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Multiple weighted estimates for commutators of multilinear singular integrals with non-smooth kernels[☆]

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ABSTRACT

Multilinear commutators and iterated commutators generated by the multilinear singular integrals with non-smooth kernels and BMO functions are studied. By the weighted estimates of a class of new variant maximal operators and the sharp maximal functions, the multiple weighted norm inequalities for such operators are obtained. In particular, some previous results in Anh and Duong (2010) [1], Lerner et al. (2009) [4], Pérez et al. (2011) [5] are improved or extended significantly.

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1. Introduction and main results

Commutators of multilinear singular integral operators with BMO functions have been the subject of many recent articles (see [1–7] et al.). The purpose of this paper is to extend and improve some known results for two types of commutators associated with multilinear operators.

Let $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{S}'(\mathbb{R}^n)$ be an m-linear operator. A locally integrable function $K(x, y_1, \ldots, y_m)$ defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ is called an associated kernel of T if

$$\langle T(f_1, \ldots, f_m), g\rangle = \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1)\cdots f_m(y_m) g(x) dy_1 \cdots dy_m dx \tag{1.1}$$

holds for all $f_1, \ldots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$ with $\bigcap_{j=1}^m \operatorname{supp} f_j \bigcap \operatorname{supp} g = \emptyset$. Throughout this paper, we assume that the associated kernel K satisfies the following size condition:

$$|K(y_0, y_1, \ldots, y_m)| \le \frac{A}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn}} \tag{1.2}$$

for some $A > 0$ and all $(y_0, y_1, \ldots, y_m)$ with $y_0 \ne y_j$ for some $j \in \{1, 2, \ldots, m\}$.

For the multilinear operator T and $\vec{b} = (b_1, \ldots, b_m)$ in BMO^m, we define the m-linear commutator $T_{\Sigma\vec{b}}$:

$$T_{\Sigma\vec{b}}(\vec{f}) := \sum_{j=1}^{m} T_{b_j}^j(\vec{f}) := \sum_{j=1}^{m}[b_j, T]_j(f_1, \ldots, f_m), \tag{1.3}$$

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and the iterated commutator $T_{\Pi\vec{b}}$:

$$T_{\Pi\vec{b}}(\vec{f}) := [b_1, [b_2, \ldots, [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2]_1(f_1, \ldots, f_m), \tag{1.4}$$

where $\vec{f} = (f_1, \ldots, f_m)$, each $T^j_{b_j}$ is the commutator of b_j and T in the j-th entry of T, that is,

$$T^j_{b_j}(\vec{f}) = [b_j, T]_j(f_1, \ldots, f_m) = b_j T(f_1, \ldots, f_j, \ldots, f_m) - T(f_1, \ldots, b_j f_j, \ldots, f_m).$$

If T is associated with a distribution kernel, which coincides with the function K defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, then, at formal level,

$$T^j_{b_j}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} (b_j(x) - b_j(y_j))K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \tag{1.5}$$

and

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} (b_j(x) - b_j(y_j))K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \tag{1.6}$$

whenever $x \notin \cap_{j=1}^m \operatorname{supp} f_j$ and $f_1, \ldots, f_m$ are C^∞ functions with compact support. Here the notations of our commutators are taken from [5], also see [4,6] et al. for the definitions and notations of these commutators.

Clearly, for $m = 1$, $T_{\Sigma\vec{b}} = T_{\Pi\vec{b}}$ coincides with the linear commutator $[b, T]$ introduced by Coifman et al. in [8]. The original interest in the study of $[b, T]$ was related to generalizations of the classical factorization theorem for Hardy spaces. Further applications have then been found in partial differential equations [9–12]. Some multiparameter versions have also received renewed attention; see [13,14].

For the multilinear setting, Pérez and Torres [6] showed that if T is the m-linear Calderón–Zygmund operator, then $T_{\Sigma\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ whenever $1 < p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$ with $p > 1$. Tang [7] obtained the weighted boundedness of $T_{\Pi\vec{b}}$ for the classical A_p classes (in fact, Tang established the corresponding result for the vector-valued version of $T_{\Pi\vec{b}}$). Very recently, Lerner et al. [4] and Pérez et al. [5] have extended the above results to the following:

Theorem A. *Let T be an m-linear Calderón–Zygmund operator; $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ and $v_{\vec{w}} = \Pi_{j=1}^m w_j^{p/p_j}$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 < p_j < \infty$, $j = 1, \ldots, m$; and $\vec{b} \in \mathrm{BMO}^m$. Then, there exists a constant C such that*

$$\text{(i) (cf. [4])} \quad \left\| T_{\Sigma\vec{b}}(\vec{f}) \right\|_{L^p(v_{\vec{w}})} \leq C \left(\sum_{j=1}^{m} \|b\|_{\mathrm{BMO}} \right) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}; \tag{1.7}$$

$$\text{(ii) (cf. [5])} \quad \left\| T_{\Pi\vec{b}}(\vec{f}) \right\|_{L^p(v_{\vec{w}})} \leq C \left(\prod_{j=1}^{m} \|b\|_{\mathrm{BMO}} \right) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}. \tag{1.8}$$

Here the classes $A_{\vec{P}}$ were introduced by Lerner et al. in [4] and are the largest classes of weights for which all m-linear Calderón–Zygmund operators are bounded (see Section 2 for definitions). An important fact is that

$$\prod_{j=1}^{m} A_{p_j} \subsetneq A_{\vec{P}}. \tag{1.9}$$

An m-linear operator T associated with K is said to be an m-linear Calderón–Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $1/q = 1/q_1 + \cdots + 1/q_m$, and the kernel K satisfies (1.2) and the regularity condition

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y'_j, \ldots, y_m)| \leq \frac{A|y_j - y'_j|^\epsilon}{\left(\sum_{k,l=0}^{m} |y_k - y_l| \right)^{mn+\epsilon}}, \tag{1.10}$$

whenever $0 \leq j \leq m$ and $|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$. We denote by $m - CZK(A, \epsilon)$ the collection of all kernels K satisfying (1.2) and (1.10).

In this paper, we will continuously focus on the two types of commutators $T_{\Sigma\vec{b}}$ and $T_{\Pi\vec{b}}$ by replacing the regularity condition (1.10) by weaker regularity conditions on the kernel K given by Assumptions (H1) and (H2) described below. These assumptions were introduced by Duong et al. in [15,16]. An important example for satisfying these assumptions is the m-th Calderón commutator.

Let $\{A_t\}_{t>0}$ be a class of integral operators, which play the role of the approximation to the identity (see [17]). We always assume that the operators A_t are associated with kernels $a_t(x, y)$ in the sense that for all $f \in \cup_{p\in[1,\infty]} L^p$ and $x \in \mathbb{R}^n$,

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y) dy,$$

and that the kernels $a_t(x, y)$ satisfy the following conditions

$$|a_t(x, y)| \le h_t(x, y) := t^{-n/s} h\left(\frac{|x-y|}{t^{1/s}}\right), \tag{1.11}$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying that for some $\eta > 0$,

$$\lim_{r\to\infty} r^{n+\eta} h(r^s) = 0. \tag{1.12}$$

Recall that the j-th transpose $T^{*,j}$ of the m-linear operator T is defined via

$$\langle T^{*,j}(f_1, \dots, f_m), g\rangle = \langle T(f_1, \dots, f_{j-1}, g, f_{j+1}, \dots, f_m), f_j\rangle \tag{1.13}$$

for all $f_1, \dots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$. Notice that the kernel $K^{*,j}$ of $T^{*,j}$ is related to the kernel K of T via the identity

$$K^{*,j}(x, y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_m) = K(y_j, y_1, \dots, y_{j-1}, x, y_{j+1}, \dots, y_m).$$

If an m-linear operator T maps a product of Banach spaces $X_1 \times \cdots \times X_m$ to another Banach space X, then the transpose $T^{*,j}$ maps $X_1 \times \cdots \times X_{j-1} \times X \times X_{j+1} \times \cdots \times X_m$ to X_j. Moreover, the norms of T and $T^{*,j}$ are equal. To maintain uniform notation, we may occasionally denote T by $T^{*,0}$ and K by $K^{*,0}$.

Assumption (H1). Assume that for each $i = 1, \dots, m$ there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x, y)$ that satisfy conditions (1.11) and (1.12) with constants s and η and that for every $j = 0, 1, 2, \dots, m$, there exist kernels $K_t^{*,j,(i)}(x, y_1, \dots, y_m)$ such that

$$\langle T^{*,j}(f_1, \dots, A_t^{(i)} f_i, \dots, f_m), g\rangle = \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^m} K_t^{*,j,(i)}(x, y_1, \dots, y_m) f_1(y_1)\cdots f_m(y_m) g(x) dy_1 \cdots dy_m dx \tag{1.14}$$

for all $f_1, \dots, f_m$ in $\mathcal{S}(\mathbb{R}^n)$ with $\cap_{k=1}^m \operatorname{supp} f_k \cap \operatorname{supp} g = \emptyset$. Also assume that there exist a function $\phi \in C(\mathbb{R})$ with $\operatorname{supp}\phi \subset [-1, 1]$ and a constant $\epsilon > 0$ so that for every $j = 0, 1, \dots, m$ and every $i = 1, 2, \dots, m$, we have

$$\left|K^{*,j}(x, y_1, \dots, y_m) - K_t^{*,j,(i)}(x, y_1, \dots, y_m)\right| \le \frac{A}{\left(\sum_{k=1}^m |x-y_k|\right)^{mn}} \sum_{k=1,k\ne i}^m \phi\left(\frac{|y_i - y_k|}{t^{1/s}}\right) + \frac{A t^{\epsilon/s}}{\left(\sum_{k=1}^m |x-y_k|\right)^{mn+\epsilon}} \tag{1.15}$$

whenever $t^{1/s} \le |x - y_i|/2$.

If T satisfies Assumption (H1) we will say that T is an m-linear operator with generalized Calderón–Zygmund kernel K. The collection of functions K satisfying (1.14) and (1.15) with parameters m, A, s, η and ϵ will be denoted by $m - GCZK(A, s, \eta, \epsilon)$. We say that T is of class $m - GCZO(A, s, \eta, \epsilon)$ if T has an associated kernel K in $m - GCZK(A, s, \eta, \epsilon)$.

Theorem B (*Cf. [15, Theorem 3.1]*)**.** *Assume that T is a multilinear operator in $m - GCZO(A, s, \eta, \epsilon)$. If there exist some $1 \le q_1, \dots, q_m < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$ such that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then for $1/m \le p < \infty$, $1 \le p_j \le \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, all the following statement are valid:*

(i) *when all $p_j > 1$, then T can be extended to be a bounded operator from the m-fold product $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$;*
(ii) *when some $p_j = 1$, then T can be extended to be a bounded operator from the m-fold product $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.*

Moreover, there exists a constant $C(n, m, p_j, q_j)$ such that

$$\|T\|_{L^1\times\cdots\times L^1 \to L^{1/m,\infty}} \le C(n, m, p_j, q_j)(A + \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^q}).$$

Assumption (H2). Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x, y)$ that satisfy conditions (1.11) and (1.12) with constants s and η. Let

$$K_t^{(0)}(x, y_1, \dots, y_m) = \int_{\mathbb{R}^n} K(z, y_1, \dots, y_m) b_t(x, z) dz \tag{1.16}$$

whenever $2t^{1/s} \le \min_{1\le j\le m} |x-y_j|$, and

$$\left|K(x,y_1,\ldots,y_m) - K_t^{(0)}(x',\ldots,y_1,\ldots,y_m)\right| \le \frac{A}{\left(\sum_{k=1}^m |x-y_k|\right)^{mn}} \sum_{k=1}^m \phi\left(\frac{|x-y_k|}{t^{1/s}}\right) + \frac{At^{\epsilon/s}}{\left(\sum_{k=1}^m |x-y_k|\right)^{mn+\epsilon}} \tag{1.17}$$

whenever $2|x-x'| \le t^{1/s}$ and $2t^{1/s} \le \max_{1\le j\le m} |x-y_j|$.

From the proof of Proposition 2.1 in [15], we know that condition (1.15) is weaker than, and indeed a consequence of the Calderón–Zygmund kernel condition (1.10). Similarly, we can verify that (H2) is also weaker than the condition (1.10) for $K(x,y_1,\ldots,y_m)$.

For T in $m - GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption (H2), the study on the corresponding commutators $T_{\Sigma\vec{b}}$ and $T_{\Pi\vec{b}}$ have attracted much attention. In [2] (resp., [3]), Gong and Li (resp., Lian, Li and Wu) proved that $T_{\Sigma\vec{b}}$ (resp., $T_{\Pi\vec{b}}$) is bounded from $L^{p_1}(w) \times \cdots \times L^{p_m}(w)$ to $L^p(w)$ for $1 < p, p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $w \in A_p$. Moreover, Anh and Duong [1] established the following result:

Theorem C (*Cf.* [1, *Theorem 4.5*]). *Let T be a multilinear operator in $m - GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption (H2). Assume that there exist some $1 \le q_1,\ldots,q_m < \infty$ and some $0 < q < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$, such that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. Then for $\vec{b} \in \mathrm{BMO}^m$, $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_j < \infty$, $w_j \in A_{p_j}$, $j = 1,\ldots,m$, there exists a constant C such that*

$$\left\|T_{\Sigma\vec{b}}(\vec{f})\right\|_{L^p(v_{\vec{w}})} \le C\sum_{i=1}^m \|b_i\|_{\mathrm{BMO}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)},$$

where $v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$.

Compared with the results in Theorems A and C, (1.9) shows an obvious gap between Theorems A and C. It is natural to ask whether the results in Theorem A are also true for the corresponding commutators $T_{\Sigma\vec{b}}$ of T in Theorem C, and what about $T_{\Pi\vec{b}}$? This problem will be addressed by our next theorems:

Theorem 1.1. *Assume that T is a multilinear operator in $m - GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption (H2), $\vec{b} = (b_1,\ldots,b_m) \in \mathrm{BMO}^m$. If there exist some $1 \le q_1,\ldots,q_m < \infty$ and some $0 < q < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$, such that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$, then for $1/m < p < \infty$, $1 < p_1,\ldots,p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{P} = (p_1,\ldots,p_m)$, and $\vec{w} \in A_{\vec{P}}$,*

$$\left\|T_{\Sigma\vec{b}}(\vec{f})\right\|_{L^p(v_{\vec{w}})} \le C\sum_{i=1}^m \|b_i\|_{\mathrm{BMO}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}, \tag{1.18}$$

where $v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$.

Theorem 1.2. *Assume that T is a multilinear operator in $m - GCZO(A,s,\eta,\epsilon)$ with kernel K satisfying Assumption (H2), $\vec{b} = (b_1,\ldots,b_m) \in \mathrm{BMO}^m$. If there exist some $1 \le q_1,\ldots,q_m < \infty$ and some $0 < q < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$, such that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. Then for $1/m < p < \infty$, $1 < p_1,\ldots,p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{P} = (p_1,\ldots,p_m)$, and $\vec{w} \in A_{\vec{P}}$,*

$$\left\|T_{\Pi\vec{b}}(\vec{f})\right\|_{L^p(v_{\vec{w}})} \le C\prod_{i=1}^m \|b_i\|_{\mathrm{BMO}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}, \tag{1.19}$$

where $v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$.

Remark 1.1. Since $\Pi_{j=1}^m A_{p_j} \subsetneq A_{\vec{P}}$, Theorem 1.1 is an essential improvement of Theorem C (i.e.,Theorem 4.5 in [1]). Meanwhile, note that the regularity in our conditions (H1) and (H2) is significantly weaker than those of the standard Calderón–Zygmund kernels, Theorems 1.1 and 1.2 can be considered as an extension to Theorem A (i.e., Theorem 3.18 in [4] and Theorem 1.1 in [5]).

We would like to remark that the main method employed in this paper is a combination of ideas and arguments from [4,5,18], among others. One of the main ingredients of our proofs is the introduction of a class of new variant maximal operators and the multiple weighted estimates for such maximal operators (see (3.2)–(3.4) and Proposition 3.1), which is the key leading to the improvement of the results in [1,4,5].

Remark 1.2. We remark that under standard regularity conditions, the weak type end-point estimates on $T_{\Sigma\vec{b}}$ and $T_{\Pi\vec{b}}$ are established by Lerner et al. in [4, Theorem 3.16] and Pérez et al. in [5, Theorem 1.2], respectively. However, under the (H1) and (H2) conditions, our methods do not work to the end-point cases. It will be very interesting to establish the corresponding weak end-point estimates under our conditions.

Furthermore, Theorems 1.1 and 1.2 can be given in the following more general forms. To state the results, we need to introduce some notations. Following [5], for positive integers m and j with $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements, where we always take $\sigma(k) < \sigma(l)$ if $k < l$. For any $\sigma \in C_j^m$, we associate the complementary sequence $\sigma' \in C_{m-j}^m$ given by $\sigma' = \{1, \dots, m\} \setminus \sigma$ with the convention $C_0^m = \emptyset$. Given an m-tuple of functions $\vec{b}$ and $\sigma \in C_j^m$, we also use the notation $\vec{b}_\sigma$ for the j-tuple obtained from $\vec{b}$ given by $(b_{\sigma(1)}, \dots, b_{\sigma(j)})$. Similarly to $T_{\Pi\vec{b}}$, we define for T in $m-GCZO(A, s, \eta, \varepsilon)$, $\sigma \in C_j^m$ and $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ in BMO^j, the j-th order iterated commutator

$$T_{\Pi\vec{b}_\sigma}(f_1, \dots, f_m) = [b_{\sigma(1)}, [b_{\sigma(2)}, \cdots [b_{\sigma(j-1)}, [b_{\sigma(j)}, T]_{\sigma(j)}]_{\sigma(j-1)} \cdots]_{\sigma(2)}]_{\sigma(1)}(\vec{f}),$$

that is, formally

$$T_{\Pi\vec{b}_\sigma}(\vec{f})(x) = \int_{(R^n)^m} \left(\prod_{i=1}^{j} \left(b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)}) \right) \right) K(x, y_1, \dots, y_m) \prod_{i=1}^{m} f_i(y_i) d\vec{y}. \tag{1.20}$$

Clearly $T_{\Pi\vec{b}_\sigma} = T_{\Pi\vec{b}}$ when $\sigma = \{1, \dots, m\}$, while $T_{\Pi\vec{b}_\sigma} = T_{b_j}^j$ when $\sigma = \{j\}$. Then we have the following theorem.

Theorem 1.3. *For $1 \le j \le m, \sigma \in C_j^m$ and $\vec{b}_\sigma \in \mathrm{BMO}^j$, let $T_{\Pi\vec{b}_\sigma}$ be given in (1.20). Under the same assumptions as in Theorem 1.2, we have*

$$\left\| T_{\Pi\vec{b}_\sigma}(\vec{f}) \right\|_{L^p(v_{\vec{w}})} \le C \prod_{i=1}^{j} \|b_{\sigma(i)}\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{L^{p_j}(w_j)}. \tag{1.21}$$

We remark that Theorem 1.3 is Theorem 1.2 when $j = m$, and Theorem 1.3 implies Theorem 1.1 when $j = 1$. The proof of Theorem 1.3 will be omitted since it follows from the same way in proving Theorem 1.2 and it is simpler than the latter.

The rest of this paper is organized as follows. In Section 2, we recall some standard definitions and lemmas. In Section 3, we introduce a class of new multi(sub)linear maximal operators and prove some useful estimates which will play key roles in the proofs of our theorems. Section 4 is devoted to the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 5.

Throughout the rest of the paper, the letter C will stand for a positive constant not necessarily the same one at each occurrence but is independent of the essential variables.

2. Preliminaries

2.1. Multiple weights

The following class of weights was introduced in [4].

Let $1 \le p_1, \dots, p_m < \infty$ and $1/m \le p < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{P} = (p_1, \dots, p_m)$. A multiple weight $\vec{w} = (w_1, \dots, w_m)$ is said to satisfy the multilinear $A_{\vec{P}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{w}}(x) dx \right)^{1/p} \prod_{j=1}^{m} \left(\frac{1}{|Q|} \int_Q w_j^{1-p_j'}(x) dx \right)^{1/p_j'} < \infty, \tag{2.1}$$

where $v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}$. When $p_j = 1$, $(|Q|^{-1} \int_Q w_j(x)^{1-p_j'} dx)^{1/p_j'}$ is understood as $(\inf_Q w_j(x))^{-1}$.

It is easy to check that $A_{(1,\dots,1)}$ is contained in $A_{\vec{P}}$ for each $\vec{P}$, however $A_{\vec{P}}$ are not increasing with the natural partial order. As mentioned in the introduction, these are the largest classes of weights for which the multilinear Calderón–Zygmund operators are bounded on Lebesgue spaces and an important fact is (1.9) in Section 1. Moreover, we will use the following results:

Lemma 2.1 (*Cf. [4, Theorem 3.6]*). *Let $\vec{w} = (w_1, \dots, w_m)$ and $1 \le p_1, \dots, p_m < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if and only if $v_{\vec{w}} \in A_{mp}$ and $w_j^{1-p_j'} \in A_{mp_j'}$, $j = 1, \dots, m$, where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$.*

Lemma 2.2 (*Cf. [4, Lemma 6.1]*). *Assume that $\vec{w} = (w_1, \dots, w_m)$ satisfies $A_{\vec{P}}$ condition. Then there exists a finite constant $r > 1$ such that $\vec{w} \in A_{\vec{P}/r}$.*

2.2. Orlicz norms

For $\Phi(t) = t(1+\log^+ t)$ and a cube Q in $\mathbb{R}^n$, we will consider the average $\|f\|_{\Phi,Q}$ of a function f given by the Luxemburg norm

$$\|f\|_{L\log L,Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}. \tag{2.2}$$

We will use the following two basic estimates in several occasions without further comment.

$$\|f\|_{\Phi,Q} > 1 \quad \text{if and only if} \quad \frac{1}{|Q|}\int_Q \Phi(|f(x)|)dx > 1; \tag{2.3}$$

$$\frac{1}{|Q|}\int_Q |b(y) - b_Q|\,|f(y)|dy \le C\|b\|_{\mathrm{BMO}}\|f\|_{L(\log L),Q}, \tag{2.4}$$

where $b_Q = |Q|^{-1}\int_Q b(y)dy$.

2.3. Sharp maximal functions

We recall the definitions of the Hardy–Littlewood maximal function and the sharp maximal function:

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy,$$

$$M^\sharp(f)(x) = \sup_{Q\ni x}\inf_c \frac{1}{|Q|}\int_Q |f(y) - c|dy \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

and their variants $M_\delta(f)(x) = M(|f|^\delta)^{1/\delta}(x)$ and $M_\delta^\sharp(f)(x) = M^\sharp(|f|^\delta)^{1/\delta}(x)$. We will use the following inequality (see [19]):

$$\int_{\mathbb{R}^n} (M_\delta(f)(x))^p\, w(x)dx \le C\int_{\mathbb{R}^n}\left(M_\delta^\sharp(f)(x)\right)^p w(x)dx \tag{2.5}$$

for all functions f for which the left-hand side is finite, and where $0 < p, \delta < \infty$ and $w \in A_\infty$.

3. Multilinear maximal operators

In this section, we will introduce certain variant multilinear maximal operators and establish the multiple weighted estimates for such maximal operators, which are ones of the main novelties in this paper.

Let us recall the definitions of the multilinear maximal functions, which are introduced by Lerner et al. in [4].

$$\mathcal{M}(\vec{f})(x) = \sup_{Q\ni x}\prod_{j=1}^m \frac{1}{|Q|}\int_Q |f_j(x)|dx,$$

$$\mathcal{M}_r(\vec{f})(x) = \sup_{Q\ni x}\prod_{j=1}^m \left(\frac{1}{|Q|}\int_Q |f_j(x)|^r dx\right)^{1/r},$$

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{Q\ni x}\prod_{j=1}^m \|f_j\|_{L(\log L),Q}.$$

By the fact that for $r > 1$, there exists a constant $c > 0$ such that

$$\|f\|_{L(\log L),Q} \le c\left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{1/r},$$

it is easy to check that

$$\mathcal{M}(\vec{f})(x) \le \mathcal{M}_{L(\log L)}(\vec{f})(x) \le c\,\mathcal{M}_r(\vec{f})(x). \tag{3.1}$$

In the next, we introduce the following modified multilinear maximal operators. Let $r > 1$, $1 \le l < m$, $\varrho = \{j_1, j_2, \ldots, j_l\} \subseteq \{1, \ldots, m\}$ and $\varrho' = \{1, 2, \ldots, m\}\setminus\varrho$. We define the following multilinear maximal functions:

$$\mathcal{M}_\varrho(\vec{f})(x) = \sup_{Q\ni x}\sum_{k=0}^\infty 2^{-knl}\prod_{j\in\varrho}\frac{1}{|Q|}\int_Q |f_j(x)|dx\prod_{j\in\varrho'}\frac{1}{|2^kQ|}\int_{2^kQ}|f_j(x)|dx, \tag{3.2}$$

$$\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x) = \sup_{Q\ni x}\sum_{k=0}^\infty 2^{-knl}\prod_{j\in\varrho}\|f_j\|_{L(\log L),Q}\prod_{j\in\varrho'}\|f_j\|_{L(\log L),2^kQ}, \tag{3.3}$$

and

$$\mathcal{M}_{\varrho,r}(\vec{f})(x)=\sup_{Q\ni x}\sum_{k=0}^{\infty}2^{-knl}\prod_{j\in\varrho}\left(\frac{1}{|Q|}\int_Q|f_j(x)|^r dx\right)^{1/r}\prod_{j\in\varrho'}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|f_j(x)|^r dx\right)^{1/r}. \quad (3.4)$$

We remark that when $\varrho=\{1,2,\ldots,\ell\}$, $\mathcal{M}_\varrho$ is first introduced by Grafakos et al. in [18] and denote $\mathcal{M}_\varrho$ by $\mathcal{M}_\ell$. Similarly to (3.1), for any $r>1$, we have

$$\mathcal{M}_\varrho(\vec{f})(x)\le\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\le c\,\mathcal{M}_{\varrho,r}(\vec{f})(x). \quad (3.5)$$

The following proposition will play a key role in proving our theorems.

Proposition 3.1. *Let $1<p_1,\ldots,p_m<\infty$, $1/p=1/p_1+\cdots+1/p_m$, $\vec{P}=(p_1,\ldots,p_m)$, and $\vec{w}\in A_{\vec{P}}$. Let $1\le l<m$ and $\varrho=\{j_1,\ldots,j_l\}\subseteq\{1,\ldots,m\}$. Then for some $r>1$ (r depending only on $\vec{w}$), $\mathcal{M}_r$ and $\mathcal{M}_{\varrho,r}$ is bounded from $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ to $L^p(v_{\vec{w}})$.*

Proof. The boundedness of $\mathcal{M}_r$ is contained in the proof of Theorem 3.18 in [4, p. 1258]. We need only to prove the boundedness of $\mathcal{M}_{\varrho,r}$. Without loss of generality, we may assume $\varrho=\{1,\ldots,l\}$. Since $\vec{w}\in A_{\vec{P}}$, by Lemma 2.2 (i.e., [4, Lemma 6.1]), there exists a finite constant $r>1$ such that $\vec{w}\in A_{\vec{P}/r}$. Let $s_i=p_i/r$, $\vec{S}=(s_1,\ldots,s_m)$ and $1/s=1/s_1+\cdots+1/s_m$. Then $s=p/r$, $\prod_{j=1}^m w_j^{p/p_j}=\prod_{j=1}^m w_j^{s/s_j}$ and $\vec{w}\in A_{\vec{S}}$. Consequently, $w_j^{-1/(s_j-1)}$ satisfies the reverse Hölder inequality, that is, there exist $u_j>1$ and $C>0$ such that for all $u\in(0,u_j]$ and all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q w_j^{-u/(s_j-1)}(x)dx\right)^{1/u}\le\frac{C}{|Q|}\int_Q w_j^{-1/(s_j-1)}(x)dx.$$

Let $\eta=\min_{1\le j\le m}u_j$ and

$$t=\max_{1\le j\le m}\frac{sm}{sm+(1-1/\eta)(s_j-1)}.$$

Observe that $t<1$ and $ts_j>1$ for $1\le j\le m$. Then the proof of Proposition 2.1 in [18] implies that for all cubes Q and $1\le j\le m$,

$$\frac{1}{|Q|}\int_Q|f_j|^r(x)dx\le C\left\{M^c_{v_{\vec{w}}}\left(\left[|f_j|^{rs_j}w_j/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/ts_j}\left(\frac{v_{\vec{w}}(Q)}{|Q|}\right)^{1/s_j}\left(\frac{1}{|Q|}\int_Q w_j^{-s_j'/s_j}(x)dx\right)^{1/s_j'},$$

where $M^c_{v_{\vec{w}}}$ denotes the weighted centered Hardy–Littlewood maximal function, that is,

$$M^c_{v_{\vec{w}}}(f)(x):=\sup_{r>0}\frac{1}{v_{\vec{w}}(Q(x,r))}\int_{Q(x,r)}|f(y)|v_{\vec{w}}(y)dy,$$

where $Q(x,r)$ denotes the cube centered at x and of side length r. Hence,

$$\left(\frac{1}{|Q|}\int_Q|f_j|^r(x)dx\right)^{1/r}\le C\left\{M^c_{v_{\vec{w}}}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/tp_j}\left(\frac{v_{\vec{w}}(Q)}{|Q|}\right)^{1/rs_j}\left(\frac{1}{|Q|}\int_Q w_j^{-s_j'/s_j}(x)dx\right)^{1/rs_j'}.$$

This together with the definition of $\mathcal{M}_{\varrho,r}$ implies that

$$\begin{aligned}\mathcal{M}_{\varrho,r}(\vec{f})(x)&\le C\sup_{x\in Q}\sum_{k=0}^{\infty}2^{-knl}\prod_{i=1}^{l}\left\{M^c_{v_{\vec{w}}}\left(\left[|f_i|^{p_i}w_i/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/tp_i}\left(\frac{v_{\vec{w}}(Q)}{|Q|}\right)^{1/rs_i}\\&\quad\times\left(\frac{1}{|Q|}\int_Q w_i^{-s_i'/s_i}(x)dx\right)^{1/rs_i'}\prod_{j=l+1}^{m}\left\{M^c_{v_{\vec{w}}}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/tp_j}\\&\quad\times\left(\frac{v_{\vec{w}}(2^kQ)}{|2^kQ|}\right)^{1/rs_j}\left(\frac{1}{|2^kQ|}\int_{2^kQ}w_j^{-s_j'/s_j}(x)dx\right)^{1/rs_j'}\\&\le C\prod_{j=1}^{m}\left\{M^c_{v_{\vec{w}}}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/tp_j}\sup_{x\in Q}\sum_{k=0}^{\infty}2^{-knl}\prod_{i=1}^{l}\left(\frac{v_{\vec{w}}(Q)}{|Q|}\right)^{1/rs_i}\\&\quad\times\left(\frac{1}{|Q|}\int_Q w_i^{-s_i'/s_i}(x)dx\right)^{1/rs_i'}\prod_{j=l+1}^{m}\left(\frac{v_{\vec{w}}(2^kQ)}{|2^kQ|}\right)^{1/rs_j}\left(\frac{1}{|2^kQ|}\int_{2^kQ}w_j^{-s_j'/s_j}(x)dx\right)^{1/rs_j'}.\end{aligned}$$

Denote

$$I_{l,k}=\prod_{i=1}^{l}\left(\frac{v_{\vec{w}}(Q)}{|Q|}\right)^{1/s_i}\left(\frac{1}{|Q|}\int_Q w_i^{-s_i'/s_i}(x)dx\right)^{1/s_i'}\prod_{j=l+1}^{m}\left(\frac{v_{\vec{w}}(2^kQ)}{|2^kQ|}\right)^{1/s_j}\left(\frac{1}{|2^kQ|}\int_{2^kQ} w_j^{-s_j'/s_j}(x)dx\right)^{1/s_j'}.$$

We have

$$\begin{aligned}I_{l,k}&\le\prod_{j=1}^{m}\left(\frac{v_{\vec{w}}(2^kQ)}{|2^kQ|}\right)^{1/s_j}\left(\frac{1}{|2^kQ|}\int_{2^kQ} w_j^{-s_j'/s_j}(x)dx\right)^{1/s_j'}\prod_{i=1}^{l}\left(\frac{v_{\vec{w}}(Q)}{|Q|}\right)^{1/s_i}\left(\frac{|2^kQ|}{v_{\vec{w}}(2^kQ)}\right)^{1/s_i}\left(\frac{|2^kQ|}{|Q|}\right)^{1/s_i'}\\&\le\left([\vec{w}]_{A_{\vec{P}}}\right)^r 2^{knl}\left(\frac{v_{\vec{w}}(Q)}{v_{\vec{w}}(2^kQ)}\right)^{\Sigma_{i=1}^{l}1/s_i}.\end{aligned}$$

Since $v_{\vec{w}}\in A_\infty$, there exists $\theta>0$ such that for all cubes Q and all sets $E\subseteq Q$,

$$\frac{v_{\vec{w}}(E)}{v_{\vec{w}}(Q)}\le C\left(\frac{|E|}{|Q|}\right)^{\theta}.$$

Therefore,

$$\begin{aligned}\mathcal{M}_{\varrho,r}(\vec{f})(x)&\le C\prod_{j=1}^{m}\left\{M_{v_{\vec{w}}}^{c}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/tp_j}\sum_{k=0}^{\infty}2^{-knl}[\vec{w}]_{A_{\vec{P}}}2^{knl/r}2^{-kn\theta(\Sigma_{i=1}^{l}1/s_i)/r}\\&\le C\prod_{j=1}^{m}\left\{M_{v_{\vec{w}}}^{c}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)(x)\right\}^{1/tp_j}.\end{aligned}$$

Then,

$$\begin{aligned}\left\|\mathcal{M}_{\varrho,r}(\vec{f})(x)\right\|_{L^p(v_{\vec{w}})}&\le C\left\|\prod_{j=1}^{m}\left\{M_{v_{\vec{w}}}^{c}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)\right\}^{1/tp_j}\right\|_{L^p(v_{\vec{w}})}\\&\le C\prod_{j=1}^{m}\left\|\left\{M_{v_{\vec{w}}}^{c}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)\right\}^{1/tp_j}\right\|_{L^{p_j}(v_{\vec{w}})}\\&=C\prod_{j=1}^{m}\left\|M_{v_{\vec{w}}}^{c}\left(\left[|f_j|^{p_j}w_j/v_{\vec{w}}\right]^t\right)\right\|_{L^{1/t}(v_{\vec{w}})}^{1/tp_j}\\&\le C\prod_{j=1}^{m}\left\|\left(|f_j|^{p_j}w_j/v_{\vec{w}}\right)^t\right\|_{L^{1/t}(v_{\vec{w}})}^{1/tp_j}\\&\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)}.\end{aligned}$$

This completes the proof of Proposition 3.1. □

From (3.1), (3.5) and Proposition 3.1 we have the following corollary.

Corollary 3.1. *Under the same assumptions as in Proposition 3.1, $\mathcal{M}$, $\mathcal{M}_{L(\log L)}$, $\mathcal{M}_\varrho$ and $\mathcal{M}_{\varrho,L(\log L)}$ are bounded from $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ to $L^p(v_{\vec{w}})$ for $\vec{w}\in A_{\vec{P}}$.*

We remark that the results of $\mathcal{M}$ and $\mathcal{M}_{L(\log L)}$ can be found in Theorem 3.7 and the proof of Theorem 3.18 in [4], and one of $\mathcal{M}_\ell$ (the special case of $\mathcal{M}_\varrho$ for $\varrho=\{1,2,\ldots,\ell\}$) is Proposition 2.1's (i) in [18].

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. At first, we present the estimates on the sharp Fefferman–Stein maximal operators acting on $T(\vec{f})$ and $T_{\Sigma\vec{b}}(\vec{f})$ in terms of the multilinear maximal functions given in Section 3.

Proposition 4.1. *Let T be an m-linear operator in GCZO(A,s,η,ϵ) that satisfies the assumptions in Theorem 1.1. Let $\varrho_0=\{1,\ldots,m\}$ and $0<\delta<1/m$. Then for any $\vec{f}$ in the product of $L^{q_j}(R^n)$, with $1\le q_j\le\infty$,*

$$M_\delta^\sharp(T(\vec{f}))(x)\le C\left(\mathcal{M}(\vec{f})(x)+\sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\mathcal{M}_\varrho(\vec{f})(x)\right).\tag{4.1}$$

Proposition 4.2. *Let T be an m-linear operator in $GCZO(A, s, \eta, \epsilon)$ that satisfies the assumptions in Theorem 1.1, and $T_{\Sigma\vec{b}}$ be the corresponding multilinear commutator with $\vec{b} \in \mathrm{BMO}^m$. Let $0 < \delta < \min\{\varepsilon, 1/m\}$, $r > 1$ and $\varrho_0 = \{1, \ldots, m\}$. Then, there exists a constant $C > 0$, depending on δ and ε, such that*

$$M_\delta^\sharp\left(T_{\Sigma\vec{b}}(\vec{f})\right)(x) \le C\sum_{i=1}^m \|b_i\|_{\mathrm{BMO}}\left(M_\varepsilon\left(T(\vec{f})\right)(x) + \mathcal{M}_{L(\log L)}(\vec{f})(x) + \sum_{\emptyset\neq\varrho\subsetneq\varrho_0}\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right) \tag{4.2}$$

for all bounded measurable vector functions $\vec{f} = (f_1, \ldots, f_m)$ with compact supports.

Remark 4.1. Propositions 4.1 and 4.2 improve the following inequalities obtained in Theorems 4.1 and 4.3 of [1], respectively:

$$M_\delta^\sharp(T(\vec{f}))(x) \le C\prod_{j=1}^m M(f_j)(x),$$

and

$$M_\delta^\sharp\left(T_{\Sigma\vec{b}}(\vec{f})\right)(x) \le C\sum_{i=1}^m \|b_i\|_{\mathrm{BMO}}\left(\prod_{j=1}^m M_{L(\log L)}(f_j)(x) + M_\varepsilon\left(T(\vec{f})\right)(x)\right),$$

since $\mathcal{M}$ and $\mathcal{M}_\varrho$ (resp., $\mathcal{M}_{L(\log L)}$ and $\mathcal{M}_{\varrho,L(\log L)}$) are trivially controlled by the m-fold product of M (resp., $M_{L(\log L)}$), where $M_{L(\log L)}(f)(x) = \sup_{Q\ni x}\|f\|_{L(\log L),Q}$.

The ideas and arguments used in the proofs are similar, although with some modifications. For completeness, we give the proofs as follows.

Proof of Proposition 4.1. We employ the ideas taken from [4,6]. Fix a point x and a cube $Q \ni x$. Since $|\,|\alpha|^r - |\beta|^r\,| \le |\alpha-\beta|^r$ for $0 < r < 1$, to obtain (4.1), it suffices to prove for $0 < \delta < 1/m$

$$\left(\frac{1}{|Q|}\int_Q \left|T(\vec{f})(z) - c_Q\right|^\delta dz\right)^{1/\delta} \le C\left(\mathcal{M}(\vec{f})(x) + \sum_{\emptyset\neq\varrho\subsetneq\varrho_0}\mathcal{M}_\varrho(\vec{f})(x)\right)$$

for some constant c_Q to be determined later. Let $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j\chi_{Q^*}$ for $j = 1, \ldots, m$, and $Q^* = (8\sqrt{n}+4)Q$. Then

$$\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m\left(f_j^0(y_j) + f_j^\infty(y_j)\right) = \sum_{\alpha_1,\ldots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m)$$
$$= \prod_{j=1}^m f_j^0 + {\sum}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),$$

where each term of $\sum'$ contains at least one $\alpha_j \neq 0$. Then

$$T(\vec{f})(z) = T(\vec{f}^0)(z) + {\sum}' T\left(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(z).$$

By Theorem B and the Kolmogorov inequality, it is actually shown in the proof of Theorem 4.1 in [1] that

$$\left(\frac{1}{|Q|}\int_Q |T(\vec{f}^0)(z)|^\delta dz\right)^{1/\delta} \le C\mathcal{M}(\vec{f})(x).$$

To estimate the remaining terms, employing the arguments in the proof of Theorem 3.2 of [4] (also see [6]), we choose $c_Q = \sum' T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x)$ and will show that

$${\sum}'\left|T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x)\right| \le C\left(\mathcal{M}(\vec{f})(x) + \sum_{\emptyset\neq\varrho\subsetneq\varrho_0}\mathcal{M}_\varrho(\vec{f})(x)\right). \tag{4.3}$$

For the case of $\alpha_1 = \cdots = \alpha_m = \infty$, taking $t = (2\sqrt{n}l(Q))^s$ ($l(Q)$ denotes the side-length of Q) and by Assumption (H2), Anh and Duong in the proof of Theorem 4.1 of [1] actually showed that

$$\left|T(\vec{f}^\infty)(z) - T(\vec{f}^\infty)(x)\right| \le C\mathcal{M}(\vec{f})(x).$$

It remains to estimate the terms in (4.3) with $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for some $\varrho = \{j_1, \ldots, j_l\} \subset \varrho_0$ and $1 \le l < m$. We have

$$\begin{aligned}
&\left|T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x)\right| \\
&\le \int_{(\mathbb{R}^n)^m} \left(\left|K(z,\vec{y}) - K_t^{(0)}(z,\vec{y})\right| + \left|K_t^{(0)}(z,\vec{y}) - K(x,\vec{y})\right|\right) \prod_{j=1}^m \left|f_j(y_j)\right| d\vec{y} \\
&= \int_{(\mathbb{R}^n)^m} \left|K(z,\vec{y}) - K_t^{(0)}(z,\vec{y})\right| \prod_{j=1}^m \left|f_j(y_j)\right| d\vec{y} + \int_{(\mathbb{R}^n)^m} \left|K_t^{(0)}(z,\vec{y}) - K(x,\vec{y})\right| \prod_{j=1}^m \left|f_j(y_j)\right| d\vec{y} \\
&:= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Set $\varrho' = \varrho_0 \setminus \varrho$, by (1.17) of Assumption (H2), we get

$$\begin{aligned}
\mathrm{I} &\le C \prod_{j\in\varrho} \int_{Q^*} \left|f_j(y_j)\right| dy_j \left(\int_{(\mathbb{R}^n\setminus Q^*)^{m-l}} \frac{t^{\epsilon/s} \prod_{j\in\varrho'} \left|f_j(y_j)\right| dy_j}{\left(\sum_{j\in\varrho'} |x - y_j|\right)^{mn+\epsilon}} + \int_{(\mathbb{R}^n\setminus Q^*)^{m-l}} \frac{\prod_{j\in\varrho'} \left|f_j(y_j)\right| dy_j}{\left(\sum_{j\in\varrho'} |x - y_j|\right)^{mn}} \right) \\
&\le C \prod_{j\in\varrho} \int_{Q^*} \left|f_j(y_j)\right| dy_j \left(\sum_{k=1}^{\infty} \frac{|Q^*|^{\epsilon/n}}{\left(2^k |Q^*|^{1/n}\right)^{mn+\epsilon}} \int_{(2^{k+1}Q^*)^{m-l}} \prod_{j\in\varrho'} \left|f_j(y_j)\right| dy_j \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \frac{1}{\left(2^k |Q^*|^{1/n}\right)^{mn}} \int_{(2^{k+1}Q^*\setminus 2^k Q^*)^{m-l}} \prod_{j\in\varrho'} \left|f_j(y_j)\right| dy_j \right) \\
&\le C \left(\sum_{k=1}^{m} \frac{1}{2^{k\epsilon}} \prod_{j=1}^{m} \frac{1}{|2^k Q^*|} \int_{2^k Q^*} \left|f_j(y_j)\right| dy_j + \sum_{k=1}^{m} \frac{1}{2^{knl}} \prod_{j\in\varrho} \frac{1}{|Q^*|} \int_{Q^*} \left|f_j(y_j)\right| dy_j \prod_{j\in\varrho'} \frac{1}{|2^k Q^*|} \int_{2^k Q^*} \left|f_j(y_j)\right| dy_j \right) \\
&\le C \left(\mathcal{M}(\vec{f})(x) + \mathcal{M}_{\varrho}(\vec{f})(x) \right).
\end{aligned}$$

Similarly,

$$\mathrm{II} \le C \left(\mathcal{M}(\vec{f})(x) + \mathcal{M}_{\varrho}(\vec{f})(x) \right).$$

This finishes the proof of Proposition 4.1. □

Proof of Proposition 4.2. It suffices to consider the operator:

$$T_b(\vec{f})(x) := b(x) T(f_1, \ldots, f_m)(x) - T(bf_1, \ldots, f_m)(x)$$

for fixed $b \in \mathrm{BMO}(\mathbb{R}^n)$. Note that for any constant λ we have

$$T_b(\vec{f})(x) = (b(x) - \lambda) T(f_1, \ldots, f_m)(x) - T((b - \lambda) f_1, \ldots, f_m)(x).$$

Fix x. For any cube Q centered at x and a constant c determined later, we estimate

$$\begin{aligned}
\left(\frac{1}{|Q|} \int_Q \left| |T_b(\vec{f})(z)|^\delta - |c|^\delta \right| dz \right)^{1/\delta} &\le \left(\frac{1}{|Q|} \int_Q \left| T_b(\vec{f})(z) - c \right|^\delta dz \right)^{1/\delta} \\
&\le \left(\frac{C}{|Q|} \int_Q \left| (b(z) - \lambda) T(\vec{f})(z) \right|^\delta dz \right)^{1/\delta} \\
&\quad + \left(\frac{C}{|Q|} \int_Q \left| T\left((b - \lambda) f_1, \ldots, f_m\right)(z) - c \right|^\delta dz \right)^{1/\delta} \\
&:= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Let $Q^* = (8\sqrt{n} + 4)Q$ and $\lambda = b_{Q^*}$. Note that $0 < \delta < \varepsilon$. By John–Nirenberg's inequality and the Hölder inequality, it follows from the proof of Theorem 4.3 of [1] that

$$\mathrm{I} \le C \|b\|_{\mathrm{BMO}} M_\varepsilon \left(T(\vec{f}) \right)(x).$$

To estimate II, we split each f_j as $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f\chi_{Q^*}$ and $f_j^\infty = f - f_j^0$ for $j = 1, \ldots, m$. Then

$$\prod_{j=1}^m f_j(y_j) = \sum_{\{\alpha_1,\ldots,\alpha_m\}\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m)$$
$$= \prod_{j=1}^m f_j^0(y_j) + {\sum}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),$$

where each term in $\sum'$ contains at least one $\alpha_j \neq 0$.

Take $c = \sum' c_{\alpha_1,\ldots,\alpha_m}$ with $c_{\alpha_1,\ldots,\alpha_m} = T((b-\lambda)f_1^{\alpha_1}, f_2^{\alpha_2}, \ldots, f_m^{\alpha_m})(x)$. We have

$$\mathrm{II} \le C\left(\left(\frac{1}{|Q|}\int_Q \left|T\left((b-\lambda)f_1^0, f_2^0, \ldots, f_m^0\right)(z)\right|^\delta dz\right)^{1/\delta}\right.$$
$$\left. + {\sum}'\left(\frac{1}{|Q|}\int_Q \left|T\left((b-\lambda)f_1^{\alpha_1}, f_2^{\alpha_2}, \ldots, f_m^{\alpha_m}\right)(z) - c_{\alpha_1,\ldots,\alpha_m}\right|^\delta dz\right)^{1/\delta}\right)$$
$$:= \mathrm{II}_{0,\ldots,0} + {\sum}' \mathrm{II}_{\alpha_1,\ldots,\alpha_m}.$$

In the proof of Theorem 4.3 of [1], Anh and Duong actually showed that

$$\mathrm{II}_{0,\ldots,0} \quad \text{and} \quad \mathrm{II}_{\infty,\ldots,\infty} \le C\|b\|_{\mathrm{BMO}}\, \mathcal{M}_{L(\log L)}(\vec{f})(x).$$

It remains to estimate the terms $\mathrm{II}_{\alpha_1,\ldots,\alpha_m}$ with $\alpha_{j_1} = \cdots = \alpha_{j_l}$ for some $\varrho = \{j_1, \ldots, j_l\} \subset \varrho_0$ and $1 \le l < m$. We consider only the case $1 \in \varrho$ since the other one follows in an analogous way. By (1.17) in Assumption (H2), we get

$$\mathrm{II}_{\alpha_1,\ldots,\alpha_m} = \left(\frac{1}{|Q|}\int_Q \left|T\left((b-\lambda)f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(z) - T\left((b-\lambda)f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(x)\right|^\delta dz\right)^{1/\delta}$$
$$\le \frac{1}{|Q|}\int_Q \left|T\left((b-\lambda)f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(z) - T\left((b-\lambda)f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(x)\right| dz$$
$$\le C\frac{1}{|Q|}\int_Q\int_{Q^*} |(b(y_1)-\lambda)f_1(y_1)|\,dy_1 \prod_{j\in\varrho\setminus\{1\}}\int_{Q^*}\left|f_j(y_j)\right| dy_j$$
$$\times\left(\int_{(\mathbb{R}^n\setminus Q^*)^{m-l}} \frac{t^{\epsilon/s}\prod_{j\in\varrho_0\setminus\varrho}\left|f_j(y_j)\right| dy_j}{\left(\sum_{j\in\varrho_0\setminus\varrho}|z-y_j|\right)^{mn+\epsilon}} dz + \int_{(\mathbb{R}^n\setminus Q^*)^{m-l}} \frac{\prod_{j\in\varrho_0\setminus\varrho}\left|f_j(y_j)\right| dy_j}{\left(\sum_{j\in\varrho_0\setminus\varrho}|z-y_j|\right)^{mn}} dz\right)$$
$$\le C\|b\|_{\mathrm{BMO}}\,\|f_1\|_{L(\log L),Q^*}\,|Q^*| \prod_{j\in\varrho\setminus\{1\}}\int_{Q^*}\left|f_j(y_j)\right| dy_j$$
$$\times\left(\sum_{k=1}^\infty \frac{1}{2^{k\epsilon}}\frac{1}{|2^kQ^*|^m}\prod_{j\in\varrho_0\setminus\varrho}\int_{2^kQ^*}\left|f_j(y_j)\right| dy_j + \sum_{k=1}^\infty \frac{1}{|2^kQ^*|^m}\prod_{j\in\varrho_0\setminus\varrho}\int_{2^kQ^*}\left|f_j(y_j)\right| dy_j\right)$$
$$\le C\|b\|_{\mathrm{BMO}}\left(\sum_{k=1}^\infty\frac{1}{2^{k\epsilon}}\prod_{j=1}^m \|f_j\|_{L(\log L),2^kQ^*} + \sum_{k=1}^\infty \frac{1}{2^{knl}}\prod_{j\in\varrho}\|f_j\|_{L(\log L),Q^*}\prod_{j\in\varrho_0\setminus\varrho}\|f_j\|_{L(\log L),2^kQ^*}\right)$$
$$\le C\|b\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(\vec{f})(x) + \mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right).$$

This finishes the proof of Proposition 4.2. □

Proposition 4.3. *Let T be as in Theorem 1.1 and $T_{\Sigma\vec{b}}$ be the corresponding m-linear commutator with $\vec{b} \in \mathrm{BMO}^m$, $\varrho_0 = \{1, \ldots, m\}$. Suppose that w is an A_∞ weight, $0 < p < \infty$. Then there exists a constant $C > 0$, depending on the A_∞ constant of w, such that*

$$\int_{\mathbb{R}^n}\left|T_{\Sigma\vec{b}}(\vec{f})(x)\right|^p w(x)dx \le C\left(\sum_{i=1}^m\|b_i\|_{\mathrm{BMO}}\right)^p\left(\int_{\mathbb{R}^n}\left(\mathcal{M}_{L(\log L)}(\vec{f})(x)\right)^p w(x)dx + \sum_{\emptyset\neq\varrho\subsetneq\varrho_0}\int_{\mathbb{R}^n}\left(\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right)^p w(x)dx\right) \tag{4.4}$$

for all bounded measurable vector functions $\vec{f} = (f_1, \ldots, f_m)$ with compact supports.

Proof. For simplicity, we may assume that $\sum_{i=1}^{m}\|b_i\|_{\mathrm{BMO}} = 1$. Using Propositions 4.1 and 4.2 and the Fefferman–Stein inequality (2.5), with $0 < \delta < \varepsilon < 1/m$, we have

$$\begin{aligned}\left\|T_{\Sigma\vec{b}}(\vec{f})\right\|_{L^p(w)} &\le \left\|M_\delta\left(T_{\Sigma\vec{b}}(\vec{f})\right)\right\|_{L^p(w)} \le \left\|M_\delta^\sharp\left(T_{\Sigma\vec{b}}(\vec{f})\right)\right\|_{L^p(w)}\\ &\le C\left(\left\|M_\varepsilon\left(T(\vec{f})\right)\right\|_{L^p(w)} + \left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)} + \sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\left\|\mathcal{M}_{\varrho,L(\log L)}(\vec{f})\right\|_{L^p(w)}\right)\\ &\le C\left(\left\|M_\varepsilon^\sharp\left(T(\vec{f})\right)\right\|_{L^p(w)} + \left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)} + \sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\left\|\mathcal{M}_{\varrho,L(\log L)}(\vec{f})\right\|_{L^p(w)}\right)\\ &\le C\left(\left\|\mathcal{M}(\vec{f})\right\|_{L^p(w)} + \sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\left\|\mathcal{M}_{\varrho}(\vec{f})\right\|_{L^p(w)} + \left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)} + \sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\left\|\mathcal{M}_{\varrho,L(\log L)}(\vec{f})\right\|_{L^p(w)}\right)\\ &\le C\left(\left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)} + \sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\left\|\mathcal{M}_{\varrho,L(\log L)}(\vec{f})\right\|_{L^p(w)}\right).\end{aligned}$$

We remark that to apply the inequality (2.5), in the above computations, we need to check that $\|M_\delta(T_{\Sigma\vec{b}}(\vec{f}))\|_{L^p(w)}$ and $\|M_\varepsilon(T(\vec{f}))\|_{L^p(w)}$ are finite when the terms $\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(w)}$ and $\|\mathcal{M}_{\varrho,L(\log L)}\|$ are finite. This task is not difficult. We omit the details as it is actually shown in the proof of Theorem 4.4 in [1]. □

Proof of Theorem 1.1. By Lemma 2.1, $\vec{w}\in A_{\vec{P}}$ implies that $v_{\vec{w}}\in A_\infty$. Therefore, Proposition 4.3 shows that

$$\begin{aligned}\int_{\mathbb{R}^n}\left|T_{\Sigma\vec{b}}(\vec{f})(x)\right|^p v_{\vec{w}}(x)dx \le C\left(\sum_{i=1}^{m}\|b_i\|_{\mathrm{BMO}}\right)^p\Bigg(&\int_{\mathbb{R}^n}\left(\mathcal{M}_{L(\log L)}(\vec{f})(x)\right)^p v_{\vec{w}}(x)dx\\ &+\sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\int_{\mathbb{R}^n}\left(\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right)^p v_{\vec{w}}(x)dx\Bigg).\end{aligned}\tag{4.5}$$

Then (1.18) follows from (4.5) and Corollary 3.1. Theorem 1.1 is proved. □

5. Proof of Theorem 1.2

As before, the proof of Theorem 1.2 also rely on a point-wise estimate using sharp maximal functions. For positive integers m and j with $1\le j\le m$, let C_j^m, $\sigma=\{\sigma(1),\dots,\sigma(j)\}$, $\sigma'=\{1,\dots,m\}\setminus\sigma$, $\vec{b}_\sigma$ and $T_{\Pi\vec{b}_\sigma}$ be as in Section 1. Then we have the following proposition:

Proposition 5.1. *Suppose that T is an m-linear operator in GCZO(A,s,η,ϵ) and satisfies the assumptions in Theorem 1.2, and $T_{\Pi\vec{b}}$ is the corresponding iterated commutator with $\vec{b}\in\mathrm{BMO}^m$. Let $0<\delta<\min\{\varepsilon,1/m\}$, $r>1$ and $\varrho_0=\{1,\dots,m\}$. Then, there exists a constant $C>0$, depending on δ and ε, such that*

$$\begin{aligned}M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))(x) \le\ & C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}}\left(M_\varepsilon\big(T(\vec{f})\big)(x)+\mathcal{M}_{L(\log L)}(\vec{f})(x)+\sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right)\\ &+C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i=1}^{j}\|b_{\sigma(i)}\|_{\mathrm{BMO}}M_\varepsilon\big(T_{\Pi\vec{b}_{\sigma'}}(\vec{f})\big)(x)\end{aligned}\tag{5.1}$$

for all bounded measurable vector functions $\vec{f}=(f_1,\dots,f_m)$ with compact supports.

Proof. The arguments in the proof of Theorem 3.1 in [5] can be followed with some modifications. The way to interpret (5.1) is

$$M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))(x)\le C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(\vec{f})(x)+\sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right)+\text{“lower order terms”}.$$

For simplicity in the exposition, we only present the case $m=2$. The general case is only notationally more complicated and can be obtained with a similarly procedure. Hence, we will limit ourselves to establish the following (5.2).

For $b_1, b_2 \in \mathrm{BMO}$ we will show that

$$M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))(x) \le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}(M_\varepsilon(T(\vec{f}))(x) + \mathcal{M}_{L(\log L)}(\vec{f})(x) + \mathcal{M}_{\{1\},L(\log L)}(\vec{f})(x) + \mathcal{M}_{\{2\},L(\log L)}(\vec{f})(x)) + C\,(\|b_2\|_{\mathrm{BMO}}M_\varepsilon(T^1_{b_1}(\vec{f}))(x) + \|b_1\|_{\mathrm{BMO}}M_\varepsilon(T^2_{b_2}(\vec{f}))(x)). \tag{5.2}$$

For any constants λ_1 and λ_2, write

$$\begin{aligned} T_{\Pi\vec{b}}(\vec{f})(x) &= (b_1(x)-\lambda_1)(b_2(x)-\lambda_2)T(f_1,f_2)(x) - (b_1(x)-\lambda_1)T(f_1,(b_2-\lambda_2)f_2)(x) \\ &\quad - (b_2(x)-\lambda_2)T((b_1-\lambda_1)f_1,f_2)(x) + T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(x) \\ &= -(b_1(x)-\lambda_1)(b_2(x)-\lambda_2)T(f_1,f_2)(x) + (b_1(x)-\lambda_1)T^2_{b_2-\lambda_2}(f_1,f_2)(x) \\ &\quad + (b_2(x)-\lambda_2)T^1_{b_1-\lambda_1}(f_1,f_2)(x) + T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(x). \end{aligned}$$

Also, if we fix $x \in R^n$, a cube Q centered at x and a constant c, then for $0 < \delta < 1/2$, we have

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q \left|\left|T_{\Pi\vec{b}}(\vec{f})(z)\right|^\delta - |c|^\delta\right| dz\right)^{1/\delta} &\le \left(\frac{1}{|Q|}\int_Q \left|T_{\Pi\vec{b}}(\vec{f})(z) - c\right|^\delta dz\right)^{1/\delta} \\ &\le \left(\frac{C}{|Q|}\int_Q |(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)|^\delta\, dz\right)^{1/\delta} \\ &\quad + \left(\frac{C}{|Q|}\int_Q \left|(b_1(z)-\lambda_1)T^2_{b_2-\lambda_2}(f_1,f_2)(z)\right|^\delta dz\right)^{1/\delta} \\ &\quad + \left(\frac{C}{|Q|}\int_Q \left|(b_2(z)-\lambda_2)T^1_{b_1-\lambda_1}(f_1,f_2)(z)\right|^\delta dz\right)^{1/\delta} \\ &\quad + \left(\frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z) - c|^\delta\, dz\right)^{1/\delta} \\ &:= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

We analyze each term separately selecting appropriate constants. Let $Q^* = (8\sqrt{n}+4)Q$ and let $\lambda_j = (b_j)_{Q^*}$ be the average of b_j on Q^*, $j = 1, 2$. Following the arguments used in the proof of Theorem 3.1 of [5], by the Hölder inequality and the definition of M_ε, it is easy to check that

$$\begin{aligned} &\mathrm{I} \le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_\varepsilon(T(f_1,f_2))(x), \\ &\mathrm{II} \le C\|b_1\|_{\mathrm{BMO}}M_\varepsilon(T^2_{b_2-\lambda_2}(f_1,f_2))(x) = C\|b_1\|_{\mathrm{BMO}}M_\varepsilon(T^2_{b_2}(f_1,f_2))(x), \end{aligned}$$

and

$$\mathrm{III} \le C\|b_2\|_{\mathrm{BMO}}M_\varepsilon(T^1_{b_1-\lambda_1}(f_1,f_2))(x) = C\|b_2\|_{\mathrm{BMO}}M_\varepsilon(T^1_{b_1}(f_1,f_2))(x).$$

It remains to estimate the last term IV. We split each f_i as $f_i = f_i^0 + f_i^\infty$ where $f_i^0 = f_i\chi_{Q^*}$ and $f_i^\infty = f_i - f_i^0$. Let

$$c = \sum_{j=1}^3 c_j,$$

where

$$\begin{aligned} c_1 &= T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x), \\ c_2 &= T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0)(x), \\ c_3 &= T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x). \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{IV} &= \left(\frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z) - c|^\delta dz\right)^{1/\delta} \\ &\le \left(\frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)(z)|^\delta dz\right)^{1/\delta} + \left(\frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(z) - c_1|^\delta dz\right)^{1/\delta} \\ &\quad + \left(\frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0)(z) - c_2|^\delta dz\right)^{1/\delta} \end{aligned}$$

$$+\left(\frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(z)-c_3|^{\delta}dz\right)^{1/\delta}$$
$$:= \mathrm{IV}_1+\mathrm{IV}_2+\mathrm{IV}_3+\mathrm{IV}_4.$$

Note that $\delta < 1/2$, by the Kolmogorov inequality and the boundedness of T, we easily obtain

$$\mathrm{IV}_1 \le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x).$$

Since IV_2 and IV_3 are symmetric, we consider for example IV_2, and estimate

$$\begin{aligned}&\left|T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^{\infty})(z)-T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^{\infty})(x)\right|\\ &\quad\le \int_{Q^*\times(R^n\setminus Q^*)}\left(|K(z,y_1,y_2)-K_t^0(z,y_1,y_2)|+|K_t^0(z,y_1,y_2)-K(x,y_1,y_2)|\right)\\ &\qquad\times|(b_1(y_1)-\lambda_1)f_1^0(y_1)(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)|dy_1dy_2\\ &\quad:= \mathrm{V}.\end{aligned}$$

Since $z\in Q$, and $y_2\in R^n\setminus(8\sqrt{n}+4)Q$, taking $t=(2\sqrt{n}\,l(Q))^s$, we have

$$|y_2-z| > (4\sqrt{n}+1)l(Q) > 2t^{1/s},$$

and

$$|z-x|\le\sqrt{n}\,l(Q)\le\frac{1}{2}t^{1/s}.$$

By (1.17) of Assumption (H2), we obtain

$$\begin{aligned}\mathrm{V} &\le C\int_{Q^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1\left[\int_{(R^n\setminus Q^*)}\frac{t^{\varepsilon/s}|(b_2(y_2)-\lambda_2)f_2(y_2)|}{|z-y_2|^{2n+\varepsilon}}dy_2+\int_{(R^n\setminus Q^*)}\frac{|(b_2(y_2)-\lambda_2)f_2(y_2)|}{|z-y_2|^{2n}}dy_2\right]\\ &\le C\int_{Q^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1\left[\sum_{k=0}^{\infty}\frac{|Q^*|^{\varepsilon/n}}{(2^k|Q^*|^{1/n})^{2n+\varepsilon}}\int_{2^{k+1}Q^*}|(b_2(y_2)-\lambda_2)f_2(y_2)|dy_2\right.\\ &\qquad\left.+\sum_{k=0}^{\infty}\frac{1}{(2^k|Q^*|^{1/n})^{2n}}\int_{2^{k+1}Q^*}|(b_2(y_2)-\lambda_2)f_2(y_2)|dy_2\right]\\ &\le C\|b_1\|_{\mathrm{BMO}}\,\|f_1\|_{L(\log L),Q^*}\,|Q^*|\left[\sum_{k=1}^{\infty}\frac{k}{2^{k\epsilon}}\frac{\|b_2\|_{\mathrm{BMO}}}{|2^kQ^*|}\|f_2\|_{L(\log L),2^kQ^*}+\sum_{k=1}^{\infty}\frac{1}{2^k|Q^*|}k\|b_2\|_{\mathrm{BMO}}\,\|f_2\|_{L(\log L),2^kQ^*}\right]\\ &\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(f_1,f_2)(x)+\mathcal{M}_{\{1\},L(\log L)}(f_1,f_2)(x)\right),\end{aligned}$$

which implies that

$$\mathrm{IV}_2\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(f_1,f_2)(x)+\mathcal{M}_{\{1\},L(\log L)}(f_1,f_2)(x)\right).$$

Similarly,

$$\mathrm{IV}_3\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(f_1,f_2)(x)+\mathcal{M}_{\{2\},L(\log L)}(f_1,f_2)(x)\right).$$

Finally, we estimate the term IV_4 by considering

$$\begin{aligned}&\left|T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(z)-T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(x)\right|\\ &\quad\le\int_{(R^n\setminus Q^*)^2}\left(\left|K(z,y_1,y_2)-K_t^0(z,y_1,y_2)\right|+\left|K_t^0(z,y_1,y_2)-K(x,y_1,y_2)\right|\right)\\ &\qquad\times\left|(b_1(y_1)-\lambda_1)f_1^{\infty}(y_1)(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)\right|dy_1dy_2\\ &\quad:= \mathrm{VI}.\end{aligned}$$

Since $z\in Q$, and $y_1,y_2\in R^n\setminus(8\sqrt{n}+4)Q$, we have

$$|y_j-z|>(4\sqrt{n}+1)l(Q)>2t^{1/s},\quad\text{for } j=1,2,$$

and

$$|z-x|\le\sqrt{n}\,l(Q)\le\frac{1}{2}t^{1/s}.$$

Hence $\phi(\frac{|y_j-z|}{t^{1/s}})=0$ for $j=1,2$. By (1.17) again, we obtain

$$\begin{aligned}
\mathrm{VI} &\le C\int_{(R^n\setminus Q^*)^2}\frac{At^{\varepsilon/s}}{(|z-y_1|+|z-y_2|)^{2n+\varepsilon}}\,|(b_1(y_1)-\lambda_1)f_1(y_1)(b_2(y_2)-\lambda_2)f_2(y_2)|\,dy_1dy_2\\
&\le C\sum_{k=0}^{\infty}\frac{|Q^*|^{\varepsilon/n}}{(2^k|Q^*|^{1/n})^{2n+\varepsilon}}\int_{(2^{k+1}Q^*)^2}|(b_1(y_1)-\lambda_1)f_1(y_1)(b_2(y_2)-\lambda_2)f_2(y_2)|\,dy_1dy_2\\
&\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}\frac{k^2}{2^{k\epsilon}}\,\|f_1\|_{L(\log L),2^kQ^*}\,\|f_2\|_{L(\log L),2^kQ^*}\\
&\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\mathcal{M}_{L(\log L)}(f_1,f_2)(x).
\end{aligned}$$

This finishes the proof of Proposition 5.1. □

Remark 5.1. More generally, by the same arguments as in Proposition 5.1, we can obtain analogous estimates to (5.1) for j-iterated commutators involving $j<m$ functions in $\mathrm{BMO}(\mathbb{R}^n)$. That is estimates of the form

$$M_\delta^\sharp\Big(T_{\Pi\vec{b}_\sigma}(\vec{f})\Big)(x)\le C\prod_{k=1}^{j}\|b_{\sigma(k)}\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(\vec{f})(x)+\sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)\right)+\text{“lower order terms”}\tag{5.3}$$

for $\sigma=(\sigma(1),\dots,\sigma(j))$, where the lower order terms are

$$\sum_{\varsigma\subsetneq\sigma}\prod_{k\in\varsigma'}\|b_k\|_{\mathrm{BMO}}M_\varepsilon\left(T_{\Pi\vec{b}_\varsigma}(\vec{f})\right)(x),$$

where $\varsigma'=\sigma\setminus\varsigma$.

Proposition 5.2. *Assume that T is a multilinear operator in $m-GCZO(A,s,\eta,\epsilon)$ and satisfies the assumptions in Theorem 1.2. Let w be an A_∞ weight, $0<p<\infty$ and $\vec{b}\in\mathrm{BMO}^m$, $\varrho_0=\{1,\dots,m\}$. Then, there exists a constant $C_w>0$ (independent of $\vec{b}$) such that*

$$\int_{\mathbb{R}^n}\left|T_{\Pi\vec{b}}(\vec{f})(x)\right|^p w(x)dx\le C_w\prod_{j=1}^{m}\|b_j\|_{\mathrm{BMO}}^p\left(\int_{\mathbb{R}^n}\mathcal{M}_{L(\log L)}(\vec{f})(x)^p w(x)dx+\sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\int_{\mathbb{R}^n}\mathcal{M}_{\varrho,L(\log L)}(\vec{f})(x)^p w(x)dx\right),$$

for all bounded measurable vector functions $\vec{f}=(f_1,\dots,f_m)$ with compact supports.

Proof. Our proof follows the same outline as the proof of Theorem 3.2 in [5]. We briefly indicate it in the case $m=2$. As the reader will immediately notice, an iterative procedure using (5.1) and (5.3) can be followed to obtain the general case. By the same arguments as in [4, p. 1254] (also see [5, p. 14]), we can use the Fefferman–Stein inequality and get

$$\left\|T_{\Pi\vec{b}}(\vec{f})\right\|_{L^p(w)}\le\left\|M_\delta(T_{\Pi\vec{b}}(\vec{f}))\right\|_{L^p(w)}\le C\left\|M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))\right\|_{L^p(w)}.$$

Invoking (5.2) and the Fefferman–Stein inequality again,

$$\begin{aligned}
\left\|M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))\right\|_{L^p(w)} &\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\left(\left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)}+\sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec{f})\right\|_{L^p(w)}+\left\|M_\varepsilon(T(\vec{f}))\right\|_{L^p(w)}\right)\\
&\quad+C\|b_2\|_{\mathrm{BMO}}\left\|M_\varepsilon(T_{b_1}^1(\vec{f}))\right\|_{L^p(w)}+C\|b_1\|_{\mathrm{BMO}}\left\|M_\varepsilon(T_{b_2}^2(\vec{f}))\right\|_{L^p(w)}\\
&\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\left(\left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)}+\sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec{f})\right\|_{L^p(w)}+\left\|M_\varepsilon^\sharp(T(\vec{f}))\right\|_{L^p(w)}\right)\\
&\quad+C\|b_2\|_{\mathrm{BMO}}\left\|M_\varepsilon^\sharp(T_{b_1}^1(\vec{f}))\right\|_{L^p(w)}+C\|b_1\|_{\mathrm{BMO}}\left\|M_\varepsilon^\sharp(T_{b_2}^2(\vec{f}))\right\|_{L^p(w)}.
\end{aligned}$$

Taking ε small enough and using the results in Propositions 4.1 and 4.3, we have

$$\begin{aligned}
\left\|M_\varepsilon^\sharp(T(\vec{f}))\right\|_{L^p(w)} &\le C\left(\left\|\mathcal{M}(\vec{f})\right\|_{L^p(w)}+\sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\}}(\vec{f})\right\|_{L^p(w)}\right)\\
&\le C\left(\left\|\mathcal{M}_{L(\log L)}(\vec{f})\right\|_{L^p(w)}+\sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec{f})\right\|_{L^p(w)}\right);
\end{aligned}$$

and for $\varepsilon < \varepsilon'$,

$$\begin{aligned}\left\|M_\varepsilon(T^1_{b_1}(\vec f))\right\|_{L^p(w)} &\le C\|b_1\|_{\mathrm{BMO}}\left(\left\|\mathcal{M}_{L(\log L)}(\vec f)\right\|_{L^p(w)} + \sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec f)\right\|_{L^p(w)} + \left\|M_{\varepsilon'}(T(\vec f))\right\|_{L^p(w)}\right)\\ &\le C\|b_1\|_{\mathrm{BMO}}\left(\left\|\mathcal{M}_{L(\log L)}(\vec f)\right\|_{L^p(w)} + \sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec f)\right\|_{L^p(w)} + \left\|M^\sharp_{\varepsilon'}(T(\vec f))\right\|_{L^p(w)}\right)\\ &\le C\|b_1\|_{\mathrm{BMO}}\left(\left\|\mathcal{M}_{L(\log L)}(\vec f)\right\|_{L^p(w)} + \sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec f)\right\|_{L^p(w)}\right).\end{aligned}$$

Similarly,

$$\left\|M_\varepsilon(T^2_{b_2}(\vec f))\right\|_{L^p(w)} \le C\|b_2\|_{\mathrm{BMO}}\left(\left\|\mathcal{M}_{L(\log L)}(\vec f)\right\|_{L^p(w)} + \sum_{j=1}^{2}\left\|\mathcal{M}_{\{j\},L(\log L)}(\vec f)\right\|_{L^p(w)}\right).$$

The desired inequality now follows. □

Proof of Theorem 1.2. By Lemma 2.1, $\vec w \in A_{\vec P}$ implies that $v_{\vec w} \in A_\infty$. Therefore, Proposition 5.2 tells us that

$$\int_{\mathbb{R}^n}\left|T_{\Pi\vec b}(\vec f)(x)\right|^p v_{\vec w}(x)dx \le C\prod_{j=1}^{m}\|b_j\|^p_{\mathrm{BMO}}\left(\int_{\mathbb{R}^n}\mathcal{M}_{L(\log L)}(\vec f)(x)^p v_{\vec w}(x)dx + \sum_{\emptyset\ne\varrho\subsetneq\varrho_0}\int_{\mathbb{R}^n}\mathcal{M}_{\varrho,L(\log L)}(\vec f)(x)^p v_{\vec w}(x)dx\right).$$

This together with Proposition 3.1 and Corollary 3.1 leads to the inequality (1.19) and completes the proof of Theorem 1.2. □

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