\mathbf{x}\leqslant\|\mathbf{B}\|^2_{\frac{4(1+\delta)}{1+2\delta},\Omega}\|\mathbf{v}_x\|_{2+2\delta,\Omega}$$
$$\leqslant\epsilon\|\mathbf{B}_x\|_{2,\Omega}^2+C_\epsilon\|\mathbf{B}\|_{2,\Omega}^2\|\mathbf{v}\|_{2+2\delta,\Omega}^{\frac{4(1+\delta)}{1+4\delta}},$$
$$(\mathbf{u}\cdot\nabla\mathbf{h},\mathbf{B})\leqslant\int_\Omega|\mathbf{u}||\mathbf{B}_x||\mathbf{h}|\,d\mathbf{x}\leqslant\|\mathbf{u}\|_{2,\Omega}\|\mathbf{B}_x\|_{2,\Omega}\|\mathbf{h}_{xx}\|_{2,\Omega}$$
$$\leqslant\epsilon\|\mathbf{B}_x\|_{2,\Omega}^2+C_\epsilon\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{h}_{xx}\|_{2,\Omega}^2,$$

for ϵ sufficiently small we obtain that

$$\big\|\mathbf{u}(t)\big\|_{2,\Omega}^2+\big\|\mathbf{B}(t)\big\|_{2,\Omega}^2+\int_0^t\big(\big\|\mathbf{u}_x(s)\big\|_{2,\Omega}^2+\big\|\mathbf{u}_x(s)\big\|_{2+2\delta,\Omega}^{2+2\delta}+\big\|\mathbf{B}_x(s)\big\|_{2,\Omega}^2\big)\,ds$$
$$\leqslant\lambda^2\Phi_4\big(T,\|\mathbf{g}\|_{2,\Omega},\|\operatorname{curl}\mathbf{k}\|_{2,\Omega}\big) \tag{4.19}$$

with a continuous function Φ_4 which depends on the information about the known functions $\mathbf{v}^0$, $\mathbf{h}^0$, $\mathbf{f}$, $\operatorname{curl}\mathbf{j}$ used before.

We estimate the norm $\|\cdot\|_{2,\Omega}$ of $\mathbf{B}_{xx}$ using the identity (4.17) with $\boldsymbol{\zeta}=-\Delta\mathbf{B}$ and the estimate (4.19). Choosing $\boldsymbol{\zeta}=-\Delta\mathbf{B}$, (4.18) can be transformed in the following way:

$$\frac{\mu}{2}\frac{d}{dt}\|\operatorname{curl}\mathbf{B}\|_{2,\Omega}^2+\frac{1}{\sigma}\|\Delta\mathbf{B}\|_{2,\Omega}^2$$
$$\leqslant \mu(\mathbf{u}\cdot\nabla\mathbf{B}+\mathbf{u}\cdot\nabla\mathbf{h}+\mathbf{v}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{B}\cdot\nabla\mathbf{v}-\mathbf{h}\cdot\nabla\mathbf{u},\Delta\mathbf{B})-\frac{\lambda}{\sigma}(\operatorname{curl}\mathbf{k},\Delta\mathbf{B})$$
$$\leqslant -\mu\big((\mathbf{u}_{k,j}+\mathbf{v}_{k,j})\mathbf{B}_{i,k},\mathbf{B}_{i,j}\big)+\frac{1}{2\sigma}\|\Delta\mathbf{B}\|_{2,\Omega}^2$$
$$+C\big(\|\mathbf{u}\cdot\nabla\mathbf{h}+\mathbf{B}\cdot\nabla\mathbf{u}+\mathbf{B}\cdot\nabla\mathbf{v}+\mathbf{h}\cdot\nabla\mathbf{u}\|_{2,\Omega}^2+\lambda^2\|\operatorname{curl}\mathbf{k}\|_{2,\Omega}^2\big).$$

With a similar method as for (3.8), using (2.3), (2.4) and (4.19) we conclude that

$$\max_{t\in[0,T]}\big\|\mathbf{B}_x(t)\big\|_{2,\Omega}^2+\|\mathbf{B}_{xx}\|_{2,Q_T}^2\leqslant\lambda^2\Phi_5\big(T,\|\mathbf{g}\|_{2,\Omega},\|\operatorname{curl}\mathbf{k}\|_{2,\Omega}\big). \tag{4.20}$$

Finally, we can apply Gronwall's inequality in (4.16) and use estimates (4.19) and (4.20) to obtain

$$\big\|\tilde{\mathbf{v}}(t)\big\|_{2,\Omega}^2+\big\|\tilde{\mathbf{h}}(t)\big\|_{2,\Omega}^2+\int_0^t\left(\|\tilde{\mathbf{v}}_x\|_{2,\Omega}^2+\|\tilde{\mathbf{h}}_x\|_{2,\Omega}^2+\int_0^1\int_\Omega|\varepsilon_\tau|^{2\delta}\big|\varepsilon(\tilde{\mathbf{v}})\big|^2\,d\mathbf{x}\,d\tau\right)ds$$
$$\leqslant C\lambda^2 o(\lambda),$$

from which our claim follows. □

The Gâteaux derivative gives useful information about the sensitivity of the system at a particular point $(\mathbf{v},\mathbf{h})$ in a particular direction $(\mathbf{g},\operatorname{curl}\mathbf{k})$, but complete information requires one to solve (4.1) and (4.2) for every possible direction $(\mathbf{g},\operatorname{curl}\mathbf{k})$. Fortunately, in order to minimize the functional we need only an integral over all these directions which can more easily be obtained through the solution of a single adjoint equation.

Theorem 4. *Let* $\mathbf{v}^0\in W_2^2(\Omega)\cap\mathring{\mathcal{J}}_2^1(\Omega)$ *and* $\mathbf{h}^0\in\tilde{\mathcal{J}}_2^1(\Omega)$ *and let* $(\hat{\mathbf{v}},\hat{\mathbf{h}},\hat{\mathbf{f}},\operatorname{curl}\hat{\mathbf{j}})$ *be a solution of the optimal control problem. Let* $(\mathbf{w},\mathbf{D})$ *be the solution of the adjoint problem*

$$\big(-\mathbf{w}_t-\hat{\mathbf{v}}\cdot\nabla\mathbf{w}+(\nabla\hat{\mathbf{v}})^T\mathbf{w},\boldsymbol{\eta}\big)+\left(\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon(\hat{\mathbf{v}})}:\varepsilon(\boldsymbol{\eta}),\varepsilon(\mathbf{w})\right)$$
$$-\mu\big((\nabla\hat{\mathbf{h}})^T\mathbf{w}-\hat{\mathbf{h}}\cdot\nabla\mathbf{w},\boldsymbol{\zeta}\big)=\alpha_1(\hat{\mathbf{v}}-\mathbf{v}_d,\boldsymbol{\eta}), \tag{4.21}$$
$$-\mu(\mathbf{D}_t,\boldsymbol{\zeta})+\frac{1}{\sigma}(\operatorname{curl}\mathbf{D},\operatorname{curl}\boldsymbol{\zeta})+\mu\big(-\hat{\mathbf{v}}\cdot\nabla\mathbf{D}+(\nabla\hat{\mathbf{v}})^T\mathbf{D},\boldsymbol{\zeta}\big)$$
$$+\mu\big((\nabla\hat{\mathbf{h}})^T\mathbf{D}+\hat{\mathbf{h}}\cdot\nabla\mathbf{D},\boldsymbol{\eta}\big)=\alpha_2\big(\operatorname{curl}(\hat{\mathbf{h}}-\mathbf{h}_d),\boldsymbol{\zeta}\big), \tag{4.22}$$

for any $\boldsymbol{\eta}\in\mathring{\mathcal{J}}_{2+2\delta}^1(\Omega)$ *and* $\boldsymbol{\zeta}\in L^2(\Omega)$*, with the final data*

$$\check{\mathbf{v}}|_{t=T}=\check{\mathbf{h}}|_{t=T}=\mathbf{0}\quad\text{in }\Omega.$$

Then,

$$\hat{\mathbf{f}}=-\frac{1}{\beta_1}\mathbf{w}\quad\text{and}\quad\operatorname{curl}\hat{\mathbf{j}}=-\frac{1}{\beta_2\sigma}\mathbf{D}. \tag{4.23}$$

Proof. Let $(\hat{\mathbf{v}},\hat{\mathbf{h}},\hat{\mathbf{f}},\operatorname{curl}\hat{\mathbf{j}})$ be a solution of the optimal control problem. The derivative of the cost functional $J(\hat{\mathbf{v}},\hat{\mathbf{h}},\hat{\mathbf{f}},\operatorname{curl}\hat{\mathbf{j}})$ in the direction $(\mathbf{g},\operatorname{curl}\mathbf{k})$ is then

$$\frac{dJ(\hat{\mathbf{v}},\hat{\mathbf{h}},\hat{\mathbf{f}},\operatorname{curl}\hat{\mathbf{j}})}{d(\hat{\mathbf{f}},\operatorname{curl}\hat{\mathbf{j}})}\cdot(\mathbf{g},\operatorname{curl}\mathbf{k})$$
$$=\int_0^T\int_\Omega\big(\alpha_1(\hat{\mathbf{v}}-\mathbf{v}_d)\cdot\check{\mathbf{v}}+\alpha_2(\hat{\mathbf{h}}-\mathbf{h}_d)\cdot\check{\mathbf{h}}+\beta_1\hat{\mathbf{f}}\cdot\mathbf{g}+\beta_2\operatorname{curl}\hat{\mathbf{j}}\cdot\operatorname{curl}\mathbf{k}\big)\,d\mathbf{x}\,dt,$$

where $(\check{\mathbf{v}},\check{\mathbf{h}})$ is the solution of the system (4.1) and (4.2). Since $(\hat{\mathbf{v}},\hat{\mathbf{h}},\hat{\mathbf{f}},\operatorname{curl}\hat{\mathbf{j}})$ is an optimal solution and the Gâteaux derivative of J exists, the latter must be zero on all directions $(\mathbf{g},\operatorname{curl}\mathbf{k})\in L^2(Q_T)$.

Taking $\boldsymbol{\eta}=\mathbf{w}(t)$ in (4.1) with $\mathbf{v}=\hat{\mathbf{v}}$, $\boldsymbol{\zeta}=\mathbf{D}(t)$ in (4.2) with $\mathbf{h}=\hat{\mathbf{h}}$, $\boldsymbol{\eta}=\check{\mathbf{v}}(t)$ in (4.21), $\boldsymbol{\zeta}=\check{\mathbf{h}}(t)$ in (4.22), and integrating by parts we obtain

$$\int_0^T\int_\Omega\big(\alpha_1(\hat{\mathbf{v}}-\mathbf{v}_d)\cdot\check{\mathbf{v}}+\alpha_2(\hat{\mathbf{h}}-\mathbf{h}_d)\cdot\check{\mathbf{h}}\big)\,dx\,dt=\int_0^T\int_\Omega\left(\mathbf{g}\cdot\mathbf{w}+\frac{1}{\sigma}\operatorname{curl}\mathbf{k}\cdot\mathbf{D}\right)d\mathbf{x}\,dt.$$

Therefore,

$$\int_0^T\int_\Omega\left((\beta_1\hat{\mathbf{f}}+\mathbf{w})\cdot\mathbf{g}+\left(\beta_2\operatorname{curl}\hat{\mathbf{j}}+\frac{1}{\sigma}\mathbf{D}\right)\cdot\operatorname{curl}\mathbf{k}\right)d\mathbf{x}\,dt=0\quad\forall(\mathbf{g},\operatorname{curl}\mathbf{k})\in L^2(Q_T),$$

and by the completeness of the Hilbert space, we obtain (4.23). □

The optimality condition $\operatorname{curl}\hat{\mathbf{j}}=-\frac{1}{\beta_2\sigma}\mathbf{D}$ along with $\operatorname{div}\hat{\mathbf{j}}=0$ and the boundary condition $\hat{\mathbf{j}}\cdot\mathbf{n}=0$ can be used to determine the optimal applied current $\hat{\mathbf{j}}$ (see, e.g., [4]).

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