Journal of Pure and Applied Algebra 214 (2010) 1306–1318

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Critical points between varieties generated by subspace lattices of vector spaces[☆]

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ARTICLE INFO

Article history:
Received 24 September 2008
Received in revised form 11 October 2009
Available online 1 November 2009
Communicated by M. Sapir

MSC:
Primary: 08A30
Secondary: 16E50; 51D25; 06B20

ABSTRACT

We denote by $\operatorname{Con}_c A$ the semilattice of all compact congruences of an algebra A. Given a variety $\mathcal{V}$ of algebras, we denote by $\operatorname{Con}_c \mathcal{V}$ the class of all semilattices isomorphic to $\operatorname{Con}_c A$ for some $A \in \mathcal{V}$. Given varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the *critical point* of $\mathcal{V}$ under $\mathcal{W}$ is defined as $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \min\{\operatorname{card} D \mid D \in \operatorname{Con}_c \mathcal{V} - \operatorname{Con}_c \mathcal{W}\}$. Given a finitely generated variety $\mathcal{V}$ of modular lattices, we obtain an integer ℓ, depending on $\mathcal{V}$, such that $\operatorname{crit}(\mathcal{V}; \mathbf{Var}(\operatorname{Sub} F^n)) \geq \aleph_2$ for any $n \geq \ell$ and any field F.

In a second part, using tools introduced in Gillibert (2009) [5], we prove that:

$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}(\operatorname{Sub} F^3)) = \aleph_2,$$

for any finite field F and any ordinal n such that $2 + \operatorname{card} F \leq n \leq \omega$. Similarly $\operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3)) = \aleph_2$, for all finite fields F and K such that $\operatorname{card} F > \operatorname{card} K$.

1. Introduction

We denote by $\operatorname{Con} A$ (resp., $\operatorname{Con}_c A$) the lattice (resp., $(\vee, 0)$-semilattice) of all congruences (resp., compact congruences) of an algebra A. For a homomorphism $f: A \to B$ of algebras, we denote by $\operatorname{Con} f$ the map from $\operatorname{Con} A$ to $\operatorname{Con} B$ defined by the rule

$$(\operatorname{Con} f)(\alpha) = \text{congruence of } B \text{ generated by } \{(f(x), f(y)) \mid (x, y) \in \alpha\},$$

for every $\alpha \in \operatorname{Con} A$, and we also denote by $\operatorname{Con}_c f$ the restriction of $\operatorname{Con} f$ from $\operatorname{Con}_c A$ to $\operatorname{Con}_c B$.

A *congruence-lifting* of a $(\vee, 0)$-semilattice S is an algebra A such that $\operatorname{Con}_c A \cong S$. Given a variety $\mathcal{V}$ of algebras, *the compact congruence class of* $\mathcal{V}$, denoted by $\operatorname{Con}_c \mathcal{V}$, is the class of all $(\vee, 0)$-semilattices isomorphic to $\operatorname{Con}_c A$ for some $A \in \mathcal{V}$. As illustrated by [12], even the compact congruence classes of small varieties of lattices are complicated objects. For example, in case $\mathcal{V}$ is the variety of all lattices, $\operatorname{Con}_c \mathcal{V}$ contains all distributive $(\vee, 0)$-semilattices of cardinality at most $\aleph_1$, but not all distributive $(\vee, 0)$-semilattices (cf. [15]).

Given varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the *critical point* of $\mathcal{V}$ and $\mathcal{W}$, denoted by $\operatorname{crit}(\mathcal{V}; \mathcal{W})$, is the smallest cardinality of a $(\vee, 0)$-semilattice in $\operatorname{Con}_c(\mathcal{V}) - \operatorname{Con}_c(\mathcal{W})$ if it exists, or ∞, otherwise (i.e., if $\operatorname{Con}_c \mathcal{V} \subseteq \operatorname{Con}_c \mathcal{W}$).

Let I be a poset. A *direct system indexed by* I is a family $(A_i, f_{i,j})_{i \leq j \text{ in } I}$ such that A_i is an algebra, $f_{i,j}: A_i \to A_j$ is a morphism of algebras, $f_{i,i} = \operatorname{id}_{A_i}$, and $f_{i,k} = f_{j,k} \circ f_{i,j}$, for all $i \leq j \leq k$ in I.

☆ This paper is a part of the author's "Doctorat de l'université de Caen", prepared under the supervision of Friedrich Wehrung.

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doi:10.1016/j.jpaa.2009.10.013

Denote by Sub V the subspace lattice of a vector space V, and by $\mathcal{M}_n$ the variety of lattices generated by the lattice M_n of length two with n atoms, for $3 \leq n \leq \omega$. Using the theory of the *dimension monoid* of a lattice, introduced by Wehrung in [13], together with some von Neumann regular ring theory, we prove in Section 3 that if $\mathcal{V}$ is a finitely generated variety of modular lattices with all subdirectly irreducible members of length less or equal to n, then $\operatorname{crit}(\mathcal{V}; \mathbf{Var}\,(\operatorname{Sub} F^n)) \geq \aleph_2$ for any field F. As an immediate application, $\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_3) \geq \aleph_2$ for every n with $3 \leq n \leq \omega$ (cf. Corollary 3.12). Thus, by using the result of M. Ploščica in [10], we obtain the equality $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2$ for all m, n with $3 \leq n < m \leq \omega$. Our proof does not rely on the approach used by Ploščica in [11] to prove the inequality $\operatorname{crit}(\mathcal{M}_m^{0,1}; \mathcal{M}_n^{0,1}) \geq \aleph_2$, and it extends that result to the unbounded case. We also obtain a new proof of that result in Section 4, that does not even rely on the approach used by Ploščica in [10] to prove the inequality $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) \leq \aleph_2$.

Let $\mathcal{V}$ be a variety of lattices, let $\vec{D}$ be a diagram of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms. A *congruence-lifting* of $\vec{D}$ in $\mathcal{V}$ is a diagram $\vec{L}$ of $\mathcal{V}$ such that the composite $\operatorname{Con}_c \circ \vec{L}$ is naturally equivalent to $\vec{D}$.

In Section 4, we give a diagram of finite $(\vee, 0)$-semilattices that is congruence-liftable in $\mathcal{M}_n$, but not congruence-liftable in $\mathbf{Var}\,(\operatorname{Sub} F^3)$, for any finite field F and any n such that $2 + \operatorname{card} F \leq n \leq \omega$. As the diagram of $(\vee, 0)$-semilattices is indexed by some "good" lattice, we obtain, using results of [5], that $\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}\,(\operatorname{Sub} F^3)) = \aleph_2$. This implies immediately that $\operatorname{crit}(\mathcal{M}_4; \mathcal{M}_{3,3}) = \aleph_2$. Let F and K be finite fields such that $\operatorname{card} F > \operatorname{card} K$, we also obtain $\operatorname{crit}(\mathbf{Var}\,(\operatorname{Sub} F^3); \mathbf{Var}\,(\operatorname{Sub} K^3)) = \aleph_2$.

In a similar way, we prove that $\operatorname{crit}(\mathcal{M}_\omega; \mathcal{V}) = \aleph_2$, for every finitely generated variety of lattices $\mathcal{V}$ such that $M_3 \in \mathcal{V}$.

2. Basic concepts

We denote by dom f the domain of any function f. A *poset* is a partially ordered set. Given a poset P, we put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\}, \qquad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\},$$

for any $X, Q \subseteq P$, and we will write $\downarrow X$ (resp., $\uparrow X$) instead of $P \downarrow X$ (resp., $P \uparrow X$) in case P is understood. We shall also write $\downarrow p$ instead of $\downarrow \{p\}$, and so on, for $p \in P$. A poset P is *lower finite* if $P \downarrow p$ is finite for all $p \in P$. For $p, q \in P$ let $p \prec q$ hold, if $p < q$ and there is no $r \in P$ with $p < r < q$, in this case p is called a *lower cover* of q. We denote by $P^{=}$ the set of all non-maximal elements in a poset P. We denote by $M(L)$ the set of all completely meet-irreducible elements of a lattice L.

A 2-*ladder* is a lower finite lattice in which every element has at most two lower covers. S. Z. Ditor constructs in [1] a 2-ladder of cardinality $\aleph_1$.

For a set X and a cardinal κ, we denote by:

$$[X]^\kappa = \{Y \subseteq X \mid \operatorname{card} Y = \kappa\},$$
$$[X]^{\leq\kappa} = \{Y \subseteq X \mid \operatorname{card} Y \leq \kappa\},$$
$$[X]^{<\kappa} = \{Y \subseteq X \mid \operatorname{card} Y < \kappa\}.$$

Denote by $\mathscr{P}$ the category with objects the ordered pairs (G, u) where G is a pre-ordered abelian group and u is an order-unit of G (i.e., for each $x \in G$, there exists an integer n with $-nu \leq x \leq nu$), and morphisms $f: (G, u) \to (H, v)$ where $f: G \to H$ is an order-preserving group homomorphism and $f(u) = v$.

We denote by Dim the functor that maps a lattice to its *dimension monoid*, introduced by F. Wehrung in [13], we also denote by $\Delta(a, b)$ for $a \leq b$ in L the canonical generators of Dim L. We denote by K_0^ℓ the functor that maps a lattice to the pre-ordered abelian universal group (also called Grothendieck group) of its dimension monoid. If L is a bounded lattice then (the canonical image in $K_0^\ell(L)$ of) $\Delta(0_L, 1_L)$ is an order-unit of $K_0^\ell(L)$. If $f: L \to L'$ is a 0, 1-preserving homomorphism of bounded lattices, then $K_0^\ell(f): (K_0^\ell(L), \Delta(0_L, 1_L)) \to (K_0^\ell(L'), \Delta(0_{L'}, 1_{L'}))$ preserves the order-unit.

All our rings are associative but not necessarily unital.

- We denote by $\mathbb{L}(R)$ the poset of principal right ideals of every regular ring R. The results of Fryer and Halperin in [4, Section 3.2], imply that, $\mathbb{L}(R)$ is a 0-lattice, and for any homomorphism $f: R \to S$ of regular rings, the map $\mathbb{L}(f): \mathbb{L}(R) \to \mathbb{L}(S)$, $I \mapsto f(I)S$ is a 0-lattice homomorphism (cf. Micol's thesis [9, Theorem 1.4] for the unital case). Hence $\mathbb{L}$ is a functor from the category of regular rings to the category of 0-lattices with 0-lattice homomorphisms.
- We denote by V the functor from the category of unital rings with morphisms preserving units to the category of commutative monoids, that maps a unital ring R to the commutative monoid of all isomorphism classes of finitely generated projective right R-modules and any homomorphism $f: R \to S$ of unital rings to the monoid homomorphism $V(f): V(R) \to V(S)$, $\sum_i e_i R \mapsto \sum_i f(e_i)S$.

We denote by Id R (resp., $\operatorname{Id}_c R$) the lattice of all two-sided ideals (resp., finitely generated two-sided ideals) of any ring R. We denote by Sub E the subspace lattice of a vector space E. We denote by $M_n(F)$ the F-algebra of $n \times n$ matrices with entries from F, for every field F and every positive integer n. A *matricial F-algebra* is an F-algebra of the form $M_{k_1}(F) \times \cdots \times M_{k_n}(F)$, for positive integers $k_1, \ldots, k_n$.

For a finitely generated projective right module P over a unital ring R, we denote by $[P]$ the corresponding element in $K_0(R)$, that is, the stable isomorphism class of P. We refer to [7, Section 15] for the required notions about the K_0 functor.

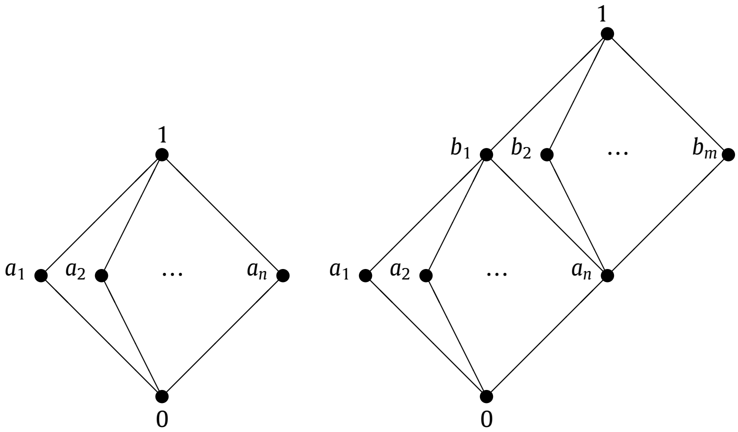

Fig. 1. The lattices M_n and $M_{n,m}$.

A K_0-*lifting* of a pre-ordered abelian group with order-unit (G, u) is a regular ring R such that $(K_0(R), [R]) \cong (G, u)$. A K_0-lifting of a diagram $\vec{G}\colon I \to \mathscr{P}$ is a diagram $\vec{R}\colon I \to \mathscr{P}$ such that $(K_0(-), [-]) \circ \vec{R} \cong \vec{G}$.

We denote by ∇ the functor that sends a monoid to it maximal semilattice quotient, that is, $\nabla(M) = M/\asymp$ where $\asymp$ is the smallest congruence of M such that $M/\asymp$ is a semilattice. We denote by $\overline{\nabla}$ the functor that maps a partially pre-ordered abelian group G to $\nabla(G^+)$ where G^+ is the monoid of all positive elements of G.

We denote by **Var** (L) (resp., $\mathbf{Var}_0(L)$, resp., $\mathbf{Var}_{0,1}(L)$) the variety of lattices (resp., lattices with 0, resp., bounded lattices) generated by a lattice L.

A lattice K is a *congruence-preserving extension of* a lattice L, if L is a sublattice of K and $\mathrm{Con}_c\, i\colon \mathrm{Con}\, L \to \mathrm{Con}\, K$ is an isomorphism, where $i\colon L \to K$ is the inclusion map.

We denote by M_n and $M_{n,m}$ the lattices represented in Fig. 1, for $3 \le m, n \le \omega$, and by $\mathcal{M}_n$ and $\mathcal{M}_{n,m}$, respectively, the lattice varieties that they generate. We also denote by $\mathcal{M}_n^0$ the variety of lattices with 0 generated by M_n, and so on.

A lattice *L satisfies Whitman's condition* if for all a, b, c, and d in L:

$$a \wedge b \le c \vee d \quad \text{implies either } a \le c \vee d \quad \text{or} \quad b \le c \vee d \quad \text{or} \quad a \wedge b \le c \quad \text{or} \quad a \wedge b \le d.$$

The lattice M_n satisfies Whitman's condition for all $n \ge 3$.

3. Lower bounds for some critical points

The following proposition is proved in [13, Proposition 5.5].

Proposition 3.1. *Let L be a modular lattice without infinite bounded chains. Let P be the set of all projectivity classes of prime intervals of L. Given* $\xi \in P$, *denote by* $|a, b|_\xi$ *the number of occurrences of an interval in* ξ *in any maximal chain of the interval* $[a, b]$. *Then there exists an isomorphism* $\pi : \mathrm{Dim}\, L \to (\mathbb{Z}^+)^{(P)}$ *such that* $\pi(\Delta(a, b)) = (|a, b|_\xi \mid \xi \in P)$ *for all* $a \le b$ *in L.*

This makes it possible to prove the following lemma, which gives an explicit description of $K_0^\ell(L)$ for every modular lattice L of finite length (in such a case the set P is finite).

Lemma 3.2. *Let L be a modular lattice of finite length, set* $X = M(\mathrm{Con}\, L)$. *Then there exists an isomorphism* $\pi'\colon K_0^\ell(L) \to \mathbb{Z}^X$ *such that*

$$\pi'(\Delta(a, b)) = (\mathrm{lh}\,([a/\theta, b/\theta]) \mid \theta \in X), \quad \textit{for all } a \le b \textit{ in } L.$$

In particular $(K_0^\ell(L), \Delta(0, 1))$ *is isomorphic to* $(\mathbb{Z}^X, (\mathrm{lh}\,(L/\theta))_{\theta \in X})$.

Proof. Denote by P be the set of all projectivity classes of prime intervals of L. For any $\xi \in P$ denote by θ_ξ the largest congruence of L that does not collapse any prime intervals in ξ. As L is modular of finite length, the congruences of L are in one-to-one correspondence with subsets of P (cf. [6, Chapter III]), and so the assignment $\xi \mapsto \theta_\xi$ defines a bijection from P onto X. Moreover any prime interval not in ξ is collapsed by θ_ξ, for any $\xi \in P$. Let $a \le b$ in L, let $\xi \in P$. Let $a_0 \prec a_1 \prec \cdots \prec a_n$ in L such that $a_0 = a$ and $a_n = b$. Let $0 \le r_1 < r_2 < \cdots < r_s < n$ be all the integers such that $[a_{r_k}, a_{r_k+1}] \in \xi$ for all $1 \le k \le s$. Thus $|a, b|_\xi = s$. Set $r_{s+1} = n$. As $[a_{r_k}, a_{r_k+1}] \in \xi$ and $[a_{r_k+t}, a_{r_k+t+1}] \notin \xi$ for all $1 \le t \le r_{k+1} - r_k - 1$, we obtain that

$$a_{r_k}/\theta_\xi \prec a_{r_k+1}/\theta_\xi = a_{r_k+2}/\theta_\xi = \cdots = a_{r_{k+1}}/\theta_\xi, \quad \text{for all } 1 \le k \le s.$$

Thus the following covering relations hold:

$$a/\theta_\xi = a_{r_1}/\theta_\xi \prec a_{r_2}/\theta_\xi \prec \cdots \prec a_{r_s}/\theta_\xi \prec a_{r_{s+1}}/\theta_\xi = b/\theta_\xi.$$

So $\mathrm{lh}\,([a/\theta_\xi, b/\theta_\xi]) = s = |a, b|_\xi$. We conclude the proof by using Proposition 3.1. $\square$

Proposition 3.3. *The following natural equivalences hold*

(i) $\nabla \circ \operatorname{Dim} \cong \operatorname{Con}_c$ *on lattices*

(ii) $\nabla \circ V \cong \operatorname{Con}_c \circ \mathbb{L}$ *on regular rings*.

Proof. (i) follows from [13, Corollary 2.3], while (ii) is contained in [7, Corollary 2.23]; see also the proof of [14, Proposition 4.6]. □

We shall always apply this result to unital regular rings R such that $V(R)$ is cancellative (i.e., R is unit-regular), so $K_0(R)^+ = V(R)$, and to lattices L such that $\operatorname{Dim} L$ is cancellative, so $K_0^\ell(L)^+ \cong \operatorname{Dim} L$. Here G^+ denotes the positive cone of G, for any partially pre-ordered abelian group G.

The following theorem is proved in [7, Theorem 15.23].

Theorem 3.4. *Let F be a field, let R be a matricial F-algebra, and let S be a unit-regular F-algebra.*

(1) *Given any morphism $f : (K_0(R), [R]) \to (K_0(S), [S])$ in $\mathscr{P}$, the category of pre-ordered abelian groups with order-unit (cf. Section 2), there exists an F-algebra homomorphism $\phi : R \to S$ such that $K_0(\phi) = f$.*

(2) *If $\phi, \psi : R \to S$ are F-algebra homomorphisms, then $K_0(\phi) = K_0(\psi)$ if and only if there exists an inner automorphism θ of S such that $\phi = \theta \circ \psi$.*

The following lemma is folklore.

Lemma 3.5. *Let F be a field, let $\boldsymbol{u} = (u_k)_{1\le k\le n}$ be a family of positive integers, let $R = \prod_{k=1}^{n} M_{u_k}(F)$. Then $(K_0(R), [R]) \cong (\mathbb{Z}^n, \boldsymbol{u})$.*

Lemma 3.6. *Let F be a field. Let I be a 2-ladder, let $G_i = (\mathbb{Z}^{n_i}, \boldsymbol{u}^i = (u_k^i)_{1\le k\le n_i})$ such that $\boldsymbol{u}^i$ is an order-unit, let $R_i = \prod_{k=1}^{n_i} M_{u_k^i}(F)$ for all $i \in I$. Let $f_{i,j} : G_i \to G_j$ for all $i \le j$ in I such that $\vec{G} = (G_i, f_{i,j})_{i\le j \text{ in } I}$ is a direct system in $\mathscr{P}$. Then there exists a direct system $(R_i, \phi_{i,j})_{i\le j \text{ in } I}$ of matricial F-algebra which is a K_0-lifting of $(G_i, f_{i,j})_{i\le j \text{ in } I}$.*

Proof. By Lemma 3.5 there exists an isomorphism $\tau_i : (K_0(R_i), [R_i]) \to G_i = (\mathbb{Z}^{n_i}, \boldsymbol{u}^i)$ in $\mathscr{P}$, for all $i \in I$. Let $g_{i,j} = \tau_j^{-1} \circ f_{i,j} \circ \tau_i$, for all $i \le j$ in I.

For $i = j = 0$ (the smallest element of I), we put $\phi_{0,0} = \mathrm{id}_{R_0}$. Let $i \in I$ with a lower cover i'. It follows from Theorem 3.4(1) that there exists $\psi_{i',i} : R_{i'} \to R_i$ such that $K_0(\psi_{i',i}) = g_{i',i}$.

If i has only i' as lower cover, assume that we have a direct system $(R_j, \phi_{j,k})_{j\le k\le i'}$ lifting $(G_j, f_{j,k})_{j\le k\le i'}$. Set $\phi_{j,i} = \psi_{i',i} \circ \phi_{j,i'}$ for all $j < i$, and $\phi_{i,i} = \mathrm{id}_{R_i}$. It is easy to see that $(R_i, \phi_{j,k})_{j\le k\le i}$ is a direct system lifting $(G_j, f_{j,k})_{j\le k\le i}$.

Let i has two distinct lower covers i' and i'', and set $\ell = i' \wedge i''$. Assume that we have direct system $(R_j, \phi_{j,k})_{j\le k\le i'}$ and $(R_j, \phi_{j,k})_{j\le k\le i''}$ lifting $(G_j, f_{j,k})_{j\le k\le i'}$ and $(G_j, f_{j,k})_{j\le k\le i''}$ respectively. The following equalities hold

$$K_0(\psi_{i',i} \circ \phi_{\ell,i'}) = K_0(\psi_{i',i}) \circ K_0(\phi_{\ell,i'}) = g_{i',i} \circ g_{\ell,i'} = g_{\ell,i}.$$

Similarly $K_0(\psi_{i'',i} \circ \phi_{\ell,i''}) = g_{\ell,i} = K_0(\psi_{i',i} \circ \phi_{\ell,i'})$, thus, by Theorem 3.4(2), there exists an inner automorphism θ of R_i such that $\theta \circ \psi_{i'',i} \circ \phi_{\ell,i''} = \psi_{i',i} \circ \phi_{\ell,i'}$. Put $\phi_{i',i} = \psi_{i',i}$ and $\phi_{i'',i} = \theta \circ \psi_{i'',i}$. Thus $\phi_{i',i} \circ \phi_{i'\wedge i'',i'} = \phi_{i'',i} \circ \phi_{i'\wedge i'',i''}$, so we can construct a direct system $(R_j, \phi_{j,k})_{j\le k\le i}$.

Hence, by induction, we obtain a direct system $(R_i, \phi_{i,j})_{i\le j \text{ in } I}$ of matricial F-algebras, such that $K_0(\phi_{i,j}) = g_{i,j}$ for all $i \le j$ in I as required. □

Lemma 3.7. *Let F be a field. Let L be a bounded modular lattice such that all finitely generated sublattices of L have finite length. Assume that $\operatorname{card} L \le \aleph_1$. Then there exists a locally matricial ring R such that $\operatorname{Con} L \cong \operatorname{Con} \mathbb{L}(R)$ and $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid n < \omega)$.*

Moreover if there exists $n < \omega$ such that $n \ge \operatorname{lh}(K)$ for each simple lattice $K \in \mathbf{Var}(L)$ of finite length, then there exists a locally matricial ring R such that $\operatorname{Con} L \cong \operatorname{Con} \mathbb{L}(R)$ and $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$.

Proof. Let I be a 2-ladder of cardinality $\aleph_1$. Pick a surjection $\rho : I \twoheadrightarrow L$ and denote by L_i the sublattice of L generated by $\rho(I \downarrow i) \cup \{0, 1\}$, for each $i \in I$. Furthermore, denote by $f_{i,j} : L_i \to L_j$ the inclusion map, for all $i \le j$ in I. Then $\vec{L} = (L_i, f_{i,j})_{i\le j \text{ in } I}$ is a direct system of modular lattices of finite length and 0, 1-lattice embeddings.

Assume that there exists $n < \omega$ such that $n \ge \operatorname{lh}(K)$ for each simple lattice $K \in \mathbf{Var}(L)$ of finite length. Let $\vec{G} = K_0^\ell \circ \vec{L}$, set $X_i = M(\operatorname{Con} L_i)$ for all $i \in I$, and set $r_x^i = \operatorname{lh}(L_i/x)$ for each $x \in X_i$. The congruence lattice of any modular lattice of finite length is Boolean (cf. [6, Chapter III]), in particular, every subdirectly irreducible modular lattice of finite length is simple. This applies to the subdirectly irreducible lattice L_i/x, which is therefore simple. Thus $r_x^i \le n$, for all $i \in I$ and all $x \in X_i$. By Lemma 3.2, $G_i \cong (\mathbb{Z}^{X_i}, (r_x^i)_{x\in X_i})$ for all $i \in I$.

Set $R_i = \prod_{x\in X_i} M_{r_x^i}(F)$. By Lemma 3.5, $(K_0(R_i), [R_i]) \cong (\mathbb{Z}^{X_i}, (r_x^i)_{x\in X}) \cong G_i$. By Lemma 3.6, there exists a direct system $\vec{R} = (R_i, \phi_{i,j})_{i\le j \text{ in } I}$ with morphisms preserving units, such that:

$$K_0 \circ \vec{R} \cong \vec{G} = K_0^\ell \circ \vec{L}. \tag{3.1}$$

Moreover:

$$\mathbb{L}(R_i) \cong \mathbb{L}\left(\prod_{x\in X_i} M_{r_x^i}(F)\right) \cong \prod_{x\in X_i} \mathbb{L}(M_{r_x^i}(F)) \cong \prod_{x\in X_i} \operatorname{Sub} F^{r_x^i} \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n).$$

Let $R = \varinjlim \vec{R}$. As $\mathbb{L}$ preserves direct limits, $\mathbb{L}(R) \cong \varinjlim(\mathbb{L}\circ\vec{R})$, but $\mathbb{L}\circ\vec{R}$ is a diagram of $\mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$, so $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$. Moreover the following isomorphisms hold:

$$\begin{aligned}
\operatorname{Con}_c\mathbb{L}(R) &\cong \overline{\nabla}(K_0(R)) \quad \text{by Proposition 3.3}\\
&\cong \overline{\nabla}(K_0(\varinjlim \vec{R}))\\
&\cong \overline{\nabla}(\varinjlim(K_0\circ\vec{R})) \quad \text{as } K_0 \text{ preserves direct limits}\\
&\cong \overline{\nabla}(\varinjlim(K_0^{\ell}\circ\vec{L})) \quad \text{by 3.1}\\
&\cong \overline{\nabla}(K_0^{\ell}(\varinjlim \vec{L})) \quad \text{as } K_0^{\ell} \text{ preserves direct limits}\\
&\cong \overline{\nabla}(K_0^{\ell}(L))\\
&\cong \operatorname{Con}_c L \quad \text{by Proposition 3.3.}
\end{aligned}$$

The other case, without restriction on finite lengths of simple lattices, is similar. □

Lemma 3.7 works for bounded lattices, however any lattice can be embedded into a bounded lattice. In the rest of this section, using this result, we extend Lemma 3.7 to unbounded lattices.

Lemma 3.8. *Let L be a lattice, let $L' = L \sqcup \{0, 1\}$ such that 0 is the smallest element of L' and 1 is the largest. Let $f : L \hookrightarrow L'$ be the inclusion map. Then $\operatorname{Con}_c f$ is a injective $(\vee, 0)$-homomorphism and $(\operatorname{Con}_c f)(\operatorname{Con}_c L)$ is an ideal of $\operatorname{Con}_c L'$.*

Proof. Let $\theta \in \operatorname{Con}_c L$, let $L'_\theta = (L/\theta) \sqcup \{0, 1\}$ such that 0 is the smallest element of L'_θ and 1 is its largest element. The following map

$$g : L' \to L'_\theta$$
$$x \mapsto \begin{cases} 0 & \text{if } x = 0\\ 1 & \text{if } x = 1\\ x/\theta & \text{if } x \in L\end{cases}$$

is a lattice homomorphism, and $\ker g = \theta \cup \{(0, 0), (1, 1)\}$, so the latter is a congruence of L'. It follows that $(\operatorname{Con}_c f)(\theta) = \theta \cup \{(0, 0), (1, 1)\}$. Thus $\operatorname{Con}_c f$ is an embedding. Let $\beta = \bigvee_{i=1}^n \Theta_{L'}(x_i, y_i) \in \operatorname{Con}_c L'$, such that $\beta \subseteq (\operatorname{Con}_c f)(\theta)$. We can assume that $x_i \neq y_i$ for all $1 \le i \le n$. Thus, as $(x_i, y_i) \in \theta \cup \{(0, 0), (1, 1)\}$, $(x_i, y_i) \in \theta$ for all $1 \le i \le n$. Let $\alpha = \bigvee_{i=1}^n \Theta_L(x_i, y_i)$, then $(\operatorname{Con}_c f)(\alpha) = \beta$. Thus $(\operatorname{Con}_c f)(\operatorname{Con}_c L)$ is an ideal of $\operatorname{Con}_c L'$. □

Wehrung proves the following proposition in [14, Corollary 4.4]; the result also applies to the non-unital case, with a similar proof.

Proposition 3.9. *For any regular ring R, $\operatorname{Con}_c\mathbb{L}(R)$ is isomorphic to $\operatorname{Id}_c R$.*

Lemma 3.10. *Let R be a regular ring, and let I be a two-sided ideal of R. Then the following assertions hold*

(1) *The set I is a regular subring of R.*
(2) *Any right (resp., left) ideal of I is a right (resp., left) ideal of R.*
(3) *In particular $\operatorname{Id}(I) = \operatorname{Id}(R) \downarrow I$, and $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$.*

Proof. The assertion (1) follows from [7, Lemma 1.3].

Let J be a right ideal of I, let $a \in J$, let $x \in R$. As I is regular there exists $y \in I$ such that $a = aya$, so $ax = ayax$, but $a \in I$, so $yax \in I$, moreover J is a right ideal of I, so $ax = ayax \in J$. Thus J is a right ideal of R. Similarly any left ideal of I is a left ideal of R. Thus $\operatorname{Id}(I) = \operatorname{Id}(R) \downarrow I$.

Let $a \in R$ idempotent. If $aR \subseteq I$, then $a \in I$, so $aI \subseteq aR = aaR \subseteq aI$, and so $aI = aR$, thus $aR \in \mathbb{L}(I)$. So $\mathbb{L}(I) = \mathbb{L}(R)\downarrow I$. □

Theorem 3.11. *Let F be a field. Let $\mathcal{V}$ be a variety of modular lattices (resp., a variety of bounded modular lattices). Assume that all finitely generated lattices of $\mathcal{V}$ have finite length. Then*

$$\operatorname{crit}(\mathcal{V}; \mathbf{Var}_0(\operatorname{Sub} F^n \mid n \in \omega)) \ge \aleph_2 \quad (\textit{resp.}, \operatorname{crit}(\mathcal{V}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid n \in \omega)) \ge \aleph_2).$$

Moreover for $L \in \mathcal{V}$ of cardinality at most $\aleph_1$, there exists a regular ring A such that $\operatorname{Con} L \cong \operatorname{Con}\mathbb{L}(A)$ and $\mathbb{L}(A) \in \mathbf{Var}_0(\operatorname{Sub} F^n \mid n \in \omega)$ (resp., $\mathbb{L}(A) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid n \in \omega)$).

If there exists $n < \omega$ such that $\operatorname{lh}(K) \le n$ for each simple lattice $K \in \mathcal{V}$ of finite length, then:

$$\operatorname{crit}(\mathcal{V}; \mathbf{Var}_0(\operatorname{Sub} F^n)) \ge \aleph_2 \quad (\textit{resp.}, \operatorname{crit}(\mathcal{V}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)) \ge \aleph_2).$$

Moreover for $L \in \mathcal{V}$ of cardinality at most $\aleph_1$, there exists a regular ring A such that $\operatorname{Con} L \cong \operatorname{Con}\mathbb{L}(A)$ and $\mathbb{L}(A) \in \mathbf{Var}_0(\operatorname{Sub} F^n)$ (resp., $\mathbb{L}(A) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$).

Observe that $\mathbb{L}(A)$ is, in addition, relatively complemented; in particular, it is congruence-permutable.

Proof. The bounded case is an immediate application of Lemma 3.7.

Let $\mathcal{V}$ be a variety of modular lattices in which finitely generated lattices have finite length. Let $L \in \mathcal{V}$ such that card $L \leq \aleph_1$, let $L' = L \sqcup \{0, 1\}$ as in Lemma 3.8 and let D be the ideal of $\mathrm{Con}_c L'$ corresponding to $\mathrm{Con}_c L$. By Chapter I, Section 4, Exercise 14 in [6] we have $L' \in \mathcal{V}$, thus, by Lemma 3.7, there exists a regular ring R such that $\mathbb{L}(R) \in \mathbf{Var}_0(\mathrm{Sub}\, F^n)$, and $\mathrm{Con}_c\mathbb{L}(R) \cong \mathrm{Con}_c L'$. By Proposition 3.9, $\mathrm{Con}_c\mathbb{L}(R) \cong \mathrm{Id}_c R$. Let I be the ideal of R corresponding to D. Then $\mathrm{Con}\, L \cong \mathrm{Id}\, D \cong \mathrm{Id}\, R \downarrow I \cong \mathrm{Id}\, I \cong \mathrm{Con}\, \mathbb{L}(I)$. Moreover $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$ belongs to $\mathcal{W}$. □

We obtain the following generalization of Ploščica's results in [11].

Corollary 3.12. *Let m, n be ordinals such that $3 \leq n < m \leq \omega$. Then the equality $\mathrm{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2$ holds.*

Proof. Every simple lattice of $\mathcal{M}_n$ has length at most two. Moreover, $\mathrm{Sub}\, \mathbb{F}_2^2 \cong M_3 \in \mathcal{M}_n$, where $\mathbb{F}_2$ is the two-element field. Thus, by Theorem 3.11, $\mathrm{crit}(\mathcal{M}_m; \mathcal{M}_n) \geq \aleph_2$.

Conversely, M. Ploščica proves in [10] that there exists a $(\vee, 0)$-semilattice of cardinality $\aleph_2$, congruence-liftable in $\mathcal{M}_m$, but not congruence-liftable in $\mathcal{M}_n$. So $\mathrm{crit}(\mathcal{M}_m; \mathcal{M}_n) \leq \aleph_2$. □

In Section 4 we shall give another $(\vee, 0)$-semilattice of cardinality $\aleph_2$, congruence-liftable in $\mathcal{M}_m$, but not congruence-liftable in $\mathcal{M}_n$.

4. An upper bound of some critical points

Using the results of [5], we first prove that if a simple lattice of a variety of lattices $\mathcal{V}$ has larger length than all simple lattices of a finitely generated variety of lattices $\mathcal{W}$, then $\mathrm{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0$.

Remark 4.1. Let $x \prec y$ in a lattice L. Let $(\alpha_i)_{i\in I}$ be a family of congruences of L, if $(x, y) \in \bigvee_{i\in I} \alpha_i$, then $(x, y) \in \alpha_i$ for some $i \in I$. In particular there exists a largest congruence separating x and y. Such a congruence is completely meet-irreducible, and in a lattice of finite height all completely meet-irreducible congruences are of this form.

Lemma 4.2. *Let L be a lattice and let $n \geq 0$. If $\mathrm{Con}_c L \cong 2^n$ then $\mathrm{lh}\,(L) \geq n$. Moreover, if C is a finite maximal chain of L, then $\mathrm{Con}_c f$ is surjective, where $f : C \to L$ is the inclusion map.*

Proof. If L has no finite maximal chain then $\mathrm{lh}\,(L) \geq n$ is immediate. Assume that C is a finite maximal chain of L. Denotes by $0 = x_0 \prec x_1 \prec \cdots \prec x_m = 1$ the elements of C. Denote by $f : C \to L$ the inclusion map.

Let $k \in \{0, \ldots, m-1\}$. We have $x_k \prec x_{k+1}$, hence $\Theta_L(x_k, x_{k+1})$ is join-irreducible in $\mathrm{Con}_c L$. As $\mathrm{Con}_c L$ is Boolean, $\Theta_L(x_k, x_{k+1})$ is an atom of $\mathrm{Con}_c L$.

Let θ be an atom of $\mathrm{Con}_c L$, we have:

$$\theta \leq \Theta_L(0, 1) = \bigvee_{k=0}^{m-1} \Theta_L(x_k, x_{k+1})$$

So there exists $k \in \{0, \ldots, m-1\}$ such that $\theta \leq \Theta_L(x_k, x_{k+1})$. As $\Theta_L(x_k, x_{k+1})$ is an atom of $\mathrm{Con}_c L$, we have $\theta = \Theta_L(x_k, x_{k+1})$. It follows that $\mathrm{Con}_c f$ is surjective, so $m \geq n$ and so $\mathrm{lh}\,(L) \geq n$. □

Theorem 4.3. *Let $\mathcal{V}$ be a variety of lattices (resp., a variety of bounded lattices), let $\mathcal{W}$ be a finitely generated variety of lattices, let D be a finite $(\vee, 0)$-semilattice. If there exists a lifting $K \in \mathcal{V}$ of D of length greater than every lifting of D in $\mathcal{W}$, then $\mathrm{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0$. Moreover if $\mathcal{V}$ is a finitely generated variety of modular lattices and $\mathcal{W}$ is not trivial, then $\mathrm{crit}(\mathcal{V}; \mathcal{W}) = \aleph_0$.*

Proof. As D is finite, taking a sublattice, we can assume that card $K \leq \aleph_0$. Let n be the greatest length of a lifting of D in $\mathcal{W}$. As $\mathrm{lh}\,(K) > n$, there exists a chain C of K of length $n + 1$ (resp., we can assume that C has 0 and 1). Let $f : C \to K$ be the inclusion map. Assume that there exists a lifting $g : C' \to K'$ of $\mathrm{Con}_c f$ in $\mathcal{W}$. As f is an embedding, g is also an embedding. As $\mathrm{Con}_c K' \cong \mathrm{Con}_c K \cong D$, $\mathrm{lh}\,(K') \leq n$. Moreover $\mathrm{Con}_c C' \cong \mathrm{Con}_c C \cong 2^{n+1}$, thus, by Lemma 4.2, $\mathrm{lh}\,(C') = n + 1$. So $n \geq \mathrm{lh}\,(K') \geq \mathrm{lh}\,(C') = n + 1$; a contradiction.

Therefore $\mathrm{Con}_c f$ has no lifting in $\mathcal{W}$. So, as card $K \leq \aleph_0$ and by [5, Corollary 7.6], $\mathrm{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0$ (in the bounded case f preserves bounds, thus the result of [5] also applies).

Moreover if $\mathcal{V}$ is a finitely generated variety of modular lattices, then the finite $(\vee, 0)$-semilattices with congruence-lifting in $\mathcal{V}$ are the finite Boolean lattices. Finite Boolean lattices are also liftable in $\mathcal{W}$. Hence $\mathrm{crit}(\mathcal{V}; \mathcal{W}) = \aleph_0$. □

The following corollary is an immediate application of Theorems 4.3 and 3.11. It shows that the critical point between a finitely generated variety of modular lattices and a variety generated by a lattice of subspaces of a finite vector space, cannot be $\aleph_1$.

Corollary 4.4. *Let $\mathcal{V}$ be a finitely generated variety of modular lattices, let F be a finite field, let $n \geq 1$ be an integer. If there exists a simple lattice in $K \in \mathcal{V}$ such that $\mathrm{lh}\,(K) > n$, then $\mathrm{crit}(\mathcal{V}; \mathbf{Var}\,(\mathrm{Sub}\, F^n)) = \aleph_0$, else $\mathrm{crit}(\mathcal{V}; \mathbf{Var}\,(\mathrm{Sub}\, F^n)) \geq \aleph_2$.*

We shall now give a diagram of $(\vee, 0)$-semilattices $\vec{S}$, congruence-liftable in $\mathcal{M}_n$, such that for every finitely generated variety $\mathcal{V}$, generated by lattices of length at most three, the diagram $\vec{S}$ is congruence-liftable in $\mathcal{V}$ if and only if $M_n \in \mathcal{V}$.

Let $n \geq 3$ be an integer. Set $\underline{n} = \{0, 1, \dots, n-1\}$, and set:

$$I_n = \{P \in \mathfrak{P}(\underline{n}) \mid \text{either card}\,(P) \leq 2 \text{ or } P = \underline{n}\}.$$

Denote by $a_0, \dots, a_{n-1}$ the atoms of M_n. Set $A_P = \{a_x \mid x \in P\} \cup \{0, 1\}$, for all $P \in I_n$. Let $f_{P,Q} : A_P \to A_Q$ be the inclusion map for all $P \subseteq Q$ in I_n. Then $\vec{A} = (A_P, f_{P,Q})_{P \subseteq Q \text{ in } I_n}$ is a direct system in $\mathcal{M}_n^{0,1}$. The diagram $\vec{S}$ is defined as $\mathrm{Con}_c \circ \vec{A}$.

Lemma 4.5. *Let $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\mathrm{Con}_c \circ \vec{A}$ by lattices, with all the maps $g_{P,Q}$ inclusion maps, for all $P \subseteq Q$ in I_n. Let $u < v$ in $B_\emptyset$. Let $P \in I_n$ then:*

$$\Theta_{B_P}(u, v) = B_P \times B_P, \quad \text{the largest congruence of } B_P.$$

Let $\vec{\xi} = (\xi_P)_{P \in I_n} : \mathrm{Con}_c \circ \vec{A} \to \mathrm{Con}_c \circ \vec{B}$ be a natural equivalence. Let $x, y \in \underline{n}$ distinct. Let $b_x \in [u, v]_{B_{\{x\}}}$ and $b_y \in [u, v]_{B_{\{y\}}}$. Set $P = \{x, y\}$. Let $c \in \{0, 1\}$. Then the following assertions hold:

(1) *If $\Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$, then $\Theta_{B_P}(u, b_x) = \xi_P(\Theta_{A_P}(c, a_x))$.*
(2) *If $\Theta_{B_{\{z\}}}(u, b_z) = \xi_{\{z\}}(\Theta_{A_{\{z\}}}(c, a_z))$ for all $z \in \{x, y\}$, then $b_x \wedge b_y = u$.*
(3) *If $\Theta_{B_{\{x\}}}(b_x, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$, then $\Theta_{B_P}(b_x, v) = \xi_P(\Theta_{A_P}(c, a_x))$.*
(4) *If $\Theta_{B_{\{z\}}}(b_z, v) = \xi_{\{z\}}(\Theta_{A_{\{z\}}}(c, a_z))$ for all $z \in \{x, y\}$, then $b_x \vee b_y = v$.*
(5) *If $\Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$ and $\Theta_{B_{\{y\}}}(b_y, v) = \xi_{\{y\}}(\Theta_{A_{\{y\}}}(c, a_y))$, then $b_x \leq b_y$.*

Proof. As $f_{P,Q}$ preserves bounds, $\mathrm{Con}_c f_{P,Q}$ preserves bounds, thus $\mathrm{Con}_c g_{P,Q}$ preserves bounds, for all $P \subseteq Q$ in I_n. Let $u < v$ in $B_\emptyset$. As $B_\emptyset$ is simple, $\Theta_{B_\emptyset}(u, v)$ is the largest congruence of $B_\emptyset$. Moreover, $\mathrm{Con}_c g_{\emptyset,P}$ preserves bounds, for all $P \in I_n$. Hence:

$$\Theta_{B_P}(u, v) = B_P \times B_P, \quad \text{the largest congruence of } B_P.$$

(1) The following equalities hold:

$$\begin{aligned}
\Theta_{B_P}(u, b_x) &= (\mathrm{Con}_c\, g_{\{x\},P})(\Theta_{B_{\{x\}}}(u, b_x)) \\
&= (\mathrm{Con}_c\, g_{\{x\},P})(\xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))) \quad \text{by assumption} \\
&= \xi_P \circ (\mathrm{Con}_c\, f_{\{x\},P})(\Theta_{A_{\{x\}}}(c, a_x)) \\
&= \xi_P(\Theta_{A_P}(c, a_x)).
\end{aligned}$$

(2) The following containments hold:

$$\begin{aligned}
\Theta_{B_P}(u, b_x \wedge b_y) &\subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(u, b_y) \\
&= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) \quad \text{by (1)} \\
&= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y)) \\
&= \xi_P(\mathrm{id}_{A_P}) = \mathrm{id}_{B_P}.
\end{aligned}$$

so $u = b_x \wedge b_y$.

(3) Similar to (1).
(4) Similar to (2).
(5) The following containments hold:

$$\begin{aligned}
\Theta_{B_P}(b_y, b_x \vee b_y) &\subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(b_y, v) \\
&= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) \quad \text{by (1) and (3)} \\
&= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y)) \\
&= \xi_P(\mathrm{id}_{A_P}) = \mathrm{id}_{B_P}.
\end{aligned}$$

so $b_y = b_x \vee b_y$, thus $b_x \leq b_y$. □

The following lemma shows that if we have some "small" enough congruence-lifting of $\mathrm{Con}_c \circ \vec{A}$ in a variety, then M_n belongs to this variety.

Lemma 4.6. *Let $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\mathrm{Con}_c \circ \vec{A}$ by lattices. Assume that $B_{\{x\}}$ is a chain of length two for all $x \in \underline{n}$. Then M_n can be embedded into $B_{\underline{n}}$.*

Proof. Let $\vec{\xi} = (\xi_P)_{P \in I_n} : \mathrm{Con}_c \circ \vec{A} \to \mathrm{Con}_c \circ \vec{B}$ be a natural equivalence. As $f_{P,Q}$ is an embedding, $\mathrm{Con}_c f_{P,Q}$ separates 0, so $\mathrm{Con}_c\, g_{P,Q}$ separates 0, hence $g_{P,Q}$ is an embedding, thus we can assume that $g_{P,Q}$ is the inclusion map from B_P into B_Q, for all $P \subseteq Q$ in I_n.

Let $u < v$ in $B_\emptyset$. By Lemma 4.5, $\Theta_{B_{\{x\}}}(u, v)$ is the largest congruence of $B_{\{x\}}$. Moreover $B_{\{x\}}$ is the 3-element chain, so u is the smallest element of $B_{\{x\}}$ while v is its largest element. Denote by b_x the middle element of $B_{\{x\}}$.

The congruence $\xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x))$ is join-irreducible, thus it is either equal to $\Theta_{B_{\{x\}}}(u, b_x)$ or to $\Theta_{B_{\{x\}}}(b_x, v)$. Set:

$$X' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(u, b_x)\},$$
$$X'' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(b_x, v)\}.$$

As $\Theta_{A_{\{x\}}}(0, a_x)$ is the complement of $\Theta_{A_{\{x\}}}(a_x, 1)$ and $\Theta_{B_{\{x\}}}(u, b_x)$ is the complement of $\Theta_{B_{\{x\}}}(b_x, v)$, we also get that:

$$X' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)) = \Theta_{B_{\{x\}}}(b_x, v)\}$$
$$X'' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)) = \Theta_{B_{\{x\}}}(u, b_x)\}.$$

Moreover $\underline{n} = X' \cup X''$. As card $\underline{n} \geq 3$, either card $X' \geq 2$ or card $X'' \geq 2$.

Assume that card $X' \geq 2$. Let x, y in X' distinct. By Lemma 4.5(2), $b_x \wedge b_y = u$. By Lemma 4.5(4), $b_x \vee b_y = v$.

Now assume that $X'' \neq \emptyset$. Let $z \in X''$. As $\xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(u, b_x)$ and $\xi_{\{z\}}(\Theta_{A_{\{z\}}}(0, a_z)) = \Theta_{B_{\{z\}}}(b_z, v)$, it follows from Lemma 4.5(5) that $b_x \leq b_z$. Similarly, as $\xi_{\{z\}}(\Theta_{A_{\{z\}}}(a_z, 1)) = \Theta_{B_{\{z\}}}(u, b_z)$ and $\xi_{\{y\}}(\Theta_{A_{\{y\}}}(a_y, 1)) = \Theta_{B_{\{y\}}}(b_y, v)$, it follows from Lemma 4.5(5) that $b_z \leq b_y$. Thus $b_x \leq b_y$. So $u = b_x \wedge b_y = b_x > u$, a contradiction.

Thus $X'' = \emptyset$, so $X' = \underline{n}$, and so $\{u, b_0, b_1, \ldots, b_n, v\}$ is a sublattice of $B_{\underline{n}}$ isomorphic to M_n. The case card $X'' \geq 2$ is similar. □

We shall now use a tool introduced in [5] to prove that having a congruence-lifting of $\mathrm{Con}_c \circ \vec{A}$ is equivalent to having a congruence-lifting of some $(\vee, 0)$-semilattice of cardinality $\aleph_2$. This requires the following infinite combinatorial property, proved by Hajnal and Máté in [8], see also [3, Theorem 46.2]. This property is also used by Ploščica in [10].

Proposition 4.7. *Let $n \geq 0$ be an integer, let α be an ordinal, let $\kappa \geq \aleph_{\alpha+2}$, let $f : [\kappa]^2 \to [\kappa]^{<\aleph_\alpha}$. Then there exists $Y \in [\kappa]^n$ such that $a \notin f(\{b, c\})$ for all distinct $a, b, c \in Y$.*

Now recall the definition of supported poset and norm-covering introduced in [5, Section 4].

Definition 4.8. A finite subset V of a poset U is a *kernel*, if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$. We denote this element by $V \cdot u$.

We say that U is *supported*, if every finite subset of U is contained in a kernel of U.

We denote by $V \cdot \boldsymbol{u}$ the largest element of $V \cap \boldsymbol{u}$, for every kernel V of U and every ideal $\boldsymbol{u}$ of U. As an immediate application of the finiteness of kernels, we obtain that any intersection of a nonempty set of kernels of a poset U is a kernel of U.

Definition 4.9. A *norm-covering* of a poset I is a pair $(U, |\cdot|)$, where U is a supported poset and $|\cdot| : U \to I, u \mapsto |u|$ is an order-preserving map.

A *sharp ideal* of $(U, |\cdot|)$ is an ideal $\boldsymbol{u}$ of U such that $\{|v| \mid v \in \boldsymbol{u}\}$ has a largest element. For example, for every $u \in U$, the principal ideal $U \downarrow u$ is sharp; we shall often identify u and $U \downarrow u$. We denote this element by $|\boldsymbol{u}|$. We denote by $\mathrm{Id}_s(U, |\cdot|)$ the set of all sharp ideals of $(U, |\cdot|)$, partially ordered by inclusion.

A sharp ideal $\boldsymbol{u}$ of $(U, |\cdot|)$ is *extreme*, if there is no sharp ideal $\boldsymbol{v}$ with $\boldsymbol{v} > \boldsymbol{u}$ and $|\boldsymbol{v}| = |\boldsymbol{u}|$. We denote by $\mathrm{Id}_e(U, |\cdot|)$ the set of all extreme ideals of $(U, |\cdot|)$.

Let κ be a cardinal number. We say that $(U, |\cdot|)$ is *κ-compatible*, if for every order-preserving map $F : \mathrm{Id}_e(U, |\cdot|) \to \mathfrak{P}(U)$ such that card $F(\boldsymbol{u}) < \kappa$ for all $\boldsymbol{u} \in \mathrm{Id}_e(U, |\cdot|)^=$, there exists an order-preserving map $\sigma : I \to \mathrm{Id}_e(U, |\cdot|)$ such that:

(1) The equality $|\sigma(i)| = i$ holds for all $i \in I$.
(2) The containment $F(\sigma(i)) \cap \sigma(j) \subseteq \sigma(i)$ holds for all $i \leq j$ in I.

Lemma 4.10. *Let X be a set, let $(A_x)_{x\in X}$ be a family of sets, let:*

$$U = \bigsqcup_{P \in [X]^{<\omega}} \prod_{x \in P} A_x.$$

We view the elements of U as (partial) functions and "to be greater" means "to extend". Then U is a supported poset.

Proof. Let V be a finite subset of U. Let $Y_x = \{u_x \mid u \in V \text{ and } x \in \mathrm{dom}\, u\}$ for all $x \in X$. Let $D = \bigcup_{u\in V} \mathrm{dom}\, u$. Let:

$$W = \{u \in U \mid \mathrm{dom}\, u \subseteq D \text{ and } (\forall x \in \mathrm{dom}\, u)(u_x \in Y_x)\}$$

the set D, and the sets Y_x for $x \in X$ are all finite, so W is finite.

Let $u \in U$, let $P = \{x \in \mathrm{dom}\, u \mid x \in D \text{ and } u_x \in Y_x\}$. Then $u \restriction P \in W$. Moreover let $w \in W$ such that $w \leq u$. Let $x \in \mathrm{dom}\, w$, then $x \in D$, and $u_x = w_x \in Y_x$, thus $\mathrm{dom}\, w \subseteq P$, so $w \leq u \restriction P$. Therefore $u \restriction P$ is the largest element of $W \downarrow u$. □

Using Lemma 4.10 and Proposition 4.7 we can construct a $\aleph_\alpha$-compatible lower finite norm-covering of I_n, the poset constructed earlier.

Lemma 4.11. *Let α be an ordinal. Let $U = \bigsqcup_{P\in\mathfrak{P}(\underline{n})} \aleph_{\alpha+2}^P$, partially ordered by inclusion. Let*

$$|\cdot|\colon U \to I_n$$
$$u \mapsto |u| = \begin{cases} \operatorname{dom} u & \text{if } \operatorname{card}(\operatorname{dom} u) \leq 2 \\ \underline{n} & \text{otherwise.} \end{cases}$$

Then $(U, |\cdot|)$ is a $\aleph_\alpha$-compatible lower finite norm-covering of I_n. Moreover $\operatorname{card} U = \aleph_{\alpha+2}$.

Proof. By Lemma 4.10, the set U is supported. Moreover $|\cdot|$ preserves order, so $(U, |\cdot|)$ is a norm-covering of I_n. The poset U is lower finite.

Extreme ideals are of the form $\downarrow u$, where $u \in U$ and $\operatorname{dom} u \in I_n$, so we identify the corresponding extreme ideal with u. Thus $\operatorname{Id}_e(U, |\cdot|) = \{u \in U \mid \operatorname{dom} u \in I_n\}$.

Let $F\colon \operatorname{Id}_e(U, |\cdot|) \to \mathfrak{P}(U)$ be an order-preserving map such that $\operatorname{card} F(\boldsymbol{u}) < \aleph_\alpha$ for all $\boldsymbol{u} \in \operatorname{Id}_e(U, |\cdot|)^=$, let

$$G\colon [\aleph_{\alpha+2}]^2 \to [\aleph_{\alpha+2}]^{<\aleph_\alpha}$$
$$s \mapsto \bigcup \left\{ \operatorname{im} v \mid u \in \bigcup_{P\in I_n - \{\underline{n}\}} s^P \text{ and } v \in F(u) \right\}.$$

By Proposition 4.7, there exists $A \subset \aleph_{\alpha+2}$ such that $\operatorname{card} A = n$ and $a \notin G(\{b, c\})$ for all distinct $a, b, c \in A$. Let $u\colon \underline{n} \to A$ be a one-to-one map. Let $\phi\colon I_n \to \operatorname{Id}_e(U, |\cdot|)$, $P \mapsto u \restriction P$. Then $|\phi(P)| = P$. Let $P \subsetneq Q$ in I_n, let $v \in F(u \restriction P) \downarrow (u \restriction Q)$. Let $x \in \operatorname{dom} v - P$. As $P \in I_n$, and $P \neq \underline{n}$, $\operatorname{card} P \leq 2$. Let $P' = \{y, z\} \subseteq \underline{n}$, such that y, z are distinct, $P \subseteq P'$, and $x \notin P'$. Let $s = \{u_y, u_z\}$, then $u \restriction P' \in s^{P'}$, as $v \in F(u \restriction P) \subseteq F(u \restriction P')$, $v_x \in G(s)$. Moreover $u_x, u_y, u_z \in A$ are distinct, thus $u_x \notin G(\{u_y, u_z\}) = G(s)$, so $v_x \neq u_x$ in contradiction with $v \leq u$, so $\operatorname{dom} v \subseteq P$, and so $v \leq u \restriction P$. □

Using the results of [5] together with Lemma 4.11, we obtain the following result.

Lemma 4.12. *Let $\mathcal{V}$ be a variety of algebras with a countable similarity type, let $\mathcal{W}$ be a finitely generated congruence-distributive variety such that* $\operatorname{crit}(\mathcal{V}; \mathcal{W}) > \aleph_2$. *Let $\vec{D}\colon I_n \to \mathcal{S}$ be a diagram of finite $(\vee, 0)$-semilattices. If $\vec{D}$ is congruence-liftable in $\mathcal{V}$, then $\vec{D}$ is congruence-liftable in $\mathcal{W}$.*

Proof. In this proof we use, but do not give, many definitions of [5]. By Lemma 4.11 there exists $(U, |\cdot|)$ a $\aleph_0$-compatible lower finite norm-covering of I_n such that $\operatorname{card} U = \aleph_2$. Let J be a one-element ordered set. By [5, Lemma 3.9], $\mathcal{W}$ is $(\operatorname{Id}_e(U, |\cdot|)^=, J, \aleph_0)$-Löwenheim–Skolem.

Let $\vec{A} = (A_P, f_{P,Q})_{P\subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\vec{D}$ in $\mathcal{V}$. As $\operatorname{Con}_c A_P$ is finite, using [5, Lemma 3.6], taking sublattices we can assume that A_P is countable for all $P \in I_n$. By [5, Lemma 6.7], there exists an U-quasi-lifting $(\tau, \operatorname{Cond}(\vec{A}, U))$ of $\vec{D}$ in $\mathcal{V}$. Moreover:

$$\operatorname{card} \operatorname{Cond}(\vec{A}, U) \leq \sum_{V\in[U]^{<\omega}} \operatorname{card}\left(\prod_{u\in V} A_{|u|}\right) \leq \sum_{V\in[U]^{<\omega}} \aleph_0 \leq \aleph_2.$$

As $\operatorname{crit}(\mathcal{V}; \mathcal{W}) > \aleph_2$, there are $B \in \mathcal{W}$ and an isomorphism $\xi : \operatorname{Con}_c\operatorname{Cond}(\vec{A}, U) \to \operatorname{Con}_c B$. So $(\tau \circ \xi^{-1}, B)$ is an U-quasi-lifting of $\vec{D}$. Moreover $\mathcal{W}$ is $(\operatorname{Id}_e(U, |\cdot|)^=, J, \aleph_0)$-Löwenheim–Skolem, hence, by [5, Theorem 6.9], with $I = I_n$, there exists a congruence-lifting of $\vec{D}$ in $\mathcal{W}$. □

A similar proof, using Lemmas 3.6, 3.7, 6.7, and Theorem 6.9 in [5] together with Lemma 4.11, yields the following generalization of Lemma 4.12.

Lemma 4.13. *Let $\alpha \geq 1$ be an ordinal. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras, with similarity types of cardinality $< \aleph_\alpha$. Let $\vec{D} = (D_P, \varphi_{P,Q})_{P\subseteq Q \text{ in } I_n}$ be a direct system of $(\vee, 0)$-semilattices. Assume that the following conditions hold:*

(1) $\operatorname{crit}(\mathcal{V}; \mathcal{W}) > \aleph_{\alpha+2}$.
(2) $\operatorname{card}(D_P) < \aleph_\alpha$, *for all* $P \in I_n - \{\underline{n}\}$.
(3) $\operatorname{card}(D_{\underline{n}}) \leq \aleph_{\alpha+2}$.
(4) *$\vec{D}$ is congruence-liftable in $\mathcal{V}$.*

Then $\vec{D}$ is congruence-liftable in $\mathcal{W}$.

The following lemma implies, in particular, that a modular lattice of length three is a congruence-preserving extension of one of its subchains.

Lemma 4.14. *Let L be a lattice of length at most three, let u, v in L such that $\Theta_L(u, v) = L \times L$. If $\operatorname{Con}_c L \cong 2^2$, then there exists $x \in L$ with $u < x < v$ such that L is a congruence-preserving extension of the chain $C = \{u, x, v\}$.*

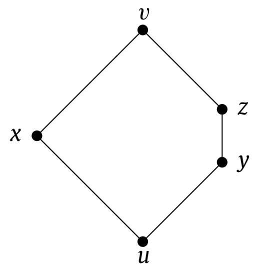

Fig. 2. The lattice N_5.

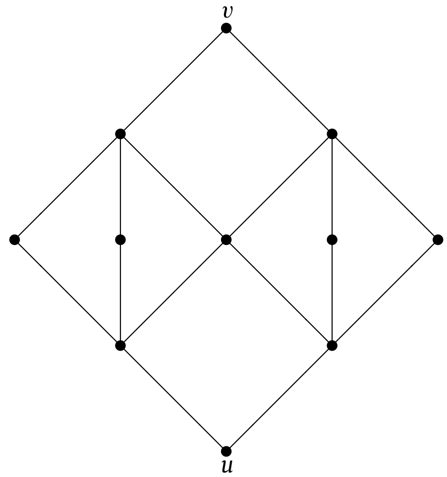

Fig. 3. Lemma 4.14 does not extend to lattices of greater length.

Proof. As $\mathrm{Con}_c L \cong 2^2$, $\mathrm{lh}([u, v]) \geq 2$. If $\mathrm{lh}([u, v]) = 2$, then let $C = \{u, x, v\}$, where x is any element such that $u < x < v$. Let $i\colon C \to L$ the inclusion map. The morphism $\mathrm{Con}_c\, i\colon \mathrm{Con}_c\, C \to \mathrm{Con}_c\, L$ is onto, moreover $\mathrm{Con}_c\, C \cong 2^2 \cong \mathrm{Con}_c\, L$, so $\mathrm{Con}_c\, i$ is an isomorphism.

Now assume that $[u, v]$ has length three. As $\mathrm{lh}(L) \leq 3$, $\mathrm{lh}(L) = 3$, u is the smallest element of L, and v is the largest element.

Assume that L has a sublattice isomorphic to N_5, as labeled in Fig. 2. Then $C = \{u, y, z, v\}$ is a maximal chain of L. Let $i\colon C \to L$ be the inclusion map. By Lemma 4.2, $\mathrm{Con}_c\, i$ is surjective. Thus, as $\mathrm{Con}\, L \cong 2^2$, and $\Theta_L(u, y)$, $\Theta_L(y, z)$, and $\Theta_L(z, v)$ are all the atoms of $\mathrm{Con}\, L$,

$$\Theta_L(y, z) \subseteq \Theta_L(u, y) \cap \Theta_L(y, z) \cap \Theta_L(z, v) = \mathrm{id}_L,$$

a contradiction. Thus L does not contain any lattice isomorphic to N_5, that is, L is modular.

As $\mathrm{Con}\, L \cong 2^2$ and $\mathrm{lh}(L) = 3$, L is not distributive. Hence there exists a sublattice of L isomorphic to M_3, let $a < x_1, x_2, x_3 < b$ be its elements. As L is modular, $[a, x_1]_L \cong [x_1, b]_L$, thus $\mathrm{lh}([a, b]_L)$ is even. But $2 \leq \mathrm{lh}([a, b]_L) \leq 3$, so $\mathrm{lh}([a, b]_L) = 2$, thus $a \prec x_1 \prec b$. This chain can be completed into a maximal chain $c \prec a \prec x_1 \prec b$ or $a \prec x_1 \prec b \prec c$. By symmetry, we may assume that $b < c$. Observe that $a = u$ and $c = v$. Set $C = \{u, b, v\}$ and $C_1 = \{u, x_1, b, v\}$. Let $i\colon C \to L$ and $i_1\colon C_1 \to L$ be the inclusion maps. As C_1 is a maximal chain, $\mathrm{Con}_c\, i_1$ is onto. As $\Theta_L(u, x_1) = \Theta_L(x_1, b) = \Theta_L(u, b)$, $\mathrm{Con}_c\, i_1$ and $\mathrm{Con}_c\, i$ have the same image, thus $\mathrm{Con}_c\, i$ is onto, so $\mathrm{Con}_c\, i$ is an isomorphism. □

The result of Lemma 4.14 does not extend to length four or more. The lattice of Fig. 3 is not a congruence-preserving extension of any chain with extremities u and v.

Lemma 4.15. *Let $n \geq 4$ be an integer, let $\mathcal{V}$ be a finitely generated variety of lattices such that $M_n \notin \mathcal{V}$. If $\mathrm{lh}(K) \leq 3$ for each simple lattice K of $\mathcal{V}$, then $\mathrm{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) \leq \aleph_2$.*

Proof. We consider the diagram $\vec{A}$ introduced just before Lemma 4.5. Assume that $\mathrm{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) > \aleph_2$. As $M_n \in \mathcal{M}_n^{0,1}$, $\vec{A}$ is a diagram of $\mathcal{M}_n^{0,1}$ indexed by I_n. By Lemma 4.12, the diagram $\mathrm{Con}_c \circ \vec{A}$ has a congruence-lifting $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ in $\mathcal{V}$. As $\mathrm{Con}\, B_{\underline{n}} \cong 2$, the lattice $B_{\underline{n}}$ is simple, thus, by assumption on $\mathcal{V}$, $\mathrm{lh}(B_{\underline{n}}) \leq 3$, and so $\mathrm{lh}(B_{\{x\}}) \leq 3$, for all $x \in \underline{n}$. The lattice $B_\emptyset$ is simple, so, taking a sublattice, we can assume that $B_\emptyset = \{u, v\}$, with $u < v$. By Lemma 4.14, we can assume that $B_{\{x\}}$ is a chain of length two, for each $x \in \underline{n}$. So by Lemma 4.6, M_n is a sublattice of $B_{\underline{n}}$, and so $M_n \in \mathcal{V}$, a contradiction. □

Theorem 4.16. *Let $\mathcal{V}$ be a finitely generated variety of modular lattices and $\mathcal{W}$ be finitely generated variety of lattices. Let $n \geq 3$ be an integer such that $M_n \in \mathcal{V} - \mathcal{W}$. If $\operatorname{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$. Moreover if either $\operatorname{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}$ and $M_3 \in \mathcal{W}$ or $\operatorname{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$ and $\operatorname{Sub} F^3 \in \mathcal{W}$ for some field F, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \aleph_2$.*

Proof. By Lemma 4.15, $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$.

Assume that $\operatorname{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}$ and $M_3 \in \mathcal{W}$. As $\operatorname{Sub} \mathbb{F}_2^2 \cong M_3 \in \mathcal{W}$, it follows from Theorem 3.11 that $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \geq \aleph_2$.

Assume that $\operatorname{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$ and $\operatorname{Sub} F^3 \in \mathcal{W}$ for some field F, it follows from Theorem 3.11 that $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \geq \aleph_2$. □

Similarly we obtain the following critical points.

Corollary 4.17. *The following equalities hold*

$$\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}^{0,1}) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}^{0}) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_{m}^{0}) = \aleph_2, \quad \textit{for all } n, m \textit{ with } 3 \leq m < n \leq \omega.$$

Proof. Let $n' \leq n$ be an integer such that $m < n' < \omega$. As $M_{n'} \notin \mathcal{M}_{m,m}$, it follows from Lemma 4.15 that $\operatorname{crit}(\mathcal{M}_{n'}^{0,1}; \mathcal{M}_{m,m}) \leq \aleph_2$, thus:

$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}) \leq \aleph_2. \tag{4.1}$$

Moreover $M_3 \in \mathcal{M}_{m,m}$, simple lattices of $\mathcal{M}_{m,m}$ are of length at most 3, and finitely generated lattices of $\mathcal{M}_n$ have finite length (and are even finite). Thus, by Theorem 3.11

$$\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_{m,m}^{0}) \geq \aleph_2. \tag{4.2}$$

Similarly:

$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{M}_{m,m}^{0,1}) \geq \aleph_2. \tag{4.3}$$

All the desired equalities are immediate consequences of (4.1)–(4.3). □

As an immediate consequence we obtain:

Corollary 4.18. $\operatorname{crit}(\mathcal{M}_{4,3}; \mathcal{M}_{3,3}) \leq \aleph_2$.

This question was suggested by Ploščica.

Lemma 4.19. *Let F be field. Then $M_n \in \mathbf{Var}\,(\operatorname{Sub} F^3)$ if and only if $n \leq 1 + \operatorname{card} F$.*

Proof. If F is infinite then the result is obvious. So we can assume that F is finite.

If $n \leq 1 + \operatorname{card} F$, then M_n is a sublattice of $M_{1+\operatorname{card} F} \cong \operatorname{Sub} F^2 \in \mathbf{Var}\,(\operatorname{Sub} F^3)$, thus $M_n \in \mathbf{Var}\,(\operatorname{Sub} F^3)$.

Now assume that $M_n \in \mathbf{Var}\,(\operatorname{Sub} F^3)$. By Jónsson's Lemma, M_n is a homomorphic image of a sublattice of $\operatorname{Sub} F^3$. As M_n satisfies Whitman's condition, it follows from the Davey–Sands Theorem [2, Theorem 1] that M_n is projective in the class of all finite lattices. Therefore, as $\operatorname{Sub} F^3$ is finite, M_n is a sublattice of $\operatorname{Sub} F^3$. Thus there exist distinct subspaces $A, B, V_1, V_2, \dots, V_n$ of F^3 such that $V_i \cap V_j = A$ and $V_i + V_j = B$, for all $1 \leq i < j \leq n$. Let i, j, k distinct. Then:

$$\dim V_i + \dim V_j = \dim B + \dim A = \dim V_i + \dim V_k.$$

Thus $\dim V_j = \dim V_k$. But $\dim A < \dim V_1 < \dim B \leq \dim F^3 = 3$. If $\dim A = 1$, then M_n is isomorphic to $\{A/A, V_1/A, \dots, V_n/A, B/A\}$ which is a sublattice of $\operatorname{Sub}(B/A)$, with $\dim B/A = 2$. If $\dim A = 0$, then:

$$\dim B = \dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2 = 2 \cdot \dim V_1.$$

Thus $\dim B$ is even, moreover $\dim B \leq 3$, hence $\dim B = 2$.

In both cases M_n is a sublattice of $\operatorname{Sub} E$ for some F-vector space E of dimension two. But $\operatorname{Sub} E \cong M_{1+\operatorname{card} F}$, thus $n \leq 1 + \operatorname{card} F$. □

Corollary 4.20. *Let F be a finite field and let $n > 1 + \operatorname{card} F$. Then:*

$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}\,(\operatorname{Sub} F^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}_0(\operatorname{Sub} F^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}\,(\operatorname{Sub} F^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^3)) = \aleph_2.$$

Proof. By Lemma 4.19, $M_n \notin \mathbf{Var}\,(\operatorname{Sub} F^3)$, moreover simple lattices of $\mathbf{Var}\,(\operatorname{Sub} F^3)$ are of length at most three. Thus, by Lemma 4.15:

$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}\,(\operatorname{Sub} F^3)) \leq \aleph_2. \tag{4.4}$$

As each simple lattice of $\mathcal{M}_n$ is of length at most two, it follows from Theorem 3.11 that

$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}_0(\operatorname{Sub} F^n)) \geq \aleph_2, \quad \text{and} \quad \operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)) \geq \aleph_2. \tag{4.5}$$

All the other desired equalities are consequences of (4.4), (4.5). □

Corollary 4.21. *Let F and K be finite fields. If* $\operatorname{card} F > \operatorname{card} K$ *then:*

$$\operatorname{crit}(\mathbf{Var}\,(\operatorname{Sub} F^3); \mathbf{Var}\,(\operatorname{Sub} K^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathbf{Var}\,(\operatorname{Sub} F^3); \mathbf{Var}_0(\operatorname{Sub} K^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}\,(\operatorname{Sub} K^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}_{0,1}(\operatorname{Sub} K^3)) = \aleph_2.$$

Proof. By Lemma 4.19, $M_{1+\operatorname{card} F} \notin \mathbf{Var}\,(\operatorname{Sub} K^3)$, moreover simple lattices of $\mathbf{Var}\,(\operatorname{Sub} K^3)$ are of length at most three. Thus, by Lemma 4.15:

$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}\,(\operatorname{Sub} K^3)) \leq \aleph_2. \tag{4.6}$$

As each simple lattice of $\mathbf{Var}\,(\operatorname{Sub} F^3)$ is of length at most three, it follows from Theorem 3.11 that:

$$\operatorname{crit}(\mathbf{Var}\,(\operatorname{Sub} F^3); \mathbf{Var}_0(\operatorname{Sub} K^n)) \geq \aleph_2, \tag{4.7}$$
$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}_{0,1}(\operatorname{Sub} K^n)) \geq \aleph_2. \tag{4.8}$$

All the other desired equalities are consequences of (4.6)–(4.8). □

Lemma 4.22. *Let $\mathcal{V}$ be a finitely generated variety of lattices (resp., a finitely generated variety of lattices with* 0*), let $m \geq 2$ an integer. Assume that for each simple lattice K of $\mathcal{V}$, there do not exist $b_0, b_1, \dots, b_{m-1} > u$ in K such that $b_i \wedge b_j = u$ (resp., $b_0, b_1, \dots, b_{m-1} > 0$ such that $b_i \wedge b_j = 0$), for all $0 \leq i < j \leq m-1$. Then $\operatorname{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_2$.*

Proof. Set $n = 2m - 1 \geq 3$. Let $\vec{A} = (A_P, f_{P,Q})_{P\subseteq Q \text{ in } I_n}$ be the direct system of $\mathcal{M}_n^{0,1}$ introduced just before Lemma 4.5. Assume that $\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) > \aleph_2$. By Lemma 4.12, there exists a congruence-lifting $\vec{B} = (B_P, g_{P,Q})_{P\subseteq Q \text{ in } I_n}$ of $\operatorname{Con}_c \circ \vec{A}$ in $\mathcal{V}$. Let $\vec{\xi} = (\xi_P)_{P\in I_n} \colon \operatorname{Con}_c \circ \vec{A} \to \operatorname{Con}_c \circ \vec{B}$ be a natural equivalence. Taking a sublattice of $B_\emptyset$, we can assume that $B_\emptyset$ is a chain $u < v$. Moreover, as the map $f_{P,Q}$ is an inclusion map, we can assume that $g_{P,Q}$ is an inclusion map, for all $P \subseteq Q$ in I_n.

Let $x \in \underline{n}$. By Lemma 4.5, $\Theta_{B_{\{x\}}}(u, v)$ is the largest congruence of $B_{\{x\}}$. Thus:

$$\Theta_{B_{\{x\}}}(u, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) \vee \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)).$$

Therefore there exist $t_0^x = u < t_1^x < \cdots < t_{r+1}^x = v$ in $B_{\{x\}}$ such that, for all $0 \leq i \leq r$:

$$\text{either } (t_i^x, t_{i+1}^x) \in \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) \text{ or } (t_i^x, t_{i+1}^x) \in \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)).$$

Set $b_x = t_1^x$. Put:

$$X' = \{x \in \underline{n} \mid \Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x))\}$$
$$X'' = \{x \in \underline{n} \mid \Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1))\}.$$

By symmetry we can assume that $\operatorname{card} X' \geq \operatorname{card} X''$ (we can replace the diagram $\vec{A}$ by its dual if required). As $\underline{n} = X' \cup X''$ and $\operatorname{card} \underline{n} = n = 2m - 1$, $\operatorname{card} X' \geq m$. Let $x, y \in X'$ distinct, it follows from Lemma 4.5(2) that $b_x \wedge b_y = u$. So we obtain a family of elements $(b_x)_{x\in X'}$ greater than u such that $b_x \wedge b_y = u$ (resp., $b_x \wedge b_y = u = 0$) for all $x \neq y$ in X', a contradiction. □

With a similar proof using Lemma 4.13 instead of Lemma 4.12 we obtain the following lemma.

Lemma 4.23. *Let $\mathcal{V}$ be a variety of lattices (resp., a variety of lattices with* 0*), let $m \geq 2$ an integer. Assume that for each simple lattice K of $\mathcal{V}$, there do not exist $b_0, b_1, \dots, b_{m-1} > u$ in K such that $b_i \wedge b_j = u$ (resp., $b_0, b_1, \dots, b_{m-1} > 0$ such that $b_i \wedge b_j = 0$), for all $0 \leq i < j \leq m-1$. Then $\operatorname{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_3$.*

Theorem 4.24. *Let $\mathcal{V}$ be either a finitely generated variety of lattices or a finitely generated variety of lattices with* 0*. If $M_3 \in \mathcal{V}$ then:*

$$\operatorname{crit}(\mathcal{M}_\omega; \mathcal{V}) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_\omega^0; \mathcal{V}) = \aleph_2.$$

Let $\mathcal{V}$ be a finitely generated variety of bounded lattices. If $M_3 \in \mathcal{V}$ then:

$$\operatorname{crit}(\mathcal{M}_\omega^{0,1}; \mathcal{V}) = \aleph_2.$$

Proof. Let $\mathcal{V}$ be a finitely generated variety of lattices, let m be the maximal cardinality of a simple lattice of $\mathcal{V}$. Thus the assumptions of Lemma 4.22 are satisfied, so *a fortiori* $\operatorname{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_2$, and so $\operatorname{crit}(\mathcal{M}_{\omega}^{0,1}; \mathcal{V}) \leq \aleph_2$.

Denote by $\mathbb{F}_2$ the two-element field. Let $\mathcal{V}$ be a variety of lattices with 0 (resp., with 0 and 1), such that $M_3 \in \mathcal{V}$. The variety $\mathcal{M}_\omega$ is locally finite, thus all finitely generated lattices of $\mathcal{M}_\omega$ are of finite length. Moreover all simple lattices of $\mathcal{M}_\omega$ have length at most two. Thus, by Theorem 3.11:

$$\operatorname{crit}(\mathcal{M}_\omega; \mathbf{Var}_0(\operatorname{Sub}\mathbb{F}_2^2)) \geq \aleph_2 \text{ (resp., } \operatorname{crit}(\mathcal{M}_\omega^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub}\mathbb{F}_2^2)) \geq \aleph_2).$$

Moreover $\operatorname{Sub}\mathbb{F}_2^2 \cong M_3$, so $\operatorname{crit}(\mathcal{M}_\omega; \mathcal{V}) \geq \aleph_2$. □

Acknowledgements

I thank Friedrich Wehrung whose help, advise, and careful reading of the paper, led to several improvements both in form and substance. I also thank Miroslav Ploščica for the stimulating discussions that led to the diagram $\vec{A}$ introduced in Section 4. Partially supported by the institutional grant MSM 0021620839.

References

[1] S.Z. Ditor, Cardinality questions concerning semilattices of finite breadth, Discrete Math. 48 (1) (1984) 47–59.
[2] B.A. Davey, B. Sands, An application of Whitman's condition to lattices with no infinite chains, Algebra Universalis 7 (2) (1977) 171–178.
[3] P. Erdös, A. Hajnal, A. Máté, R. Rado, Combinatorial Set Theory: Partition Relations for Cardinals, in: Studies in Logic and the Foundations of Mathematics, vol. 106, North-Holland Publishing Co., Amsterdam, 1984, 347 p.
[4] K.D. Fryer, I. Halperin, The von Neumann coordinatization theorem for complemented modular lattices, Acta Sci. Math. (Szeged) 17 (1956) 203–249.
[5] P. Gillibert, Critical points of pairs of varieties of algebras, Internat. J. Algebra Comput. 19 (1) (2009) 1–40.
[6] G. Grätzer, General Lattice Theory, Second edition, Birkhäuser Verlag, Basel, 1998, xx+663 p.
[7] K.R. Goodearl, Von Neumann Regular Rings, Second ed., Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991, xviii+412 p.
[8] A. Hajnal, A. Máté, Set mappings, partitions, and chromatic numbers, in: Logic Colloquium'73, Bristol, 1973, in: Studies in Logic and the Foundations of Mathematics, vol. 80, North-Holland, Amsterdam, 1975, pp. 347–379.
[9] F. Micol, On representability of ∗-regular rings and modular ortholattices, Ph.D. thesis, TU Darmstadt, January 2003. Available online at: http://elib.tu-darmstadt.de/diss/000303/diss.pdf.
[10] M. Ploščica, Separation properties in congruence lattices of lattices, Colloq. Math. 83 (2000) 71–84.
[11] M. Ploščica, Dual spaces of some congruence lattices, Topology Appl. 131 (2003) 1–14.
[12] M. Ploščica, J. Tůma, F. Wehrung, Congruence lattices of free lattices in nondistributive varieties, Colloq. Math. 76 (2) (1998) 269–278.
[13] F. Wehrung, The dimension monoid of a lattice, Algebra Universalis 40 (3) (1998) 247–411.
[14] F. Wehrung, A uniform refinement property for congruence lattices, Proc. Amer. Math. Soc. 127 (2) (1999) 363–370.
[15] F. Wehrung, A solution to Dilworth's congruence lattice problem, Adv. Math. 216 (2) (2007) 610–625.