Dynamics of the Atlantic meridional overturning circulation and Southern Ocean in an ocean model of intermediate complexity

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A steady-state, variable-density, 2-layer, ocean model (VLOM) is used to investigate basic dynamics of the Atlantic meridional overturning circulation and Southern Ocean. The domain consists of idealized (rectangular) representations of the Atlantic, Southern, and Pacific Oceans. The model equations represent the depth-averaged, layer-1 response (except for one solution in which they represent the depth-integrated flow over both layers). To allow for overturning, water can cross the bottom of layer 1 at the velocity \( w_d = w_1 + w_m + w_c \), the three parts representing: interior diffusion \( w_d \) that increases the layer-1 thickness \( h \) throughout the basin, mixed-layer entrainment \( w_m \) that ensures \( h \) is never less than a minimum value \( h_m \), and diapycnal (cooling) processes external to the basin \( w_c \) that adjust \( h \) to \( h_s \). For most solutions, horizontal mixing has the form of Rayleigh damping with coefficient \( \nu \), which we interpret to result from baroclinic instability through the closure, \( \mathbf{V} = -\left(\nu f^2\right)\mathbf{P} \), where \( \mathbf{V} = \mathbf{V}\left(\frac{1}{g}h^2\right) \) is the depth-integrated pressure gradient, \( g \) is the reduced-gravity coefficient, and \( \nu \) is a mixing coefficient; with this interpretation, the layer-1 flow corresponds to the sum of the Eulerian-mean and eddy-mean (\( \mathbf{V}^e \)) transport/widths, that is, the “residual” circulation. Finally, layer-1 temperature cools polewards in response to a surface heat flux \( Q_s \), and the cooling can be strong enough in the Southern Ocean for \( g = 0 \) south of a latitude \( y_g \), in which case layer 1 vanishes and the model reduces to a single layer 2.

Solutions are obtained both numerically and analytically. The analytic approach splits fields into interior and boundary-layer parts, from which a coupled set of integral constraints can be derived. The set allows properties of the circulation (upwelling-driven transport out of the Southern Ocean \( M \), downwelling transport in the North Atlantic, transport of the Antarctic Circumpolar Current) and stratification (Atlantic thermocline depth, and the latitudes, \( y \) and \( y_g \), where \( h \) thins to \( h_m \) or layer 1 vanishes in the Southern Ocean) to be evaluated in terms of model forcings (Southern-Ocean wind strength \( \tau_s \), \( Q_s \), entrainment due to \( w_d \)), processes (\( \mathbf{V}^e \) in the Southern Ocean, northern sinking, upwelling within the Atlantic Subpolar Gyre), and to the presence of the Pacific Ocean.

A hierarchy of solutions is reported in which forcings and processes are individually introduced. The complete solution set includes a wide variety of solution types: with \( M > 0 \) and \( M < 0 \); with and without wind forcing; with, without, and for two parameterizations of northern-boundary sinking that represent cooling external to and within the North Atlantic; for a wide range of \( \nu \) and \( \tau_s \); and for different closures. Novel aspects of the model and solutions include the following: use of VLOM, which allows \( Q_s \) forcing to be introduced realistically; the aforementioned closure, which allows \( \mathbf{V}^e \) to be determined when layer 1 represents both the surface mixed layer (\( h = h_m \)) and the depth of subsurface isopycnals (\( h > h_m \)); latitude \( y \), where layer 1 outcrops in the Southern Ocean, being internally determined rather than externally specified; and a boundary layer, based on Gill’s (1968) solution, that smoothly connects the Southern- and Atlantic-Ocean responses across the latitude of the southern tip of South America. Finally, some solutions in the set are comparable to solutions to idealized, ocean general circulation models (OGCMs); in these cases, our solutions provide insight into the underlying dynamics of the OGCM solutions, for example, pointing toward processes that may be involved in eddy saturation and compensation.

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1. Introduction

The Atlantic meridional overturning circulation (AMOC) is a key element of the general ocean circulation, among other things providing a source of deep water to all the ocean basins. Despite considerable progress, the processes that determine the AMOC strength and structure are still not well understood. One reason for this lack is that its spatial structure is complex, with several contributing upwelling and downwelling branches. Another is that the AMOC extends into the Southern Ocean, the dynamics of which are still under debate.

1.1. Background

1.1.1. AMOC spatial structure

The AMOC upwelling branch was first hypothesized to result from vertical diffusion $\kappa$ (Stommel and Arons, 1960a,b), and hence to occur relatively uniformly throughout the interior of the Atlantic Ocean (depending on the horizontal structure of $\kappa$). Subsequently, it was suggested to be driven by Southern-Ocean upwelling, with the Ekman transport across the southern boundary of the Atlantic determining the AMOC transport (Wyrtki, 1961; Toggweiler and Samuels, 1995). Although observational data are sufficient to determine overturning circulations reasonably well (Talley et al., 2015), they are not adequate to determine the split between the two branches accurately. There is now general agreement that a large part of the AMOC transport arises from Southern-Ocean upwelling (e.g., Marshall and Speer, 2012).

Another possible AMOC upwelling branch occurs within the North Atlantic Subpolar Gyre. Fig. 1a (right panel) shows the density structure of the North Atlantic, plotting a density section across the basin at 58.5°N. The section is located within the subpolar gyre, and isopycnals rise markedly to the west in response to Ekman suction. In the western Atlantic, isopycnals with densities of $\sim 27.7 \, \sigma_0$ rise close to the surface, suggesting that deep water can entrain into the upper ocean there. In contrast, in a density section within the North Pacific Subpolar Gyre at 50.5°N (left panel), isopycnal slopes are considerably smaller and deep-water isopycnals do not rise near the surface.

Tsujino and Suginohara (1999) investigated the impact of subpolar upwelling on the basin-scale, meridional overturning circulation (MOC) using an idealized OGCM. In one solution, they cooled the ocean near the southern boundary, generating a basin-wide MOC with its sinking branch in the southeastern corner of the basin and its upwelling branch determined by $\kappa$. In another, they added a patch of westerly winds in the northern hemisphere (similar to $\tau^n$ defined below in Eq. 9a). In response, there was a transport of deep water into the near-surface ocean, $W^u$, due to upwelling in the resulting subpolar gyre. This additional, upper-layer water was then carried southward in the western-boundary current, eventually downwelling in the southeastern corner to strengthen the MOC. Interestingly, the thermocline thickened throughout the basin to allow for the strengthened MOC (see their Figs. 8–11), pointing toward a linkage between $W^u$, MOC strength $\mathcal{M}$, and thermocline depth $h_\mathcal{T}$. Recently, Schloesser et al. (2014) discussed the influence of subpolar-gyre upwelling on the MOC in detail, among other things describing the dynamical connection among $W^u$, $\mathcal{M}$, and $h_\mathcal{T}$.

The impact of the AMOC on the other oceans is demonstrated by the presence of North Atlantic Deep Water (NADW) in the Pacific and Indian Oceans (e.g., Knauss, 1962), a property leading to the concept of the “global conveyor belt” (Broecker and Peng, 1982; Broecker, 1987). Conversely, modeling studies have sought to determine the impact of the other oceans on the AMOC. McDermott (1996) and Furue and Endoh (2005), for example, obtained OGCM solutions in idealized domains consisting of the Southern Ocean and either a single Atlantic Ocean or both the Atlantic and Pacific Oceans. As might be expected, the AMOC strength increased considerably in the latter geometry, due to the increased Ekman drift out of the wider Southern Ocean and to additional diffusive upwelling by $\kappa$ in the Pacific. Presumably, including the Pacific impacts other AMOC properties as well, such as the Atlantic thermocline depth.

1.1.2. Southern-Ocean dynamics

The dynamics of the Southern Ocean are complicated because part of its circulation is meridionally unbounded, and in this cyclic region there can be no zonally-averaged, geostrophic, meridional currents. As a result, ageostrophic processes are required to generate the southward flow that balances northward Ekman drift. The prevailing view is that diabatic processes (small-scale mixing and surface fluxes) are confined to the surface mixed layer whereas, as discussed next, nonlinear, adiabatic processes (baroclinic and barotropic instabilities) dominate at greater depths.

An important clue into the nature of Southern-Ocean dynamics is the density structure (stratification) across the basin. Fig. 1b shows two density sections across the Southern Ocean along 150°W (left panel) and 110°E (right panel). North of about 55°S, the latitude of the tip of South America, isopycnals are relatively flat in the deep ocean. In marked contrast, south of 55°S they slope upward to the south, where they either rise to the surface ($\leq 27.2 \, \sigma_0$) or bend to the south to form a near-surface layer of less-dense water ($\geq 27.2 \, \sigma_0$). The isopycnal slopes in the two panels are close to their maxima and minimum values: At other longitudes, the
stratification has a similar structure but with isopycnal slopes that lie between these two extremes. It is noteworthy that no isopycnals rise directly (smoothly) to the surface, but rather first bend to the south near 100–200 m. This property suggests that they are not tightly linked thermodynamically to surface buoyancy forcing.

In solutions to coarse-resolution OGCMs, isopycnals along the northern edge of the ACC tend to tilt much too strongly upward to the south (are oriented nearly vertical), a consequence of Southern-Ocean upwelling and northward Ekman drift (e.g., Vallis, 2000). In contrast, in solutions to eddy-resolving OGCMs, baroclinic eddies tend to flatten isopycnal slopes across the ACC and, in so doing, to stratify the Southern Ocean. The Gent and McWilliams (1990, GM) parameterization is a popular way of representing this flattening process in coarse-resolution OGCMs. Let \( h = h + h' \) be the depth of a particular isopycnal, where \( h \) and \( h' \) are its time-mean and time-varying parts. Then, the GM parameterization adopts the closure \( \bar{h} \bar{v} = -\kappa \bar{v} h \), which allows the introduction of an eddy-driven, “bolus” velocity, \( \bar{v} = \frac{h'}{h} / \bar{h} \). In the Southern Ocean, \( \bar{v} \) is directed southward in the upper ocean and northward at depth, thereby flattening isopycnal slopes. Velocity \( \bar{v} \) is analogous to the Stokes drift in surface waves (e.g., Lee et al., 1997) in that the mean flow of water particles, \( \bar{v} \), is the sum of the Eulerian-mean velocity \( \bar{v} \) and \( \bar{v}' \) (\( \bar{v} = \bar{v} + \bar{v}' \), the “residual circulation”.

A number of modeling studies investigate the importance of the meridional component of \( \bar{v} \) in cyclic ocean basins like the Southern Ocean. For example, in the solutions of Lee et al. (1997) and Karsten et al. (2002), obtained in a cyclic channel and circular basin, respectively, \( \bar{v} \) is essentially zero so that the entire overturning circulation is associated with \( \bar{v}' \). In solutions to models with a realistic domain, \( \bar{v}' \) is associated with an overturning cell that counteracts the Ekman-driven one, thereby reducing the transport of upper-ocean water out of the Southern Ocean into the Atlantic (e.g., Hirst and McDougall, 1998; Treguier et al., 2007). Karsten and Marshall (2002) used observations to estimate \( \bar{v}' \) in the near-surface ocean, concluding that it can dominate (reverse) the poleward flow due to Ekman drift.

The preceding discussion suggests that the Southern-Ocean stratification is not directly linked to the local buoyancy forcing \( Q \); rather, it is determined through the southward and upward extension of isopycnals from the Atlantic stratification. Other studies have argued conversely that the Southern-Ocean stratification is determined by \( Q \). In this view, \( Q \) determines the surface density field in the Southern Ocean, the surface isopycnals extend northward and downward across the Southern Ocean, and then northward into the Atlantic where they determine the mid-depth stratification there (type-1 isopycnals). In their solutions, Wolfe and Cessi (2010) note that type-1 isopycnals exist in their solutions only when they are deep enough in the North Atlantic not to be affected by diapycnal processes there, and not for shallower isopycnals that are so affected (type-2 isopycnals). Since observed subsurface isopycnals associated with the AMOC are not directly linked to surface density values in the Southern Ocean (Fig. 1b), it is not clear that type-1 isopycnals occur in the real ocean.

1.1.3. Simplified models

Solutions to OGCMs can simulate the AMOC flow field quite realistically. Given their dynamical complexity, however, a number of studies have used simpler systems to isolate fundamental processes and identify the interactions among them. Here, we summarize a few such studies that are most relevant to our own.

Stommel and Arons (1960a,b) obtained analytic solutions to a reduced-gravity model of the deep ocean layer. They forced the layer with a point source of mass to represent deep convection, and assumed that it was balanced by interior diffusion that transfers mass out of the layer into the upper ocean at a constant rate \( \omega_0 \). Although they obtained a solution in the Southern Ocean, the Drake Passage of their deep layer was closed so that the basin had no cyclic part.

Gill (1968) first explored the nature of the dynamical connection between the Atlantic and cyclic Southern Ocean using a 1-layer model. The model was wind-forced, and included viscosity in the form of Rayleigh damping. He obtained solutions in the interiors of both basins, joining them with a zonal boundary layer centered on the latitude of the tip of South America, \( y_0 \). The structure of the boundary layer is complex, a consequence of its southern half lying within the Southern Ocean and hence requiring cyclicity. Because he used a 1-layer model, his solutions were not able to develop an overturning cell. Nevertheless, his boundary layer can be extended to apply to layer 1 of a 2-layer model that allows for an MOC, an application that we utilize here (Appendix C.4).

Gnanadesikan (1999) developed a conceptual model that parameterizes and relates various mass transports into and out of the AMOC upper branch: the net transport out the Southern Ocean \( M \), that is, the sum of northward Ekman transport and the southward eddy transport due to \( \bar{v}' \): the upwelling transport due to interior diffusion \( W_E' \) driven by \( \kappa \); and the downwelling transport in the North Atlantic \( W_{AD} \). Most of the parameterizations involve \( h_a \), a measure of the Atlantic thermocline depth. Mass balance requires that the sum of the inflows and outflows vanishes, resulting in a single cubic equation for \( h_a \). The other AMOC properties are then known from \( h_a \) and model parameters (Southern-Ocean wind strength \( s \), \( \kappa \), and \( k_0 \)). Klinger et al. (2003) obtained OGCM solutions that support and extend Gnanadesikan’s (1999) results.

Recently, Samelson (2009) and Radko and Kamenkovich (2011) obtained solutions to 1-layer and 2-layer models, respectively, in a domain with an Atlantic and cyclic Southern Ocean. Both models allow for analytic solutions that relate key AMOC variables to forcings and processes, as in Gnanadesikan (1999).

In the Samelson (2009) model, the layer-1 structure and dynamics are highly constrained in the Southern Ocean: Layer 1 consists of a region of constant thickness \( h_{a} \) with vertically uniform isotherms, the zonal flow is in thermal-wind balance, and the ageostrophic, meridional flow is assumed to have the transport \( M = C r' (y_{0}) / f(y_{0}) + V' \), where \( C \) is the zonal width of the domain, the Ekman transport is evaluated at the latitude of the tip of South America \( y_{0} \), and \( V' = -z h_{a} \) is a southward eddy-driven transport. In the Radko and Kamenkovich (2011) model, the Southern Ocean is less constrained. Since its dynamics include GM mixing, \( h_{a} \) and \( h_{b} \) vary across the basin, thinning to the south. Nevertheless, the stratification is still not free to develop on its own, as the outcrop lines of layers 1 and 2 are fixed to externally specified latitudes \( y' \) and \( y'' \), respectively, and \( h(y') = h(y'') = 0 \). Transport \( M \) is specified either as a free parameter, or determined by a northern-boundary constraint, \( M = 0 h_{a} \). In both models, the Atlantic and Southern-Ocean circulations are joined by matching meridional transports and layer thicknesses along \( y = y_{0} \). Since layer thicknesses vary zonally in the Atlantic due to \( r' \), the matching of layer thicknesses is made only in an integral sense: \( h_{a} = (h(y'') r') / (y_{0}) \) in Samelson (2009), where the brackets indicate a zonal average; and \( h_{a} = h_{a} \) in Radko and Kamenkovich (2011), where \( h_{a} \) is the eastern-boundary thickness of layer 1.
The domains in all the noted models consist of a rectangular Atlantic attached to a cyclic Southern Ocean, and solutions are forced by $x$-independent, wind and $Q$ (or relaxation temperature $T$) fields. Otherwise, the models have significant dynamic differences. For example, they differ in their horizontal closure, either obtaining solutions in a coarse-resolution model with GM mixing (NV11 and NV12) or in models with eddy-permitting and eddy-resolving resolutions. Further, they differ in the nature of the processes imposed at the northern boundary: To generate the downwelling branch of an AMOC-like overturning cell ($\mathcal{M} > 0$), WC10 and MJM13 cool the northern ocean, whereas MH13 prescribe a northern-boundary sponge layer. In contrast, the HV05 and NV11 models lack any northern downwelling process, and consequently their solutions develop a “reverse” MOC ($\mathcal{M} < 0$) like that for Antarctic bottom water (AABW), with upwelling by interior diffusion in the Atlantic and sinking in the Southern Ocean.

1.2. Present research

In this study, we continue the effort to understand basic dynamics of the AMOC and its interaction with the Southern Ocean, using a model of intermediate complexity. The model is a 2-layer system in which water can transfer between layers, thereby allowing solutions to have upwelling and downwelling branches and hence to develop an MOC. Further, in response to a surface buoyancy flux $Q$, temperature (density) varies meridionally within layer 1 (a “variable-temperature layer model,” VLOM), and layer 1 can become cold enough in the Southern Ocean to eliminate the density contrast between the layers, in which case the model reduces to a single layer 2. Traditional, constant-density layer models (CLOMs) have the disadvantage that they cannot represent thermodynamics in a natural way; for example, they parameterize cooling by surface $Q$ as a change in upper-layer thickness. VLOMs overcome this deficiency by allowing $Q$ to alter $T_1$, and hence they represent polar-ocean cooling more realistically. As such, our VLOM is a “sensible “next step” in a hierarchy of intermediate models that lead to state-of-the-art OGCMs.

Most solutions (the exceptions being those discussed in Appendix A.2) are obtained using an extended version of the GM parameterization to close the $\tilde{h} \nu \tilde{v}$ term in the layer-1 continuity equation (Eq. 2 below). The closure has the advantage that it applies when $h$ represents both the thickness of the surface mixed layer and the depth of subsurface isopycnals. Using this closure, the layer-1 velocities correspond to the residual-mean circulation, and horizontal mixing has the form of Rayleigh damping with a coefficient $\nu$.

Solutions are found in idealized domains with rectangular basins that represent the Atlantic, Southern and Pacific Oceans. They are forced by an $x$-independent, zonal-wind $T(x,y)$, a heat flux $Q$ that relaxes layer-1 temperature to an $x$-independent temperature profile $T_0(y)$, and vertical diffusion $\kappa$. Northern sinking is parameterized in two different ways: In most solutions, it is represented by a relaxation of $h$ along the Atlantic northern boundary to a prescribed value $h_0$, which parameterizes cooling processes in marginal seas external to the domain; in some solutions, it is represented by strong cooling by $Q$ within the North Atlantic (Section 4.2.6). Solutions are obtained both analytically and numerically. The analytic solutions split the response into interior and boundary-layer parts, and the resulting simplification allows a set of integral constraints to be derived that directly relates MOC properties to model parameters and forcing. The numerical solutions confirm the analytic results, and provide a means for visualizing the flow fields.

Given these properties, the model can be regarded as an extension of the layer models noted above. It extends the Stommel and Arons (1960a,b) model to apply to an upper layer with forcing by $T$ and a cyclic Southern Ocean, and Gill’s (1968) 1-layer model to apply to an upper layer with an MOC. It replaces Gnanadesikan’s (1999) $h_0$-equation with a coupled set of constraints based on our model’s physics that can be solved for a single equation (Eqs. 36 below). It extends the Samelson (2009) and Radko and Kamenkovich (2011) models in several ways: by relaxing their restrictions on the Southern-Ocean stratification; replacing their matching conditions between the Southern and Atlantic Oceans with one based on the Gill (1968) boundary layer; and internally determining the eddy-driven transport $\nu$ both within and below the surface mixed layer. Finally, our model is also an extension of the Atlantic-only models of Schloesser et al. (2012, 2014) to include the Southern Ocean.

Novel aspects of the model and solutions include the following. The model allows for a rich suite of solutions that contains all the types of solutions in the aforementioned, idealized modeling studies: solutions with $\mathcal{M} > 0$ and $\mathcal{M} < 0$; with and without wind forcing; with, without, and for different parameterizations of northern-boundary sinking; for a wide range of values of $\nu$, $\kappa$, and eddy-mixing parameter $\nu$ (proportional to $k_\kappa$); and for different $\tilde{h} \nu \tilde{v}$ closures. A key property of solutions is that, depending on the choices of $T$ and $\nu$, the Southern-Ocean stratification adjusts to have three distinct states: Either layer 1 intersects Antarctica, outcrops at latitude $y_0$ or extends to a latitude $y_0$ where $T_1$ cools to $T_2$ and layer 1 vanishes (as illustrated schematically in Fig. 5 below). Further, $y$ is internally determined rather than externally specified as in the Radko and Kamenkovich (2011) model, a feature that significantly alters the dependency of MOC properties to model parameters and forcings. Finally, some solutions are directly comparable to the idealized-OGCM solutions noted above. In these cases, our solutions provide insight into the underlying dynamics of the OGCM solutions, for example, pointing toward processes that may be involved in eddy compensation (Sections 4.2.2.3 and 4.2.5) and saturation (Section 4.2.6), and that cause $\mathcal{M}$ and the Atlantic thermocline depth $h_0$ to have different sensitivities for large and small values of $\kappa$ and $\nu$ (Section 5.2).

Section 2 provides an overview of our ocean model. It is supplemented by two appendices: Appendix A, which derives the residual-mean equations for our standard closure, and explores the impacts of other closures in the Southern Ocean; and Appendix B that describes our method for obtaining exact solutions when there are no diapycnal processes and no overturning circulation, the situation considered by Gill (1968). Section 3 derives the integral constraints for the steady-state solutions, leaving a discussion of the specific structure of each boundary layer to Appendices C and D. Section 4 provides a hierarchy of solutions that are primarily wind-driven $(\mathcal{M} \geq 0)$, and Section 5 discusses solutions that are primarily diffusion-driven $(\mathcal{M} \leq 0)$. Section 6 provides a summary and discussion of results. Finally, Appendix E provides an alphabetical list and brief description of the variables and parameters defined in the text, and Table 1 lists the solutions reported.

2. Ocean model

In this section, we describe our ocean model and experimental design. To allow for analytic solutions, the model requires several simplifications. They are noted and their potential impacts discussed.

2.1. Equations of motion

The model ocean consists of two layers with densities $\rho_1$ and $\rho_2$, in which the layer-1 density varies meridionally in response to a surface heat flux $Q$ (Schloesser et al., 2012, 2014; Sections 2.5.2
and 2.6). We interpret layer 1 to correspond to intermediate and surface waters and layer 2 to represent bottom and deep waters.

Equations of motion for the 2-layer model can be split into sets for the depth-integrated flows in the entire water column (barotropic response) and in layer 1 (related to the baroclinic response); see Greatbatch and Lu (2003), Appendix A of Schloesser et al. (2012), and Section 2 of Schloesser et al. (2014) for detailed derivations of similar sets. The layer-1 set is linked to the depth-integrated set through terms proportional to \(h/D\), where \(h\) is the thickness of layer 1 and \(D\) is the ocean depth. We assume that \(h/D \ll 1\), in which case the two sets are decoupled. This assumption is only marginally satisfied in our solutions for which \(h \sim 1000\) m and \(D = 5000\) m. We therefore expect the lack of coupling to distort our layer-1 solutions somewhat, but not to alter their basic dynamics.

When \(h/D \ll 1\), the steady-state response for layer 1 can be written in the form

\[
-fV + P_x = \tau^x - vU, \quad fU + P_y = -vV, \quad U_x + V_y = \omega_c, \tag{1}
\]

where \(V = (U, V)\) is the “residual-mean” layer-integrated velocity (Section 2.2; Appendix A.1), \(\omega_c\) specifies the rate at which water transfers between the two layers (Section 2.3), and \(\tau^x\) is zonal wind stress divided by the density of seawater (Section 2.5.1). (The unit of \(\tau^x\) is therefore \(\text{cm}^2/\text{s}^2\), although we report values in \(\text{dyn}/\text{cm}^2\).) Horizontal mixing has the form of Rayleigh damping, which we interpret to result from baroclinic instability (Section 2.2). Variable \(VP = (P_x, P_y)\) is the depth-integrated, pressure gradient in layer 1, where \(P = gh^2/2\) and \(g^* = g(\rho_2 - \rho_1)/\rho_2\). Note that, because \(\rho_1\) varies with latitude, so does \(g^*\).

Since \(\rho_1\) varies in \(y\), thermal wind requires the pressure-gradient and zonal-flow fields in layer 1 to vary linearly with depth. Eqs. (1), however, only describe the vertically-integrated part of the circulation. Because \(T_1\) is fixed to \(T'(y)\), the thermal-wind part is completely determined by, and does not impact, the depth-integrated response (see Eq. A15 of Schloesser et al., 2012). In principle, then, we can add the thermal-wind part of the circulation to our solutions. For our purposes, its inclusion is not necessary since we are interested in layer-1 transports, and by definition the integral of the thermal-wind flow across layer 1 vanishes.

For one solution (Section 4.1), we view (1) as describing the barotropic response of the system as in Gill (1968). In this case, \(\omega_c = 0\) and \(P = gdh\), where \(d\) is sea level. For this solution, we view the Rayleigh damping to result from either barotropic instability or bottom drag.

### 2.2. Residual-mean velocity and horizontal viscosity

For a steady flow field without eddies, \(V = hW\), where \(W = (u, v)\) is the layer-averaged velocity. When there are eddies, it is useful to separate fields \(q\) into time-mean \(\bar{q}\) and time-varying (eddy) \(q'\) parts. (Eqs. 1 describe the time-mean response, where for convenience overbars are neglected.) It follows that the time-averaged continuity equation involves an additional term, \(V \cdot (\bar{h} \mathbf{\bar{v}})\), and this term defines a time-mean, eddy-driven velocity, \(\mathbf{\bar{v}}'' = (\bar{h} \mathbf{\bar{v}})/\bar{h}\) (Eqs. A2 and A4, respectively). In Eqs. (1), \(V\) is rewritten to include \(\mathbf{\bar{v}}\), that is, \(\mathbf{\bar{v}} = hW + h\mathbf{\bar{v}}''\), the sum (residual) of the Eulerian- and eddy-mean velocities (Lee et al., 1997; Ferrari and Plumb, 2003). Then, with the closure,

\[
\bar{h} \mathbf{\bar{v}}'' = -\frac{1}{\bar{f}} \nabla \mathbf{\bar{P}}, \tag{2}
\]

the impact of the eddies takes the form of Rayleigh damping in the momentum equations (Greatbatch and Lu, 2003; see Appendix A for a derivation).

According to (2), \(\bar{h} \mathbf{\bar{v}}''\) is proportional to the gradient of the time-mean, available potential energy of layer 1, \(P = g\bar{h}^2/2\), which is sensible since the presumed source of the perturbations is baroclinic instability. Unless specified otherwise, \(v = 2 \times 10^{-6} \text{ s}^{-1}\) (Aiki and Yamagata, 2006), or it is allowed to vary over a range of values. For a few solutions, \(v\) either increases with \(\tau_0\) (Section 4.2.5) or is inversely proportional to the square of the Rossby radius of deformation (\(v = \kappa_h/R^2\), \(R^2 = g\bar{h}H^2\); Section 5.2: Appendix A).

Closure (2) is unusual in that it involves \(\nabla \mathbf{\bar{P}} = g\bar{h} \nabla \mathbf{\bar{V}} + \bar{h} \nabla \mathbf{\bar{V}}'/2\) rather than \(\mathbf{\bar{V}}\). Mathematically, it is a natural choice for our model since mixing then has the form of Rayleigh damping. Physically, it parameterizes the property that baroclinic instability arises from two sources: the traditional type proportional to \(\mathbf{\bar{V}}\) associated with sloping isopycnals at the layer bottom (e.g., Phillips, 1954); and a frontal type proportional to \(\mathbf{\bar{V}}'\) associated with sharply tilted isopycnals in the overlying stratification (Stone, 1966, 1970; Fukamachi et al., 1993; Fox-Kemper et al., 2008; Qiu et al., 2014). As such, it applies when \(\bar{h}\) represents both the thickness of the surface mixed layer (\(\bar{h} = h_m\)) as well as the depth of subsurface isopycnals (\(\bar{h} > h_m\)).

Note that when \(g'\) is constant and \(v = \kappa_h/R^2\), (2) reduces to the GM parameterization, \(\bar{h} \mathbf{\bar{v}}'' = -\kappa_h \mathbf{\bar{V}}\). We explore the impact of using GM and other closures in Appendix A2, concluding that (2) is the best choice in comparison to the others.

### 2.3. Across-interface velocities, \(W_e\)

The across-interface velocity, \(W_e\), parameterizes all processes in the model that cause water to transfer across the bottom of layer 1, thereby allowing solutions to develop overturning cells. It is the sum of three terms,

\[
W_e = W_d + W_m + W_n, \tag{3}
\]
the three terms representing diffusion in the interior ocean, mixed-layer entrainment, and diapycnal processes that occur external to the basin (in the Arctic Ocean and the GIN and Labrador Seas), respectively.

2.3.1. Interior diffusion, \( w_d \)

As in Stommel and Arons (1960a,b), we assume that \( w_d \) is a positive constant throughout the domain. As such, it acts to thicken layer 1 continuously, and steady-state solutions are possible only when the domain contains a compensating detrainment process. It is useful to relate \( w_d \) to a vertical diffusion coefficient \( \kappa \). Based on the balance \( w_d f = \kappa T_2 \), the simplest correspondence is

\[
   w_d = \frac{\kappa}{f} \tag{4}
\]

where \( H \) is typical depth of the bottom of the surface AMOC branch. For the solutions shown in Sections 4 and 5, we set \( w_d = 2.5 \times 10^{-6} \) cm/s, which with \( H = 1000 \) m corresponds to \( \kappa = 0.25 \) cm²/s, on the large end of \( \kappa \) values typically used in OGCMS (\( \kappa = 0.1-0.2 \) cm²/s).

Other choices for \( w_d \) are possible. For example, to compare our results to those from OGCMS solutions, we allow \( w_d \) to have the form \( w_d = \kappa \delta h \) where \( \delta h \) is the Atlantic thermocline depth determined by the model (Section 5.2). Further, it is also possible to allow \( w_d \) to vary spatially. For example, Schloesser et al. (2012) and Samelson (2009) allow diffusion to vary spatially by setting \( w_d = \kappa \delta h \) and \( w_d = -\alpha_0 (h^2 - h^2_m) \), respectively; for our purposes, this additional complexity is unnecessary.

2.3.2. Mixed-layer entrainment, \( w_m \)

In principle, wind-driven upwelling can force \( h \) to thin to zero thickness. In the real ocean, however, subsurface isopycnals can only rise until they intersect the bottom of the surface mixed layer. To represent the mixed layer, whenever \( h \) becomes thinner than \( h_m \), we allow water from layer 2 to entrain into layer 1 at the rate,

\[
   w_m = \frac{1}{\delta t_m}(h_m - h)/\theta(h_m - h), \tag{5a}
\]

where \( \delta t_m \) is the time scale of the entrainment, \( h_m = 50 \) m is the mixed-layer thickness, and \( \theta(\zeta) \) is a step function \( \theta(\zeta = 1 \text{ for } \zeta \geq 0 \text{ and is zero otherwise}) \). For numerical solutions, \( \delta t_m = 1 \) day. For analytic solutions, we assume that \( \delta t_m \rightarrow 0 \), in which case \( w_m \) ensures that

\[
   h \geq h_m, \tag{5b}
\]

that is, \( h \) can never be less than \( h_m \).

2.3.3. Northern-boundary processes, \( w_n \) vs. \( Q \)

The other across-interface velocity,

\[
   w_n = \frac{1}{\delta t_n}(h_n - y)Y_n(y), \tag{6a}
\]

parameterizes northern sinking by relaxing \( h \) along the northern \((y = y_n)\) boundary of the Atlantic to \( h_n \). In the time-stepping numerical model, \( \delta t_n = 1 \) day and \( Y_n(y) \) varies from 1 at the coast to zero within a few grid points offshore. In the analytic model, \( \delta t_n \rightarrow 0 \) and \( Y_n(y) = \delta(y - y_n) \) so that \( w_n \) ensures that

\[
   h = h_n \text{ at } y = y_n, \tag{6b}
\]

a statement that \( h \) is fixed to \( h_n \) at the northern boundary. We set \( h_n = 1500 \) m, roughly the bottom of intermediate water in the North Atlantic. Boundary constraint (6a) provides a simple way to represent diapycnal (cooling) processes external to the Atlantic that are associated with the AMOC downwelling branch, with all the cooling occurring along \( y = y_n \) where layer-1 water detains into layer 2. As such, \( w_n \) is analogous to a sponge layer in OGCMS without the Arctic Ocean that relaxes temperature and salinity to observed values (e.g., Nonaka et al., 2006; Mf. 13). In Sections 4 and 5, we report solutions with and without \( w_n \).

It is possible to replace Eqs. (6) with a parameterization of northern sinking based on cooling by \( Q \) within the North Atlantic. In the 2-layer model of Schloesser et al. (2012), for example, \( Q \) cools layer 1 until \( T_1 = T_2 \) for \( y > y' = 50 \) N so that layer 1 vanishes in the northern ocean \((g' = 0)\). In that situation, constraint (6b) is replaced by

\[
   \left[ \frac{M_n}{\alpha} = \frac{f}{f'} P_n \right] \tag{7}
\]

where \( f' \) has a value close to \( f^{-1}(y') \), \( P_n = \frac{1}{2} g h_n^2 \) is a measure of the depth of the Atlantic thermocline along the eastern boundary (defined in Eq. 14b below), and \( x \) increases with the strength of subsurface diapycnal mixing (their Eq. 23). (See Schloesser et al., 2014, for an extension of Eq. 7 that allows for forcing by subpolar westerly.) According to (7), \( M_n \times g h_n^2 \) a relationship that arises from a variety of scalings (see Appendix 1 of Fürst and Levermann, 2012), occurs in idealized OGM solutions (e.g., WC10) and is adopted in simpler models (e.g., Samelson, 2009; Radko and Kamenkovich, 2011).

MJf. 13 argue that a \( Q \)-forced constraint like (7) is better (more physically realistic) than a northern-boundary sponge layer like (6a). On the other hand, the \( Q \)-forced constraint ignores all processes within marginal seas (Spall, 2004, 2010, 2011), as well as any overflow entrainment that thickens the AMOC upper branch and contributes significantly to \( M_n \) (Döschner and Redler, 1997; Eldevik et al., 2009, Kösters et al., 2005). Furthermore, in a 2-layer-model configuration like ours, (7) implies that the stratification vanishes in the North Atlantic \((g' = 0; \) Schloesser et al., 2012), inconsistent with the real ocean where the densest waters are formed in the Southern Ocean. Here, then, we use (6) for most solutions, commenting on the impact of using (7) in Section 4.2.6.

2.4. Basin and boundary conditions

Fig. 2 illustrates the domain in its most general configuration (all three oceans). When the model domain has only an “Atlantic Ocean” and a “Southern Ocean,” it consists of a rectangular basin
that extends zonally from \( x = -L_A = -5000 \) km to \( x = 0 \) and meridionally from \( y_1 = 60^\circ S \) to \( y_0 = 60^\circ N \). The sole continent, “South America,” is then a line segment along \( x = 0 \) (and \(-L_A\)) that extends from \( y_0 = 45^\circ S \) to \( y_0 = 45^\circ N \), thereby providing both the eastern and western boundaries of the Atlantic basin, and the Southern Ocean extends from \( y_0 \) to \( y_0 = 60^\circ N \). When a “Pacific Ocean” is included, the domain extends from \( x = -L_A = -15000 \) km to \( x = 0 \), and the Atlantic and Pacific Oceans are separated by “Africa,” a line segment along \( x = -L_A = -10000 \) km that extends from \( y_0 \) to \( y_0 = 35^\circ S \), with the Pacific located from \( x = -L_A \) to \( x = 0 \) and the Atlantic from \( x = -L_A \) to \( x = -L_A \).

No-normal-flow conditions are applied at all continental boundaries. In the Southern Ocean \((y < y_0)\) and at the tip of South America \((y = y_0)\), the condition

\[
P(0, y) = P(-L, y), \quad y < y_0, \tag{8}
\]

imposes cyclicity there, where \( L = L_A \) (\( L = L_T \)) is the width of the Southern Ocean when the domain includes only the Atlantic (Atlantic + Pacific) basin(s).

2.5. Forcing

2.5.1. Wind stress

Most solutions are forced by x-independent, zonal wind-stress fields \( \tau^x(y) \), the exceptions being for some solutions in Sections 4.2.4 and 5 with \( \tau^x = 0 \). Fig. 3 illustrates the specific profiles used to force solutions. They have the general form

\[
\tau^x(y) = \tau_0 + \tau^*_1(y) + \tau^*_2(y), \tag{9a}
\]

where

\[
\tau^*_1(y) = \begin{cases} 
\tau_0 \sin \left( \frac{y - y_0}{Y_2 - y_0} \right) - \tau_0, & y \leq y_0, \\
(y_0 - \tau_0) \frac{y}{Y_2}, & y > y_0,
\end{cases} \tag{9b}
\]

\[
\tau^*_2(y) = \begin{cases} 
\frac{\tan \frac{y}{Y_2}}{1 - \cos 2 \pi \frac{y}{Y_2}} - \tau_0, & y_1 \leq y < y_2, \\
0, & \text{otherwise},
\end{cases} \tag{9c}
\]

\( Y_2(y) \) is any meridionally broad function that varies monotonically from \( \tau_0 \) at \( y = y_0 \) to zero at any latitude south of \( y_1 \) (its precise form is not needed), \( y_1 = 20^\circ N \), \( y_2 = 50^\circ N \), and \( L_T = y_0 - y_0 \). Note that \( \tau^*_2 = 0 \) along \( y = y_0 \), a useful restriction that simplifies (35a), the boundary constraint that joins the Southern and Atlantic (Atlantic + Pacific) Oceans.

Solutions are obtained when \( \tau^x = \tau_0 \), \( \tau_0 + \tau^*_1 \), and contains all three components (gray, red and magenta, and dashed curves in Fig. 3, respectively). Wind stress \( \tau_0 = 1 \) dyn/cm\(^2\) represents constant background westerlies. Wind-stress \( \tau^x = \tau_0 + \tau^*_1(y) \) weakens from a maximum of \( \tau_0 \) at the tip of South America \((y = y_0)\) to zero at Antarctica and to \( \tau_0 \) where \( Y_2(y) = 0 \). Almost all solutions that include \( \tau^*_2 \) set \( \tau_0 = \tau_0 \), the exceptions being for those discussed in Section 4.2.4 for which \( \tau_0 \) has a range of values. Approximate analytic solutions are also obtained when \( \tau^x = \tau_0 + \tau^*_1 \) varies linearly across the Southern Ocean so that \( \tau^x = (\tau_0/L_0)(y - y_0) \). Wind-stress \( \tau^*_2(y) \) is included to explore the sensitivity of solutions to wind forcing in the North Atlantic and North Pacific. It generates subpolar gyres in the northern oceans, and upwelling can occur there for sufficiently strong \( \Delta x_w \). In the Atlantic, we set \( \Delta x_w = 3 \) dyn/cm\(^2\), a large value that emphasizes subpolar upwelling there (Section 4.4). We also obtain solutions with subpolar upwelling in the Pacific (Section 4.5), but for the solutions shown (Figs. 12 and 13), \( \Delta x_w \) in the Pacific has a value low enough to prevent any upwelling. We ignore winds in other regions since they do not affect any of the model constraints that determine MOC properties.

2.5.2. Surface buoyancy flux

Solutions are also forced by a surface buoyancy flux \( Q \). For simplicity, we assume that density depends only on temperature according to

\[
\rho_i = \rho_o (1 - \alpha_i T_i), \tag{10}
\]

where \( i = 1, 2 \) is a layer index, \( \rho_o = 1.028 \) g/cm\(^3\) is a background density, and \( \alpha_i = 0.00015 \) K\(^{-1}\) is the coefficient of thermal expansion. The buoyancy (heat) flux and layer-1 temperature have the forms

\[
Q(x, y) = \lim_{\Delta t \to 0} \frac{T_1 - T^*(y)}{\Delta t} \Rightarrow T_1 = T^*(y), \tag{11a}
\]

where

\[
T^*(y) = \begin{cases} 
T_1, & |y| \leq y_c, \\
T_1 + (T_n - T_1) \frac{y - y_c}{y_o - y_c}, & y > y_c, \\
T_1 + (T_n - T_1) \frac{y - y_c}{y_o - y_c}, & y < y_c, \\
T_n, & y \leq y_c.
\end{cases} \tag{11b}
\]

with \( y_c = 40^\circ C, T_1 = 23^\circ C, T_n = 3^\circ C, \) and either \( T_1 = 3^\circ C \) and \( y_0 = y_c \) or \( T_1 = 0^\circ C \) and \( y_0 = 55^\circ S \). We set \( T_2 = 0^\circ C \).

Fig. 4 plots the two \( T^* \) profiles used in the model. When \( T_1 = 0^\circ C \) and \( y_0 = y_c \) (warm \( T^* \)). \( T^* \) cools symmetrically poleward of \( y_c \) (black curve). With this symmetricity, the regions where \( T_1 \) cools \( T^* \) in the Southern Ocean are assumed to be confined to the Antarctic marginal seas, similar to the North Atlantic. When \( T_1 = 0^\circ C \) and \( y_0 = 55^\circ S \) (cold \( T^* \)), however, \( T^* \) is cooler in the southern hemisphere (gray curve). In this case, \( T_1 = T_2 \) and \( g = 0 \) in the Southern Ocean south of \( y_0 \); thus, layer 1 no longer exists for \( y < y_c \), and the model reduces to a 1-layer system there. Solution properties differ depending on the choice of \( T^* \) in the Southern Ocean (Sections 2.6 and 5). They are also sensitive to \( T^* \) in the North Atlantic through the dependence of \( P_n = -\frac{1}{2} g' T_n y_0^2 / a \) on \( T_n \) (Section 3.3.2.1).

Eq. (11a) provides a simple way to include thermodynamic processes in the model physics. In so doing, however, it eliminates temperature advection as a process that can alter \( T_1 \) and requires isotherms within layer 1 (within layer 2 south of \( y_0 \)) to be oriented vertically wherever \( T_1 \neq 0 \). In the Southern Ocean, we do not expect the lack of mean temperature advection to be a serious problem.
detriment, given the zonality of the ACC \((V \ll U)\). Likewise, the lack of time-dependent advection is not likely serious; its absence eliminates the growth of baroclinic instability, but its effects are represented by horizontal mixing (Section 2.2). In OGCM solutions and the real ocean, isotherms in the upper Southern Ocean are not oriented vertically but rather tilt upward to the south with a finite slope (Fig. 1b). Consequences of this model simplification are less clear. In its support we note that, consistent with the observed stratification, the model stratification still allows the available potential energy of the upper ocean to decrease southward across the Southern Ocean. Furthermore, the simplification may not be as restrictive as it appears: Schloesser et al. (2014, their Section 6.2) define an upper-layer thickness for their North Atlantic OGCM solution that agrees remarkably well with the \(h\) field from an analogous VLOM solution.

### 2.6. Southern-Ocean stratification

Fig. 5 illustrates the possible layer-1 structures (stratifications) when \(g' \neq 0\) everywhere (warm-\(T\) forcing, left panel) and \(g' = 0\) south of \(y_0\) (cold-\(T\) forcing, right panel). In all cases, the model's idealized stratification consists of a sharp pycnocline at the bottom of layer 1 that gradually weakens to the south as isotherms within layer 1 outcrop; this structure is similar to the observed stratification, except that the Atlantic thermocline has finite thickness and all Southern-Ocean isopycnals rise to the surface with a finite slope (Fig. 1b). When \(g' = 0\) (left panel), the bottom of layer 1 either outcrops \((h = h_{m})\) along a latitude \(y'\) (upper curve) or extends to Antarctica (lower curve). When \(g'(y < y_0) = 0\) (right panel), layer 1 only exists north of \(y_0\), and it either outcrops (upper curve) or extends to \(y_0\) (lower curve). Note that, by assuming the outcrop line occurs on a latitude \(y'\), it has no \(x\)-dependence, a statement that \(y'\) is unaffected by the \(x\)-dependent boundary layer along \(y_0\) (see Appendix C.4.4).

A comparison of the possible layer-1 structures with the density profiles in Fig. 1b suggests that the most realistic model state occurs when layer 1 outcrops at \(y'\). On the other hand, the weakly-sloping, observed stratification (Fig. 1b, right panel) could indicate a model state in which layer 1 extends to Antarctica, and the strongly-sloping one (Fig. 1b, left panel) might correspond to a state where the available potential energy of layer 1, \(P_x\), essentially vanishes at \(y_0\). Additionally, strongly-sloping isopycnals commonly occur in OGCM solutions without freshwater forcing, since the Southern Ocean is not capped by a thin, fresh layer (e.g., Fig. 7 of WC10), a stratification analogous to that in our model when layer 1 extends to \(y_0\).

In Sections 4 and 5, we discuss solutions that attain all the model states shown in Fig. 5. One general result is that, for primarily wind-driven solutions with realistic parameter values, MOC properties are not very sensitive to whether layer 1 outcrops at \(y'\) or vanishes at \(y_0\) (Section 4.2.3). For these solutions, a constant-density (constant \(g'\)) model is sufficient to represent their dynamics. On the other hand, for solutions in which there is Southern-Ocean detrainment \((\dot{M} \leq 0)\) or that are primarily diffusion-forced \((w_0 = 0\) so that there is no northern sinking), the vanishing of layer 1 at \(y_0\) is essential and a variable-density (variable-\(g'\)) layer model is required (Sections 4.1.2 and 5.2).

#### 2.7. Solution methods

In Section 3 and Appendix C, we derive and discuss the various parts of the analytic solutions. We obtain numerical solutions to (1) using two different methods. To obtain most of them, we include the terms \(U_e, V_e, R_e, h_0\) in the first, second, and third equations of (1), respectively, and integrate the resulting system from a state of rest for 50 years, by which time solutions are near equilibrium. Figs. 7, 11, and 13, and the data points in Figs. 8a–8c and 12 are obtained by this method, and they are all taken from the last year of the integration. We also obtain steady-state, numerical solutions by solving the diffusion equation for streamfunction \(\psi\) (Section 4.1.1 and Appendix B).

For simplicity, the analytic solutions are derived in Cartesian coordinates and assume that \(\rho_e = 0\). The constraints that determine MOC properties (boxed equations in Section 3) only require that solutions are known at mid-to-high latitudes \((\geq 45^\circ)\), with southern and northern solutions linked by matching their meridional transports and eastern-boundary \(P\) values. Therefore, the curves in Figs. 8–10 and 12 are evaluated with

\[
f = f_0 + \beta(y - y_0),
\]

where \(y_0 = \pm(\pi/4)/R_e\), \(R_e = 6371\) km is the earth’s radius, \(f_0 = \pm 2\Omega \sin(\pi/4) = \pm 1.03 \times 10^{-4}\) \(s^{-1}\), \(\Omega = 2\pi \times 10^{-4}\) rad \(s^{-1}\), and unless specified otherwise \(\beta = (2\Omega/R_e) \cos(\pi/4) = 1.65 \times 10^{-10}\) cm \(s^{-1}\). (To obtain analytic solutions throughout the entire domain requires that \(f = f_0 + \beta(y - y_0)\).)

To allow a close comparison to the analytic solutions, solutions to the time-stepping, numerical model are also found in Cartesian coordinates. Further, since \(f\) must be known everywhere, \(f\) is set to (11c) for \(|y| > y_0\) and \(f = 2\Omega \sin(y/R_e)\) for \(|y| < y_0\). This “split” definition of \(f\) ensures that properties of the numerical and analytic solutions are as close as possible, but the numerical solutions are hardly changed if \(f = 2\Omega \sin(y/R_e)\) for all \(y\). For historical reasons, \(R_e = 6300\) km in the numerical solutions.

#### 3. Analytic solutions

To obtain analytic solutions, we adopt the common simplification that the flow field can be divided into interior responses within the basins and boundary layers along their edges that close the interior circulations. The split allows for dynamical simplifications in each region: Interior solutions lack horizontal mixing in the Atlantic and Pacific Oceans and are \(x\)-independent in the Southern Ocean, and the along-boundary velocity field is in geostrophic balance in boundary layers. Let \(q\) be any of the model variables; then, throughout the text we label the interior part \(q'\) and boundary-layer part \(q'^*\). (In Section 2 and Appendix A, \(q'\) is also used to indicate the time-varying part of \(q\).)

We begin with an overview of the structure of the analytic solutions (Section 3.1). Next, we derive solutions for the interior circulations in the three oceans (Section 3.2) and for the along-boundary structures of the boundary layers (Section 3.3), deferring a discussion of their across-boundary structures until Appendix C. These solutions lead to a set of constraints (boxed equations) that allow MOC properties to be expressed in terms of model parameters and forcings.
3.1. Overview

3.1.1. Horizontal structure

Fig. 2 schematically illustrates the horizontal structure of solutions when all three oceans are present. The Atlantic western- and northern-boundary layers (Boxes $W_A$ and $N_A$) close the wind-driven circulation and provide a pathway for the water that entrains into layer 1 to flow either to the North Atlantic or, for solutions with a reverse or double-celled MOC (Sections 4.2.3 and 5), to the Southern Ocean. Similarly, Pacific western- and northern-boundary layers (Boxes $W_P$ and $N_P$) close the wind-driven circulation and channel all the Pacific water that flows across $y_B$ or entrains into layer 1 north of $y_B$ to the tip of Africa; subsequently, this water flows westward across the Atlantic within Box $S_A$ to merge with the Atlantic western-boundary current in Box $W_A$ (bottom panel of Fig. 13). The boundary layer centered on $y_B$ (Box $S_A$) smoothly joins the Atlantic and Southern Ocean circulations. Regions $B$ ($B_A$ and $B_P$) designate where upwelling (with transports $V_{rA}$ and $V_{rP}$) can occur due to Ekman suction in subpolar gyres (see Fig. 11). Finally, the dashed curves, $X_{st}$, $X_{sf}$, and $y'$ (defined precisely below) indicate where layer 1 can outcrop ($h = h_{id}$); the dashed line along latitude $y_B$ indicates that $g'$ can also decrease to zero at and south of $y_B$, where layer 1 vanishes (Figs. 4, 5 and 7).

In the aforementioned boundary layers, the assumption of along-boundary geostrophy is accurate because the along-boundary spatial scale is much greater than the layer width (Appendix C). For the solutions in corner areas (darker shading), however, the spatial scales are small in both directions, so that both flow components are inherently ageostrophic. For our purposes, it is not necessary to obtain solutions in these regions since they only serve to join the pressures and transports in adjacent boundary layers smoothly. (We do, however, take into account effects of the ageostrophic region at the tip of South America in deriving the $y_B$-boundary constraint; see Appendix C.4.5.)

3.1.2. Overturning transports, $M$ and $M_A$

For most solutions in Section 4.2, $M > 0$ so that water entrains into layer 1 across its southern edge at $y = y'$ or $y_B$ (Figs. 2 and 5). The transport of the entrained water is the residual flow across $y$, $M = C^* \langle y / f \rangle + C^* V_{d}$, where $V_{d}$ is the eddy-driven transport/width across $y$. The entrained water first circulates around the Southern Ocean several times, and eventually the $y_B$-boundary layer (Region $S_A$) where it flows to the tip of South America. It then flows northward in the Atlantic western-boundary current (Region $W_A$), circulates about the North Atlantic upwelling region if it exists (Region $B_A$), flows eastward in a northern-boundary current (Region $N_A$), and finally downwells in the northeast corner of the Atlantic (Fig. 11). The total transport of the water that detains there is $M_A = M + V_{d}^r + V_{d}^s + V_{d}^d$, the sum of all the water that entrains into layer 1 anywhere in the domain, where the latter term is the entrainment due to interior diffusion by $w_d$.

There are also solutions in which $M < 0$ (reverse or double-celled MOCs). They only occur for cold-$T$ forcing when layer 1 extends to $y_B$; then only is the eddy-driven circulation across $y_B$, $V_{d}^r$, large enough to overwhelm the northward Ekman drift there. Such solutions occur for large $v$ (Section 4.2.3 and Fig. 8b) and for primarily diffusion-forced solutions (Section 5).

3.1.3. Boundary-layer widths

The dynamics, and hence widths, of the boundary layers in each box differ (Gill, 1968). The western-boundary currents in Boxes $W_A$ and $W_P$ have the structure of the well-known Stommel (1948) layer of width

$$r = v/\beta$$

(Appendix C.1). In contrast, the boundary currents in Boxes $N_A$ and $N_P$ broaden to the west, with a width scale $\Delta(L) = 2 \sqrt{r|x|}$ (a “zonal Stommel layer”; Appendix C.2), and the average value of $\Delta$ across the basin,

$$\langle \Delta(L) \rangle^2 = 4 \sqrt{rL},$$

(13b)

$L = L_A$ or $L_P$ provides a measure of their overall width. The boundary layer in Box $S_A$ also broadens like $\Delta(L)$ but, since it spreads both north and south of $y_B$ (Appendix C.3), $\langle \Delta(L) \rangle^2$ measures its halfwidth. With $v = 2 \times 10^{-6} \text{S}^{-1}$ and $\beta = \beta_0 = 1.65 \times 10^{-13} \text{cm}^{-1} \text{S}^{-1}$, $r = 121 \text{km}$ and $\langle \Delta(L) \rangle^2 = 1038 \text{ (1978) km}$ in the Atlantic (Pacific) Ocean.

The $y_B$-boundary layer (Box-$S_A$) is more complex than the others because its northern half ($y > y_B$) is blocked by South America whereas its southern half satisfies cyclic conditions (8). As a result, it has different widths north and south of $y_B$. North of $y_B$, its width $\langle \Delta(L) \rangle^2$ is defined after (8) and in Appendix E. South of $y_B$, since the Southern Ocean is zonally unbounded, one might expect the boundary layer to broaden indefinitely, eventually extending to the coast of Antarctica. Because of the cyclic nature of the solution there, however, the only part of the boundary layer that continues to broaden indefinitely is its $x$-independent part. This property can be understood by expanding $P^*$ into a Fourier series in $x$ south of $y_B$ (Gill, 1968). Each Fourier contribution satisfies (C2), and hence has the form $a_m \exp[-(1 - i)(y - y_B)/\delta_0] \exp(i k_0 x)$, $n = 0, 1, 2, \ldots$, where $h_0 = 2 \pi n / L$ and $\delta_0 = 2 \sqrt{2} r / k_0$. The decay scale of the $n = 0$ component ($k_0 = 0$) is infinity ($\delta_0 = \infty$), but for all the other components it is finite and $\delta_n \leq \delta_0 = 2 \sqrt{2} r / k_1 = 2 \sqrt{L / \pi}$. In our solutions, the $x$-independent part of the Southern-Ocean response is all contained in $P^*$, since we ensure that $P^*$ has no $x$-independent part by imposing (C28). A measure of the width of the southern half of the layer is therefore

$$\delta(L) \equiv \delta_0 = \sqrt{\frac{L}{\pi}},$$

(13c)

which with $r = 121 \text{ km}$ is $\delta = 439760 \text{ km}$ in the Atlantic (Atlantic + Pacific) Ocean, thinner by a factor of $\delta_0 / \langle \delta(L) \rangle^2 = 0.42$ than its northern part, $\langle \Delta(L) \rangle^2 = 1038 \text{ (1978) km}$.

3.2. Interior solutions

3.2.1. Atlantic and Pacific Oceans

For simplicity here (and elsewhere), we first write down the solution in a general ocean basin of width $L$ with its eastern boundary located at $x = 0$. The Atlantic (Pacific) solution is then obtained by making the following replacements in (14)-(17) below: $P_x = P_A$ ($P_x = P_P$), $L_x = L_A$ ($L_x = L_P$), and $V_{d}^r = V_{d}^r_A$ ($V_{d}^r = V_{d}^r_P$). In addition, when there is a Pacific, the Atlantic solution must be shifted westward by the width of the Pacific, that is, by setting $x \rightarrow x + L_P$ in (14).

3.2.1.1. Without interior entrainment. Neglecting the momentum damping terms and (for the moment) the mixed-layer entrainment term $w_d$ in (1), the solution is

$$U' = \frac{1}{\beta} (\tau_{yB}^r + f w_d_A) x + w_d x, \quad V' = -\frac{\tau_{yB}^r}{\beta} - \frac{f}{\beta} w_d, \quad w_e = w_d,$$

(14a)

$$P' = P_x + \frac{f}{\beta} (w_d_A - w_d) x = P_x + \left(\tau_{yB}^s - \frac{\tau_{yB}^r}{\beta} - \frac{f}{\beta} w_d\right) x,$$

(14b)

where $w_d = -\tau_{yB}^s / f$ is the Ekman-pumping velocity. According to (14), the interiors of both oceans adjust to the steady-state Stommel balance, generalized to include forcing by $w_d$; indeed, when the
forcing includes only \( w_d (\tau^* = 0) \) the response simplifies to the Stommel and Arons (1960a,b) solution for an upper layer. The constant of integration, \( P_e = gr_c^2/2 \), is the value of \( P \) along the eastern-ocean boundary, and its value provides a measure of the pycnocline depth in each basin. The property that \( P_e \) is constant follows from the second of Eqs. (1), since \( v = 0 \) for the Atlantic and Pacific interior solutions and \( \tau^* = 0 \) at their closed eastern boundaries. Note that, although \( P_e \) is constant, \( h_s \) is not since \( g^* \) varies with latitude; for example, for the \( T^* \) defined in (11b), \( h_s (y_h)/h_s (0) = \sqrt{g^* (0)/g^* (y_h)} = 2.56 \) (Schloesser et al., 2012).

3.2.1.2. With interior entrainment. In solutions forced by \( \tau^*_s (y) \), for which \( w_{xh} \) has a positive part that thins \( P \) to the west, it is possible for \( h^* = h_m \) within the domain. Let \( x_h (y) \) designate the longitude where \( h^* \) first thins to \( h_m \) in either the Atlantic or Pacific (\( x_h = x_{hA} \) or \( x_{hP} \)), in which case it follows from (14b) that

\[
x_h (y) = -\frac{P_e - P_m}{\tau^*} - \left( \frac{f / \beta}{\tau^*_s} - \left( \frac{f}{\beta} \right) \right) w_d .
\]

(15)

where \( P_m (y) = \frac{1}{2} g^* (y) h_m^2 . \) Within the area west of \( x_h \) (Region B), there is upwelling into the mixed layer driven by Ekman pumping, which ensures that \( h^* = h_m \). Eqs. (1) without mixing then give

\[
U' = -\frac{P_w}{f} = -\frac{1}{2} \frac{g^* (y) h_m^2}{f} \quad \Rightarrow \quad V' = \frac{\tau^*}{f} \quad \Rightarrow \quad P' = P_m \quad \Rightarrow \quad w_e = w_m + w_d = w_{ek} .
\]

(16)

According to (16), the flow consists of Ekman drift and a thermally-driven, geostrophic, zonal current. It follows from continuity that \( w_e = U_e + V_e = w_{ek} \), so that \( w_m = w_{ek} - w_d \); thus, the entrainment rate into layer 1 changes from \( w_d \) outside Region B to \( w_{ek} \) within it. Note that Region B does not exist if \( x_h (y) < -L \) for all \( y \), which can happen if \( P_e \) is sufficiently large, \( L \) is too small, or \( \tau^*_s \) too weak (see Section 4.5).

Let \( y_{xh} \) and \( y_{xh} \) be the latitudes where \( x_h (y) \) intersects the western boundary. With \( \tau^* \) known, can be found by setting \( x_h = -L \) in (15), defining them as functions of \( P_e \). Then, the area integral of \( w_{ek} \) throughout Region B is

\[
W' (P_e) = \int_{y_{xh}^1}^{y_{xh}^2} \left( \int_{y_1}^{y_2} w_{ek} (y) dy \right) dy = \int_{y_2}^{y_1} \left( \frac{P_e - P_m}{f} - \left( \frac{\tau^*_s}{f} \right) + \frac{P_w}{f} + w_{ek} \right) dy
\]

\[
= -\frac{L}{f} \left( \tau^*_s + f w_{ek} \right)_{y_{xh}^1}^{y_{xh}^2} + \frac{1}{2} \int_{y_1}^{y_2} g^* (y) h_m^2 dy + \int_{y_{xh}^1}^{y_{xh}^2} x_h (y) w_{ek} dy .
\]

(17)

where the second line follows from using (15) to replace \( w_{xh} \), and the third from the property that \( x_h (y_{xh}^1) = x_h (y_{xh}^2) = -L . \) According to (17), \( W' \) is a known function of \( P_e \), albeit not a simple one because \( x_h, y_{xh}^1, y_{xh}^2 \) all depend on \( P_e \) in nontrivial ways. Generally, \( W' \) is inversely related to \( P_e \). As \( P_e \) decreases, \( x_h (y) \) shifts eastward, \( y_{xh}^1, y_{xh}^2 \) shifts southward (northward), and the area of Region B and hence \( W' \) increases. Nonaka et al. (2006) obtained a version of (17) with \( g^*_s = w_{ek} = 0 . \)

3.2.2. Southern Ocean

The dynamics of the Southern Ocean differ fundamentally from those in the Atlantic; Because the Southern Ocean has no meridional boundaries, there is no mechanism to generate Rossby waves and hence circulations cannot adjust to Sverdrup balance; instead, they adjust to equilibrium via horizontal mixing, a much slower adjustment process. South of the boundary layer along \( y_s \), the interior solution is \( x \)-independent so that Eqs. (1) reduce to

\[
- f V' = \tau^* - v U', \quad f U' + P_s = 0 , \quad V'_s = w_s .
\]

(18)

We neglect the damping in the meridional momentum equation because \( |U'| \gg |V'| \) in the Southern Ocean and \( f |f| \gg v \). On the other hand, we retain it in the zonal momentum equation since the same scalings indicate that \( |V'| \ll |U'| \). Interestingly, sensible layer-1 solutions still exist in the limit that \( v \to 0 \) (see the end of Section 4.2.2.1).

It is useful to define the quantity

\[
V' (y, h') = \int f U' = -\frac{V}{f} w' ,
\]

(19)

According to the first of (18), it is the difference between \( V' \) and the Ekman driven \( \tau^*/f \), and thus equals the eddy-driven component of the meridional flow across the Southern Ocean.

The Southern-Ocean equations, structure of \( V' \), and hence the solutions and constraints obtained below, follow directly from closure (2). See Appendix A.2 for a discussion of Southern-Ocean equations and solutions that utilize different closures.

3.2.2.1. Solutions. Solutions to (18) differ depending on the stratification, that is, on whether \( h > h_m, h = h_m \), or layer 1 doesn’t exist (Fig. 5). In particular, constraints (22), below, are obtained from the solution in the first region, and constraint (22b) requires knowledge of the solution in the second. For completeness, we briefly summarize the response in the third.

Solution where \( h = h_m \); Thickness \( h \) is greater than \( h_m \) in the latitude band that extends from \( y_s \) to either Antarctica at \( y \), the outcrop latitude \( y' \), or latitude \( y_0 \) where layer 1 vanishes. Let \( y = y_s, y' \), or \( y_0 \). Since \( w_m = 0 \) when \( h > h_m \), the solution to (18) in this band is

\[
U' = \frac{\tau^*}{f} \left[ \frac{M}{L} + w_d (y - y') \right], \quad V' = \frac{M}{L} + w_d (y - y'), \quad w_e = w_d ,
\]

(20a)

\[
P' = P_w - \frac{1}{f} \int_{y_0}^{y} \left( \frac{\tau^* (y)}{f} + \frac{M}{L} + w_d (y - y') \right) dy , \quad y > y_0 ,
\]

(20b)

where \( \int f = f (\bar{y}), P_w (y_s) = P (y_s) \) is the value of \( P \) just south of \( y_s \), and \( L \) is the zonal extent of the Southern Ocean.

The slope of \( P', P_w \), is given by the integrand of (20b). Note that it depends on \( M \) and \( w_d \) as well as \( v \) and \( \tau^* \). As a result, only when \( M = w_d = 0 \) \( P_w \) proportional to the amplitude of \( \tau^* \) and inversely proportional to \( v \), as might be expected. Although the \( w_d \) term is negligible for realistic \( w_d \) values, \( M \) need not be (see Eqs. 22b and 22c below for its non-zero values). When \( M \neq 0 \), it significantly impacts \( P_w \) since it is of the order of the Ekman transport. Generally, \( M \) weakens the sensitivity of \( P_w \) to both \( v \) and \( \tau^* \).

Another noteworthy property of solution (20) is that, when \( \tau^* \neq 0 \) and \( w_d \) has a realistic value, the integrand in (20b) is negative definite so that \( P_w > 0 \) and \( P' \) decreases realistically to the south. When \( \tau^* = 0 \), the integrand is determined only by the \( M \) and \( w_d \) terms, and so \( P_w \) is positive only if \( M < 0 \), that is, when there is a reverse MOC (Section 5).

Region where \( h = h_m \); For our choices of \( \tau^* \) and \( w_d \), Ekman pumping is positive across the Southern Ocean and its magnitude is larger than the entrainment rate due to \( w_d \). Thus, \( h \) necessarily thins to \( h_m \) almost everywhere south of \( y' \), the exception being the very narrow region adjacent to Antarctica noted in the next paragraph. The solution to (18) with \( h = h_m \) is then
\[ \begin{align*}
U' &= -\frac{P_{my}}{T} = -\frac{1}{2D}g_s h_m^2, \quad V' = -\frac{\tau_y}{T} + V_m', \\
W_e &= -\left(\frac{\tau_h}{T}\right)_y + V_{my}', \quad P' = P_m, \quad h = h_m, 
\end{align*} \]

where \( V_{m}'(y') = V'(y', h_m) = \frac{1}{2} (v^f g_s h_m^2 \Delta) \) is the southward, eddy-driven transport due to \( v \). For our purposes, the key property of (21) is that \( V' \) just south of \( y' \) is known.

When \( g'(y < y_0) = 0 \), solution (21) holds everywhere in the region \( y_0 < y' < y_1 \) where layer 1 exists. When \( g' \neq 0 \) so that layer 1 extends to \( y_1 \), however, it requires some modification. If \( \tau^x = \tau_0 \), there must be additional upwelling at the Antarctic coast to balance the offshore divergence due to \( V'(y_1) \); that upwelling is not provided for in (18), and must be supplied by including a horizontal Ekman layer (see Appendix C.2.1.2). If \( \tau^x = \tau_0 + \tau^s \), solution (21) only extends southward to the latitude \( y_1 \), where \( V' \) in (21) equals \( w_0(y_1 - y) \), which for standard model parameters lies very close to Antarctica. South of \( y_1 \), the solution is (20) with \( M = 0, P = P_0, P_s = P_m(y_1) \). Interestingly, \( h' \) thickens to the south in this region even though \( P' \) decreases. This opposite behavior happens because \( P' \) depends on \( g' \), so that \( h'_y = \left( P'_y - \frac{1}{2} g' h'^2 \right) / (g' h') \). Thus, \( h'_y \) and \( P'_y \) have opposite signs if \( 0 < P'_y < g' h'^2 \), an inequality that holds when \( y < y_1 \).

**Region without layer 1** \((y < y_0)\). For \( y < y_0 \), the model consists of a single layer, \( V_2 = \tau^x f + V_2 = 0 \) owing to mass continuity and the boundary condition \( V_1(y_0) = 0 \). In this region, then, the near-surface, northward Ekman drift is balanced by a southward, eddy-driven (or bottom–drag–driven) flow \( V_2 \) at depth. Mass balance is achieved by upwelling throughout the domain, which is driven by Ekman pumping in the interior ocean and by offshore Ekman drift at the Antarctic coast.

3.2.2.2. **Southern-Ocean constraints.** The three, possible, layer-1 structures result in three different Southern-Ocean constraints that link the values of \( P' \) at the southern and northern edges of layer 1, namely, \( P' = P(y) \) and \( P_s' = P_m(y_1) \). When layer 1 extends to Antarctica, it follows from Eqs. (20) that

\[ \begin{align*}
V'(y) &= \mathcal{M} = 0, \quad P = P_0, \quad \mathcal{M} = \int_{y_0}^{y_1} \frac{\tau^x}{f} \left[ \frac{\tau^x}{f} + W_0(y-y_1) \right] dy, 
\end{align*} \]

where the first equation in (22a) is required since there can be no flow across the continental boundary, and the second is (20b) with \( y = y_1 \) and \( P' = P(y_1) \). If layer 1 outcrops, then

\[ \begin{align*}
\mathcal{M} &= -\frac{\tau^x(y)}{f} \mathcal{L} + V'(y', h_m) \mathcal{L}, \\
P_m &= P_0 + \int_{y_1}^{y_0} \frac{\tau^x}{f} \left[ \frac{\tau^x}{f} + \frac{\mathcal{M}}{\mathcal{L}} + W_0(y-y_1) \right] dy, 
\end{align*} \]

where \( \mathcal{M} = \int_{y_0}^{y} \mathcal{V}'(y') dx = \mathcal{V}'(y) \mathcal{L} \) from (21), and the second equation is (20b) with \( y = y_1 \). \( P'(y') = P_0 \), and \( f' \equiv f(y') \). When \( g'(y < y_0) = 0 \) and layer 1 extends southward to \( y_0 \)

\[ \begin{align*}
\mathcal{M} &= -\frac{\tau^x(y_0)}{f_0} \mathcal{L} + V'(y_0, h_m_0) \mathcal{L}, \\
0 &= P_0 + \int_{y_0}^{y_0} \frac{\tau^x}{f_0} \left[ \frac{\tau^x}{f_0} + \frac{\mathcal{M}}{\mathcal{L}} + W_0(y-y_0) \right] dy, 
\end{align*} \]

where in the first equation, \( f_0 \equiv f(y_0), h_m = h'_0(y_0) \), and the second equation is (20b) with \( y = y_0 \) and \( P'(y_0) = 0 \) since \( g'(y_0) = 0 \); the first equation follows from eliminating \( U' \) from the first two of equations (18), setting \( P'(y_0) = (v/f_0) g_s h_m^2 / 2 \) since \( g'(y_0) = 0 \), and identifying \( V'(y_0) \) with \( \mathcal{M} / \mathcal{L} \). Either (22a), (22b), or (22c) provides one of the constraints needed to determine MOC properties.

The difference in the values of \( \mathcal{M} \) among the three constraints is striking. In (22a), \( \mathcal{M} \) vanishes completely. In (22b) and (22c), \( \mathcal{M} \) is small (negligible) with respect to the Ekman transport because \( h \) is fixed to \( h_m \) so that \( \mathcal{M} \approx -\frac{\tau^x(y') h'}{f} \mathcal{L} \). (With \( h'(y') = h_m = 50 \text{ m} \) and \( f' = 0.25 \text{ s}^{-1} \), \( \mathcal{M} \approx -\frac{\tau^x(y') h'}{f} \mathcal{L} \approx 400 \text{ cm}^2/\text{s} \).)

**3.3. Boundary layers**

An underlying assumption in all the boundary-layer solutions discussed in this section and in Appendix C is that they are narrow with respect to the radius of the earth. Mathematically, this assumption allows \( f \) to be set to a constant value across zonally-oriented boundary layers, considerably simplifying derivations. Because the zonal layers broaden to the west, however, their widths can increase to be \( O(R/10) \) or larger in the western ocean. To obtain good agreement between our analytic and numerical solutions, we correct the boundary-layer solution along \( y_s \) to allow for variable \( f \) (Appendix D).

**3.3.1. Atlantic western-boundary layer**

The Atlantic western-boundary transport, \( V_{y}^w(y) \), is obtained by integrating all the sources of inflow/outflow to the boundary from south to north. Eqs. (23)–(25) assume that the domain includes both the Pacific and Atlantic Oceans. Their Atlantic-only form is obtained by setting \( \mathcal{L} = L, u' = 0, \) and, redefining the area, \( A, \) to involve only the Atlantic (\( \mathcal{L}^w, \) and \( A^w \) defined below).

Assume for the moment (for convenience in writing down the following expression) that the boundary layers along \( y_s, y_{w}, \) and \( y_o \) are infinitesimally thin (i.e., their \( y \)-structure is a Dirac delta function). Then, in the latitude range from \( y_{w} < y < y_o \),

\[ \mathcal{M} = \int_{x_{y,w,0}}^{x_{y,0}} \mathcal{L} V'(y) dx = \int_{y_{w}}^{y_{w}} U(-L, y) dy, \]

where \( \mathcal{L} = L \) and \( y_{w}^w(y_{w}^w) \) is a latitude just south (north) of \( y_{w} \). Transport \( \mathcal{L} V'(y_{w}) = M + \mathcal{L}^w \mathcal{L} w_{dy} \mathcal{L} \), \( y = y_{w} \), \( y \), or \( y_o \) is the net transport that enters into (or detraints from) layer 1 in the Southern Ocean, all of which is channeled to the western boundary by the boundary current along \( y_s \). The second term on the right-hand side of (23) balances the meridional transport of the flow across \( y_{w} \) from the interior Atlantic, which is also carried to the western boundary by the \( y_{w} \)-boundary layer. The third and fourth terms on the right-hand side result from the zonal flow into the western boundary from the interior ocean, with the latter due to a jet that extends across the Atlantic from the tip of Africa along \( y_{o} \) (see Fig. 2 and the discussion of Eq. 34 below).

It is possible to evaluate \( V_{y}^w \) for all \( y \). For our purposes, it is sufficient to determine its value at \( y = y_{w} \). Integrating the continuity equation in (1) over the areas \((-L < x < 0, y_{w} \leq y < y_{w}^w) \) and \((-L < x < -L, y_{w} \leq y < y_{w}) \) to eliminate the \( U' \) integral, using
the second of Eqs. (14a) to evaluate the zonal integrals of \( V(y_n) \), \( V'(y_n) \), and \( V''(y_n) \), and using (34) below to replace \( \mathcal{U}''_P \) gives

\[
Y''_A(y_n) = M + W'_A + W''_A + \frac{L_n}{\beta} \left( \tau''_x + f w_d \right), \tag{24}
\]

where \( W'A \) and \( W''_A \) are defined in (17) and \( \tilde{\alpha} \) is the area of the domain north of \( y_0 \) or \( y_0 \) excluding Regions B in the Atlantic and Pacific Oceans. (As noted above, when Regions B exist the upwelling rate within them is \( w_A - w_{\text{et}} \) not \( w_A \); thus, the upwelling transport within them is \( W'_A + W''_A \) \( w_A - w_{\text{et}} \tilde{\alpha} \) rather than \( w_A \tilde{\alpha} \). where \( \tilde{\alpha} \) is the area of Regions B.) According to (24), \( Y''_A(y_n) \) is the sum of: all the water that entrains into layer 1 anywhere in the basin (terms 1–4 on the right-hand side); and a term that balances the transport out of (or into) the northern boundary, \( \int_{L_n}^x V'(x, y_n)dx \), which is channeled to the western boundary by the Atlantic interior circulation (last term).

It is straightforward to find \( F(-y, y) \), the value of \( F \) along the Atlantic western boundary. Assuming along-boundary geostrophy, an integral of the first of (1) across the boundary layer gives \( P(-L, y) = P(-L, y) - fV''_n \). With the aid of (14b) and (24), its value at the northern boundary is

\[
P_{\text{out}}(y_n) = P_A - \frac{L_n}{\beta} M - W'_A - W''_A - \frac{L_n}{\beta} \tilde{\alpha}, \tag{25}
\]

where \( \tau_n \equiv \tau''(y_n) \) and \( P_{\text{out}}(y) \equiv P(-L, y) \).

The derivation of (25) assumes there is no upwelling within the western-boundary layer, that is, \( P_{\text{out}} \geq P_n \) so that \( w_A = w_{\text{et}} = 0 \) in the layer for all \( y \). This property (as well as \( P_{\text{in}} \geq P_n \)) holds for all the solutions reported in this paper. See Schlosser et al. (2012, 2014) for examples of western-boundary layers with upwelling.

### 3.3.2. Atlantic northern-boundary layer

Properties of the Atlantic northern-boundary layer differ depending on whether \( w_n \neq 0 \) or \( w_n = 0 \). For notational simplicity, we let the Atlantic eastern boundary be located at \( x = 0 \). When the basin includes the Pacific, \( x \rightarrow x + L_P \) in (26) and (29) below. (Although not relevant to the present discussion, when \( w_n \neq 0 \) the northern-boundary layer has inner and outer parts, and the inner one contains a zonal overturning cell; see Sections C.2.1.2 and C.2.1.3.)

3.3.2.1. Solution with \( w_n \neq 0 \). Assuming along-boundary geostrophy and that \( f = f_s \) across the narrow boundary layer, the integral of the second of the second of (1) across the boundary current is

\[
P(x, y_n) = P(x, y_n) - f \mathcal{U}'_P, \tag{29a}
\]

Then, using (14b) and (6b) to set \( P(x, y_n) = P_n \), gives

\[
\mathcal{U}'_n(x) = -\frac{1}{\beta} \left[ P_n - P_A - \left( \tau_n - f_s \tau_{ny} - \frac{f_s^2}{\beta} w_d \right) x \right], \quad x < 0, \tag{26}
\]

where \( \tau''_n \equiv \tau''(y_n) \). At the northeastern corner of the basin, we impose continuity of transport between the western- and northern-boundary layers, that is,

\[
\mathcal{U}'_n(-L_n) = V''_n(y_n). \tag{27}
\]

Eqs. (26) and (24) then imply that

\[
P_A = P_n + \tau_n L_n + f_s M_n, \quad w_n \neq 0 \tag{28}
\]

where \( M_n = M + W'_n + W''_n + \frac{L_n}{\beta} \tilde{\alpha} \), the net sinking within the northern boundary layer, is the sum of all the entrainment transports everywhere else in the basin. Eq. (28) provides another of the constraints needed to obtain solutions. In a few solutions we replace (28) with the Q-forced constraint (7), which differs from (28) in that \( P_n = \tau_n = 0 \) and the presence of factor \( x \) (Section 4.2.6).

Constraint (28) clearly shows the influence of \( P_A = \frac{1}{\beta} g''(y_n) L_n^2 \) in setting the Atlantic thermocline depth, \( P_A \); indeed, for standard parameters, \( P_A \) dominates the other terms. Further, the value of \( P_A \) is sensitive to \( T_n \) through \( g''(y_n) \). For example, with \( h_n = 1500 \text{ m} \) and \( T_n = 3 \text{ C} \), then \( P_A = 5.1 \times 10^5 \text{ m}^3/\text{s}^2 \); whereas if there is no northern cooling \( T_n = T_s = 23 \text{ C} \), \( P_A \) increases to \( 38 \times 10^3 \text{ m}^3/\text{s}^2 \). Such a large increase in \( P_A \) would impact all MOC properties significantly; it would, for example, eliminate all upwelling in the North Atlantic Subpolar Gyre by eliminating Region \( B_n \) and deepen the Southern-Ocean stratification considerably.

Finally, note that the derivation of constraint (28) assumes that there are jumps in \( \mathcal{U}'_P \) and \( P \) at the northeast corner of the Atlantic basin, where the eastern-boundary conditions require that \( \mathcal{U}'_P(0) = 0 \) and \( P(0, y_n) = P_n \). These jumps are required to provide a sink for \( M_n \), but it is not obvious in our steady-state model that they are located in the northeast corner. In fact, they must occur there (Nonaka et al., 2006). Suppose, for example, that we try to construct a steady-state solution in which \( P_A = P_n \), so that the jumps occur at the northwest corner. In our time-stepping model, these jumps will be quickly removed by the southward propagation of Kelvin waves along the western boundary; the resulting pressure change will spread throughout the basin (both basins when there is a Pacific) via familiar wave adjustments, eventually altering \( P_n \) until it is given by (28). Conversely, northeastern jumps cannot be so eliminated because coastal Kelvin waves propagate northward along the eastern boundary.

A corollary of the preceding argument is that the downwelling location differs depending on whether the northern boundary of the basin is north or south of the equator: If it is south of the equator, coastal Kelvin waves propagate in the opposite direction, and hence the downwelling region must shift to the northwest corner. In this regard, MH13 impose a sponge layer analogous to constraint (6a) in their OGCM solutions when the northern boundary lies along the equator. They did not mention where downwelling occurred, but it is likely concentrated in the western, equatorial ocean.

3.3.2.2. Solution with \( w_n = 0 \). When \( w_n = 0 \), there is no northern-boundary entrainment or detrainment, and no large-scale MOC or northern-boundary zonal cell is possible. An integral of the continuity equation gives

\[
\mathcal{U}'_n = -V''(x, y_n), \tag{29b}
\]

a statement that water diverges from (or converges into) the northern-boundary current uniformly everywhere along the boundary. Assuming alongshore geostrophy with \( f = f_s \), the integral of the second of equations (1) across the boundary current, together with (14b) and (29a), gives

\[
P(x, y_n) = P_A + \tau_n x, \quad x \leq 0, \tag{29b}
\]

that is, \( P(x, y_n) \) tilts along the northern boundary to balance the wind.

3.3.3. Pacific boundary layers

Since \( w_n = 0 \) in the Pacific, its northern-boundary layer is given by (29) with \( P_A = P_s \). The Pacific western-boundary transport is obtained similarly to that for the Atlantic except that, because there is no northern-boundary sink, it is convenient to fix the upper limit of the boundary integral to \( y = y_n \). The integration gives

\[
V''_P(y) = V''_P(y_n) + \int_{y_n}^y U(-L_P, y')dy', \tag{30}
\]
where \( V_p'(y_b) \) balances the Pacific-interior flow \( V'(x,y_b) \), and \( (or \ out \ of) \) the northern boundary. Integrating the continuity equation in Eqs. 1 over the area \((-L_p \leq x \leq 0, y_b \leq y \leq y_b)\), to eliminate \( U' \) integral and using the second of Eqs. (14a) to evaluate the zonal integrals of \( V'(y_b) \) and \( V'(y_a) \), gives

\[
V_p'(y_b) = \frac{L_p}{\beta} \left( \tau_p + f w_d \right) \bigg|_{y_b} - \int_{y_b}^{y_a} \left( -\tau_p - f w_d \hat{A}_p \right) dy
\]

(31)

where \( \hat{A}_p \) designates the area of the Pacific north of \( y_b \) and without Region B (since \( w_e = w_d \) not \( w_e \) there). According to (31), all the water that upwells in the Pacific north of \( y_b \) plus the water that flows northward across \( y_b \) in the interior of the Pacific flows southward to the tip of Africa. Along-boundary geostrophy gives,

\[
P(\tau_p - L_p, y_b) = P(\tau_p - L_p, y_b) - f V_p'(y_b), \text{ so that}
\]

\[
P_w(y_b) = P_b = L_p \tau_b + f_y W_p + w_d \hat{A}_p
\]

(32)

where \( \tau_b = \tau'(y_b) \) and \( P_w(y_b) = P(-L_p, y_b) \). The governing equations require that \( P \) is continuous around the tip of Africa. Since \( \hat{P} \) everywhere along the west coast of Africa, the condition requires that \( \hat{P}_s = P_w(y_b) \), and hence (32) gives

\[
\hat{P}_s = \hat{P} + L_p \tau_b - f_y \left( W_p + w_d \hat{A}_p \right)
\]

(33)

Eq. (33) provides the constraint needed to link the Pacific and Atlantic Oceans.

3.3.4. Boundary layer along \( y_b \)

There is a zonal current along \( y_b \) across the Atlantic (Appendix C.3), with a transport \( \hat{U}_c(x) \) determined by the mismatch in the \( \hat{P}' \) fields across \( y_b \). Assuming along-boundary geostrophy,

\[
\hat{U}_c'(x) = \frac{-P(x,y_b) - P(x,y_n)}{f_y} \bigg|_{y_b} = \int_{y_b}^{y_n} \left( \tau_p - L_p \right) dy
\]

(14b)

with \( P_b = \hat{P} \), and along \( y_n \) is (14b) with \( x \rightarrow -x \) and \( L_p = P_b - \hat{P} \). With the aid of (33), it follows that

\[
\left( \frac{L_p}{\beta} \tau_p + f w_d \right) \bigg|_{y_b} - \hat{W}_p + w_d \hat{A}_p
\]

(34)

a constant transport across the basin that contributes to the Atlantic western-boundary current (23). The first term on the right-hand side of (34) is the transport of water that flows into (or out of) the Pacific across \( y_b \) due to the interior Sverdrup and Stommel-Arons (1966a,b) flows; the second and third terms ensure that the water upwelled in the Pacific north of \( y_b \) flows across the Atlantic. Note that \( \hat{U}_c'(x) = V_p'(y_b) \), as required by mass conservation.

3.3.5. Boundary layer along \( y_a \)

The final constraint links the Southern Ocean to the Atlantic and Pacific Oceans along \( y_a \). Specifically, it relates \( P_a \) to \( P_b \), the values of \( \hat{P}' \) for the Southern-Ocean interior solution evaluated at \( y_a \) and along the eastern boundary of the Atlantic (Pacific) Ocean, respectively. In contrast to the other boundary layers, it is not possible to derive the constraint simply by integrating across the layer, because of the cyclicity requirement (8) in the southern half of the layer. Consequences of this complexity are that the constraint depends on the structure of the boundary layer itself and that an exact expression for it cannot be found. Nevertheless, it is still possible to obtain an accurate approximation,

\[
P_a = P_b - a \left( \lambda \frac{L}{\beta} \right) \left( \tau_a - \frac{L}{\beta} \right) \hat{A}_a + b \left( \frac{4}{3} \lambda \right) \left( \frac{L}{\beta} \right) \hat{A}_a \hat{A}_a
\]

(35a)

where

\[
a = 1 + \frac{3}{4} \frac{L}{\beta} - 2 \frac{2 \sqrt{2} L}{\beta} \left( \frac{L}{\beta} \right) \hat{A}_a
\]

\[
b = 1 - \frac{3}{4} \frac{L}{\beta} + 2 \frac{2 \sqrt{2} L}{\beta} \left( \frac{L}{\beta} \right) \hat{A}_a
\]

(35b)

\[
\tau_a = \tau'(y_a), f_a = f(y_a), \delta \text{ is defined in (13c),} \quad \ell \text{ is the width of the Southern Ocean, and} \quad \Delta y = y_a - y_b \text{ is the distance from the southern edge of layer 1 to} \quad y_a \text{ (Appendix C.4). Specifically, (35a) with} \quad a = b = 1 \text{ is the first-order expression for} \quad P_a \text{ in a sequence that rapidly converges (Appendix C.4.3), and the terms in} \quad a \quad \text{and} \quad b \quad \text{are proportional to} \delta/R \quad \text{and} \quad r/\ell \quad \text{are corrections that arise from} \quad f \quad \text{varying across the boundary layer and the finite width of the western-boundary current, respectively. Finally,} \quad (35a) \quad \text{is valid only if reflections of the} \quad y_a \quad \text{boundary solution from the southern edge of layer 1} \quad \text{are small at} \quad y_a \text{ (Appendix C.4.4), which is true for most of our solutions.} \]

As we shall see, values of quantities predicted using (35) are close to their exact numerical counterparts (‘‘s in Figs. 8a–8c and 12). This good agreement supports the approximations that lead to (35). At the same time, given their extent, the agreement is remarkable: It must result from errors in one part of the expression canceling those in another (see Appendix C.4).

As noted in the introduction, previous studies have used simpler relations than (35a) to connect the Southern and Atlantic Oceans, equivalent either to setting \( P_b = P_e \) (Radko and Kamenkovich, 2011; \( a = b = 0 \)) or \( P_b = P_e - \tau_e \ell /2 \) (Samelson, 2009; \( a = 1, b = w_d = 0 \)). In our solutions, however, these simple relationships don’t hold because the \( a \) and \( b \) terms are not both negligible. For example, in solutions with \( w_d \neq 0 \) and for standard parameter values, \( P_e \geq P_b \geq 5.1 \times 10^9 \text{m}^2/\text{s}^2 \) is the largest term on the right-hand side of (35a): With \( \tau_e = \tau_e = f_a \ell ^1/2 \approx 100 \) and \( \delta / \ell \approx 0.1 \), although the \( a \) term is \( O(P_e \ell ^1/2) \) and hence negligible, the \( b \) term is \( O(P_e) \) and is not. The magnitude of the latter term, as well as its dependency on \( v \), is a mathematical statement of the sensitivity of (35a) to the structure of the \( y_a \)-boundary layer.

4. Wind-driven solutions

In this section, we discuss solutions that are primarily wind forced \( (M \geq 0) \), organizing them as much as possible in a hierarchical manner. For each solution, we use the concepts and constraints developed in Section 3 to describe its flow field and to determine its MOC properties. For some, we illustrate their flow fields with plots from numerical versions of the model; in so doing, it is understood that unless stated otherwise we use the standard model parameters listed in Section 2. Solutions are reported for Southern-Ocean stratifications both when \( g' \neq 0 \) (warm-\( T^* \) forcing) and \( g' (y \leq y_b) = 0 \) (cold-\( T^* \) forcing); the two sets are labeled Solutions \( n \) and \( n' \) (Solutions \( Ph \) and \( Pn' \)), \( n = 1 \), when the basin has only the Atlantic and Southern Oceans (includes the Pacific), respectively (Table 1).

Most solutions are obtained in the basin with only the Atlantic and Southern Oceans (Sections 4.1–4.4). We begin by reporting solutions without an MOC (Section 4.1). Then, we discuss a suite of solutions with MOCs that sequentially add entrainment processes: Southern Ocean upwelling \( M \) (Section 4.2), interior diffusion \( \mathcal{N} \) (Section 4.3), and upwelling in the North Atlantic Subpolar Gyre \( \mathcal{V} \) (Section 4.4). We conclude by commenting on the impacts of including a Pacific Ocean (Section 4.5).

4.1. Solutions without an MOC

Solutions without an MOC \( (M = 0) \) are possible for the barotropic response, the case considered by Gill (1968), and, with some restrictions, for the layer-1 response.
4.1.1. Barotropic response

4.1.1.1. Numerical solution with $\psi_{xx}$. For the barotropic response (Solution 1a), Eqs. (1) hold with $\psi_{xx}$ = 0. Fig. 6a shows the resulting solution for the streamfunction $\psi$ ($V = \psi_x$, $U = -\psi_y$) when $\nu = 2 \times 10^{-6}$ s$^{-1}$, $\tau^i(y) = \tau_0 = 1$ dyn/cm$^2$, and $\beta = 1.6 \times 10^{-11}$ cm$^2$ s$^{-1}$, obtained by the numerical method discussed in Appendix B.2 when the $\psi_{xx}$ term is retained in (B1). Consistent with the analytic solution, there is an ACC that extends across the Southern-Ocean interior (Eq. 20a with $M = w_0 = 0$), a boundary layer along $y = y_a$ that smoothly joins the Southern and Atlantic circulations (Appendix C.4), and no flow in the Atlantic interior (Eq. 14 with $\tau^i(y) = w_0 = 0$). There is no zonally-averaged meridional flow, a statement that the southward, eddy-driven flow $V'\approx(y,h) = \langle V/f \rangle U'$ everywhere balances the northward Ekman drift. Finally, the value of the ACC transport determined from (B4) is $\psi_0 = 1.32(\tau_0/\nu)L_y = 109$ Sv.

The spatial structure of the $y_a$-boundary layer is also consistent with the analytic solution. It is noticeably thinner south of $y_a$ than north of it, with widths close to the analytic values ($\delta = 445$ km and $\langle \Delta L_4 \rangle = 1584$ km). Furthermore, part of the ACC extends northward along the east coasts of South America for a distance of about $\Delta L_4 = 1584$ km as a western-boundary current, and its width is close to that of a meridional Stommel (1948) layer, $r = 125$ km. The western-boundary current then bends offshore to flow eastward and weakly southward across the Atlantic, and finally circulates around the tip of South America to close the ACC circulation. As noted by Gill (1968), northward bending of a part of the actual ACC does occur along the east coasts of continents, but the degree to which mixing is the cause of the bending in the real ocean is still not clear. Boundary currents along $y_a$ have a similar structure in all of our solutions.

4.1.1.2. Numerical solution without $\psi_{xx}$. In deriving our analytic solution for the $y_a$-boundary layer (Appendix C.4), we neglect the term $\psi_{xx}$ in the $P'$-equation, thereby eliminating the ageostrophic region at, and the western-boundary current near, the tip of South America (Fig. 2). To demonstrate the impact of this neglect, we obtained a solution to (B1) without $\psi_{xx}$, $\psi$, using the numerical method of Appendix B.3. Fig. 6b (left panel) illustrates the response, showing $\psi(x,y)$ near the tip of South America. The solution lacks the aforementioned subregions, and as a result there is a jump in $\psi$ from the west to the east coast near its tip ($y = y_a$). Fig. 6b (right panel) illustrates the jump more clearly, plotting $\psi(0,y)$ and $\psi(0',y)$ (red and green curves, respectively) and $\psi(0,y)$ for comparison (blue curve). The $\psi$ profile is the same along $x = 0'$ and $x = 0$, decreasing continuously to zero as $y$ approaches $y_a$ from below and vanishing identically for $y > y_a$. In contrast, $\psi(0,y)$ and $\psi(0',y)$ coincide for $y < y_a$, satisfying the periodicity condition, but do not coincide for $y > y_a$, since the boundary condition $\psi = 0$ is satisfied along the western edge of the barrier but not along the eastern edge.

Fig. 6c shows the difference $\Delta \psi = \psi - \psi$; the contribution to $\psi$ that emanates from the two subregions. It is confined primarily near the tip of South America, validating our neglect of $\psi_{xx}$. Nevertheless, $\Delta \psi$ does modify the large-scale flow field to some extent. Let $\psi_0$ and $\psi_0$ be the values of $\psi$ and $\psi$ along Antarctica ($y = y_a$). Then, one measure of the impact of $\Delta \psi$ is that $\psi_0 = 1.39(\tau_0/\nu)L_y$, whereas $\psi_0 = 1.32(\tau_0/\nu)L_y$ so that $(\psi_0 - \psi_0)/\psi_0 = 5\%$. Its impact is also demonstrated by the need to correct constraint (35a) to account for the finite width of the western-boundary current.

\[ \Delta \psi = \psi - \psi \]

\[ |\Delta \psi| = |(\psi_0 - \psi_0)/\psi_0| = 5\% \]

\[ \psi(0,y) \]

\[ \psi(0',y) \]

\[ \psi(0,y) \]

\[ \psi(0',y) \]
compensate for the neglect of the barotropic component of the ACC transport, that largest along the eastern edge, and reaches its maximum value of 0.34 at the tip. Away from the barrier Δψ is much smaller, less than 0.03 over 90% of the basin.

(Appendix C.4.5). Gill (1968) also commented on the smallness of Δψ away from South America (see the discussion of his Fig. 8).

4.1.1.3. ACC transport. The analytic model provides an expression for the barotropic component of the ACC transport, \( \tilde{U} = \tilde{U}^b + \tilde{U}^r \), where \( \tilde{U}^b \) and \( \tilde{U}^r \) are the transports across the interior of the Southern Ocean and within the \( y \)-boundary layer, respectively. (To avoid confusion with the baroclinic part of the ACC transport, we designate the barotropic part with a circumflex.) According to (20a) with \( M = w_\theta = 0 \),

\[
\tilde{U}^r = \frac{1}{V} \int_{y_s}^{y_0} \tau^b dy = \frac{\tau_b}{V} L_y.
\]  

(36a)

Since we don't calculate the boundary solution exactly, we relate \( \tilde{U}^r \) geostrophically to the jump in \( P \) from \( P_s \) to \( P_\Lambda \) at the tip of South America by setting \( \tilde{U}^r = -(P_s - P_\Lambda) / f \omega \), and, with the aid of (35a),

\[
\tilde{U}^r = \frac{1}{V} \left[ \frac{4}{3} \tau_\omega L_y + \frac{1}{3} \frac{L_x}{f} \right] \tau_s L_y.
\]  

(36b)

With \( V = 2 \times 10^6 \) and \( \tau_s = \tau_o \), (36b) gives \( \tilde{U}^r = 0.33(\tau_o/v)L_y \), so that \( \tilde{U} = 1.33(\tau_o/v)L_y = 110 \) Sv. This value is close to \( \psi_9 \) and hence supports the validity of the approximations built into (35a) that compensate for the neglect of Δψ.

Note that \( \tilde{U} \) is proportional to \( \tau_s \) and is almost inversely proportional to \( v \) (not exactly because of \( \tilde{U}^r \)). These strong sensitivities contrast with solutions to highly resolved OGCMs in which \( \tilde{U} \) varies more weakly with each parameter. A possible explanation for the weak sensitivities is that \( \tilde{U} \) is weak because the barotropic response is blocked by the Drake–Passage sill (Munk and Palmen, 1951), so that the total ACC transport is dominated by its baroclinic part \( \tilde{U} \) (Sections 4.2.2.5, 4.2.4.3, and 5.2).

4.1.2. Layer-1 response

No-MOC solutions for layer 1 are only possible if \( w_\theta = 0 \), and the solutions discussed here all adopt that restriction. They are possible both when layer 1 extends to Antarctica (\( g' = 0 \)) and when it extends to \( y_0 \) (\( g' = 0 \) for \( y < y_0 \)).

4.1.2.1. Response when \( g' \neq 0 \). When \( w_\theta = 0 \) (Solution 1b), the solution is essentially the same as Solution 1a. It is possible, however, that the initial state of \( P \) allows \( h \) to become less than \( h_m \) during the spin-up. In that case, \( w_m \) becomes active, and there is upwelling into layer 1 until the smallest value of \( h \) in the domain is \( h_m \); as a result, the value of \( P_s \) is not arbitrary as it is in Solution 1a, but rather is constrained by the condition \( \min(h) \geq h_m \).

When \( w_m \neq 0 \) (Solution 1c), the response differs from that of Solution 1b in that (28) applies so that \( P_s \) is linked to \( P_n \); in this case, there is entrainment into (or detrainment from) layer 1 in the northeast corner until the value of \( P_n \) ensures that \( M_n = 0 \). A result of fixing \( P_n \), however, is that Solution 1c does not exist for all values of \( v \). To ensure that \( M = 0 \), layer 1 must extend to Antarctica, which only occurs if \( v \) is larger than a critical value \( v_1 \). An equation for \( v_1 \) can be obtained by eliminating \( P_n \) and \( P_m \) from (22a), (28), and (35a), and setting \( P_n = P_m \) and \( M = M_b = w_m = \Psi'_m \). With \( \tau^b = \tau_0 \) \( \tau^s = \tau + \tau^s \), \( \tau_s = \tau_m \), its solution gives \( v_1 = 5.15 \times 10^{-6} \) s\(^{-1} \) \( \psi_1 = 3.56 \times 10^{-6} \) s\(^{-1} \), the large difference in values arising largely from the change in the slope of \( P \) (integral of Eq. 22a) due to the different wind structures. (See Section 4.2.2.2 for discussion of a related critical value \( v_0 \).)

4.1.2.2. Response when \( g'(y < y_0) = 0 \). A particularly interesting, no-MOC solution exists when \( w_\theta = 0 \), \( g'(y < y_0) = 0 \), and layer 1 extends to \( y_0 \) (Solution 1b). Similar to Solution 1d, the first of Eqs. (22c) with \( M = 0 \) requires the balance \( \tau^s(y_0) / f \omega = V(y_0) h'_0 = 0 \) \( V(y_0 - 0) = \tau_0 / f \omega \). In this case, though, the balance determines \( h'_0 \) rather than \( y' \). With \( h'_0 \) \( V_2 \) known, constraints (22c) and (35a) then determine the thermocline thickness \( h'_0 \) throughout the Atlantic. Thus, \( h \) is determined entirely from the Southern Ocean by \( Q \) through its specification of \( y_0 \) and \( g' \) and by \( V \) through its impact on \( h'_0 \) and \( h'_2 \). Solution 1b is the only solution to our model that has this property. As noted in Section 1, the same property holds for type-1 isopycnals in WC10s OGCM solutions because they are deep enough not to be affected by \( Q \) in the North Atlantic (analogous to setting \( w_s = 0 \) in our model).

Solution 1b is in fact the only possible no-MOC state when \( g'(y < y_0) = w_\theta = 0 \). Let \( h'_0 = 2(\tau_0 / v) \tau^s(y_0) / g_b' \) be the value of \( h'_0 \) in Solution 1b. Suppose initially that \( h = h_m \) throughout the basin; then, if \( h_0 < h'_0 \), \( h_0 > h'_0 \), water will entrain into (detrain from) layer 1 across \( y_0 \) during the spinup, eventually adjusting the solution to the state, \( h'_0 \), \( h'_2 \), of Solution 1b. Furthermore, if \( w_\theta = 0 \) (Solution 1c) a no-MOC solution is possible only as a special case of Solution 1b. Let \( P_n \) be the value of \( P_n \) from Solution 1b. Then, according to (28) a no-MOC state is possible only when \( P_n = P_n - h'_0 L_a = \psi'_m \) \( P_\Lambda \) \( P_n \). Any other choice of \( P_n \) ensures that \( M = 0 \): Suppose that \( P_n < P_m \); then, \( P_n < P_\Lambda \), and hence \( h'_0 < h'_2 \) and \( M > 0 \); conversely, if \( P_n > P_m \), a reverse MOC.

4.2. Solutions with Southern-Ocean upwelling

Here, we discuss layer-1 solutions with \( w_\theta = 0 \) and \( w_d = 0 \) driven by \( \tau^s = \tau_0 \) (Solutions 2 and 3) and \( \tau^s = \tau_s + \tau^s \) (Solutions 3 and 3). For these solutions, Southern-Ocean entrainment (or detrainment) is balanced by northern-boundary processes (i.e., \( M_n = M \)). We first describe solutions to the numerical model, which illustrate the spatial structure of solutions and demonstrate the accuracy of the analytic ones (Section 4.2.1). Then, we discuss the sensitivities of MOC properties to \( v \) for Southern-Ocean stratifications when \( g' \neq 0 \) (Section 4.2.2) and \( g'(y < y_0) = 0 \) (Section 4.2.3), and to \( \tau_s \) (Section 4.2.4). We conclude by reporting their sensitivities when \( v \) increases with \( \tau_\omega \) (Section 4.2.5) and for different northern-boundary constraints (Section 4.2.6). In Sections 4.2.2 and 4.2.4, we derive approximate expressions for the MOC properties that capture their dominant characteristics and highlight the processes that determine them.
4.2.1. Numerical solutions

Fig. 7 illustrates two solutions to the time-stepping, numerical model when \( v = 2 \times 10^{-6} \) s\(^{-1}\) and \( \tau^+ = \tau_0 + \tau^*_s \), plotting \( P \) for Solution 3 (top panel) and \( h \) for Solution 3' (bottom panel). In Solution 3, layer 1 outcrops along a line (white curve) that has a zonal-mean latitude of \( \varphi = 56.6^\circ \) S, close to the value of 56.8\(^\circ\) S for the analytic solution predicted by (37) below. Note that the line has a weak \( x \)-dependence whereas the analytic solution assumes there is none; it has \( x \)-dependence because the \( y_0 \)-boundary layer is vanishingly small at \( y' \) and because \( \delta_0(x) \) in (5a) has a finite value of 1 day (see Appendix C.4.4). In Solution 3', layer 1 vanishes south of \( y_0 \) (white area). Although \( P \) vanishes along \( y_0 \) where \( g^* = 0 \), \( h_0 \) does not. As for \( y', h_0 \) is not \( x \)-independent, again a consequence of the \( y_0 \)-boundary solution not vanishing at \( y_0 \) (Appendix C.4.4). The curious \( h \) structure centered on 40\(^\circ\) S is caused by the abrupt change in slope of \( g^* \) at that latitude.

In both solutions, the Southern-Ocean flow field is similar to that in Fig. 6a, except that layer 1 extends only to \( y' = y^* \) or \( y_0 \) rather than \( y_0 \), and the flow also contains a zonal-mean, meridional transport \( M \) due to entrainment across \( y \). As noted in Section 3.1.2, the entrained water flows northward within the ACC to join the \( y_0 \)-boundary layer where it is channeled to the tip of South America; there, it enters the Atlantic western-boundary current, and is carried to the northern ocean to downwell in the northeast corner of the basin.

To validate the analytic solutions, Figs. 8a–8c and 12 plot data points (\( \times \)’s) from solutions to the time-stepping, numerical model (Section 2.7). The points all lie very close to the analytic curves, confirming the accuracy of the analytic constraints and the corrections built into (35a).

4.2.2. Sensitivity to \( v \) when \( g^* \neq 0 \)

4.2.2.1. Solution method and general properties. When \( g^* \neq 0 \) (warm-\( T \) forcing) and layer 1 outcrops, the applicable constraints for the analytic model are (22b), (28), and (35a), providing a set of 4 equations for the 4 unknowns, \( M, P, P_\nu, \) and \( y' \). (The number of constraints and unknowns is somewhat arbitrary since, with a Pacific Ocean and northern-ocean upwelling, Eq. 28 also involves \( W_\nu, M_\nu, \) and \( P_\nu \) through \( M_\nu \). Stating that there are 4 constraints and unknowns assumes that \( W_\nu, M_\nu, \) and \( P_\nu \) are known functions of \( P_\nu \) through Eqs. 17 and 33.)

Generally, it is not possible to solve the set analytically, but it is straightforward to do so numerically. Keeping all terms (for later reference), elimination of the first three variables from the equation set gives

\[
\begin{align*}
\int y' f^2 \left[ \left( \frac{\tau^+}{f} - \tau_0 \right) - V'' - \omega_d (y' - \tilde{y}) \right] dy' \\
+ \frac{4}{3} b h^2 f^2 \left[ \left( \frac{\tau^+}{f} - \tau_0 \right) - V'' - \omega_d (y_0 - \tilde{y}) \right] \\
= v \left[ P_n - P_m + \tau_n b a + f a \left( \frac{f}{f} L + V' L + W'_\nu + W'_p + W'_d \right) \right] \\
- a L \left( \frac{1}{2} \tau_0 - \frac{f^2}{f} w_d \right),
\end{align*}
\]

(37)

where for Solutions 2 and 3, \( y' = y, \tilde{f} = f(y), \tilde{\tau} = \tau^+(y, \tilde{y}), \tilde{V'} = V'(y, \tilde{y}), \tilde{h}, \tilde{h} = h_m, \) and \( W'_\nu = W'_\nu = W'_p = 0 \). For a given \( v \) and \( \tau'^*, y' \) is iterated numerically until (37) is satisfied. With \( y' \) known, (22b), (28), and (35a) provide \( \mathcal{M}, P_n, \) and \( P'_\nu \) to complete the solution. (The same

---

**Fig. 7.** Maps of \( P \) for Solution 3 (top panel) and \( h \) for Solution 3' (bottom panel) from solutions to the numerical model using standard model parameters. The white curve in the top panel indicates the curve \( y'(x) \) where layer 1 outcrops, and \( P = P_n \) south of the curve. The white area in the bottom panel indicates where layer 1 does not exist \( (\tau^+ < \tau_0) \), and its northern edge lies along \( y' = y_0 \). Values of \( h \) are determined by \( h = \sqrt{2P/g} \) everywhere except along \( y_0 \), where they are given by \( h_0 = \sqrt{2P_0/g} \). The \( P \) and \( h \) fields in both panels are similar, differing primarily in their values along \( y' \) and \( y_0 \). Thickness \( h = h_0 \) and \( P = P_n \) along \( y' \) in Solution 3, whereas \( h \gg h_0 \) and \( P = 0 \) along \( y_0 \) in Solution 3'.

**Fig. 8a.** Curves of MOC properties as a function of \( v \) for Solutions 2–5 when \( \tau_0 = \tau_0 \) in \( \tau^*_s \) and \( g^* = 0 \), showing \( \gamma \) (top-left panel), \( h_m \) (top-right panel), \( \mathcal{M} \) (middle-left panel), \( M_\nu \) (middle-right panel); they are linearly related through Eq. (28), \( \tau^+(y)/f \) and \( V' \) (solid and dashed curves, respectively; bottom-left panel), and \( W'_\nu \) (solid and dashed curves, respectively; bottom-right panel). The solutions are forced by \( \tau^+ = \tau_0 \) (Solution 2, red curves), \( \tau^+ + \tau^*_s \) (Solution 3, black curves), \( \tau^+ + \tau_n + \tau^*_s \) (Solution 4, blue curves), and by \( \tau^+ + \tau_n + \tau^*_n + W'_d \) (Solution 5, magenta curves). The dashed curve (top-left panel) is \( \delta = \sqrt{h_0/g} \), the thickness of the \( y_0 \)-boundary layer in the Southern Ocean. The dot-dashed curve (middle-left panel) indicates the Ekman transport, \( \tau_0/f \), that flows into the Atlantic across \( y_\gamma \). Crosses (\( \times \)) indicate values of variables from corresponding solutions to the numerical model. Data points of \( y' \) for \( v < v_0 \) are the zonal average of the \( x \)-dependent outcrop latitude \( (y')^* \); those of \( h \) for \( v > v_0 \), are given by \( \sqrt{\langle h \rangle} \).
Eq. (37) provides a concise summary of the model physics in a single equation for $y$. (As such, it is analogous to the $h_m$ equation of Gnanadesikan, 1999), albeit more complex owing to the different representation of, and inclusion of additional, processes.) Several general properties of solutions are evident in (37). First, it involves all model parameters and forcings, a statement that the outcrop latitude $y'$ is determined globally. Second, it involves $Q$ (more specifically $T'$) only through the values of $g'$ in $p_m$ and $g'_y$ in $V'$, so that there is no direct linkage of $T'$ to the outcrop latitude. (There is, however, a direct linkage for cold-$T'$ forcing in that layer 1 can extend southward only to latitude $y = y_0$ where $T' = T_2$; see Sections 4.1.2.2 and 4.2.3.) Finally, solutions still exist when $v = 0$, in which case only the integral in (37) remains, and its solution is then $y' = y_0$. In this limit, it follows from the first of (22b) that $M = r_s A / f x$ the Ekman transport into the Atlantic across the latitude of the tip of South America, since $V' = 0$ as $v = 0$. Further, all the boundary currents have the structure of a Dirac $\delta$-function, that is, they are infinitesimally narrow and infinitely strong; in particular, $P$ jumps from $p_m$ to $P = p_m + f y / \beta w_0 x$ across $y = y_0$, so that the baroclinic transport of the ACC, $u^x = (P' - p_m) / f x$ flows along $y_0$. These properties mimic the response of coarse-resolution models without GM mixing, in which isopycnals tilt strongly upwards south of South America.

4.2.2.2. Outcrop latitude. For both Solutions 2 and 3, the outcrop latitude $y'$ (top-left panels of Figs. 8a and 8b; red and black curves) shifts southward until $v$ reaches a critical value $v_1$, at which layer 1 first extends to Antarctica; thereafter, Solution 2 no longer exists for the range of $v$ values in the plot (discussed next) whereas in Solution 3 the southern edge of layer 1 remains at $y_1$. The southward shift is considerably slower in Solution 3 than in Solution 2 (compare black and red curves, respectively, in Fig. 8a). This marked difference in $\Delta y = y_0 - y$ between the two solutions indicates the dynamical importance of the weakening of $\tau^s$ across the Southern Ocean, through its impact on the slope of $P$ across the basin (integrand of Eq. 20b); the slope is weaker for Solution 2 ($\tau^s = \tau_o$) because $\tau^s / f$ depends only on the variation of $f$ and hence the factor, $M / L + \tau^s / f = \tau^s / f y' + V'(y', h_m) = \tau^s / f y'$, in the integrand is smaller. In both solutions, the width of the southern half of the $y_o$-boundary layer, $\delta = \sqrt{4 L / \beta |f|}$ (gray-dashed curve), is always less than $\Delta y$ (much less than $2 \Delta y$) for all $v$, so that inequality (C32) is satisfied as required for the accuracy of constraint (35a).

An equation for $v_1$ can be found similarly to that for $v_1$ (Section 4.1.2.1), except with (22a) replaced by (22b) in the limit that $y' = y - y_0$; thus, values of $v_1$ and $v_1$ differ only because $M \neq 0$ and $M = 0$ in their respective equations. For Solution 2, $v_1 = 0.428 \times 10^6$ s$^{-1}$ is much less than $v_1 = 5.15 \times 10^6$ s$^{-1}$ because the slope of $P$ across the Southern Ocean is smaller with $M \neq 0$ (compare Eq. 22b with $p_m = p_m$ with Eq. 22a). From the definitions of $v_1$ and $v_1$, Solution 2 now exists with $M \neq 0$ only for $v \geq v_1$ ($v \leq v_1$). Thus, it does not exist in the interval $v_1 < v < v_1$ and the red curves end in Fig. 8a when $v = v_1$. In contrast, for Solutions 3–5 which $\tau^s = \tau_o + \tau^s_r$, $M - V' L \leq 0$ as $y' \rightarrow y_0$; consequently, $v_1 \leq 0$, there is no solution gap, and their curves don’t end for $v > v_1$. (Note that, when $v_1 > v$, so that layer 1 extends to Antarctica and $M = 0$, the solutions are the same as Solution 1c.)

It is instructive to obtain an approximate solution to (37) that captures the behavior of $y'$. For Solutions 2 and 3, $L = L_o$ and $W_L = W_L = W_L = 0$, and $p_m$ and $V'$ are both negligible since $p_m \approx p_m$ and $|V'(y', h_m)| = |\tau^s / f y'|$. For Solution 3, we replace $\tau^s = \tau_o + \tau^s_r$ with the linear form $\tau^s = \tau_o [1 + (y - y_0) / L]$, linearize the Ekman drift terms using (11c) with $y_o = y_0$ to $\tau^s / f = (\tau_o / f_o) [1 + (y - y_0) / L]$ where $L = L_o R_s / (L_s + R_s)$, and set

**Fig. 8b.** As in Fig. 8a, except for Solutions 3 ($g' = 0$, black curves) and 3' ($g' = 0$ for $y < y_0$, gray curves), showing $y'$ (top-left panel), $h'$ (top-right panel), $M$ (middle-left panel), $u_{max}$ and $u'$ (solid and dashed curves; middle-right panel), $\tau^s (y') / f'$ and $V'$ (solid and dashed curves; bottom-left panel), and $P_o$ and $P'_o$ (solid and dashed curves; bottom-right panel).

**Fig. 8c.** As in Fig. 8b, except as a function of $y_o$ when $v = 2 \times 10^6$ s$^{-1}$. In the middle-left panel, $M$ (black curve) asymptotes to a straight line (dashed line) for large $y_o$ and the value of $M$ is always less than the Ekman transport across $y_o$ (dash-dot line).
$f^2 = f_0^2$ in the integral. With these simplifications, (37) reduces to a quadratic equation for $y'$. The complete expression for $\Delta y = y_0 - \bar{y}$, $y = y'$ or $y_n$, is then

$$\frac{\Delta y}{\delta} = \left( \frac{16 b^2 \bar{z}^2}{\pi} + 2 \pi \left( \frac{f_0 - \bar{a}}{f_0 - \bar{a}} \right) \right)^{1/2} - \frac{4 b \bar{z}}{\pi}, \quad v \leq \bar{v},$$

(38)

where $\bar{z} = 1 + (3 n/4 b) |f_0|/|f_{0a}| (\delta/ar{R}_e)$, $\bar{a} = \bar{a}_c$, and $\bar{L} = \bar{R}_e/\bar{a} = 0.21$. For Solution 2 ($t^* = \bar{a}$), the variation of $t^*$ in the Ekman-drift differences can be dropped by taking the limit $L_0 \to \infty$ in $L$, in which case $L = \bar{R}_e$ and (37) simplifies to (38) with $\varepsilon = 1$. The approximate critical value $\bar{v}_c$ the value of $y$ where $\Delta y$ first increases to $L_0$ in (38), is obtained by equating the top and bottom expressions; with $a = 0 = b = 1$, $\bar{v}_c = 0.453 \times 10^{-6} \text{s}^{-1}$ ($4.057 \times 10^{-6} \text{s}^{-1}$) for Solution 2 (Solution 3), close to the exact values of $y_1$ in Fig. 8a.

Since $\dot{a}$, $\dot{a}$, and $\dot{b}$ in (35b) are slowly varying functions of $v$ with values close to 1, the right-hand side of (38) is approximately independent of $v$, and hence $\Delta y \propto \delta \times \sqrt{v}$. With $\bar{t}_0 = 1 \text{dyn/cm}^2$ and $\bar{b} = \bar{a} = 1 = \bar{a}_c = 7.3 \bar{a}_0$ for Solution 2 ($\bar{e} = 1$) and $\Delta y = 3 \bar{a}_0$ for Solution 3 ($\bar{e} = 0.21$), so that $y'$ in Solution 2 shifts southward with $v$ almost 2) times faster than Solution 3 does, consistent with the exact curves. The property that $\Delta y \propto \sqrt{v}$ is noteworthy: It indicates the importance of $M$ in weakening the across-basin tilt of $P'$ (integrand of (20b)), since $P' \propto v^2$ when $M = 0$ (also see Section 4.2.2.4). Finally, the impact of the northern-boundary condition (terms with subscript $n$) on $\Delta y$ is evident in (38); in this regard, for realistic values of $v$ and $\bar{a}_c$, the $P_n$ term is an order of magnitude larger than the other terms in square brackets.

4.2.2.3. Overturning strength. To simplify $M = L_a(t^*/y)'/f' + L_a V' (y', h_m)$ weakens markedly with $v$ (middle-left panels of Figs. 8a and 8b). It is essentially entirely determined by the Ekman-drift contribution since $V'$ is negligible when $h = h_m$ (compare solid-black and dashed-black curves, bottom-left panels), and it decreases because $f'$ increases and (for Solution 3) $t^*$ decreases as $y'$ shifts southward. Note that $M$ is always less than $f_0 L_a|f_{0a}|/f_0$, the Ekman drift at the latitude of the tip of South America (compare the solid and dotted-dashed curves in the middle-left panels of Figs. 8a and 8b); the difference, $\Delta M = M - f_0 L_a|f_{0a}|/f_0$, thus measures the strength of the eddy-driven circulation at $y_m$, $V'(y_m)$. Transport $M$, is equal to $M$ (middle-right panel of Fig. 8a), because $M$ is the only entrainment source for Solutions 2 and 3. The values of $M$ and $M_n$ are smaller than observed because the width of the Southern Ocean is only $L = L_a$ (compare to Fig. 12, where its width, $L = L_a + L_s$, three times larger).

To obtain an approximate expression for $M$ when layer 1 outcrops, we write $M = t^*(y'/L_a f')$ since $V'(y', h_m) \approx 0$ and linearize the Ekman drift to get

$$M = \begin{cases}
\frac{f_{0a}}{|f_{0a}|} \left( 1 - \frac{|f_0|}{|f_{0a}|} \right) & \text{if } v \leq \bar{v}_c, \\
0 & \text{if } v > \bar{v}_c.
\end{cases}$$

(Eq. 45a, below, is another version of Eq. 39 obtained by combining it with Eq. 38.) According to (39), $M$ is identical to $y$ with a rescaled and shifted vertical axis, a property consistent with the similarity of the exact $y$ and $M$ curves for Solution 3 (compare black curves in the top-left and middle-left panels of Figs. 8a and 8b).

In OGCM solutions, $M$ also decreases as model resolution, and hence eddy activity, increase (e.g., MH13, their Fig. 3a; MJM13, their Figs. 9b and 11b). The decrease is generally viewed as being a direct response to stronger eddies, which increase $|V'|$ throughout the Southern Ocean (presumably at the outcrop latitude as well), thereby shifting the system closer to eddy compensation. Similarly, in our $g' \neq 0$ solutions increased eddy activity (larger $v$) weakens $M$, not directly by increasing $|V'(y', h_m)|$, but rather indirectly by causing $y'$ to shift southward.

4.2.2.4. Stratification. In the Atlantic, the model stratification (layer-1 thickness) is measured by $P_a$. In the Southern Ocean, it is specified by the values of $P$ at the northern and southern edges of layer 1, $P_a^N$ and $P_a^S$, and $P = P(y)$ where $y = y'$ or $y_n$. Fig. 8b plots $P_a$ for Solution 3 (bottom-right panel, solid- and dashed-black curves), and Figs. 8a and 8b plot $\hat{H} = \sqrt{2/g' P}$ (top-right panels, black curves).

From constraint (28) with $M_a = M$, $P_a$ is given by

$$P_a = P_a + t_0 L_a + f_{0a} M,$$

(40a)

According to (40a), the $P_a$ curve is the same as the $M_a$ curve, except with the vertical axis rescaled by $f_{0a}$ and shifted by $P_a + t_0 L_a$ (middle panels of Fig. 8a; middle-left and bottom-right panels of Fig. 8b). From (40a) and (35a), $P_a$ is related to $M_a$ by

$$P_a = P_a + t_0 L_a - \bar{a} \tau L_a - \bar{a} \tau L_a + |f_{0a} + \bar{a} \tau L_a|M,$$

(40b)

where $\bar{a} = b/(3 n/4 b) R_e/\bar{a}$ (also see Eq. 47). According to (40b), the structure of $P_a$ is modified from $P_a$ by the terms proportional to $a$, $\bar{a}$, and $\tau$, primarily by the latter which varies like $v^{-1}$ and has a significant amplitude (for $v = 2 \times 10^{-5} \text{s}^{-1}$, $\tau = 5 \gg a/2 = 0.55$).

Several properties of $P_a$ and $P_n$ visible in Fig. 8b follow from Eqs. (40). First, their sensitivities to $v$ are weak (much less than for $M_a$), a consequence of the large $P_n$ term dominating the others. Second, $P_a$ is considerably less than $P_n$ with, for example, $P_a - P_n \approx -1782 \text{m}^3/\text{s}^2$ at $v = 0$ (bottom-right panel of Fig. 8b). This large difference, almost half of $P_a$ itself, demonstrates the importance of using constraint (35a) to join the Southern and Atlantic Oceans rather than, for example, setting $P_a = P_n$. Finally, the sensitivities of $P_a$ and $P_n$ to $v$ change abruptly when $v > v_c$ and $M = 0$; in that case, (40a) requires that $P_a$ has the constant value $P_a + t_0 L_a$, and (40b) then requires that $P_a$ increases because the term $\tau L_a \propto \delta^{-1}$ decreases.

At the southern edge of layer 1, $P_a = P_n$ when $v < v_c$ and layer 1 outcrops. In contrast, $P = P_a$ is a variable to be determined when $v > v_c$ and layer 1 extends to Antarctica. Constraint (22a) with $w = 0$ relates $P_a$ to $P_m$, stating that its value is determined by $P_m$ at the northern edge of layer 1 and by the weakening of the tilt of layer 1 across the basin with $v$. In (22a), $P_m$ is expressed as the difference of two large terms, and consequently its behavior for $v \gtrsim v_c$ is not apparent. To isolate that dependence, we subtract (22a) evaluated at $v = v_c$ from (22a) to get

$$P = \begin{cases}
P_m & v \leq v_c, \\
P_m + P_m^{v_c} + \frac{t_0 L_a}{|f_{0a}|} f & v > v_c.
\end{cases}$$

(41)

where $f = \int_{v_c}^{v} |f/f_c| f^{2/3} dy$ and $P_m^{v_c}(v = v_c) = P_m$. According to (41), the growth of $P_m$ for $v \gtrsim v_c$ is determined by two terms: a rapid quasi-linear growth by $(v - v_c)f/F_v$, and a more gradual response due to $P_m^{v_c}$, which, from (40b) with $M = 0$ and the property that $a$ and $b$ are slowly varying functions of $v$, is given by $-\tau L_a \sim (\delta - \delta h)/\delta$. The rapid rise of $\hat{H} = \sqrt{2/g' P}$ in Fig. 8b (top-right panel) is consistent with (41).

Two measures of the average slopes of $P$ across the Southern Ocean are

$$\langle P'_y \rangle \equiv \frac{P'_y - \bar{P}}{\Delta y}, \quad \langle P''_y \rangle \equiv \frac{P''_y - \bar{P}}{\Delta y}.$$
where $\bar{P} = \bar{P}'(\bar{y})$ and $\bar{y} = y - y_a$. The first expression is the mean slope associated with the Southern-Ocean interior solution, whereas the second takes into account the jump across the $y_a$-boundary layer. When $v < v_0$ and layer 1 outcrops, $\bar{P} = \bar{P}_m$ is negligible; therefore, since $\bar{P}_a$ and $\bar{P}_b$ are roughly constant, the slopes vary like $\Delta y^{-1} \times V^+$. In contrast, when $v > v_0$, $\bar{P} = \bar{P}_a'$ is not negligible and $\Delta y = h_m$; in this case, $(P_a')'$ and $(P_b')$ vary more strongly with $v$, owing to the term $(v - v_0) F / v$ in $\bar{P}_a'$.

4.2.2.5. ACC transport. In the analytic solve, the baroclinic part of the ACC transport $\bar{u}$ has components associated with the Southern-Ocean interior response $\bar{u}'$ and the $y_a$-boundary layer $\bar{u}''$. Although $\bar{u}'$ is $x$-independent, $\bar{u}''$ is not: When $\mathcal{M} \neq \mathcal{M}_0$, mass continuity requires that $\bar{u}'$ absorbs the influx of water due to $\mathcal{M}$, and so it decreases linearly to the west from a maximum value of $\bar{u}_\text{max}$ just west of South America to $\bar{u}_\text{max} - \mathcal{M}$ just east of the Atlantic western-boundary current. Fig. 8b (middle-right panel) plots curves for $\bar{u}'$ and $\bar{u}_\text{max} = \bar{u}' + \bar{u}_\text{max}$. Transport $\bar{u}$ is obtained by numerically evaluating the integral, $\bar{u}' = \int f_0 (\bar{P}_a / |f|) dy$, and $\bar{u}_\text{max}$ is determined by the jump in $\bar{P}$ from $\bar{P}_a$ to $\bar{P}_b$ at the tip of South America, that is, by $\bar{u}_\text{max} = (\bar{P}_a - \bar{P}_b) / |f_0|$. Both transports weaken with $v$, but relatively much more weakly than $\mathcal{M}$ does.

An approximate expression for $\bar{u}''$ is obtained by setting $f = f_0$ in the integral for $\bar{u}'$, in which case

$$\bar{u}'' = \frac{\bar{P}_a - \bar{P}_b}{|f_0|}, \quad \bar{u}_\text{max} = \bar{u}' + \bar{u}_\text{max} = \frac{\bar{P}_a - \bar{P}_b}{|f_0|}. \quad (43)$$

When $v < v_0$, so that $\bar{P} = \bar{P}_m$ is negligible in (43), $\bar{u}'$ and $\bar{u}_\text{max}$ are proportional to $\bar{P}_a$ and $\bar{P}_b$, respectively, a property that visually holds for the exact curves as well (compare curves in the middle- and bottom-right panels of Fig. 8b). Thus, $\bar{u}'$ and $\bar{u}_\text{max}$ are insensitive to $v$ because both $\bar{P}_a$ and $\bar{P}_b$ are, owing to the large value of $\bar{P}_m$ in the northern-boundary constraint (Section 4.2.2.4). When $v > v_0$ and $\mathcal{M} = \mathcal{M}_0$, $\bar{u}'$ and $\bar{u}_\text{max}$ decrease because of the rapid rise of $\bar{P}$.

The close connection between $\bar{u}$ and $\bar{P}_b$ also occurs in OGC solutions (e.g., Fučkar and Vallis, 2007; compare Figs. 3a and 9c of MH13 and Figs. 3b and 11c of MJM13), an indication that their ACC strength is controlled baroclinically. Further, although $\bar{u}$ decreases with model resolution as eddy activity increases, the sensitivity is weak (e.g., compare Figs. 2 and 4b of MH13, and Figs. 9a and 9b and Figs. 11a and 11b of MJM13). The cited authors attribute the weak $\bar{u}$ dependency to the system tending toward eddy saturation, rather than to the impact of the northern-boundary constraint as in our model. (See Section 4.2.6 for a detailed discussion of how $\bar{u}$ depends on the northern-boundary constraint.)

4.2.3. Sensitivity to $v$ when $g'(y < y_0) = 0$

Fig. 8b illustrates MOC properties when $g'(y < y_0) = 0$ (cold-$T'$ forcing) for Solution 3' (gray curves). Curves for Solutions 2', 4', and 5', not shown, behave similarly to those for Solution 3'; the similarity also extends to Solution 2' because layer 1 never extends to Antarctica and hence it exists for all $v$. For sufficiently small $v$, layer 1 outcrops at $y$ in Solution 3' and the curves are essentially the same as those for Solution 3 (gray, and hence $V$ differ somewhat in the two cases, which has a negligible impact in Eq. 37 since $V^+$ is so small). As $v$ increases, however, it reaches a critical value, $v_0 = 0.129 \times 10^{-6}$ s$^{-1}$, at which layer 1 first extends to $y_b$ and layer 1 vanishes; thereafter, the southern edge of layer 1 remains at $y_b$ (top-left panel, gray curve), $h > h_{m}$, and the other curves begin to diverge. Note that, although $\delta$ begins to approach $y_b$ (dashed and solid curves in the top-left panel), it is still significantly less than $2a y$ as required by inequality (C32) for the validity of (35a).

The solution method is a modification to that described in Section 4.2.2.1. When $v < v_0$ and layer 1 outcrops, the method is unchanged, that is, (37) is iterated to determine $\bar{y}$. The critical value $v_0$ is obtained by setting $g'(y = y_0)$, $P_m = P_m'' = P_m' = w_0 < 0$, and $t = t^+ (y_0)$ in (37), and solving for $v = v_0$. When $v > v_0$ and layer 1 extends to $y_b$, $\bar{y} = y_b$ is known but $\bar{V}' = V'(y_0, h_{m})$ is not since $h_{m} = h'(y_0)$ is no longer fixed to $h_{m}$. The value of $V_0$ is found by making the above replacements, setting $V' = V'_0$ in (37), and solving for $V'_0$. With $V'_0$ known, the other MOC properties follow.

The most significant way that Solution 3' differs from Solution 3 is that $\bar{V}'$ at the southern edge of layer 1, which is always negligible when layer 1 outcrops and $h' = h_{m}$, increases nearly linearly for $v > v_0$ in Solution 3' as $h_{m}$ increases (dashed-curve, bottom-left panel; gray curve, top-right panel). Constraint (22c) with $w_0 = 0$ determines $\bar{V}' = V'_0$ when layer 1 extends to $y_0$, stating that $\bar{V}'$ adjusts the layer-1 slope (the integrand in the constraint) to ensure that $P_m'$ vanishes. As in the derivation of (41), to isolate the behavior of $V'_0$ for $v \geq v_0$ we subtract (22c) evaluated at $v = v_0$ from (22c) to get

$$V'_0 = \begin{cases} \frac{v_0 a}{c_0 - y_0} \bar{V}'_0 - (v - v_0) \frac{a}{2}, & v > v_0, \\
0, & v < v_0. \end{cases} \quad (44)$$

where $\bar{A} = \int_{y_0}^{y_b} f_0 / |f| \, dy$, $\bar{V}'_0 = \bar{V}'(y_0, h_{m})$. Its growth is dominated by the linear term in $V'_0$ so that to a good approximation $h_{m} \propto (v - v_0) / \bar{A}$. Accordingly, $h_{m}$ initially rises rapidly and then levels off as $v$ increases beyond $v_0$, a behavior apparent in Fig. 8b (gray-curve, top-right panel).

Another noteworthy property of Solution 3' is the similarity of its $\bar{P}_a$, $\bar{P}_b$, and $\bar{u}$ curves to those for Solution 3 even when $v > v_0$, despite the large change in stratification between the two solutions (compare black and gray curves in the middle- and bottom-right panels of Fig. 8b). The behavior of the two components of $\mathcal{M}$ (bottom-left panels) suggests why they remain so close: In Solution 3' (black curves), $\mathcal{M}$ decreases because the Ekman drift continues to weaken as $y'$ shifts southward; by contrast, in Solution 3' (gray curves), although the Ekman drift no longer decreases after $y'$ reaches $y_b$ (solid-gray curve), further reduction of $\mathcal{M}$ is accomplished by an increase in $|V'_0|$ (dashed-gray curve). Note that a consequence of $|V'_0|$ increasing is that it becomes larger than $\bar{V}'(y_0) / c_0$ for large $v$, in which case $\mathcal{M}$ and $\mathcal{M}_0$ change sign and the MOC reverses direction: Such a reversal is possible because our specification of $w_0$ in (6a) allows both detrainment and entrainment in the northern ocean. (See Section 5 for a discussion of solutions with a reverse MOC in which the entrainment is supplied by $w_0$.)

4.2.4. Sensitivity to $x^+$

Fig. 8c illustrates the dependency of MOC properties to the Southern-Ocean wind strength $\tau_n$, plotting curves for Solution 3
Outcrop latitude. For \( \tau_a > \tau_m \), the outcrop latitude \( \nu \) shifts northward, first relatively quickly and then slowly (Fig. 8c, top-left panel, black curve). When \( \tau_a \) is increased to unrealistically large values, \( \nu \) asymptotes to 47.9° S (\( \Delta \nu = 2.9° \)), a value less than \( \nu_0 \). These properties are evident in the top expression of approximate solution (38), which varies like \( \tau_a^{-1/4} \) for small \( \tau_a \) and asymptotes to a constant value as \( \tau_a \to \infty \); with \( v = 2 \times 10^{-6} \text{s}^{-1} \), the approximate asymptotic value of \( \Delta \nu \) is 2.4°, somewhat less than the exact asymptotic value because the asymptotic solution assumes that \( \gamma \) varies linearly. The property that \( \gamma \) never reaches \( \nu_0 \) is counterintuitive, as one expects \( P_a \) to continue to increase with \( \tau_a \). In fact, \( P_a \) does increase: Although the value of \( P_a \) at the southern edge of the layer 1 is fixed (\( \tilde{P} = P_0 \)), its value at the northern edge, \( P_a' \), continues to increase (see below).

Overturning strength. Transport \( \mathcal{M} \) grows nearly linearly with \( \tau_a \), and is always less than the Ekman drift at the latitude of the tip of South America, \( \tau_a/|s| \) [compare the solid-black and dot-dashed curves in the middle-left panel of Fig. 8c], a consequence of the layer-1 tilt across the Southern Ocean increasing with \( \tau_a \) and decreasing with \( v \). There are also critical values of \( \tau_a, \tau_m = 0.605 \text{dyn/cm}^2 \) and \( \tau_{o0} = 1.455 \text{dyn/cm}^2 \) in Fig. 8c, which correspond to \( v_2 \) and \( v_0 \) and are found by similar procedures. Because of these similarities, here we only discuss the properties of Solution 3 (\( g' = 0 \)) when layer 1 outcrops (\( \tau_a \geq \tau_m \)).

Approximate solution (39) also exhibits these properties. With the aid of (38), it can be rewritten

\[
\mathcal{M} = \lambda \mathcal{M}_a \frac{\tau_a}{\tau_a - \sigma \mathcal{M}_a + \sigma \mathcal{M}_a \chi(\tau_a)},
\]

where

\[
\chi(\tau_a) = \left[ (\gamma^2/d + \tau_a/\tau_d)^2 + (\gamma^2/d^2) \right]^{-1},
\]

\( \lambda = 1 - (\gamma - 4b_\lambda/3)(\delta/\tilde{\nu}), \quad \mathcal{M}_a = \mathcal{M}_a f_a/(|s|), \quad \sigma = (\tilde{\nu}/\nu)(d/(2\gamma)), \quad \gamma^2 = (16/9)b_\lambda^2 + 2\pi |f_a/|s| - a/2), \quad d = 2\pi c P_a + \tau_a \mathcal{M}_a/(\tau_a \mathcal{M}_a). \)

The first two terms on the right-hand side of (45a) define the approximate expression of the asymptotic line. Consistent with the exact line, its slope is \( \mathcal{M}_a/|s| \), less than that of the Ekman transport at \( \nu_b \) by the small factor \( \lambda = 0.895 \), and its \( \tau_a \) intercept is 0.854 dyn/cm². The last term in (45a) is a correction to the asymptotic line that, according to \( \chi(\tau_a) \), decreases monotonically from a maximum value of \( \mathcal{M}_a \) at \( \tau_a = 0 \) to 0 as \( \tau_a \to \infty \).

NV12, MH13 and MJM13 report the sensitivity of \( \mathcal{M} \) to \( \tau_a \) in their OGCM solutions. Consistent with \( \mathcal{M} \), waves rise quasi-linearly with \( \tau_a \) in NV12’s solutions to a coarse-resolution OGCM, provided that \( \tau_a \) is large enough for the response to lie outside a diffusion-dominated regime (their Fig. 11). In contrast to our solutions, \( \mathcal{M} \) does not continue to rise linearly in MH13’s and MJM13’s solution to eddy-resolving OGCMs, a property we can simulate by allowing \( v \) to vary with \( \tau_a \) (see Section 4.2.5).

Stratification and ACC transport. As evident in (40a), \( P_a \) is a rescaled and shifted version of \( \mathcal{M} \), and the similarity of the two curves is apparent in Fig. 8c (solid-black curves in the middle-left and bottom-right panels). According to (40b), \( P_a \) is also closely related to \( P_n \) and \( \mathcal{M} \), but that relationship is not obvious in Fig. 8c (bottom-right panel) as \( P_a \) first decreases with \( \tau_a \) and then remains roughly constant whereas \( P_n \) increases.

The different structure of \( P_a \) happens because the \( a \) and \( b \) terms in (40b), defined in (35b), are proportional to \( \tau_a \). To illustrate this point, it is useful to write down approximate forms for both \( P_a \) and \( P_n \).

\[
P_a = \lambda \frac{f_a}{|s|} \tau_a L_A + P_n + \tau_a L_A - \sigma \frac{f_n}{|s|} \tau_a L_A + \sigma \frac{f_n}{|s|} \tau_a L_A \chi(\tau_a),
\]

and with (40b) gives

\[
P_a = \mu \frac{f_a}{|s|} \tau_a L_A + P_n + \tau_a L_A - \omega \tau_a L_A + \omega \tau_a L_A \chi(\tau_a),
\]

where \( \mu = \frac{f_a}{|s|} \frac{1}{\lambda} - \frac{1}{\mu} \) and \( \sigma = \frac{f_n}{|s|} \frac{1}{\lambda} + \frac{\xi}{\omega} \). As in (45a), (46) and (47) have terms that grow linearly with \( \tau_a \) (first terms on the right-hand sides), shift terms (terms 2–4 on the right-hand sides), and corrections proportional to \( \chi \) (last terms). Primarily because of the large \( \xi \) term, the terms in \( \mu \) tend to cancel so that the asymptotic slope of \( P_a \) is much less than that of \( P_a \) (e.g., with \( v = 2 \times 10^{-6} \text{s}^{-1} \) and \( \tau_a = 1 \text{dyn/cm}^2 \), \( \mu = 0.077 < \frac{f_n}{|s|} / |s| = 1.129 \)). Furthermore, although not apparent in (46), it can be shown that \( \mu > 0 \) so that \( P_a \) increases asymptotically with \( \tau_a \). Finally, \( \omega \gg \sigma \frac{f_n}{|s|} \) because of \( \xi \) so that, since \( \chi(\tau_a) \leq 1, P_a \) is always less than \( P_a \).

The \( P_n \) and \( P_a \) curves vary much more slowly with \( \tau_a \) than \( P_n \) does, again owing to the large contribution from \( P_n \) in relations (40a) and (40b). Likewise, since \( \tilde{P} = P_n \) when \( \tau_a > \tau_m \), the baroclinic ACC transports, \( U' \) and \( U_{\text{max}} \) estimated by (43) are proportional to \( P_a \), and \( P_n \), respectively, and hence also vary slowly with \( \tau_a \) (middle-right panel of Fig. 8c). The layer slopes, \( \langle P_a' \rangle \) and \( \langle P_a \rangle \), estimated by (42), vary with \( \tau_a \) like \( \gamma \Delta \nu^{-1} \), for \( \tau_a \gtrsim 1 \text{dyn/cm}^2 \) is a weak dependence (upper-left panel of Fig. 8c).

MH13 and MJM13 also report that \( U' \) and the Atlantic stratification \( h_a \) are insensitive to \( \tau_a \) in their eddy-resolved OGCM solutions (Figs. 4b and 7a in MH13; Figs. 3a and 9c in MJM13); in striking contrast, MJM13 note further that both variables vary strongly with \( \tau_a \) in their coarse-resolution model with GM mixing (their Figs. 3a and 9c). In both studies, the authors suggest that the weak \( \langle U' \rangle \) dependency in the eddy-resolving models happens because eddy activity strengthens with \( \tau_a \) to the point where near- or total-eddy saturation is established (see next subsection), a process absent in the coarse-resolution model.

Given the geostrophic linkage between \( U' \) and \( h_a \), it then follows that, if the weak \( \langle U' \rangle \) dependency is determined by eddy saturation, then so must that of \( h_a \) be. This conclusion seems extreme, yielding too much dynamical influence on \( h_a \) to Southern-Ocean processes. In our solutions, the direction of the dynamical linkage is reversed: \( P_n \) varies weakly with \( \tau_a \) (Fig. 8c) regardless of the dependence of \( v \) on \( \tau_a \), a consequence of northern-boundary condition (28) fixing \( P_n \) to a value close to \( P_0 \), and hence \( U' \) does as well through (43). (See Section 4.2.6 for further discussion of this issue.)

Sensitivity of \( \mathcal{M} \) to \( v(\tau_a) \): 4.2.5. The MH13 and MJM13 report the sensitivity of \( \mathcal{M} \) to \( v(\tau_a) \) in their suite of OGCM solutions. In contrast to our \( \mathcal{M} = \mathcal{M}_a \) curves (Fig. 8c, middle-left panel), their \( \mathcal{M}_a \) curves are not quasi-linear but rather increase less rapidly with \( \tau_a \), a slower increase they attribute to increased eddy activity (eddy compensation). For example, in the MH13 solutions with 1/8° and 1/12° resolutions,
\( M \) increases roughly like \((\tau_a)^n\) with \(n = 0.9\) and 0.8, respectively (blue and red curves in their Fig. 3). Similarly, in the MJM13 solutions \( M_0 \) increases from about 2.5 \( \text{Sv}\) to 5–10 \( \text{Sv}\) as \( \tau_a \) strengthens from 0 to 10 \( \text{dyn/cm}^2 \) (solid curves in their Fig. 9b), whereas in Fig. 8c \( M \) varies much more strongly from 0–13 \( \text{Sv} \) as \( \tau_a \) goes from 0–4 \( \text{dyn/cm}^2 \).

Our model exhibits a similar behavior if \( v \) is allowed to increase with \( \tau_a \) in the constraints. Fig. 9 (right panel) plots \( M \) when \( v = v_0(\tau_a/\tau_0)^n \) and \( v_0 = 2 \times 10^{-6} \text{ s}^{-1} \), with \( n \) values ranging from 0 to 3/2. Consistent with the results of MH13 and MJM13, \( M \) increases more slowly with \( \tau_a \) for \( n > 0 \), with the curves for \( n = 1/2, 3/4, \) and 1 comparing best with theirs. In our model, the slower increase happens because the increase in layer slope \( M \) (integrand of Eq. 20b) due to stronger \( \tau_a \) is partially counteracted by the increase in \( v \); as a result, \( v \) shifts more slowly northward with \( \tau_a \) or even southward if \( n > 1 \) (Fig. 9, left panel), thereby weakening the Ekman transport that entrains into layer 1. The precise processes that account for eddy compensation in OGCMs are not clear. It may result from the same process (southward shift of \( v' \)) as in our model.

The sensitivity of \( v' \) to \( n \) is captured by \((38)\), and is most easily seen in the limit that \( \tau_a \rightarrow 0 \). After setting \( v = v_0(\tau_a/\tau_0)^n \) in \((38)\), \( \Delta v \) for small \( \tau_a \) simplifies to

\[
\lim_{\tau_a \to 0} \Delta v = (\tau_a)^{n+1} (A + B\sqrt{\tau_a}).
\]

where \( A = 2\pi \alpha (P_n + \tau_n h_A)/ \tau_n L_A^2, \) \( B = (4/3)\beta b, \) and \( \alpha = (v_0 L_A/\beta b)^2. \) In agreement with the plotted curves, \((48)\) states that \( \Delta v \) tends toward three values as \( \tau_a \) goes to zero: \( -\infty \) for \( n < 1, \) 0 for \( n > 1, \) and \( A \) when \( n = 1. \)

4.2.6. Sensitivity of \( \mathcal{U} \) and \( P \) to the northern-boundary constraint

As mentioned at the end of Section 4.2.4.3, MH13 and MJM13 report that both \( \mathcal{U} \) and \( h_\mathcal{U} \) are insensitive to \( \tau_a \) in their eddy-resolved models, a property they attribute to eddy saturation. In contrast, they vary strongly in MJM13’s and NV12’s coarse-resolution models. In our solutions that use northern-boundary constraint \((28)\), \( \mathcal{U} \) and \( P \) are always insensitive to \( \tau_a \) owing to the larger term \( P_n \). As discussed next, they can exhibit strong sensitivity to \( \tau_a \) provided that \((28)\) is replaced by \((7)\).

Constraint \((28)\) is a simple parameterization of cooling processes external to the basin via sponge layer \((6b)\). Another possible constraint is \((7)\), which results from cooling by \( Q \) within the North Atlantic (e.g., WC10; MJM13; NV12; Schloesser et al., 2012, 2014). Fig. 10 illustrates differences in curves for \( M \) (left panels) and \( P \) (right panels), obtained using one or the other of the constraints when \( g' (y < y_0) = 0 \). (Curves when \( g' \neq 0 \) are similar, differing only slightly when \( \tau_a < \tau_{a0} \).) The figure plots curves for different values of the constant term, \( P_\Pi + \tau_n h_\Pi \), in \((28)\), multiplying it by the factor \( \phi = 1, 1/2, \) and 0 (black, gray, and light-gray curves, respectively). It also plots curves for constraint \((7)\) when \( \alpha = 1 \), 1/2, and 1/10 (light-gray, gray-dashed, and black-dashed curves). Note that the two constraints are the same when \( \phi = 0 \) and \( \alpha = 1 \), so that the light-gray curves apply to each case. For the curves using \((28)\), \( P \) shifts negatively as \( \phi \) decreases with little change in slope. For the curves using \((7)\), \( P \) slopes more strongly as \( \alpha \) increases and \( M \) slopes less. In both cases, \( M \) and \( P \) have opposite tendencies: An increase in \( P \) causes either \( y' \) to shift southward if layer 1 outcrops or \( V_0 \) to increase if it extends to \( y_0 \), thereby decreasing \( M \).

Generally, the spread of the \( M \) curves is much less than for the \( P \) curves; in particular, note the very large difference in the structures of \( P \) when \( \phi = 1 \) and \( \alpha = 1/10 \) (solid-black curve) and \( \phi = 0 \) and \( \alpha = 1/10 \) (dashed-black curve).

As shown in Fig. 10, our model can mimic a large sensitivity of \( \mathcal{U} \) and \( P \) to \( \tau_a \) only by using constraint \((7)\) with small \( \alpha. \) Schloesser et al. (2012) find that their analytic solutions with \( \alpha = 0.8 \) compare well with analogous OGCM solutions. Further, they find that the value of \( \alpha \) increases with the strength of horizontal mixing. We can parameterize this dependency in the present model by allowing \( y' \) near the northern boundary to increase with \( M \), a process representing increased eddy mixing due to a stronger northern-boundary current. In this case, the sensitivities of \( \mathcal{U} \) and \( P \) to \( \tau_a \) will decrease markedly as \( M, v, \) and \( x \) change from small (\( x \approx 1 \)) to large (\( x \approx 1 \)) values. Thus, the large difference in their sensitivities in solutions to eddy-resolving (e.g., MJM13) and coarse-resolution (e.g., NV12 and MJM13) models may result from an analogous difference in the dynamics of their Q-forced, northern-boundary constraint, with \( x \) being small (large) in their coarse-resolution (eddy-resolving) models. (Another, and perhaps more likely, possibility is that eddy mixing lowers the northern-boundary density, leading to a shallower MOC; as a result, the bottom of the MOC in the eddy-resolving solutions may no longer be captured by their definition of \( h_\Pi \)).

The differences among the curves in Fig. 10 highlight the importance of the parameterization of the northern-boundary constraint. The two types, \((28)\) and \((7)\), have their own advantages and disadvantages. Constraint \((28)\) is practical in that it arises from the application of sponge layer \((6b)\), but it does not specify the processes that determine \( P_\Pi \) (e.g., marginal-sea and overflow processes). In contrast, constraint \((7)\) results from a well-defined process, namely, Q-forcing within North Atlantic. On the other hand, for realistic parameter values \( P_\Pi \) (and \( h_\Pi \)) are too small in Q-forced solutions; further, when \( \alpha \approx 1 \), \( M \approx P_\Pi/h_\Pi \) and \( \mathcal{U} \approx P_\Pi/f_n \) are of the same order, whereas in reality \( \mathcal{U} \) is an order of magnitude greater than \( M \) (Allison et al., 2011).
4.3. Solutions with diffusive forcing

When \( w_0 \neq 0 \) (Solutions 3 and 4'), there is an additional entrainment process in the system with an upwelling transport of \( W_0' = w_0 A \), where \( A \) is defined after (24). With \( w_0 = 2.5 \times 10^{-6} \text{ cm/s} \) and \( A = \frac{1}{3}(y_n - y_e) = 60 \times 10^{16} \text{ cm}^2 \), \( W_0' \sim 1.5 \text{ Sv} \), a non-negligible amount in comparison to the Ekman drift across the Southern Ocean (\( \lesssim 5 \text{ Sv} \)). Velocity \( w_0 \) modifies the horizontal structure of the response by generating an additional Stommel-Arons (1960a,b) circulation in the interior of the Atlantic (except for Region B_5), which channels the diffusively-entrained water to the western boundary where it flows northward in the western-boundary current; otherwise, the response is essentially the same as shown in Fig. 7.

The procedure to solve the constraints is essentially the same as that for Solutions 3 and 3', except that it retains the \( w_0 \) terms in (37). Specifically, we iterate (37) with the \( w_0 \) terms to find \( y' \) when layer 1 outcrops (\( v \leq v_0 \) or \( v_0 \)), \( P_s' \) when it extends to Antarctica (\( v > v_e \)), or \( V(P_s) \) when it ends at \( y_0 \) (\( v > v_0 \)). Fig. 8a illustrates how solution properties vary with \( v \) for Solution 4 when \( w_0 = 2.5 \times 10^{-6} \text{ cm/s} \) (blue curves). The curves are similar to their counterparts for Solutions 3 (black curves), differing due to the additional upwelling transport \( W_0' \) (bottom-right panel). The contribution from \( W_0' \) increases \( M_0 \) (middle-right panel) in comparison to that for Solution 3 and, according to (28), \( P_s \) (middle-right panel) increases to allow for that entrainment. Because \( P_s \) is larger, so is \( P_s' \) (Eq. 35a); consequently, at each \( v \) value either \( y' \) shifts farther southward (top-left panel) or \( h'_O \) and hence \( V'(y_0, h'_O) \) are larger (not shown), so that \( M \) decreases (middle-left panel). In the MJO13 solutions \( M_0 \) decreases as resolution increases from 1/2° to 1/6° (i.e., as \( v \) increases), possibly for the same reason as in our model, that is, because \( M \) decreases. Note that when \( v > v_e \) and \( M = 0 \), \( M \) is entirely determined by \( V(P_s) \) (middle panel of Fig. 8a); in this regime, then, the response is primarily diffusion-forced solution, essentially an extension of the Stommel and Arons (1960a,b) solution discussed in Section 5.1 to allow for \( \tau^s \neq 0 \).

4.4. Solutions with North-Atlantic upwelling

When the model is forced by \( \tau^s = \tau^s_n + \tau^s_m + \tau^s_y \) (Solutions 5 and 5'), solutions develop a North Atlantic Subpolar Gyre, and \( h' \) thins in its northern half in response to Ekman suction. If \( \Delta \tau_{\text{en}} \) is weak enough for \( h' \) to remain thinner than \( h_n \), MOC properties are unchanged from those of Solutions 4 and 4'. On the other hand, if \( \tau^s_n \) is large enough for \( h' \approx h_n \) in Region B_n, layer-2 water entrains into layer 1 there (\( \nabla V_x = 0 \)), and an additional MOC forms that is confined to the northern hemisphere. This situation is similar to that considered by Tsujino and Sugino-hara (1999), except that their sinking region is located in the southern hemisphere.

To solve the analytic constraints when Region B_n exists, we use the same iteration procedure as for Solutions 4 and 4', except with (37) modified to allow for the possibility that \( \nabla V_x = 0 \). Specifically, for each \( y \) constraint (22b) and (35a) determine \( P_n \) (17) then gives \( V'_x(P_n) \), and the iteration proceeds as before. Fig. 8a illustrates how properties of Solution 5 vary with \( v \) (magenta curves) when \( \Delta \tau_{\text{en}} = 3 \text{ dyn/cm}^2 \). A comparison of the curves to those for Solution 4 (blue curves) shows that the impact of including \( W_0' \) is small, except of that including \( V'_x \). The additional entrainment requires that \( M_0 \) and \( P_n \) increase (middle-right panel), which in turn either shifts \( y' \) southward (top-left panel) or increases \( V'(y_0, h) \) if \( g'(y < y_0) = 0 \) (as in the bottom-left panel of Fig. 8b), thereby reducing \( M_0 \).

Interestingly, the numerical version of Solution 5 with \( v = 2 \times 10^{-6} \text{ s}^{-1} \) lacks Region B_n, and hence its data points are identical to those for Solution 4 in Fig. 8a. It is absent because the Atlantic interior response in the numerical solution is not given by the inviscid form (14b), but rather includes effects due to Rayleigh damping: The damping smooths the edges of Region B_n so much that \( h \) never thins to \( h_m \). To illustrate the circulation due to \( W_0' \), then, Fig. 11 shows the numerical version of Solution 5 with the damping coefficient reduced to \( v = 0.25 \times 10^{-6} \text{ s}^{-1} \). The white curves indicate the edges of Region B_n in the numerical (solid) and analytic (dashed) models, the difference between them indicating the impact of the damping. Consistent with (16), there is upwelling within Region B_n with a transport of 1.78 Sv equal to the area integral of Ekman pumping. The upwelled water flows southward across Region B_n via Ekman drift as described in (16), eventually crossing the Region-B_n boundary to circulate about the Subpolar Gyre. When it reaches the western boundary, it flows northward in a western-boundary current, as in (23) with \( M = 0 \). Finally, it flows eastward in a northern-boundary current given by (26) to downwell in the northeastern corner of the basin (as in Fig. 11).

Properties of the northern-boundary layer are also consistent with the analytic solution. The westward broadening of the northern-boundary layer is apparent in Fig. 11, and its average width is close to the value in (13b) with \( v = 0.25 \times 10^{-6} \text{ s}^{-1}, \) \( (\Delta \tau_{\text{en}})^{1/3} = 367 \text{ km} \), consistent with its being a zonal Stommel (1948) layer (Appendix C.2). Within the northern-boundary sponge layer where (6a) is imposed (the analog of the inner boundary layer in the analytic model discussed in Appendix C.2.1), there is upwelling everywhere across the basin with downwelling confined to the northeast corner (not shown), an indication of the presence of a zonal overturning cell (Appendix C.2; Nonaka et al., 2006). Similar northern-boundary layers are present in all of our solutions.

4.5. Solutions with a Pacific Ocean

It is straightforward to find analogs of all prior solutions when the domain includes a Pacific Ocean (Solutions Pn and Pn'). The most striking difference from the Atlantic-only solutions is that, because the domain width is increased from \( L_n \) to \( L_r \), the Southern-Ocean and diffusive entrainments, \( M \) and \( W_0' \), increase roughly by the factor \( L_r/L_n \), thereby increasing the Atlantic overturning transport \( M_n \). Another possible source of entrainment is upwelling within Region B_n of the North Pacific Subpolar Gyre, although such upwelling doesn't appear to occur in the real Pacific (Fig. 1a).

With the Pacific, we use the same procedure to solve the constraints as for Solutions 5 and 5', except that (37) is modified by setting \( L^* \rightarrow L_r^* \) and by allowing for the possibility that \( W_0' \neq 0 \). In this case, for a given \( y \), constraints (22b) and (35a) determine \( P_n \),...
\[ W_x(P_x) \] is provided by (17) with \( P_x = P_\nu \) and \( L = L_\nu \), and then \( P_x \) is then known from constraint (33), which allows \( W_x(P_x) \) to be determined. Values for all quantities in (37) are then known, allowing the iteration to proceed.

If no change is made to the standard model parameters, the analytic solution has a Region B \( P \) with a significant \( W_\nu \). Eq. (15) with \( P_x = P_\nu \) identifies the processes that impact the location of \( \chi_{\nu y}(y) \) and hence the existence of Region B \( P \). They are \( P_x, w_d \), and \( w_e \), with increases in the first two (third) tending to shift \( \chi_{\nu y} \) westward (eastward). Eq. (15) also depends on local \( g' \) through \( \sqrt{g' h_m^0} \), but that dependence is negligible since \( P_m \ll P_\nu \). Even though (33) requires \( P_\nu \) to be larger than \( P_m \), it is not sufficiently large to eliminate Region B \( P \) because the basin width is also larger (\( L_\nu = 2L_x \)). If the width of the North Pacific is reduced by half, however, to represent the narrowing of longitudinal lines on a spherical earth, Region B \( P \) no longer exists. Equivalently, Region B \( P \) also vanishes with \( A_\nu \) reduced by half to 1.5 dyn/cm².

It is surprising that \( \chi_{\nu y}(y) \) is hardly influenced at all by \( g' \), given that the across-basin tilt of layer 1, \( \left( h^2(0,y) = 2f w_0 g'/g' \right. \) from (20b), varies inversely with \( g' \). One might expect, then, that Region B \( P \) doesn’t exist in the real Pacific because its upper-ocean waters are less dense (\( g' \) is greater) than those in the Atlantic, so that deep isopycnals cannot rise close to the surface in the western Pacific (Fig. 1a). This intuitive idea doesn’t hold in our model, however, because the value of \( h' \) along the eastern boundary thins with increasing \( g' \), according to \( h^2(0,y) = 2f_w g'/g' \) (Schloesser et al., 2012, 2014); thus, for larger \( g' \), \( \chi_{\nu y}(y) \) remains unchanged because \( h'(0,y) \) thins in a manner that balances the decreased slope.

We obtained curves of MOC properties like those in Fig. 8a when there is a Pacific Ocean. Fig. 12 shows the curves when \( g' \neq 0 \) and \( A_\nu \) is reduced enough in the Pacific to eliminate Region B \( P \). The curves vary similarly to their counterparts in Fig. 8a, differing most prominently in the larger values of \( M \), \( M_\nu \), and \( M_i \), and \( M_p \), Transport \( V_{\nu y} \), however, does not increase since it does not scale with \( \mathcal{L} \). Note that, because \( \delta \) is increased by the factor \( \sqrt{\mathcal{L}/k_\nu} = \sqrt{3} \) with the Pacific, \( \delta = \Delta y \) and is greater than \( \Delta y \) for large \( y \) (top-left panel). Nevertheless, \( \delta \) is still less than \( 2\Delta y \) as required by inequality (C32), constraint (35a) is still valid, and data points lie close to the analytic curves.

Fig. 13 plots \( P \) for Solution P4 (top panel) and the difference between Solution P4 and Solution P3 (bottom panel) when \( \nu \) is reduced to \( 0.25 \times 10^{-6} \text{s}^{-1} \). The difference map measures the \( w_{\nu y} \)-driven part of Solution P4, and illustrates the structure of the \( y_0 \)-boundary layer that extends westward from the tip of Africa. Because \( \nu \) is so small, the western-boundary currents are not apparent; they are directed northward (southward) along the east coasts of South America (Africa).

\[ P_{\nu y} \] Transport \( V_{\nu y} \), however, does not increase since it does not scale with \( \mathcal{L} \). Note that, because \( \delta \) is increased by the factor \( \sqrt{\mathcal{L}/k_\nu} = \sqrt{3} \) with the Pacific, \( \delta = \Delta y \) and is greater than \( \Delta y \) for large \( y \) (top-left panel). Nevertheless, \( \delta \) is still less than \( 2\Delta y \) as required by inequality (C32), constraint (35a) is still valid, and data points lie close to the analytic curves.

5. Diffusion-forced solutions

For realistic choices of \( \tau_0 \) and \( w_0 \), wind forcing dominates diffusive forcing of the MOC so that there is entrainment into layer 1 across its southern edge (\( M \geq 0 \); Section 2.3.1). Nevertheless, it is instructive to consider solutions in which forcing by interior diffusion \( w_0 \) dominates (\( M \leq 0 \)). Clear examples of such solutions occur when \( \tau^x = 0 \) or there is no northern detrainment (\( w_0 = 0 \)). Solutions with \( \tau^x = 0 \) extend the Stommel and Arons (1960a,b) solutions to include a cyclic Southern Ocean, a case not considered in their papers. Solutions with no northern detrainment have \( M < 0 \), avoid complications due to the parameterization of the northern-boundary constraint, are useful for studying Southern-Ocean eddy dynamics, and are applicable to the circulation of AABW (e.g., HV05 and NV11). For simplicity, we assume the domain contains only the Atlantic and Southern Oceans.

5.1. Solutions with \( \tau^x = 0 \) and \( w_0 \neq 0 \)

In our model, the most straightforward extension of the Stommel and Arons (1960a,b) solutions occurs when \( \tau^x = 0 \),
$g' = 0$, and layer 1 extends to Antarctica (Solution 6), in which case (22a) holds so that $M = 0$. The MOC strength is then

$$M = \frac{W_A}{w_A},$$

where $A$ is the total area of the basin. With $M = 0$, known, (28) and (35a) give $P_0$ and $P'_s$, and (14b) and (20b) then give $P'$ across the Atlantic and Southern Oceans. Because $\tau^u = M = 0$ for Solution 6, the integrand of (20b) is positive and hence $P'_s < 0$; as a result, $P'$ increases (thick lines) monotonically to the south, an unrealistic property. Since $P'_s < 0$, it follows from the second of Eqs. (18) that $U' < 0$, and from the first that there is then a northward, eddy-driven flow across the Southern Ocean ($V' = u/f > 0$). This northward, ageostrophic transport merges with the Atlantic western-boundary current, and eventually downwells in the North Atlantic.

When $\tau^u = 0$, $g'(y < y_0) = 0$, and layer 1 extends to $y_0$ (Solution 6'), the response is more complicated because $M = V' = 0$ so that $M_A = w_A A$, where $A$ is the area of the basin north of $y_0$. For realistic parameter values, $M > -w_A A$ and hence $M_A > 0$; in this case, the MOC consists of two counter-rotating cells with entrainment in both the northeast corner and across $y_0$. When $V$ and $P_0$ are increased to unrealistically large values, $M < -w_A A$, $M_A < 0$, and there is a single reverse MOC. A similar, double-celled solution occurs in MJM13’s solutions with $\tau^u = 0$ (their Fig. 10, top panels). Interestingly, such a state also occurs in MJM13’s solutions with $\tau^u \neq 0$ provided that $\kappa$ is large (see their Fig. 12, bottom panels), and a corresponding solution also exists in our model.

Solutions 6 and 6' are in fact the only solutions with $\tau^u = 0$. The only other possible state is one in which layer 1 outcrops in the Southern Ocean ($y > y_0$). To have an outcrop requires that $M = V' = 0$; however, they conclude that the southward, an unrealistic property. Since $M = 0$, known, (28) and (35a) give $P_0$ and $P'_s$, and (14b) and (20b) then give $P'$ across the Atlantic and Southern Oceans. Because $\tau^u = M = 0$ for Solution 6, the integrand of (20b) is positive and hence $P'_s < 0$; as a result, $P'$ increases (thick lines) monotonically to the south, an unrealistic property. Since $P'_s < 0$, it follows from the second of Eqs. (18) that $U' < 0$, and from the first that there is then a northward, eddy-driven flow across the Southern Ocean ($V' = u/f > 0$). This northward, ageostrophic transport merges with the Atlantic western-boundary current, and eventually downwells in the North Atlantic.

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Interestingly, $K^a$ has a similar form using Closure 2 as it does in (50a), resulting in the same regimes. This similarity suggests that setting $K_b$ to a constant in (2), rather than fixing $g'$ to a constant, is the key change that transforms (2) to a closure that mimics standard GM mixing. Conversely, the ability of the closure (2) to represent the eddy statistics of the HV05 model suggests that standard GM mixing will represent eddy mixing more faithfully if $K_b$ is allowed to vary so as to make $v$ constant, that is, by making $K_b \propto R^2$ where $R = \sqrt{g' h'/f^2}$ is the local Rossby radius of deformation, a conclusion reached in several other studies (e.g., Bryan et al., 1999; Gent, 2011).

6. Summary and discussion

We use a variable-density, 2-layer model (VLOM) to investigate basic dynamics of the AMOC, the Southern Ocean, and their interaction. In particular, we consider how key properties of the circulation (Southern-Ocean upwelling $\mathcal{M}$, North-Atlantic downwelling $\mathcal{M}_n$, ACC transport $\mathcal{I}$) and stratification (Atlantic thermocline depth $P_x$, and latitudes, $y'$ and $y_0$, where layer 1 outcrops or vanishes in the Southern Ocean) depend on model forcings (Southern-Ocean wind strength $v_{\text{w}}$, surface heat flux $Q$, entrainment due to interior diffusion $w_0$, processes (mesoscale mixing $v$, Southern-Ocean $V'$, northern sinking, North-Atlantic upwelling), and to the presence of the Pacific Ocean.

6.1. Ocean model

Equations of motion: Model equations (1) represent the depth-integrated, layer-1 (baroclinic) response of the system, and for one solution the barotropic response, that is, the response integrated over both layers (Section 4.1.1). To parameterize vertical mixing and allow for overturning circulations, water can transfer into layer 1 at the velocity, $w_2 = w_2 + w_m + w_p$, the three components representing vertical diffusion, entrainment into a surface mixed layer of thickness $h_m$ (upwelling), and cooling processes external to the North Atlantic, respectively (Eqs. (4)–(6)). The model is forced by an $x$-independent, wind stress field $\tau^x$ (Eqs. 9 and Fig. 3) and the layer-1 temperature is fixed to an $x$-independent profile $T'(y')$ that cools polewards, allowing for a simple representation of thermodynamic processes (Eqs. 11 and Fig. 4).

Domain: The most general model domain consists of rectangular Atlantic, Pacific, and Southern Oceans (Fig. 2). Most solutions discussed, however, are obtained in a basin without the Pacific. No-normal-flow conditions are imposed at all continental boundaries, and cyclic boundary conditions (8) are applied to the Southern Ocean.

Northern sinking: Northern sinking is imposed using two different northern-boundary constraints. In most solutions, it is imposed by relaxing layer-1 thickness $h$ to $h_0 = 1500$ m along the northern boundary (Eqs. 6), a parameterization that represents diapycnal (cooling) processes external to the Atlantic. Some solutions are found by allowing strong cooling by $Q$ within the North Atlantic (Eq. 7; Section 4.2.6). It is not clear which of the two constraints is more physically realistic: The Q-forced constraint is directly linked to surface cooling, but it ignores all processes within marginal seas as well as any overflow entrainment.

Eddy parameterization: Horizontal mixing in (1) has the form of Rayleigh damping with coefficient $v$. For the barotropic response, the damping results from barotropic instability or bottom drag. For the baroclinic response, we interpret it to arise from baroclinic instability through closure (2), in which case the layer-1 velocity field of (1) describes the “residual-mean” flow, that is, the sum of the Eulerian- and eddy-mean velocity fields (Appendix A).

Closure (2) is unusual in that it involves $VP$, where $P = \frac{1}{2} g' h^2$ is the available potential energy of layer 1, rather than $Vh$. It has the advantage that it parameterizes both traditional ($x \times Vh$) and frontal ($x \times Vg'$) baroclinic instability, the latter a consequence of the layer-1 temperature $T'(y')$ decreasing polewards. As a result, it applies when $h$ represents the mixed-layer thickness ($h = h_m$) as well as the depth of subsurface isopycnals ($h > h_m$). We explore the impacts of other closures in Appendix A.2, concluding that closure (2) is the best overall choice for our model. Further, given its properties noted above, it may provide a useful extension to the GM parametrization in OGCMs.

Stratification: A consequence of the layer-1 temperature being set to $T'(y')$ is that the reduced-gravity coefficient, $g'(y')$, decreases polewards. In the Southern Ocean, $T'$ has two states: it is either warm enough for $g' \neq 0$ everywhere (Fig. 5, left panel), or it is cool enough for $g'(y' < y_0) = 0$, in which case layer 1 vanishes south of $y_0$ and the model reduces to a single layer 2 (Fig. 5, right panel). Thus, layer 1 can have three different structures in the Southern Ocean: It can extend to Antarctica at latitude $y_1$ if $g' \neq 0$, intersect $y_0$ if $g'(y' < y_0) = 0$, and outcrop ($h' = h_m$) within the basin at latitude $y'$ for either stratification.

If layer 1 outcrops, $y'$ is an internal model variable that it is not directly linked to $Q$ in the Southern Ocean (see the discussion of Eq. 37). By contrast, in the Radko and Kamenevich (2011) model $y'$ is fixed to an externally specified latitude, essentially a statement that $Q$ determines $y'$ (Section 1); as a consequence, MOC properties in their solutions are more strongly linked to Southern-Ocean processes than they are in ours. In our model, $Q$ affects $h$ only if layer 1 extends to $y_0$, but even then the value of $h(y_0)$ involves Atlantic diapycnal processes ($w_m$ and $w_p$). The basin-wide structure of $h$ is determined entirely from the Southern Ocean only when $w_m = w_p = 0$ (Section 4.1.2; Solution 1b), in which case $h$ is analogous to type-2 (deeper) isopycnals in the Wolfe and Cessi (2010) solutions (Section 1.1.2).

6.1.1. Transport $\mathcal{M}$

When layer 1 outcrops at $y'$ or extends to $y_0$ (Fig. 5), water can either entrain into, or detrain from, layer 1 across its southern edge. In these situations, there is a residual transport $\mathcal{M}$ across the southern edge given by

$$\mathcal{M} = \mathcal{L} \left( \frac{\tau^x(y')}{f(y')} + LV'(y') \right), \quad V'(y') = -\frac{1}{2} \frac{v}{f(y')} g' h^2(y'),$$

(51)

where $y' = y'$ or $y_0$ (Eqs. 22b and 22c). When layer 1 outcrops, $V'(y')$ is negligible because $h(y') = h_m$, and then $\mathcal{M}$ is essentially just the northward Ekman drift across $y'$. When $g'(y' < y_0) = 0$ and layer 1 extends to $y_0$, $h(y_0)$ can be much larger than $h_m$; then, $V'(y_0)$ can increase to values that significantly impact $\mathcal{M}$, even values large enough for $\mathcal{M} \leq 0$ (Sections 4.1.2, 4.2.3 and 5).

6.2. Analytic solutions and constraints

To obtain analytic solutions, we separate fields $q$ into interior $q'$ and boundary-layer $q''$ responses (Sections 3.2 and 3.3, Fig. 2). The separation allows a set of integral constraints to be derived that directly relates MOC properties to model parameters and forcings (boxed equations in Sections 2.3.3 and 3 that provide four equations in the four unknowns, $\mathcal{M}$, $P_x$, $P_y$, and either $V'$ or $V''$). The constraint for the $y_0$-boundary layer is unique because its southern half is cyclic. It is obtained by requiring that the along-boundary integral of $P$ vanishes (Eqs. 22b and 34, the latter including corrections to the former). This requirement ensures that all of the $x$-independent flow is contained in $P$ and hence that the boundary layer has a finite width (see the end of Section 3.1.3).
Although it is not possible to evaluate the integral exactly analytically, it is possible to obtain an accurate analytical approximation (Eq. 35a and Section C.4.5).

6.3. Wind-driven solutions

In Section 4, we discuss a hierarchy of primarily wind-driven solutions. At its bottom are solutions with no MOC ($M = M_0 = 0$ (Section 4.1). The other solutions all have an MOC ($M > 0$), and sequentially add Southern-Ocean upwelling, entrainments by diffusion and within the North Atlantic Subpolar Gyre (Sections 4.2.4.3.3.4.4, and the Pacific Ocean (Section 4.5).

6.3.1. Solutions without an MOC

For the barotropic response with $w_b = 0$ in (1), solutions necessarily have no MOC (Section 4.1.1, Solution 1a), and the solution forced by a uniform zonal wind illustrates the basic structure of the $y_a$-boundary layer in all solutions (Fig. 6a). Solutions without an MOC are also possible for the layer-1 response (Section 4.1.2). When $g' \neq 0$ (warm-$T^*$ forcing) and $\nu$ or $P_a$ is large enough for layer 1 to extend to Antarctica, they are the same as the barotropic solution (Solutions 1b and 1c). More interesting, layer-1 solutions occur when $g'(y < y_0) = 0$ (cold-$T^*$ forcing) and $w_b = w_a = 0$. In these solutions, the eddy-driven flow $V'(y_a, h)$ is strong enough to cancel the Ekman drift across $y_0$ so that $M = 0$ in (51) (Solutions 1b and 1c).

6.3.2. Solutions with an MOC

Our hierarchy of solutions with an MOC ($M > 0$) illustrates the sensitivity of MOC properties to the forcings, state of the Southern-Ocean stratification (either $g' \neq 0$ or layer 1 vanishes for $y < y_0$), a range of values of $\nu$ and $\tau_a$, and the parameterization of northern sinking.

**Outcrop latitude:** In all solutions, the outcrop latitude $y'$ shifts southward as $v$ increases (top-left panel of Fig. 8a), primarily because the slope of $P'$ across the Southern Ocean contains the factor $v^{-1}$ (integrand of Eq. 20b). In comparison to the $v = \tau_a$ solution (Solution 1, red curve), the southward shift in the other solutions is much slower because $v = \tau_a + \tau^*_n$ weakens to the south; as a result, $P_n$ in (20b) is greater for a given $v$ since the factor, $M / L + \tau^*_n = -\tau^*(y)/f + C(\nu, h_0) + \tau^*_n/f \approx \tau^*(y)/f - \tau^*(y)/f'$, becomes much larger as $\tau^*_n$ decreases southward. This sensitivity highlights the dynamical importance of the southward weakening of $\tau^*_n$ across the Southern Ocean.

**Transports $M$ and $M_s$, vs. $v$:** Transports $M$ and $M_s$ decrease with $v$ (middle panels of Fig. 8a). When layer 1 outcrops, $M_s$ weakens because the Ekman drift in (51) decreases as $y'$ shifts southward; the linkage between $M$ and $y'$ is evident in approximate solution (39) in that $M$ is proportional to $1 - \Delta y/L$, $\Delta y = y_a - y'$. Surprisingly, the $M$ curves in the $g' = 0$ and $g'(y < y_0) = 0$ solutions are similar, despite the marked difference in their stratifications (Fig. 8b, middle-left panel); this similarity happens because, although the Ekman-drift contribution to $M$ is fixed to $\tau^*o(y)/f_o$ for $v > v_b$ (solid-gray curve in the bottom-left panel of Fig. 8b), $M$ continues to decrease because $V_b$ decreases (dashed-gray in the bottom-left panel of Fig. 8b). In response to additional entrainments by $w_b$ and upwelling in the North-Atlantic Subpolar Gyre, $M_s$ increases as expected (blue and magenta curves in the middle- and bottom-right panels of Fig. 8a); at the same time, $M$ decreases somewhat (middle-left panel of Fig. 8a) because as $M_s$ increases constraints (28) and (35a) require that $P_a$ and $P_n$ also increase, so that $y'$ shifts southward and $M$ decreases.

**Transports $M$ and $M_s$, vs. $\tau_a$:** Transports $M$ and $M_s$ increase quasi-linearly with $\tau_a$ (Fig. 8c, middle-left panel). By contrast, in solutions to eddy-resolving OGCMs, they increase more weakly (e.g., MH13 and MJM13), a property attributed to the intensification of eddy strength with $\tau_a$ (eddy compensation). We simulate this behavior by allowing $v$ to increase with $\tau_a$ like $v = v_0(\tau_a/\tau_a^0)^{0.5}$ (Section 4.2.5, Fig. 9), with the curves for $n = 1/2$, 3/4, and 1 in Fig. 9 comparing best with OGCM results. In our solutions, the slower increase happens because $y'$ shifts more slowly northward with $\tau_a$ due to the increase in $v$, thereby weakening the Ekman transport that entrains into layer 1. Eddy compensation in OGCMs may result from a similar process.

**Stratification and ACC transport:** Variables $P_o$, $P'_o$, $U'$, and $U_{max}$ all vary weakly with $v$ and $\tau_a$ (Fig. 8a, middle-right panel; Figs. 8b and 8c, middle- and bottom-right panels). This weak dependence results from the northern-boundary constraint being given by (6) and hence (28), a consequence of the large value of $P_o$ (Sections 4.2.2.4, 4.2.2.5 and 4.2.4.3). To test the sensitivity of solutions to the northern-boundary constraint, we obtain solutions using (28) for a range of values of $P_p + \tau_dA$ and (7) with different values of $x$ (Fig. 10). Only for constraint (7) with small $x$ values (black-dashed curve) does $P_a$ vary strongly with $\tau_d$. In the MJM13 solutions, $P_a$ and $U$ vary strongly with $\tau_d$ in their coarse-resolution model but not in their eddy-resolving ones (their Figs. 3a and 9c). They conclude that near-eddy saturation in the eddy-resolving models accounts for the different behaviors of the two systems. Our solutions suggest that differences in their northern-boundary constraints, which are analogous to (7), may also be involved.

6.3.3. Solutions with a Pacific Ocean

When the Pacific is added (Section 4.5), the increase in the domain width increases $M$, $M_1$, and hence $M_s$ roughly by the factor $L_1/L_4$ (compare corresponding curves in Figs. 8a and 12). In the Pacific, the flow that flows northward across the latitude of the tip of Africa $y_p$ due to a Sverdrup or Stommel-Arons (1960a,b) interior flow, or diffuses into layer 1 north of $y_p$ due to $w_a$, circulates to the Pacific western-boundary current where it flows southward to the tip of Africa, and finally flows westward across the basin in a boundary layer centered on $y_p$ to join the Atlantic western-boundary current (Section 3.3.4; Fig. 13).

6.4. Diffusion-forced solutions

Solutions in which the MOC is primarily diffusion forced ($M \leq 0$) occur when $\tau^*_n$ is unrealistically weak, $w_o$ is unrealistically large, or there is no northern sinking ($w_n = 0$). Their properties compare well with analogous solutions to idealized OGCMs (e.g., HV05, NV11, and MJM13).

**Solutions with $w_n = 0$:** When $w_n = 0$ and $\tau^*_n = 0$, solutions are extensions of the Stommel and Arons (1960a,b) solutions that include a cyclic Southern Ocean (Section 5.1). When $g' = 0$, the MOC consists of a single overturning cell in which, unrealistically, $P_o < 0$ (thickenings to the south) across the Southern Ocean. When $g'(y < y_o) = 0$, it has two cells, with sinking both in the north and in the south. MJM13 report similar double-celled MOCs in their solutions with $\tau^*_n = 0$ and with strong $k$ (their Fig. 10, top panels, and Fig. 12, bottom panels).

**Solutions with $w_o = 0$:** When $w_o = 0$ and $g'(y < y_o) = 0$, a steady-state solution is possible in which all the water that entrains into layer 1 by $w_d$ detrains across $y_o$, that is, there is a single, reverse MOC (Section 5.2, Solution 7'). HV05 and NV11 report solutions to idealized OGCMs that lack northern sinking, finding that the sensitivities of $h_0$ and $M$ to $k$ and $\tau_a$ fall into two distinct regimes, depending on the relative strength of the two parameters; further, the sensitivities differ in each study because the HV05 model is eddy resolving whereas NV11 utilizes standard GM closure (constant $k_0$). With some modifications, our model is able to...
simulate the dependencies in both studies. One change is to set \( w_h = k/H_h \) (rather than \( \kappa/H \)), a better representation of the advective-diffusive balance that determines the Atlantic stratifications in the OGCMs. With the additional modification that \( \psi \) depends on \( \tau \), such that \( \psi = \psi_0 \) and \( \psi_0 (\tau_0/\tau) \) for small and large \( \tau_0 \), Solution 7' has the same regimes as in the HV05 solutions, suggesting that closure (2) adequately represents the eddy statistics in their solutions. With the modification that \( \psi = k_h (\theta^h/g^h) \), a closer representation of standard GM mixing (Appendix A.2), the regimes are the same as in the NV11 solutions.

6.5. Conclusions

In conclusion, we have developed an intermediate model of the AMOC and Southern Ocean that includes parameterizations of many processes thought to be important in their dynamics. In particular, the model includes an eddy-driven circulation in the Southern Ocean; represents effects of a surface buoyancy flux by setting \( T_1 = T'(\gamma) \), thereby allowing Southern Ocean stratifications in which layer 1 either exists everywhere or vanishes south of latitude \( y_0 \); allows for mixing strength \( \psi \) to vary with wind strength; considers different representations of northern-boundary sinking; and explores the consequences of different \( \bar{h} \bar{\psi} \) closures in the Southern Ocean. Due to its generality, solutions are able to simulate basic properties of a broad range of solutions to idealized OGCMs, providing insights into how their properties depend on process parameterizations, parameter values, and forcings.

As is the case for all intermediate models, our model has a number of simplifications and limitations. Notable simplifications are the lack of AABW (although our solutions with \( \psi_0 = 0 \) and \( \psi_4 \neq 0 \) shed light on its dynamics), Antarctic promontories, and bottom topography. It should be straightforward to extend the model to a 2-layer system that can explore the first two issues. A major limitation is the parameterization of the northern-boundary constraint, which likely involves processes in the North Atlantic marginal seas and Arctic Ocean that are not likely well represented by either (7) or (28). Finally, given the useful properties of closure (2), it would be interesting to extend it to apply to continuous stratification and to study its impacts in non-eddy-resolving OGCMs in regions where stratification gradually vanishes and the mixed layer becomes deeper (as near our \( y_0 \)).

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Appendix A. Parameterization of eddy mixing

Eddy-driven circulation is an essential part of Southern-Ocean dynamics, allowing isopycnals to slope across the basin and generating a southward transport that counteracts northward Ekman drift. In the main text, the impact of eddies is parameterized by closure (2). Here, we first demonstrate that closure (2) leads to Eqs. (1) where the velocities are defined to be the residual flow (Section A.1). Then, we explore the impact of other closures, showing how they alter our Southern-Ocean interior solution (Section A.2). We conclude that closure (2) is the best choice for our model. Further, because closure (2) allows \( V' \) to be defined when layer 1 represents both the response of subsurface isopycnals (\( h > h_m \)) and the surface mixed layer (\( h = h_m \)), it may also be a useful extension of the GM parameterization in coarse-resolution OGCMs.

A.1. Residual-mean equations

We begin with the layer-1 equations of the 2-layer model of Schloesser et al. (2012; their Eqs. (A.11a) and (A.11b)), with wind forcing, \( T_1 = T'(\gamma) \), and without horizontal mixing forcing. Then, in the limit that \( h_1 \ll D \), their equations become

\[
\psi'' + f k \times \psi' = - \left[ \left( \frac{g}{\rho_0} + \frac{\psi''}{\bar{h}} \right) + \frac{\tau \psi''}{\bar{h}} \right] + \frac{\tau}{\bar{h}}, \quad \frac{h_1''}{\bar{h}} + \nabla \cdot (\bar{h} \psi') = \psi_h(h'),
\]

(A1)

where \( \psi' = (u', v') \) is the depth-averaged instantaneous velocity, the expression in parentheses is the depth-averaged pressure gradient when \( g' \) varies, \( k \) is a unit vector that points vertically, and \( \tau = (\tau', \tau'') \). Separating variables \( q' \) into time-averaged \( \bar{q} \) and time-varying (eddy) \( q' \) parts (\( q' = \bar{q} + q' \)), and averaging in time then gives

\[
\psi'' + f k \times \bar{\psi} = - \left[ \left( \frac{g}{\rho_0} \bar{\psi} + \frac{\psi''}{\bar{h}} \right) + \bar{\tau} \right] + \frac{\bar{\tau}}{\bar{h}}, \quad \nabla \cdot (\bar{h} \bar{\psi}) = \bar{\psi}_h,
\]

(A2)

where \( \bar{\psi} = \bar{h}(\bar{\tau}/\bar{h}) \) and overbars indicate time averaging.

A closure is needed to express the transport due to eddies, \( \bar{h} \bar{\psi} \), in terms of mean quantities, and we choose

\[
\bar{h} \bar{\psi} = - \frac{\bar{\psi}}{\bar{T}} \bar{V} \bar{P},
\]

(A3)

where \( \bar{V} \bar{P} \) is specified below. Then, \( \bar{h} \bar{\psi} \) defines a mean velocity,

\[
\psi' = \frac{\bar{h} \bar{\psi}}{\bar{h}} = - \frac{\bar{\psi}}{\bar{T}} \frac{\bar{V} \bar{P}}{\bar{h}},
\]

(A4)

the eddy-driven (“bolus”) velocity.

It is useful to rewrite (A2) in terms of the total (residual) velocity field,

\[
\psi = \bar{\psi} + \psi'.
\]

(A5)

The continuity equation then simplifies to

\[
\nabla \cdot (\bar{h} \bar{\psi}) = \bar{\psi}_h,
\]

(A6)

and the momentum equations become

\[
f \int \left( k \times \bar{h} \bar{\psi} + \frac{\bar{\psi}}{\bar{h}} \bar{h} \bar{\psi} \right) = - \bar{V} \bar{P} - \bar{h} \bar{\psi} + \left( \bar{\psi} - \frac{q'}{\bar{T}} k \times \bar{\psi} \right),
\]

(A7)

where \( \bar{P} = g \bar{h} \bar{\psi}^2/2 \).

Closure (2) assumes that \( \bar{V} \bar{P} = \bar{V} \bar{P} \). Using the first of equations (A2) multiplied by \( \bar{h} \) to eliminate \( \bar{V} \bar{P}, (A7) \) becomes

\[
f \int \left( \bar{h} \bar{\psi} + \frac{\bar{\psi}}{\bar{h}} \bar{h} \bar{\psi} \right) = - \bar{V} \bar{P} - \bar{h} \bar{\psi} + \left( \bar{\psi} - \frac{q'}{\bar{T}} k \times \bar{\psi} \right),
\]

(A8)

which, under the restrictions that \( |\psi'| \lesssim |\psi| \) and \( \psi \ll |f| \), reduces to

\[
f \int \bar{h} \bar{\psi} = - \bar{V} \bar{P} - \bar{h} \bar{\psi} + \bar{\psi} - \frac{q'}{\bar{T}} k \times \bar{\psi}.
\]

(A9)

With the definition \( \bar{V} = (U, V) \equiv \bar{h} \bar{\psi} \) and the replacements \( \bar{h} \rightarrow h, \bar{\psi} \rightarrow \psi \), and \( \bar{\tau} \rightarrow \tau \), (A6) and (A9) are the set (1).
A.2. Impact of different closures

Closure (2) is unusual in that it involves the gradient of available potential energy $P$. An alternate closure, comparable to the GM parameterization in OGCMs, is $h\nu = -\kappa\psi h$, which can be obtained from (A3) by setting $\nu = \kappa_0\psi^2/(g'h)$.

Based on this pair, we consider the four possible closures that result when $\tilde{P}_1 = P_0$ or $\tilde{P}_2 = \int h\nu h/2$ and either $\nu$ or $\kappa_0$ are assumed constant (i.e., are independent of $g'h$, and $f$). For each closure, we obtain Southern-Ocean solutions when layer 1 outcrops at $y^*$ and when it extends to the latitude $y_0$ where $g'h$ vanishes (Section 3.2.2).

A.2.1. Southern-Ocean equations

For notational convenience, we drop the overbars and primes that indicate time averaging and interior fields from all variables. Assuming $x$-independence and (for simplicity) setting $u_0 = 0$, continuity equation (A6) requires that $V = M/L$ is constant north of the southern edge of layer 1 $y^*(y = y' < y_0)$, where $M$ is the transport that entrains into layer 1 across $y$. The zonal momentum equation in (A7) can then be rewritten

$$g'h_V + \frac{1}{2} eg'h = \frac{f^2}{v} \left( \frac{\tau_y}{f} + \frac{M}{L} \right), \quad y > y^*, \tag{A10a}$$

where $\epsilon = 1$ if $\tilde{P}_1 = P_0$ and $\epsilon = 0$ if $\tilde{P}_2 = \int h\nu h/2$, and $\nu$ is either constant or $\nu = \kappa_0\psi^2/(g'h)$ with $\kappa_0$ constant. With the aid of (A4) and (A10a), we define the useful variable

$$V' = h\nu = \frac{-\int \tilde{P}_y}{f} = \frac{-\tau_y}{f} - \frac{M}{L}, \tag{A10b}$$

the eddy-driven, meridional transport per width.

When layer 1 outcrops at $y^*$, the solution south of $y'$ corresponds to a mixed layer of thickness $h_{m0}$ (Eq. 21). Here, we only need to know its properties just south of $y'$ ($y = y'$) where $V'(y') = M$. There, (A10a) applies with $h = h_{m0}$ and $h_y = h_{m0} = 0$, providing an independent expression for $M$.

A.2.2. Solutions

Our analytic model (without the Pacific) requires 4 equations for the 4 unknowns: $M$, $P_1$, $P_0$, and either $\psi$ if layer 1 outcrops or $h_0$ if layer 1 extends $y_0$. The northern- and $y_0$-boundary constraints (Eqs. 28 and 35a) provide two of the required equations, so that the Southern-Ocean solution must provide the other two. One constraint is the solution to (A10a) evaluated at $y_0$ (second expressions in either Eq. 22b or 22c). The other is an expression for $M(y')$ or $M(h_0)$, and both arise from the application of boundary conditions (first expressions in either Eq. 22b or 22c): When layer 1 outcrops, the boundary conditions match $h$ and $V$ across $y'$; when layer 1 extends to $y_0$, they are provided by the property that $g'(y_0) = 0$ and the requirement that $h_0$ remains bounded.

**Closure 1 ($\tilde{P}_1 = P_0$, $\psi$ constant):** Closure 1, which is closure (2) used for all the solutions except those in Section 5.2, sets $\epsilon = 1$ and $\nu$ constant in (A10a). The solution to (A10a) is then

$$g'(y)h^2(y) = g'(y')h^2(y') - \frac{2}{v} \int \frac{\tau_y}{f} + \frac{M}{L} \ dy, \quad y > y^*, \tag{A11}$$

where $h(y)$ is a constant of integration to be determined.

When layer 1 outcrops, $\dot{y} = y'$ and $h(y) = h_{m0}$ in solution (A11). Transport $M(y') = LV(y')$ is given by (A10a) with $h = h_{m0}$ and $h_y = 0$,

$$\frac{M}{L} = -\frac{\tau_y}{f} + V'(y') = -\frac{\tau_y}{f} + \frac{1}{2} \frac{v}{f} \int g'(y')h_{m0}^2. \tag{A12a}$$

When layer 1 extends to $y_0$, solution (A11) has $\dot{y} = y_0$, $h = h(y_0)$, and, assuming that $h(y_0)$ is bounded, $g'(y_0)h_{m0}^2/2 = 0$ since $g'(y_0) = 0$. Solution (A11) provides $h'$ across the Southern Ocean, and at $y_0$ it is defined by taking the limit $y \to y_0$ of (A10a) and (A10b). With the aid of the L'Hopital's rule, the resulting expression can be rewritten

$$\frac{M}{L} = -\frac{\tau_y}{f} + V'(y'), \tag{A12b}$$

which provides $M(h_0)$ and $V'$. Solution (A11) evaluated at $y_0$ together with either (A12a) or (A12b), are constraints (22b) and (22c).

As stated in the main text, $V'(y')$ is negligible in (A12a) because $h = h_{m0}$ is small. In contrast, because $h_0$ can be much larger than $h_{m0}$, $V'$ can be significant in (A12b), even large enough to reverse the sign of $M$ (Section 5).

**Closure 2 ($\tilde{P}_1 = P_0$, $\kappa_0$ constant):** Closure 2 sets $\epsilon = 1$ and $v = \kappa_0\psi^2/(g'h)$ in (A10a). After dividing (A10a) by $h\sqrt{\psi}/2$, the left-hand side can be expressed in terms of $\sqrt{\psi}$, leading to the solution

$$\sqrt{\psi}(y)h = \sqrt{\psi}(y')h(y') - \frac{1}{\kappa_0} \int \sqrt{\psi}(\frac{\tau_y}{f} + \frac{M}{L}) dy, \quad y > y^*, \tag{A13}$$

Note the difference in the response from (A11): Although the integrals on the right-hand sides are similar, the left-hand sides provide $h^2$ in (A11) and $h$ in (A13).

When layer 1 outcrops, we set $\dot{y} = y'$ and $h(y') = h_{m0}$ in (A13). Transport $M$ is given by (A12a) with $v = \kappa_0\psi^2/(g'h_{m0})$, that is,

$$\frac{M}{L} = -\frac{\tau_y}{f} + V'(y') + \frac{\kappa_0}{f} \int \sqrt{\psi}(\frac{\tau_y}{f} + \frac{M}{L}) dy, \tag{A14}$$

In comparison to (A12a), $V'(y')$ in (A14) can be large even though $h(y') = h_{m0}$, provided that $g'(y')$ is small.

Assume, for the moment, that layer 1 extends to $y_0$. Then, solution (A13) has $\dot{y} = y_0$, $h(y_0) = h_0$, and, provided that $h_0$ is bounded, $\sqrt{\psi}(y)h = 0$. It then follows from (A13) and the L'Hopital's rule that $h_0 = \lim_{y \to y_0} \sqrt{\psi}(y)h(y) = 0$ since $g'(y_0) = 0$. Thus, layer 1 can never reach $y_0$, but rather must outcrop at a latitude somewhat north of $y_0$ where $h$ first thins to $h_{m0}$. In that case, (A14) holds and $V'$ can be quite large since $y'$ is close to $y_0$ where $g'(y')$ is small.

In the interval $y_0 < y < y'$, layer 1 corresponds to a mixed layer with $h = h_{m0}$. Consequently, $V' = (\kappa_0/2)/(g'h_{m0})$ there, which blows up as $y \to y_0$. Therefore, Closure 2 does not allow a physically sensible solution if $g'(y_0) = 0$. This problem can be avoided by allowing $\kappa_0$ south of $y'$ (within the mixed layer) to have a different form than it does north of $y'$. Its form is arbitrary except that it must satisfy the constraints that $\kappa_0 \to 0$ as $g' \to 0$ and $\kappa_0(y') = \kappa_0(y')$, the latter condition required to ensure $V'$ is continuous across $y'$. A simple example of a replacement that satisfies both constraints is $\kappa_0 \to \kappa_0(g'(y'))/(g'(y'))$.

**Closure 3 ($\tilde{P}_1 = h\nu h$, $\nu$ constant):** Closure 3 sets $\epsilon = 0$ and $\nu$ constant in (A10a), and the solution is then

$$h^2(\dot{y}) = \frac{2}{v} \int \frac{\tau_y}{f} + \frac{M}{L} \ dy, \tag{A15}$$

When layer 1 outcrops, $\dot{y} = y'$ and $h(y') = h_{m0}$ in (A15). Then, (A10a) with $h = h_{m0} = 0$ and $\epsilon = 0$ gives $V'(y') = 0$ so that from (A10b),

$$\frac{M}{L} = -\frac{\tau_y}{f}. \tag{A16a}$$
When layer 1 extends to \( y_0 \), the solution is (A15) with \( \dot{y} = y_0 \) and \( h(y) = h_0 \). To ensure that \( h_0 \lim_{\dot{y} \to y_0} h \) remains finite as \( g' \times (y - y_0) \to 0 \) in the denominator of the integrand, its numerator must also vanish, that is,

\[
M \frac{\partial}{\partial z} = -\frac{\tau^x(y_0)}{f_0},
\]

(A16b)

a statement that \( V''_c = 0 \).

An issue with Closure 3 (in comparison to Closures 1 and 2) is that, because \( V''_c = 0, M/\mathcal{L} \) is fixed to \( \tau^x(y_0)/f \) and hence \( V''_c \) can never increase to appreciable values. Using Closure 3, then, our model cannot represent OGCM solutions with \( M < 0 \) (Section 5).

Closure 4 (\( P_y = h_0, \kappa_n \) constant): Closure 4 sets \( \epsilon = 0 \) and \( \kappa_n \) constant in (A10a), in which case

\[
h = h(y) = -\frac{1}{\kappa_n} \int_0^y \frac{\tau^x}{\mathcal{L}} \, dy.
\]

(A17)

When layer 1 outcrops, \( M \) is still given by (A16a). When it extends to \( y_0, (A17) \) is always bounded, and hence the boundedness requirement does not provide an additional constraint. In this case, we conclude that the model is “degenerate” for Closure 4 in that the boundary conditions do not allow the solution to be completely determined: either \( h_0 \) or \( M \) must be externally specified.

A2.2.3. Conclusions

When \( g' \neq 0 \), all four closures produce similar solutions. When \( g' \neq 0 \), however, solution properties differ considerably. Closure 1 allows \( V''(y_0) \) to be large enough to impact \( M \), and hence has solutions that compare favorably with a range of solutions to idealized OGCMs (with \( M > 0 \) and \( M \leq 0 \); further, Closure 1 is valid when the bottom of layer 1 corresponds to the depths of both subsurface isopycnals \( h > h_m \) and the surface mixed layer \( h = h_m) \). Closure 2 also allows \( V''(y) \) to be large, but it does so when \( g'(y) \) is small, a property that seems inconsistent with it resulting from baroclinic instability. Closure 3 has \( V''(y_0) = 0 \), and so cannot simulate OGCM solutions in which \( M \leq 0 \). Finally, Closure 4 is degenerate in that, when layer 1 extends to \( y_0 \), it is not possible to determine both \( h_0 \) and \( M \) internally. Given the above, we conclude that Closure 1 is the best choice for our model.

In Section 5.2, we compare our solutions to those of NV11, which are obtained using GM mixing with a constant coefficient in the subsurface ocean. The closest analog to that parameterization in our model is Closure 4, since it has \( h'' = -\kappa_n h_y \) with \( \kappa_n \) constant. Indeed, the NV11 analytic model (as well as the RK11 model) is essentially the same as our model using Closure 4 when \( y' \) outcrops, except that their \( y' \) is externally fixed. (Equivalently, their model is analogous to ours using Closure 4 when layer 1 extends to \( y_0 \) and \( h_0 \) is set to \( h_m \).) By fixing \( y' \), their system has one less degree of freedom than ours, and hence requires one less constraint. The neglected constraint is (A16a), that is, \( M \) is no longer specified by matching to the mixed-layer solution south of \( y' \). By contrast, in the NV11 numerical model \( M \) is necessarily linked to the mixed-layer response south of \( y' \); however, its value differs from (A16a) because GM mixing is altered near the ocean surface where isopycnals are nearly vertically oriented. As such, the dynamics of the NV11 OGCM may be more closely related to our analytic model using Closure 3.

Appendix B. Solutions with and without \( vV \)

One of the approximations built into our analytic solution is the neglect of the term \( -vV \) in (1) in the boundary layer along \( y_{gr} \), which leads to a \( P''_r \) equation that lacks the zonal mixing term \( vP''_r \) (Appendix C.4). Here we illustrate the impact of this simplification by solving (1) for the streamfunction \( \psi \) \((V = \psi_v, U = -\psi_s)\) with and without the term \( -vV \), assuming that \( w_e = 0 \), \( \tau^* = \tau_0 \), and that the domain has an Atlantic and Southern Ocean. The same set of equations has been studied by Gill (1968), who allowed \( \tau^* \) to vary with \( y \).

For convenience, we define \( \eta = y - y_{gr}, \quad a = y_0 - y_{gr} \) and \( \ell = y_n - y_{gr} \). The basin extends zonally from \( x = -L \) to \( x = 0 \).

B.1. Differential equation and boundary conditions

Setting \( w_e = 0 \) and \( \tau^* = \tau_0 \) in Eqs. (1) leads to the differential equation

\[
\rho \dot{\psi}_x + v(\psi_x + \psi_y) = 0
\]

(B1)

for the streamfunction \( \psi(x, \eta) \). We impose the boundary conditions

\[
\psi(x, 0) = \psi_a, \quad \psi(x, L) = 0, \quad -L < x < 0,
\]

(B2a)

\[
\psi(-L, \eta) = \psi(0, \eta), \quad \psi(-L, 0) = \psi_a(0, \eta), \quad 0 < \eta < a.
\]

(B3)

Conditions (B3) and the differential equation (B1) guarantee the periodicity of all the other derivatives of \( \psi(\eta), \psi_x(\eta), \psi_y(\eta), \psi_{xx}(\eta), \psi_{yy}(\eta), \psi_{xy}(\eta) \ldots \).

Integrating the first of Eqs. (1) zonally across the basin at \( \eta = 0 \) and making use of the constancy of \( \psi(x, 0) \) and periodicity of \( P \), yields

\[
\int_0^L \psi_x(x, 0) \, dx + \frac{\tau_0}{V} L = 0.
\]

(B4)

As noted next, we obtain solutions assuming that \( \psi_a = 1 \). Condition (B4) provides the additional constraint needed to determine \( \psi_a \) in terms of model parameters.

B.2. Solution with \( vV \)

It is useful to introduce the scaled streamfunction, \( \dot{\psi} = \psi_a \psi \), and to satisfy the second of boundary conditions (B2b) by writing

\[
\dot{\psi}(0, \eta) = \phi(\eta) \theta(a - \eta),
\]

(B5)

where \( \theta \) is the standard step function. In terms of these quantities, the general solution to (B1) with boundary conditions (B2a) is given by the Fourier series

\[
\phi(x, \eta) = 1 - \frac{\eta}{\ell} + \sum_{n=1}^{\infty} a_n X_n(x) \sin \left( \frac{n\pi \eta}{\ell} \right)
\]

(B6a)

where

\[
a_n = \frac{2}{\ell} \int_0^\ell \left[ \phi(\eta') \theta(a - \eta') - \left(1 - \frac{\eta'}{\ell} \right) \sin \left( \frac{n\pi \eta'}{\ell} \right) \right] \, d\eta'.
\]

(B6b)

and, with no loss of generality, we have set \( X_0(0) = 1 \). Substituting (B6a) into (B1), we find that \( X_n(x) \) is a superposition of the exponential functions, \( \exp[-(\gamma \pm \gamma'_a)x] \), where \( \gamma = \beta/(2V) \) and \( \gamma'_a = \gamma^2 + (\pi V/(a \ell)) ^{1/2} \). The linear combination that satisfies the first of boundary conditions (B3), as well as \( X_0(0) = 1 \), is then

\[
X_n(x) = e^{-\gamma_n x} \frac{\sin \gamma'_n (x + L)}{\sin \gamma'_n L} - e^{-\gamma_n x} \frac{\sin \gamma'_n x}{\sin \gamma'_n L}
\]

(B6c)

Eqs. (B6a)–(B6c) express \( \phi(x, \eta) \) as a superposition of eastward- and westward-decaying Rossby waves, \( \exp[-(\gamma \pm \gamma'_a)] \sin(n\pi \eta/\ell) \), each of which satisfies differential equation (B1).
Solution (B6) explicitly fulfills boundary conditions (B2) and the first of periodicity conditions (B3) for an arbitrary function \( \phi(\eta) \). The precise form of \( \phi(\eta) \) is determined by the second of the periodicity conditions, which can be expressed as

\[
\bar{\psi}_s(-L, \eta) - \bar{\psi}_s(0, \eta) = 2 \sum_{n=1}^{\infty} a_n = \frac{\cosh \gamma L - \cosh \gamma L}{\sinh \gamma L} \sin \left( \frac{n \pi \eta}{\ell} \right) = 0, \quad \eta < a.
\]

(B7)

Elminating \( a_n \) in (B7) using (B6b) results in an integral equation for \( \phi(\eta) \). We solve it numerically by a least-squares procedure, substituting an adjustable trial function \( \tilde{\phi}(\eta) \) for \( \phi(\eta) \) in (B6b) and varying it to minimize the corresponding integral \( \int_0^{\infty} |\tilde{\psi}_s(-L, \eta) - \tilde{\psi}_s(0, \eta)|^2 \, d\eta \). With \( \phi(\eta) \) given by the adjusted trial function \( \tilde{\phi}(\eta) \), (B6a)–(B6c) then give \( \tilde{\psi}(\eta) \).

The trial function is best represented in terms of polar coordinates \( r = \sqrt{x^2 + (\eta - a)^2} \) and \( \varphi = \arctan[x/(\eta - a)] \) relative to the tip of the barrier. In these coordinates, (B1) takes the form

\[
2 \gamma (\sin \varphi \varphi_r + r^{-1} \cos \varphi \varphi_\varphi) + \varphi_{\varphi_r} + r^{-1} \varphi_r + r^{-2} \varphi_{\varphi_\varphi} = 0.
\]

(B8)

and has the general solution

\[
\psi(x, \eta) = \sum_{j=1}^{\infty} r^j f_j(\varphi).
\]

(B9)

Requiring that the angular functions \( f_j(\varphi) \) in (B9) vanish at \( \varphi = 0 \) and \( \varphi = 2\pi \), in accordance with boundary condition (B2b), we obtain

\[
f_1(\varphi) = c_1 \sin(\varphi/2), \quad f_2(\varphi) = c_2 \sin \varphi.
\]

(B10)

Substituting (B9) into (B8) yields a differential equation that determines \( f_j(\varphi) \) from \( f_2(\varphi) \). By solving it recursively, beginning with (B10), the other \( f_j(\varphi) \), which contain constants of integration \( c_j \), may be calculated.

Since \( \phi(\eta) \) and \( \psi(0, \eta) \) coincide on the line segment, \( x = 0, \eta < a \), corresponding to \( r = a - \eta, \varphi = \pi \) in polar coordinates, (B9) implies

\[
\phi(\eta) = \sum_{j=1}^{\infty} b_j \left( 1 - \frac{\eta}{a} \right)^{j/2}, \quad 0 < \eta < a.
\]

(B11)

where \( b_j = \psi_0^{-1} f_j(\pi) a^{j/2} \). Since each of the \( f_j(\pi) \) contains an arbitrary constant of integration \( c_j \), the \( \{ b_j \} \) form an equivalent set of arbitrary constants, linearly related to the set \( \{ c_j \} \). Only odd values of the summation index \( j \) are included in (B11) since \( f_j(\pi) \) vanishes for \( j \) even.

Eq. (B9) satisfies differential equation (B8) for arbitrary values of the coefficients \( c_j \). However, in our problem the \( \{ c_j \} \), and, accordingly, the \( \{ b_j \} \), are not arbitrary but are fixed by the boundary and periodicity conditions (B2) and (B3). The second of the periodicity conditions (B3) is imposed with a numerical least-squares procedure described below (B7). For the trial function \( \tilde{\phi}(\eta) \) we use the first \( J \) terms of (B11), and \( b_1, \ldots, b_J \) are obtained by the least-squares optimization, subject to the constraint

\[
\phi(0) = b_1 + b_2 + \cdots + b_J = 1 \text{ implied by (B2a)}.
\]

Because (B11) accurately represents the small-scale structure of \( \bar{\psi}_s \) near the tip of the barrier, excellent results are obtained with only a few terms in (B11): Keeping 200 components in the Fourier series (B6a) and only 10 terms in the trial function (B11), the optimization largely eliminates discontinuities in \( \bar{\psi}_s \) at \( x = -L \) and 0, as can be seen in the contour plot of \( \psi \) in Fig. 6a.

B.3. Solution without \( vV \)

On dropping \( -vV \) from the second of Eqs. (1), (B1) simplifies to the backwards diffusion equation

\[
\beta \bar{\psi}_s + \bar{\psi}_\eta\eta = 0.
\]

(B12)

Its solution also has the form (B6) but with

\[
X_s(x) = \exp \left[ -\frac{V}{\beta} \left( \frac{\pi}{\ell} \right)^2 x \right].
\]

(B13)

which, in contrast to (B6c), contains only westward-decaying Rossby waves.

To determine \( \phi(\eta) \), we set \( x = -L \) in (B6a) and impose the first of the periodicity requirements (B3), which leads to

\[
\phi(\eta) = \left( 1 - \frac{\eta}{a} \right)^2 + \sum_{n=1}^{\infty} a_n \exp \left[ -\frac{V}{\beta} \left( \frac{n \pi}{\ell} \right)^2 L \right] \sin \left( \frac{n \pi \eta}{\ell} \right).
\]

(B14)

On eliminating \( a_n \) in (B14) using (B6b), one obtains an integral equation for \( \phi(\eta) \) that is readily solved numerically by iteration. With \( \phi(\eta) \) known, \( \psi(x, \eta) \) follows from (B6a), (B6b) and (B13).

The \( \psi(x, \eta) \) obtained in this way does not vanish at \( x = -L \), \( \eta > a \), in violation of the first of boundary conditions (B2b), a deficiency due to the absence of eastward-decaying Rossby waves in (B12). On the other hand, the other boundary conditions and both of the periodicity conditions are satisfied. The first periodicity condition (B3), which is built into (B14), and the backwards diffusion equation (B12) guarantee the periodicity of \( \psi_s, \psi_t \), and all higher derivatives. Keeping 200 Fourier components leads to the results for \( \bar{\psi}_s(x, \eta) \) shown in Fig. 6b.

Appendix C. Boundary-layer structures

In this appendix, we solve for the structures of the boundary layers discussed in the main text. For completeness, we begin by noting the familiar structure of the western-boundary layers in the Atlantic and Pacific Oceans (Section C.1). The structures of the other boundary layers are less well known: They occur along zonal boundaries, broaden to the west, and are mathematically more complex (Sections C.2–C.4).

C.1. Western-boundary layer

Assuming that \( V' \) is geostrophic (\( vV \) and \( \tau' \) are neglected in Eqs. 1), the equation for the western-boundary streamfunction \( \psi' \) (\( V' = \psi_s, U' = -\psi_t \)) and its general solution are

\[
\beta \psi_s' = vV \psi_t \quad \Rightarrow \quad \psi' = \psi_s'(y) e^{-(x-L)/\beta},
\]

(C1)

where \( -L \) is the location of the western boundary, and \( r = \sqrt{\beta} \) is the boundary-layer width. The value of \( \psi' \) at \( x = -L, V'(y) \), is the transport of the boundary current, and is given by (23) or (30) in the Atlantic and Pacific.

C.2. Northern-boundary layer

Northern-boundary layers exist because the interior solution doesn’t generally satisfy the no-normal-flow condition there. The structure of the Atlantic northern-boundary layer changes markedly depending on whether \( w_n \) is active. When \( w_n = 0 \), it consists of two parts: an outer layer where \( w_n = 0 \), and an inner one where \( w_n \neq 0 \) and entrainment and detrainment can occur (see Schloesser et al., 2012, for discussion of a similar two-part boundary layer). When \( w_n = 0 \), which is always true for the Pacific northern-boundary layer, no inner boundary layer is needed.
C.2.1. Atlantic northern-boundary layer when $w_n \neq 0$

C.2.1.1. Outer boundary layer: Assuming that $w_n = w_o = 0$ and that $U$ is geostrophic ($vV = 0$ in Eqs. 1), the $P''$ equation is

$$P''_r = r^2 \left( \frac{p''_r}{r^2} \right) + rP''_\eta - 2r \frac{p}{r^2} P''_\eta = rP''_\eta,$$  \hspace{1cm} (C2)

where for convenience we introduce the new coordinates, $\xi = -x$, $\eta = y - y_o$. Reversing the direction of the zonal axis reflects the property that the establishment of the steady-state response involves the radiation of Rossby waves, which are damped by mixing as they propagate (extend) westward. The term proportional to $\beta f = O(1/R_e)$ is negligible because we assume that $\nu$ is small enough for $r \to$ be much thinner than the Earth's radius $R_e$, with this simplification, \((C2)\) has the form of a diffusion equation. It is possible to use perturbation theory to correct for the neglect of the $P''_\eta$ term; see Appendix D for a derivation of the correction to the $y_o$-boundary layer.

Boundary conditions for $P''$ are

$$P''(\xi, 0^-) = P_n - P_A + A_0 \xi, \hspace{1cm} P''(\xi, -\infty) = 0, \hspace{1cm} P''(0, \eta) = 0,$$  \hspace{1cm} (C3)

where $A_0 = \tau_0 (f/\beta) \tau_A^m - (f/\beta) w_o$ from \((14b)\), $A_0(\eta) = A_0(0^-)$, and $\eta = 0^- = 0$ is a latitude just outside the infinitesimally thin, inner boundary layer. According to \((14b)\), these conditions ensure that $P(\xi, 0^+ = P_n$, that $P''$ is outside the boundary layer, and that $P''(0, \eta) = P_0$.

We use the method of Laplace transforms (with $s \rightarrow \zeta$) to obtain the solution to \((C2)\). Taking the Laplace transform of \((C2)\) gives

$$\hat{P''}_\eta = \frac{s \hat{P'}_r}{s^2 + \nu^2},$$  \hspace{1cm} (C4)

where $\hat{q}$ designates the Laplace transform of variable $q$. Note that in evaluating $\hat{P''}_\eta$, we used the last of boundary conditions \((C3)\) to set $\hat{P''}(s, 0^-) = s\hat{P'} - \hat{P''}(0, \eta) = \hat{s} \hat{P'}$. Imposing the second condition \((C3)\) requires that $B = 0$ (since the real part of $s$ is required to be positive in the inverse transform), and the first then gives

$${\hat{P'}_r} = \frac{\hat{P}_r}{s^2 + \nu^2} \left( \frac{P_n - P_A}{s} + \frac{A_0}{s^2} \right) e^{\sqrt{\nu \zeta} \eta}.$$  \hspace{1cm} (C5)

The inverse Laplace transform of \((C5)\) is

$$P'(\xi, \eta) = (P_n - P_A) \text{erfc} \left( \frac{\eta}{\sqrt{4 \nu \zeta}} \right) + A_0 \int_0^\xi \text{erfc} \left( \frac{\eta}{\sqrt{4 \nu \zeta}} \right) d\zeta'. $$  \hspace{1cm} (C6)

where $\text{erfc}(z) = 1 - (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt$ is the complementary error function. (The inverse transforms of the terms in Eq. \((C5)\), or relevant versions of them, can be found in most tables of Laplace transforms.)

According to \((C6)\), the width of the boundary layer broadens to the west like $\sqrt{\zeta}$, and a measure of its width is $\Delta = 2\sqrt{\zeta}$. As such, $P'$ has a cusp in the northwest corner of the basin $(\xi = \eta = 0)$ where $\Delta = 0$. This cusp is clearly visible in our numerical solutions, albeit smoothed by the additional friction terms retained in the numerical version of the model \((14)\).

Consistent with the assumption that $\Delta/R_e < 1$, we assume that $f \approx f_n$ across the boundary layer. It then follows from along-boundary geostrophy that $\psi'' = P'_n f_n$, and hence from \((C3)\) that

$$\psi''(\xi, 0^-) = \frac{1}{f_n} (P_n - P_A + A_0 \xi) - \frac{1}{f_n} (P_n - P_A)(0^-(\xi)).$$  \hspace{1cm} (C7)

The last term in \((C7)\) is required because $\lim_{\xi \rightarrow -\infty} P''(\xi, 0) = P_n - P_e$, whereas the eastern-boundary condition requires that $P''(0, \eta) = 0$.

With the aid of constraint \((28)\) and \((14a)\), it follows that $V(\xi, 0^-) = V(\xi, 0^-) - \psi''(\xi, 0^-) = -\frac{\tau_n}{f_n} + \left( \frac{\tau_n L_a \psi''}{f_n^2} + M_n \right) \delta(\xi)$. \hspace{1cm} (C8)

(Note that $V'' = -\psi''$ in Eq. \((C8)\) since $\xi = -x$). According to \((C8)\), when $\tau_n > 0$ there is uniform southward flow across the basin along $\eta = 0^-$ with a net northward transport of $\tau_n L_a f_n / M_n$ in the northeast corner of the basin.

C.2.1.2. Inner-boundary layer: Because $V(\xi, 0^-) \neq 0$, an infinitesimally thin, inner boundary layer is required from $0^- < \eta < 0$ in order to satisfy the no-normal-flow boundary condition at $\eta = 0^-$. Within the inner layer, there is entrainment and detrainment by $w_n$ that provides a source and sink to balance the flow across its southern edge $V(\xi, 0^-)$. Integrating the continuity equation in \((1)\) across the inner layer gives

$$w_n(x) = \int_0^\infty w_n(z) dz = \int_0^\infty U_d d\eta - V(\xi, 0^-) = -V(x, 0^-) = \frac{\tau_n}{f_n} - \left( \frac{\tau_n L_a \psi''}{f_n^2} + M_n \right) \delta(\xi),$$  \hspace{1cm} (C9)

where the integral of $U_d$ is negligible because the inner layer is thin. According to \((C9)\), the uniform divergence along the boundary is compensated for by the coastal entrainment $\tau_n f_n$ for $x < 0$, and there is a net detrainment of $-(\tau_n L_a f_n / M_n)$ at $x = 0$. The net across-interface transport along the boundary is then

$$W_n(0) = \int_{-L_n}^0 w_n d\xi = -M_n.$$  \hspace{1cm} (C10)

According to \((C10)\), all the water that enters layer 1 anywhere in the interior ocean detrains in the northeast corner of the basin.

The northern boundary layer can be broadened to finite thickness by including horizontal, Laplacian viscosity in \((1)\), in which case it has the form of a horizontal Ekman layer \((P. E. 1987; S h l o s e r e r  a l ., 2012)\). With $\nu = 10^{-5}/10^7 c m^2/s$, its width is 1–4 km at 50°N, justifying the assumption that the inner layer is much thinner than the outer one $(\Delta(\Lambda L_4)) = 1038$ km).

C.2.1.3. Local overturning cell: A further implication of \((C9)\) and \((C10)\) is that, in addition to providing a conduit for $M_n$, the northern-boundary layer also contains a local (zonal) overturning cell. It consists of: an upwelling transport per width, $\tau_n / f_n$, spread uniformly along the boundary within the inner layer, which flows southward into the outer layer; an accelerating surface branch in the outer layer with transport, $\tau_n (x + L_n) / f_n$; and a compensating northward transport into, and downwelling transport within, the inner layer at the northeast corner. This cell is noteworthy as similar cells exist in OGCM solutions that represent northern diapycnal processes by a sponge layer \((N o n a k a  e t  a l ., 2006)\), but it is not dynamically important for our purposes.

C.2.2. Northern-boundary layers when $w_n = 0$

When $w_n = 0$, there is no entrainment or detrainment along the northern boundary, and hence no need for an inner boundary layer. In this case, it is easier to solve for the northern-boundary layer in terms of $\phi$. The equation for $\psi'' = \beta \psi'' / \psi'' \phi$ and appropriate boundary conditions are $\psi''(\xi, -\infty) = 0$, $\psi''(0, \eta) = 0$, and $\psi''(\xi, 0^-) = -\psi''(\xi, 0^-)$, where $\psi''(\xi, 0^-) = \xi (\tau_n L_a f_n / \beta) / \beta$ as determined from \((14b)\) with the identification $\psi'' = -V$. Using the method of Laplace transforms, it is then straightforward to show that the solution is

$$\psi''(\xi, \eta) = -\frac{1}{\beta} (\tau_n L_a f_n + M_n) \int_0^\xi \text{erfc} \left( \frac{\eta}{\sqrt{4 \nu \zeta}} \right) d\zeta'.$$  \hspace{1cm} (C11)

and then $P'' = f_n \psi''$. Note that $V''(\xi, 0^-)$ gives \((29a)\) and $P''(\xi, 0^-) = f_n \psi''(\xi, 0^-)$ gives \((29b)\), as they should.
The boundary layer along $y_b$ ensures that the Atlantic circulations north and south of $y_b$ join smoothly without jumps in $P'$ and $P_o$. It differs from the northern-boundary layer in that it does not extend along a solid wall and so spreads both north and south of $y_b$ (Fig. 2). Solution $P'(\xi, \eta)$ satisfies (C2), in this case subject to boundary conditions,

$$P'(0, \eta, 0) = 0, \quad P(0, \eta, 0) = 0,$$

where $\xi = 0$ is the location of the Atlantic eastern boundary and $\eta = y - y_b$. Solution $P'$ must also ensure that $P$ and $P_o$ are continuous across $\eta = 0$.

According to (C13), $P'$ and $P''_o$ must also jump across $\eta = 0$ in such a way that they cancel the corresponding jumps in the interior fields.

The general solution to (C2) is

$$\tilde{P} = Be^{-\sqrt{\epsilon}/\eta}(\eta) + Ce^{\sqrt{\epsilon}/\eta}(\eta),$$

which ensures that it decays to either side of $\eta = 0$. To find the constants, B and C, we take the Laplace transform of the complete solution, $\tilde{P} = \tilde{P}' + \tilde{P}''_o$ to get

$$\tilde{P} = \begin{cases} \frac{P_0}{s} - \frac{1}{s} + Be^{-\sqrt{\epsilon}/\eta}, & \eta > 0, \\ \frac{P_0}{s} - \frac{1}{s} - \frac{4}{2} \mathcal{L}_p \left. + Ce^{\sqrt{\epsilon}/\eta}, \eta < 0, \right. \end{cases}$$

where $A(\eta)$ is defined after (C3). To obtain (C15), we determine $\tilde{P}$ from (14b) with $x = -\xi$ and $P_o = P_o$ for $\eta > 0$ and with $x = -\xi + L_o$ and $P_o = P_o$ for $\eta < 0$; the term, $-\mathcal{L}_p / s$, for $\eta < 0$ then arises because the Sverdrup flow is integrated from $\xi = -L_p$, rather than $\xi = 0$.

Applying (C13) across $\eta = 0$ yields

$$2 \left( \frac{B}{C} \right) = \frac{P_o - P_a - \frac{A_b}{s} + \sqrt{\epsilon} \mathcal{A}_0(0) \mathcal{L}_p}{\frac{A_b}{s} + \sqrt{\epsilon} \mathcal{L}_p},$$

where $A_0 \equiv A(0)$ with $\tau^s$, $\tau^r$, and $f$ evaluated at $y = y_a$ and $A_0(0) = (-f / \beta) \tau^s_{y_a}(y_a) - 2f \mathcal{W}_d$. Combining the above pieces gives

$$\tilde{P} = \frac{1}{2} \left[ \frac{P_0 - P_a - \frac{A_b}{s}}{s} + \frac{\sqrt{\epsilon} \mathcal{A}_0(0) \mathcal{L}_p}{s^2} \right] e^{-\sqrt{\epsilon}/\eta}, \eta \geq 0,$$

where the plus (minus) sign corresponds to $\eta > 0$ ($\eta < 0$). The inverse transform of (C17) is

$$P'(\xi, \eta) = \frac{1}{2} \left[ \frac{P_0 - P_a - \frac{A_b}{s}}{s} + \frac{\sqrt{\epsilon} \mathcal{A}_0(0) \mathcal{L}_p}{s^2} \right] e^{-\sqrt{\epsilon}/\eta}, \eta \geq 0.$$
where the plus (minus) sign corresponds to \( \eta > 0 \) \( (\eta < 0) \), and its inverse Laplace transform is

\[
P^C(\zeta, \eta) = \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi \xi}} \exp \left( -\frac{\eta^2}{4r^2} \right) \phi(\eta')d\eta' + \frac{1}{2} (P_a - P_c) \text{erfc} \left( \frac{\eta}{\sqrt{4\xi}} \right) \frac{1}{2} A_0 \int_0^\zeta \text{erfc} \left( \frac{\eta}{\sqrt{4\xi}} \right) d\zeta' + \frac{1}{2} B_0 \sqrt{\zeta} \left[ 2 \sqrt{\frac{\xi}{\pi}} \exp \left( -\frac{\eta^2}{4\xi} \right) - \frac{\eta}{\sqrt{\xi}} \text{erfc} \left( \frac{\eta}{\sqrt{4\xi}} \right) \right] - \frac{1}{2} C \sqrt{\zeta} \int_0^\zeta \left[ 2 \sqrt{\frac{\xi}{\pi}} \exp \left( -\frac{\eta^2}{4\xi} \right) - \frac{\eta}{\sqrt{\xi}} \text{erfc} \left( \frac{\eta}{\sqrt{4\xi}} \right) \right] d\zeta',
\]

where \( \phi(\eta) = \Phi(L; \eta; \eta') \phi(\eta') + \lambda(\zeta, \eta, P_a) \),

\[
P^C(\zeta, \eta) = \Phi(L; \eta; \eta') \phi(\eta') + \lambda(\zeta, \eta, P_a).
\]

We summarize \((C26a)\) as

We expect the iteration to converge for several reasons: the adjustment of \( P_a \) by \((C30)\), which requires that the positive and negative contributions of the forcing \( \lambda(\zeta, \eta; P_a) \) tend to cancel; the smoothing properties of \( \gamma(\zeta, \eta; \eta') \); and the boundary condition along \( \zeta = 0 \) that limits the upper bound of the \( \eta' \)-integration in \((C30)\) to 0, thereby eliminating any contribution north of \( \eta' = 0 \).

To demonstrate the convergence, and further that \( P_a^i \) and \( P_a^i \) are already good approximations to the equilibrium solutions, we iterate the sequence numerically. (It is possible to demonstrate the convergence analytically by carrying out the above procedure beyond \( i = 1 \), but that approach is tedious.) We first assume that \( \phi(\eta) \) is defined at discrete \( \eta \) points for \( \eta \in [-Y, Y] \), where \( Y \) is far enough from the origin for all terms to be negligibly small near \( \eta = Y \). We then evaluate the double integral in \((C30)\) numerically and compute \( P_a^i \) from the equation. Finally, we numerically determine the integral of \( \gamma(L; \eta; \eta') \phi(\eta') \) in \((C30)\) at \( \zeta = L \) to obtain \( \phi(\eta) = P_a^i(L; \eta) \) at discrete \( \eta \) points. Starting from \( \phi_0(\eta) = 0 \), this procedure determines \( \phi(\eta) \) at discrete \( \eta \) points and \( P_a^i \) for \( i = 1, 2, \ldots \). If desired, we can use \((C30)\) to obtain \( P_a^i(\zeta, \eta) \) on a two-dimensional grid.

As an example, we assume that there is no Pacific Ocean so that \( L = L_a \) and set \( v = 2 \times 10^{-6} \text{ s}^{-1} \), \( M = 1.744 \text{ Sv} \), and \( \tau^* = \tau_0 \), the latter three entering \((C30)\) through \( A_a, B_0, \) and \( C \). (The value of \( M \) is that for Solution 3, red curve, in Fig. 8a when \( v = 2 \times 10^{-6} \text{ s}^{-1} \).) Fig. 14 (left panel) plots \( P_a^i - P_c \) for \( i = 1, \ldots, 10 \), and the rapid convergence is apparent in that \( P_a^i - P_c \) changes very little for \( i \geq 2 \). Fig. 14 (right panel) plots \( \phi(\eta) \) for \( i = 1, 2, 9, 10 \). The similarity of the curves (curves \( \phi_0 \) and \( \phi_0 \) are indistinguishable in the plot) demonstrates that the sequence is converging to a cyclic solution; further, the closeness of \( \phi_i(\eta) \) to \( \phi_0(\eta) \) demonstrates that it is already a good approximation to the fully converged response.

C.4.4. Edge response

The derivation of solution \((C26a)\) assumes that the southern edge of layer 1 is far enough away from \( y_a \) for the boundary response to be negligible there, so that the \( \eta \)-integrals can be extended to \(-\infty \). In some of our solutions, however, that property does not hold (compare gray-dashed and solid curves in the top-left panels of Figs. 8a, 8b, and 12). Here, we argue that, despite the presence of the edge, the Southern-Ocean and \( y_a \)-boundary constraints (Eqs. 22 and 35a) are not affected provided that inequality \((C32)\) holds, in which case the MOC properties of our solutions are unchanged.

Suppose the \( y_a \)-boundary solution is not vanishingly small at the southern edge of layer 1, \( \eta = \eta_a, \eta_0 \), or \( \eta_c \) (corresponding to \( y' = y_0 \), \( y_0 \), and \( y_c \)). Then, since the system is linear, we can add a reflection, \( P_1^i(\zeta, \eta) \), to \((C26a)\) such that \( P^i(\zeta, \eta) + P_1^i(\zeta, \eta) = 0 \). Because \( P_r \) satisfies the approximate form of differential equation \((C2)\), which
has coefficients that are independent of $\eta$, $P_0'$ is the reflection of $P'$ about the edge, that is,
\begin{equation}
P_0'(\xi, \eta) = -P'(\xi, 2\eta - \eta). \tag{C31}
\end{equation}

(Even if $f$ varies in Eq. C2, however, the argument in the next paragraph still holds, as a northward-decaying, homogeneous solution to the exact from of Eq. C2 still exists. It is just no longer expressible in terms of the southward-decaying solution as in Eq. C31.)

Let $\bar{q}$ be the value of variable $q$ at $\eta = \eta_0$. Then, with the choice (C31), the total pressure at $\eta = \eta_0$ is $P = P_0 + (P_0 + P_0') = \bar{P}_0$, and hence the pressure equations in constraints (C22) are unchanged. When $\eta = \eta_0'$ or $\eta_0$, the expressions for $M$ in (C22b) and (C22a) are clearly independent of $P$; $M$ still must vanish in the former case, and $P(\xi, \eta') = P(\eta') = \bar{P}_0$ implies that $h(\eta') = \eta_0$ in the latter. When $\eta = \eta_0$, it follows from (C31) that the contribution of $P$ at $\eta = \eta_0$ is $\frac{1}{2}g\eta_0^2h^2(\eta_0, \eta_0) = \frac{1}{2}g\eta_0^2h^2(\eta_0) + 2\bar{P}_0(\xi, \eta_0)$ since $g(\eta_0) = 0$. Multiplying by $\sqrt{\bar{f}}\bar{g}$ and integrating around the basin then gives $\nu' = \frac{1}{2}\int_0^\infty \frac{\bar{f}}{\bar{g}}\sqrt{h^2(\eta_0, \eta_0)}d\xi = \frac{1}{2}\sqrt{\eta_0^2h^2(\eta_0)L}$, where the integral of $\bar{P}_0'$ vanishes because $\int_0^\infty P'd\xi = 0$ for all $\eta$. Therefore, the expression for $M$ in constraint (C22c) is also unchanged.

Provided that $P_0'$ is vanishingly small at $\eta = 0$, there is no impact on constraint (35a). Since the northward decay scale of $P_0'$ is $\delta$, the same as the southward decay scale of $P'$, it follows that constraint (35a) is unaffected if $\delta \approx \nu'$. $\nu_0$, or $\nu_0'$, holds. Inequality (C32) holds very well for all of our solutions, except possibly when the basin includes the Pacific Ocean for which $\delta \approx \nu'$ or $\nu_0'$ (gray-dashed and solid curves in the top-left panels of Figs. 8a and 12).

C.4.5. Boundary constraint

Because the iteration converges so rapidly, it is sensible to evaluate (C30) at first order ($i = 1$). Setting $\phi_0 = 0$ in (C30) then gives
\begin{equation}
P_{0i} = P_x - \frac{1}{2}A_0\frac{L}{\delta} - \frac{4}{3}B_0\eta - \frac{8}{15}C_0\delta, \tag{C33}
\end{equation}
providing the simplest version of constraint (35a). We used (C33) instead of (35a) to evaluate the MOC curves in Figs. 8a and 12, and found that they deviated systematically from the data points taken from numerical solutions. We determined that the error arose primarily from two differences between the numerical and analytic models: In the numerical model, $f$ varies across the $y_0$-boundary layer, and the western-boundary layer has a finite width.

The two differences alter the structure of $P'$ along $y_0$, and so impact the zonal integral in (C30). Let $P_0'(\xi)$ and $P_0''(\xi)$ be the corrections to the analytic solution that arise from variable $f$ and the finite-width, western-boundary layer, respectively. Then, replacing (C30) at first order with
\begin{equation}
\int_0^\xi [P'_1(\xi, 0) + P_0'(\xi) + P_0''(\xi)]d\xi = 0 \tag{C34}
\end{equation}
provides an improved approximation for the $y_0$-boundary constraint.

Correction $P_0'(\xi)$ is derived in Appendix D and is given in (D6). We assume that correction $P_0''(\xi) = \Delta P\exp(\xi/\sqrt{\tau'})$, where $\tau' = \sqrt{\tau}$. Width-scale $\tau'$ is larger than $\tau$ because the correction lies at the tip of South America in the region where the width scale of the response is small in both $x$ and $y$, that is, the overall scale is $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\tau^2 - \tau'}$. (The numerical solutions confirm that $\tau$ is a better measure of the decay scale than $\tau$ there.) Amplitude $\Delta P$ is set by the requirement that $P$ is continuous around the tip of South America; as a consequence, $P$ along $\eta = 0$ increases from its value of $P$ just outside the boundary layer to $P_x$ right at the coast, that is, $\Delta P = P_x - [P_0' + P_0''(L/\Delta x)]$.

Integral (C34) then gives
\begin{equation}
P_x' = P_x - \frac{1}{2}A_0\frac{L}{\delta} - \frac{1}{7} \left(1 + \frac{16}{5}B_0 - \frac{2\sqrt{2}}{3}\right) + \frac{4}{3}B_0\eta - \frac{8}{15}C_0\eta - \frac{1}{7} \left(1 - \frac{2\sqrt{2}}{3}\right), \tag{C35}
\end{equation}
where $\tau' = 1 + (4/3)/\sqrt{\tau'^2\tau}$. In (C35), the correction terms for variable-\(f\) and the finite-width, western-boundary current are proportional to $\Delta P/\tau'$ and $\tau/C_0$, respectively. Note that (C35) reduces to (C33) when $\delta$ and $\tau$ are zero, as it should.

Constraint (35a) is a version of (C35) with $\tau^2 = \tau^2_0(\tau_0^2) = 0$, which is true for the wind forcing considered in this paper. Further, (35a) neglects the term proportional to $\Delta P$, which is smaller by a factor of $\delta/\tau'$ than a corresponding expression (involving $\tau_0$) in the term proportional to $A_0$.

Appendix D. Correction for $f(\eta)$ in the $P$ equation

In obtaining the solution along $y_0$ (Appendix C.4), we assume that $f$ is constant in the $P'$ equation. Although this error is small, it is nevertheless large enough to shift the data points in Figs. 8a–8c away from the analytic curves. Here, we use perturbation theory to reduce the discrepancy.

The exact equation of motion expands into
\begin{equation}
P''_x - \tau f''_0(\eta) = -\frac{2\pi}{f} \int_0^\xi f' P_x' \tag{D1}
\end{equation}
where we include a factor of $\epsilon$ to indicate that the term on the right-hand side is small relative to those on the left-hand side, that is, $\epsilon = O(\delta/\tau')$. We look for solutions in a perturbation series of the form $P' = P'_0 + \epsilon P'_1 + \cdots$, keeping only the two lowest-order terms. The first-order equation is $P'_0 - \tau P''_0 = 0$, the same equation we solved in Appendix C.4. The second-order equation is then
\begin{equation}
P_0''_x - \tau f''_0 = -2\frac{\pi}{f} \int_0^\xi f' P_x', \tag{D2}
\end{equation}
where we can safely replace $f$ with its value at $y_0$, since the error of that change is of order $\epsilon^2$.

Note that by differentiating the first-order equation with respect to $\eta$, it follows that $P'_0$ is also a solution. Thus, the forcing of (D2) is itself a solution of the operator on the left-hand side (since $f$ and $\eta$ constant), a resonant situation. In that case, a particular solution to (D2) is simply
\begin{equation}
P_1''_x - \tau f''_0 = -2\frac{\pi}{f} \int_0^\xi f' P_x', \tag{D3}
\end{equation}

To complete the solution, we need to add on solutions to the homogeneous version of (D2) that eliminate any jumps in $P''_x$ or its $\eta'$-derivative across $\eta = 0$, that is in $P''_x(\xi, 0)$ and $P''_{10}(\xi, 0)$.

The first-order solution is given by (C26a) without the first term on its right-hand side since $\phi = 0$. With the aid of the identity
\begin{equation}
\text{erfc} \left( \frac{\pm \eta}{\sqrt{4\tau'}} \right) = \mp \frac{1}{\sqrt{\pi\tau'}} \exp \left( -\frac{\eta^2}{4\tau'} \right), \tag{D4}
\end{equation}

it follows that
\begin{equation}
P_{10}(\xi, \eta) = \frac{P_x - P_x'}{2\sqrt{\pi\tau'}} \int_0^\xi \exp \left( -\frac{\eta^2}{4\tau'} \right) \frac{A_0}{2\sqrt{\pi\tau'}} \int_0^\xi \exp \left( -\frac{\eta^2}{4\tau'} \right) d\zeta' + \frac{B_0}{2} \text{erfc} \left( \frac{\pm \eta}{\sqrt{4\tau'}} \right) + \frac{C}{2} \int_0^\xi \text{erfc} \left( \frac{\pm \eta}{\sqrt{4\tau'}} \right) d\zeta', \tag{D5}
\end{equation}
which has jumps only in the terms proportional to $B_0$ and $C$. Taking
an $\eta$ derivative of (D5) shows that $P_{10}^{\eta}(\xi, \eta)$ has no jumps at all.
Thus, the homogeneous solution is needed only to eliminate the
jump in the last two terms of (D5). These terms, however, are
antisymmetric about $\eta = 0$. Because the diffusion operator in (D2)
is symmetric in $\eta$, it follows that $P_{20}^{\eta}(\xi, \eta)$ is antisymmetric about
$\eta = 0$ so that $P_{20}^{\eta}(\xi, 0) = 0$. Although we can determine the homoge-
nous solution for all $\eta$, for our purposes it is sufficient to know
that it vanishes at $\eta = 0$.

Solution $P_x^2(\xi, 0)$ is therefore given by $P_x^2 = -2r(j/f) \xi P_x^0$, except
without the $B_0$ and $C$ terms in (D5), that is,

$$P_x^2(\xi, 0) = -\frac{\sqrt{\xi^2}}{R_e \sqrt{\pi}} [(P_e - P_x) + 2A_x^1 \xi]. \quad (D6)$$

Solution (D6) is the correction term $P_x^2$ that is included in (C34).

**Appendix E. List of variables**

For each variable, the first column gives its name and the second
provides a brief description. The top block defines labeling
conventions that apply to almost all variables $q$. The rest of the
table alphabetically lists variables that are exceptions to the con-
tentions or that are referenced at more than one location in the text.

| $q'$ ($q''$) | interior (boundary-layer) part of variable $q$ |
| $q_x$ ($q_x^2$) | value of variable $q$ at $y$ ($y_x^2$) |
| $\tilde{q}$ | value of variable $q$ at $y$ (except for $\tilde{u}$) |
| $\tilde{q}^t$ ($\tilde{q}^s$) | instantaneous (time-averaged) value of $q$ |
| $\langle q \rangle$ ($\langle q \rangle^T$) | zonally (meridionally) averaged value of variable $q$ |
| $\tilde{A}$ ($\tilde{A}_P$) | area of the domain (north of $\tilde{y}$ and not including Region $A$) |
| $A(\tilde{A})$ | area of the Pacific Ocean north of $y$ and not including Region $B_P$ |
| $A_0$ ($A_0^\eta$) | function $A(\eta)$ evaluated at $\eta = 0$ ($\eta = 0^+$) |
| $B_0$ ($B_0^\eta$) | regions enclosed by $x_{0B}$ ($x_{0B}$) in the North Atlantic (Pacific) |
| $\tilde{B}$ | Region $B$ or $B_P$ |
| $\tilde{B}(\eta)$ | function defined in Appendix C in the interior SO |
| $\tilde{B}_0$ ($\tilde{B}_0^\eta$) | function $B(\eta)$ evaluated at $\eta = 0$ ($\eta = 0^-$) |
| $\delta$ | width of $y$-boundary layer in the Southern Ocean |
| $\Delta y$ | $y_{1} - y$, width of layer 1 across the Southern Ocean |
| $\langle \Delta (\tilde{L}) \rangle^x$ | average width of a zonal Stommel layer in a basin of width $\tilde{L}$ |
| $f'$ ($f_0$) | Coriolis parameter evaluated at $y$ ($y_0$) |
| $g' \neq 0$ | notation indicating use of warm-Q forcing |
| $g(y < y_0) = 0$ | notation indicating use of cold-Q forcing |
| $h_m$ | minimum layer-1 thickness, corresponding to mixed-layer thickness |
| $h_n$ ($h_s$) | layer-1 thickness along $y_s$ ($y_s$) |
| $h_0$ | layer-1 thickness along $y_0$ |
| $\tilde{h}$ | layer thickness of the Southern-Ocean interior solution along $\tilde{y}$ |
| $H_A$ ($H_{AP}$) | layer-1 thickness along the Atlantic (Pacific) eastern boundary |
| $\kappa$ | coefficient of vertical diffusion |
| $\kappa_h$ | horizontal-mixing coefficient for GM closure |
| $L_A$ ($L_P$) | width of the Atlantic (Pacific) Ocean |
| $L_T$ | $L_A + L_P$ |
| $L$ | $L_A$ or $L_P$ |
| $L_A$ or $L_A + L_P$ | $y_s - y$, width of the Southern Ocean |
| $L_A R_e / (L_A + R_e)$ | transport that crosses the southern edge of layer 1 (across $y'$ or $y_0$) |
| $v$ | horizontal-mixing coefficient |
| $v_1$, $v_s$ | critical values of $v$ when $g' \neq 0$ |
| $v_0$ | critical value of $v$ when $g'(y < y_0) = 0$ |
| $P$ | $\frac{1}{2} g' h^2$, the available potential energy of layer 1 |
| $P_x^2$ | value of $P$ along $y_x$ (exception to the general convention of $q_x^2$) |
| $P_A (P_P)$ | value of $P$ along the Atlantic (Pacific) eastern boundary |
| $P_e (P_{ew})$ | value of $P$ along the Atlantic (Pacific) western boundary |
| $P_m$ | value of $P$ when $h = h_m$ |
| $Q$ | surface buoyancy forcing |
| $r$ | width of Stommel layer |
| $R$ | Rossby radius of deformation |
| $R_e$ | radius of the earth |
| $\rho_1 (\rho_2)$ | density of layer 1 (layer 2) |
| $\tau^x (\tau^P)$ | layer-1 (layer-2) temperature |
| $\tau^s$ ($\tau^w$) | wind forcing |
| $\tau_{s_0}$ ($\tau_{s_0}^w$) | three parts of $\tau^x$ |
| $\tau_{s_0}$ ($\tau_{s_0}^w$) | $\tau^x$ evaluated at $y_s$ |
| $\tau_{s_0}$ ($\tau_{s_0}^w$) | critical values of $\tau_{s_0}$ when $g' \neq 0$ |
| $\langle g'(y < y_0) = 0 \rangle$ | baroclinic ACC transport across the Southern Ocean |
| $U^P (U^w)$ | stepfunction ($\theta = 1$ if $\tilde{y} > 0$ and 0 otherwise) |
| $U_{max}$ | maximum baroclinic ACC transport in the $y_s$-boundary layer |
| $U_{max}$ | $U^P + U_{max}$ |
| $U^P (U^w)$ | barotropic ACC transport in the interior SO |
| $U_{max}$ | $U^P + U_{max}$ |
| $\tilde{U}$ | barotropic ACC transport in the interior SO |
| $\tilde{U}' (\tilde{U}'^w)$ | barotropic ACC transport in the interior SO |
| $\tilde{V}$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $\tilde{V}' (\tilde{V}'^w)$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $\tilde{W}$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $\tilde{W}' (\tilde{W}'^w)$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $\tilde{W}' (\tilde{W}'^w)$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $\tilde{W}' (\tilde{W}'^w)$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $\tilde{W}' (\tilde{W}'^w)$ | eddy-driven transport/width (transport) in the Southern Ocean |
| $W_{0d}$, $W_{0m}$, $W_n$ | three parts of $w_e$ |
| $W_{0d} + W_{0m} + W_n$ | three parts of $w_e$ |
| $W'_{d}$ | entrainment transport due to interior diffusion |
| $W_{d}$ | upwelling transport in the North Atlantic (Pacific) Subpolar Gyre |
| $W_{d}$ | $W_{d}$ or $W_{d}'$ |
| $W_{d}$ | eastern boundary of the North Atlantic (Pacific) upwelling region |
| $\tilde{X}$ | $x_{0A}$ or $x_{0B}$ |
| $y_n (y_s)$ | latitude of the northern (southern) boundary of the basin |
| $y_n (y_s)$ | latitude of the tip of South America (Africa) |
| $y_n (y_s)$ | latitude just north (south) of $y_s$ |
| $y_l (y_s)$ | latitude at the southern (northern) edge of $y_s$ |
| $y_0 (y_s)$ | latitudes where $x_0(y)$ intersects the western boundary |
| $y_e (-y_e)$ | latitude where $T$ first begins to cool in the northern (southern) ocean |
y' \text{ latitude where layer 1 outcrops (h first thins to h_n)}

y_0 \text{ latitude where layer 1 vanishes for strong (cold-T) cooling}

y, y_0, or y'

References


