J. Math. Anal. Appl. 368 (2010) 623–635

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mean curvatures and Gauss maps of a pair of isometric helicoidal and rotation surfaces in Minkowski 3-space

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ARTICLE INFO

Article history:
Received 25 August 2009
Available online 27 March 2010
Submitted by H.R. Parks

Keywords:
Minkowski space
Helicoidal surface
Rotation surface
Isometry
Mean curvature
Gauss map

ABSTRACT

It is proved that, in Minkowski 3-space, a *CSM*-helicoidal surface, i.e., a helicoidal surface under cubic screw motion is isometric to a rotation surface so that helices on the helicoidal surface correspond to parallel circles on the rotation surface. By distinguishing a *CSM*-helicoidal surface as three cases, that is, the case of type *I*, the case of type *II* with negative and positive pitch, the relations are discussed between the mean curvatures or Gauss maps of a pair of isometric helicoidal and rotation surface. A *CSM*-helicoidal surface of Case 1 or 2 and its isometric rotation surface with null axis have same mean curvatures (resp. Gauss maps) if and only if they are minimal. But each pair of isometric *CSM*-helicoidal surface of Case 3 and rotation surface with spacelike axis have different Gauss maps.

1. Introduction

A helicoidal surface in R^3 is defined as the orbit of a plane curve under a screw motion, which is a natural generalization of a rotation surface. A well-known result about a helicoidal and rotation surface in R^3 is

Bour's Theorem. *(See [2,7].) A helicoidal surface is isometric to a rotation surface so that helices on the helicoidal surface correspond to parallel circles on the rotation surface.*

Particularly, a right helicoid is isometric to a catenoid. Moreover, a pair of these two surfaces have interesting properties. That is, they are both members of a one-parameter family of isometric minimal surfaces and have the same Gauss map.

Denote by E^3_1 the *Minkowski* 3-*space* with an inner product of signature $(1,2)$ given by

$$g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. Many of the classical results from Euclidean geometry have a Minkowski counterpart, like the existence of Delaunay surface. But the presence of null vectors often causes important and interesting differences.

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1 Supported by BK21.
2 Supported by KOSEF R01-2007-000-20014-0 (2007).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.03.054

As the screw motion in R^3, a Lorentzian screw motion can be defined as *a Lorentzian rotation around an axis together with a translation in the direction of the axis*. Depending on the axis being spacelike, timelike or null, there are three types of so-called Lorentzian screw motions as following

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}, \tag{i}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}, \tag{ii}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 + \frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1 - \frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}. \tag{iii}$$

There are many papers about helicoidal surfaces under a so-called Lorentzian screw motion. Sasahara [11] studied spacelike helicoidal surfaces with constant mean or Gauss curvatures. Beneki et al. [1], Ji and Hou [7] constructed the helicoidal surfaces with prescribed mean or Gauss curvatures. In [1,7], it was also proved that, locally, there exist helicoidal surfaces with prescribed smooth functions as mean or Gauss curvatures. Choi et al. [3,4] classified helicoidal surfaces with pointwise 1-type Gauss maps and harmonic Gauss maps. Ikawa [10], Güler and Vanli [6] gave Bour's theorem in Minkowski 3-space. Especially, Ikawa proved that, other than a helicoidal surface with non-null axis, *a pair of helicoidal surface with a null axis and its isometric rotation surface have same profile curves, same first and second fundamental forms. Therefore, they have same Gauss maps, mean curvatures and Gauss curvatures* (see Theorem 5.1 and Corollary 5.1, p. 394 in [10]).

In fact, *the so-called helicoidal surface with null axis has so special properties is because it is still rotation surface.* Dillen and Kühnel [5] pointed out that the so-called Lorentzian screw motion above works fine for the case with non-null axis. However, *a Lorentzian rotation around a null axis, together with a translation in the direction of the axis, is again a Lorentzian rotation around a null axis* (see Remark 2.1 below). *That means the helicoidal surfaces with null axes discussed by Ikawa and other authors of* [1,6,7,9,11] *are still rotation surfaces.*

Besides, there exist other non-trivial 1-parameter families of translations that, together with a Lorentzian rotation around a null axis, constitute a 1-parameter group of Lorentzian motions, the so-called *cubic screw motion* [5], which is expressed as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 + \frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1 - \frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} \frac{s^3}{3} + s \\ \frac{s^3}{3} - s \\ s^2 \end{pmatrix}.$$

Obviously, a cubic screw motion, which have no counterpart in R^3, is different from a non-cubic case. A non-cubic screw motion has the property that, if we take a point of the axis, then the orbit of that point is simply the axis (or the point itself if the screw motion is a rotation). A cubic screw motion does not have that property. In fact, the orbit of the origin under a cubic screw motion is a cubic null curve, just given by the translational part of the cubic screw motion.

For ease of elaboration, *a helicoidal surface under cubic screw motion* is abbreviated to *a CSM-helicoidal surface*. For more details of a *CSM*-helicoidal surface, see [8] or Section 2 of this paper. In Section 3, for ease of discussion, a *CSM*-helicoidal surface is distinguished as three cases, that is, the case of type *I*, the case of type *II* with pitch $h < 0$ and the case of type *II* with pitch $h > 0$. It is proved that Bour's theorem is still true for all the cases of *CSM*-helicoidal surfaces (see Theorems 3.1, 3.4, 3.7).

The main purpose of this paper is to study the relations between the mean curvatures or Gauss maps of a pair of isometric *CSM*-helicoidal surface and rotation surface. As for the first two cases, a minimal *CSM*-helicoidal surface is isometric to a minimal rotation surface with null axis (see Corollaries 3.1, 3.2). Moreover, a *CSM*-helicoidal surface of Case 1 or 2 and its isometric rotation surface have same mean curvatures (resp. Gauss maps) if and only if they are minimal (see Theorems 3.2, 3.3, 3.5, 3.6). As for Case 3, each pair of isometric *CSM*-helicoidal surface and rotation surface with spacelike axis have different Gauss maps (see Theorem 3.8). Besides, a minimal *CSM*-helicoidal surface of the third case is isometric to a non-minimal rotation surface with spacelike axis.

2. *CSM*-helicoidal surfaces in E_1^3

We remark that a cubic screw motion can be written in a simpler form. Now, we consider a pseudo-orthonormal basis of E_1^3, i.e., a basis $\{e_1, e_2, e_3\}$ such that

$$g(e_1, e_1) = g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_3) = 0, \qquad g(e_1, e_3) = g(e_2, e_2) = 1.$$

In such a basis, we get

$$g(x, x) = 2x_1x_3 + {x_2}^2, \quad x = \sum x_k e_k.$$

Let

$$(e_1, e_2, e_3) = (\eta_1, \eta_2, \eta_3)X, \tag{2.1}$$

where $\{\eta_1, \eta_2, \eta_3\}$ is an orthonormal basis such that

$$g(\eta_i, \eta_j) = \varepsilon\delta_{ij}, \quad \varepsilon = \begin{cases} -1, & \text{if } i = 1, \\ 1, & \text{if } i = 2, 3 \end{cases}$$

and

$$X = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the cubic screw motion around the axis e_3 can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto X^{-1} \begin{pmatrix} 1+\frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1-\frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} X \begin{pmatrix} x \\ y \\ z \end{pmatrix} + hX^{-1} \begin{pmatrix} \frac{s^3}{3}+s \\ \frac{s^3}{3}-s \\ s^2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A(v) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h\beta(v), \tag{2.2}$$

where

$$A(v) = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ -\frac{v^2}{2} & -v & 1 \end{pmatrix}, \qquad \beta(v) = \begin{pmatrix} v \\ \frac{v^2}{2} \\ -\frac{v^3}{6} \end{pmatrix}, \quad v = -\sqrt{2}s.$$

Remark 2.1. Incidentally, it is easily to see that

$$A(v) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = A(v) \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}.$$

This indicates that a rotation around the null coordinate axis Oe_3, together with a translation in the direction of Oe_3, is again a rotation around the null line l:

$$\frac{x}{0} = \frac{y-h}{0} = \frac{z}{1}.$$

Definition 2.1. Let $\gamma : I = (a, b) \subset R \to P$ be a curve in a plane P in E_1^3 and denote by L a straight line that does not intersect the curve γ. A *CMS*-helicoidal surface in E_1^3 is defined as a non-degenerate surface that is generated by a cubic screw motion around L.

We distinguish the following two cases.

Case 1. Let $\gamma_1(u) = (u, 0, f(u))$, $u > 0$ be a curve in the Oe_1e_3 plane. Suppose that S be the helicoidal surface generated by $\gamma_1(u)$ under a cubic screw motion with pitch h, the position vector r of which has the form

$$r(u, v) = \left(u + hv, uv + \frac{hv^2}{2}, f(u) - \frac{uv^2}{2} - \frac{hv^3}{6}\right), \quad u > 0, \ v \in R. \tag{2.3a}$$

Case 2. Let $\gamma_2(u) = (0, u, f(u))$, $u > 0$ be a curve in the Oe_2e_3 plane. Suppose that S be the helicoidal surface generated by $\gamma_2(u)$ under a cubic screw motion with pitch h, the position vector r of which has the form

$$r(u, v) = \left(hv, u + \frac{hv^2}{2}, f(u) - uv - \frac{hv^3}{6}\right), \quad u > 0, \ v \in R. \tag{2.3b}$$

A *CSM*-helicoidal surface S given by (2.3a) (resp. (2.3b)) is called a *CSM*-helicoidal surface of type *I* (resp. type *II*). Especially when $h = 0$, a *CSM*-helicoidal surface of type *I* is called a rotation surface with null axis e_3.

Remark 2.2. A helicoidal surface given by (2.3b) is degenerate when $h = 0$.

If S is of type I, the first and the second fundamental forms I and II of S are given by

$$I = 2f'\,du^2 + 2hf'\,du\,dv + u^2\,dv^2, \tag{2.4}$$

and

$$II = |D|^{-1/2}\big[uf''\,du^2 + 2hf'\,du\,dv + \big(h^2 f' - u^2\big)\,dv^2\big], \tag{2.5}$$

where $D = EG - F^2 = 2u^2 f' - h^2 f'^2$ and the *prime* denotes derivative with respect to u.

The mean curvature H of S is thus

$$H = \frac{u^3 f'' - 2u^2 f'}{2D|D|^{1/2}}. \tag{2.6}$$

If S is of type II, the first fundamental form I, the second fundamental form II of S are

$$I = du^2 + 2hf'\,du\,dv - 2hu\,dv^2 \tag{2.7}$$

and

$$II = |D|^{-1/2}\big(-hf''\,du^2 + 2h\,du\,dv + h^2 f'\,dv^2\big), \tag{2.8}$$

where $D = EG - F^2 = -2hu - h^2 f'^2$ and the *prime* denotes derivative with respect to u.

The mean curvature H of S is thus

$$H = \frac{h^2(2uf'' - f')}{2D|D|^{1/2}}. \tag{2.9}$$

From (2.6) and (2.9), we can see that a minimal *CSM*-helicoidal surface is independent of its pitch h. By solving two differential equations $H = 0$, we get

Proposition 2.1. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Then a minimal CSM-helicoidal surface of type I is*

$$r(u,v) = \left(u + hv, uv + \frac{hv^2}{2}, c_1 u^3 - \frac{uv^2}{2} - \frac{hv^3}{6}\right), \tag{2.10}$$

where c_1 is a non-zero integration constant and $c_1 h^2 \neq 2/3$. Especially, any rotation surface with the axis e_3 can be expressed as (2.10) with $h = 0$.

Proposition 2.2. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Then a minimal CSM-helicoidal surface of type II is*

$$r(u,v) = \left(hv, u + \frac{hv^2}{2}, c_2 u^{3/2} - uv - \frac{hv^3}{6}\right), \tag{2.11}$$

where c_2 is an integration constant and ${c_2}^2 h \neq -8/9$.

Remark 2.3. If $c_1 = 0$ or $c_1 h^2 = 2/3$ (resp. ${c_2}^2 h = -8/9$), then the helicoidal surface given by (2.10) (resp. (2.11)) is degenerate.

3. Main results

Let S and S_r be a *CMS*-helicoidal surface and a rotation surface respectively. In this section, we study an isometric relation between S and S_r. Especially, we discuss the sufficient and necessary condition of S and S_r having same mean curvature or Gauss map. For ease of discussion, we distinguish S the following three cases.

Case 1. S is of type I.

Consider the rotation surface S_r generated by $\gamma(\bar{u}) = (\bar{u}, 0, g(\bar{u}))$, $\bar{u} > 0$, i.e., the rotation surface with the following parametrization.

$$r(\bar{u}, \bar{v}) = \left(\bar{u}, \bar{u}\bar{v}, g(\bar{u}) - \frac{\bar{u}\bar{v}^2}{2}\right). \tag{3.1}$$

From (2.4), the first fundamental forms of S and S_r can be respectively expressed as

$$I = \left(2f' - \frac{h^2 f'^2}{u^2}\right) du^2 + u^2\left(dv + \frac{hf'}{u^2}\, du\right)^2 \tag{3.2}$$

and

$$I = 2g_{\bar{u}}\, d\bar{u}^2 + \bar{u}^2\, d\bar{v}^2. \tag{3.3}$$

Comparing (3.2) with (3.3), if

$$2g_{\bar{u}}\, d\bar{u}^2 = \left(2f' - \frac{h^2 f'^2}{u^2}\right) du^2, \qquad \bar{u}^2 = u^2/c^2 \quad \text{and} \quad d\bar{v} = c\left[dv + \frac{hf'}{u^2}\, du\right], \tag{3.4}$$

where c is a positive constant, then we have an isometry between S and S_r.

(3.4) implies that

$$\bar{u} = u/c, \qquad \bar{v} = c\left[v + h\int \frac{f'}{u^2}\, du\right] \quad \text{and} \quad g(\bar{u}) = c\left[f(u) - \frac{h^2}{2}\int \frac{f'^2}{u^2}\, du\right]. \tag{3.5}$$

So we have proved the following

Theorem 3.1. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Locally, a CSM-helicoidal surface of type I S*

$$r(u, v) = \left(u + hv, uv + \frac{hv^2}{2}, f(u) - \frac{uv^2}{2} - \frac{hv^3}{6}\right) \tag{3.6}$$

is isometric to the one-parametric rotation surface S_r

$$r(u, v) = \left(u/c, u\left(v + h\int \frac{f'}{u^2}\, du\right), c\left[f(u) - \frac{h^2}{2}\int \frac{f'^2}{u^2}\, du - \frac{u}{2}\left(v + h\int \frac{f'}{u^2}\, du\right)^2\right]\right), \tag{3.7}$$

so that helices on the helicoidal surface correspond to parallel circles on the rotation surface, where c is a positive constant.

Especially, when $c = 1$, S is isometric to rotation surface S_{r_0}

$$r(u, v) = \left(u, u\left(v + h\int \frac{f'}{u^2}\, du\right), f(u) - \frac{h^2}{2}\int \frac{f'^2}{u^2}\, du - \frac{u}{2}\left(v + h\int \frac{f'}{u^2}\, du\right)^2\right). \tag{3.8}$$

Remark 3.1. For any positive constant c, the rotation surface S_r can be obtained by applying a rotation σ around spacelike axis e_2 on S_{r_0}, where σ is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1/c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{3.9}$$

Corollary 3.1. *Locally, a minimal CSM-helicoidal surface of type I*

$$r(u, v) = \left(u + hv, uv + \frac{hv^2}{2}, c_1 u^3 - \frac{uv^2}{2} - \frac{hv^3}{6}\right) \tag{3.10}$$

is isometric to the minimal rotation surface

$$r(u, v) = \left(u/c, u(v + 3c_1 hu + c_3), c\left(c_1 - \frac{3}{2}{c_1}^2 h^2\right)u^3 - \frac{c}{2}u(v + 3c_1 hu + c_3)^2\right), \tag{3.11}$$

where c_1 and c_3 are integration constants, $c_1 \neq 0$ and $c_1 h^2 \neq 2/3$.

Now we discuss the mean curvatures and Gauss maps of a pair of isometric helicoidal surface S and rotation surface S_r in Theorem 3.1. From (2.6), we get the mean curvature of S_r is

$$H_r = \frac{\bar{u}^3 g_{\bar{u}\bar{u}} - 2\bar{u}^2 g_{\bar{u}}}{2\bar{D}|\bar{D}|^{1/2}}, \tag{3.12}$$

where g is given by (3.5) and $\bar{D} = 2\bar{u}^2 g_{\bar{u}}$.

By differentiating the third equation of (3.5) with respect to $\bar{u}$ twice, we get

$$g_{\bar{u}} = c^2\left(f' - \frac{h^2 f'^2}{2u^2}\right) \tag{3.13}$$

and

$$g_{\bar{u}\bar{u}} = c^3\left(f'' + \frac{h^2 f'^2}{u^3} - \frac{h^2 f' f''}{u^2}\right). \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we have

$$H_r = \left(1 - \frac{h^2 f'}{u^2}\right)H, \tag{3.15}$$

where H and H_r are mean curvatures of S and S_r in Theorem 3.1 respectively. And H is given by (2.6). This follows that $H_r = H$ is equivalent with $f'H = 0$. Notice that $f' \neq 0$ (otherwise S is degenerate). Thus we have

Theorem 3.2. *Locally, a pair of isometric CSM-helicoidal surface of type I S and rotation surface S_r in Theorem* 3.1 *have same mean curvature if and only if they are minimal.*

Remark 3.2. From (3.15), we can see that the mean curvature of S_r in Theorem 3.1 is independent of parameter c. Moreover, by a direct computation, we can see that the coefficients of the first and second fundamental form of S_r are independent of parameter c.

Let n (resp. n_r) be the normal vector field on the surface S and S_r given by (3.6) (resp. (3.7)). An easy computation leads to

$$n = |D|^{-1/2}\left(u, hf' + uv, -uf' - hf'v - \frac{u}{2}v^2\right) \tag{3.16}$$

and

$$n_r = |D|^{-1/2}\left(u/c, u\left(v + h\int \frac{f'}{u^2}\,du\right), -c\left[u\left(f' - \frac{h^2 f'^2}{2u^2}\right) - \frac{u}{2}\left(v + h\int \frac{f'}{u^2}\,du\right)^2\right]\right), \tag{3.17}$$

where $D = EG - F^2 = 2u^2 f' - h^2 f'^2$.

Comparing (3.16) with (3.17), we can see that $n = n_r$ if and only if

$$\begin{cases} c = 1, \\ hf' = hu\displaystyle\int \frac{f'}{u^2}\,du, \\ -uf' = -u\left(f' - \dfrac{h^2 f'^2}{2u^2}\right) - \dfrac{u}{2}\left(h\displaystyle\int \frac{f'}{u^2}\,du\right)^2. \end{cases} \tag{3.18}$$

The general solution of (3.18) is

$$c = 1, \qquad f = c_1 u^3 + c_4, \quad \text{and} \quad \int \frac{f'}{u^2}\,du = 3c_1 u, \tag{3.19}$$

where c_1 and c_4 are integration constants.

This implies that S and S_r have same Gauss map if and only if S and S_r are minimal surfaces given by (3.10) and (3.11) with $c = 1$ and $c_3 = 0$. In general, a pair of minimal isometric helicoidal and rotation surfaces in Corollary 3.1 have different Gauss maps.

Denote by S'_{r_0} the minimal rotation surface given by (3.11) with $c = 1$ and $c_3 = 0$ and S'_r the general case given by (3.11). Obviously, the coefficients of the first and second fundamental form of S'_r are independent of parameters c and c_3. From Remark 3.2 and (3.11), we can see that S'_{r_0} can be obtained by applying a rotation σ_1 around spacelike axis e_2 together

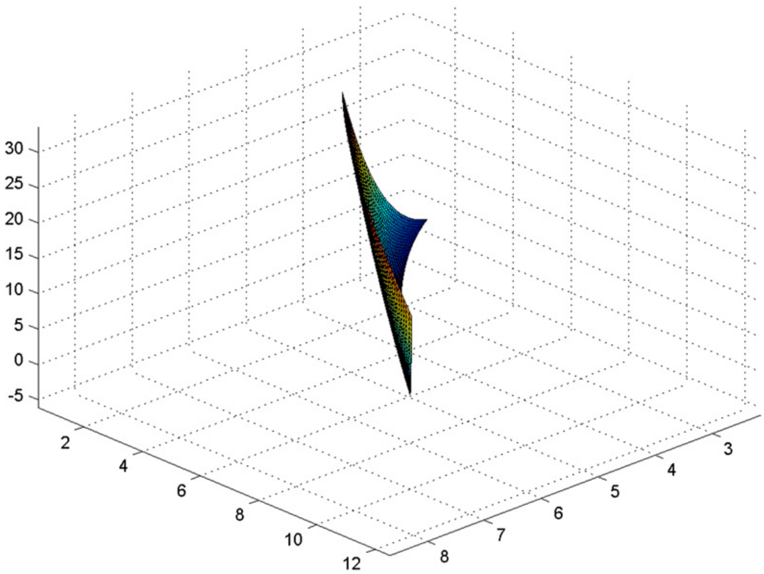

Fig. 1.

with a rotation σ_2 around null axis e_3 on S'_r. σ_1 and σ_2 are respectively written as

$$\sigma_1 = \sigma^{-1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{3.20}$$

where σ is given by (3.9) and

$$\sigma_= A(v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ -\frac{v^2}{2} & -v & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad v = -c_3. \tag{3.21}$$

Thus we prove the following

Theorem 3.3. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Locally, a pair of isometric helicoidal surface and rotation surface in Theorem* 3.1 *have same Gauss map, regardless of a rotation σ_1 around e_2 together with a rotation σ_2 around e_3 on the rotation surface, if and only if they are minimal.*

Example 3.1. A pair of isometric *CSM*-helicoidal surface of type *I* (see Fig. 1) and rotation surface (see Fig. 2) with same mean curvature ($H = 0$) and Gauss map

$$r(u, v) = \left(u + 2v, uv + v^2, u^3 - \frac{uv^2}{2} - \frac{v^3}{3}\right)$$

and

$$r(\bar{u}, \bar{v}) = \left(\bar{u}, \bar{u}\bar{v}, -5\bar{u}^3 - \frac{\bar{u}\bar{v}^2}{2}\right), \quad \bar{u} = u,\ \bar{v} = v + 6u.$$

Here the Gauss map is given by

$$n = \frac{1}{\sqrt{30}u^2}\left(u, 6u^2 + uv, -3u^3 - 6u^2 v - \frac{u}{2}v^2\right).$$

Case 2. S is of type *II* with the pitch $h < 0$.

From (2.7), the first fundamental forms of S can be expressed as

$$I = \left(1 + \frac{hf'^2}{2u}\right) du^2 - 2hu\left(dv - \frac{f'}{2u}\, du\right)^2. \tag{3.22}$$

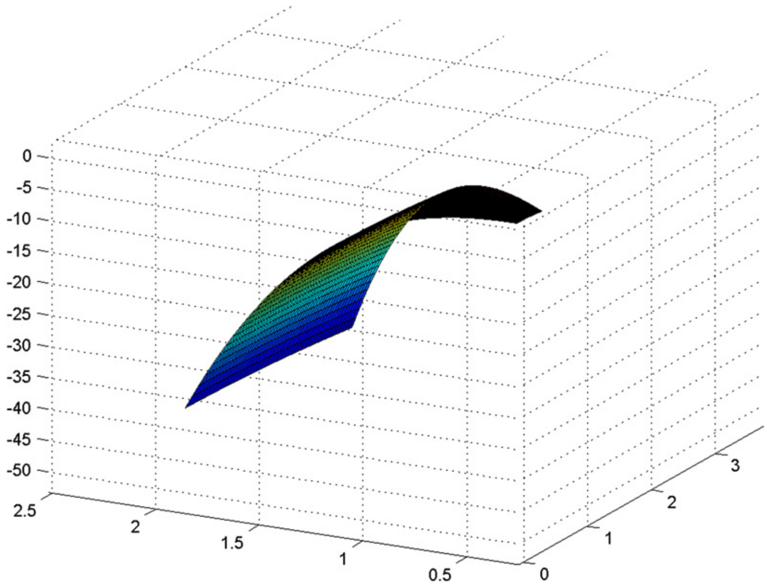

Fig. 2.

Comparing (3.22) with (3.3), if

$$2g_{\bar{u}}\,d\bar{u}^2 = \left(1 + \frac{hf'^2}{2u}\right)du^2, \qquad \bar{u}^2 = -2hu/c^2 \quad \text{and} \quad d\bar{v} = c\left[dv - \frac{f'}{2u}\,du\right], \tag{3.23}$$

where c is a positive constant, then we have an isometry between S and S_r. Here S_r is given by (3.1).

(3.23) implies that

$$\bar{u} = \sqrt{-2hu}/c, \qquad \bar{v} = c\left[v - \int \frac{f'}{2u}\,du\right] \quad \text{and} \quad g(\bar{u}) = c\left[-\frac{\sqrt{-2h}}{3h}u^{3/2} - \frac{\sqrt{-2h}}{4}\int u^{-1/2} f'^2\,du\right]. \tag{3.24}$$

So we have proved the following

Theorem 3.4. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Locally, a CSM-helicoidal surface of type II S with pitch $h < 0$*

$$r(u, v) = \left(hv, u + \frac{hv^2}{2}, f(u) - uv - \frac{hv^3}{6}\right) \tag{3.25}$$

is isometric to the one-parametric rotation surface S_r

$$r(u, v) = \left(\sqrt{-2hu}/c, \sqrt{-2hu}\left(v - \int \frac{f'}{2u}\,du\right), c\left[g_1(u) - \frac{\sqrt{-2hu}}{2}\left(v - \int \frac{f'}{2u}\,du\right)^2\right]\right) \tag{3.26}$$

so that helices on the helicoidal surface correspond to parallel circles on the rotation surface, where

$$g_1(u) = -\frac{\sqrt{-2h}}{3h}u^{3/2} - \frac{\sqrt{-2h}}{4}\int u^{-1/2} f'^2\,du,$$

and c is a positive constant.

Especially, when $c = 1$, S is isometric to rotation surface S_{r_0}

$$r(u, v) = \left(\sqrt{-2hu}, \sqrt{-2hu}\left(v - \int \frac{f'}{2u}\,du\right), g_1(u) - \frac{\sqrt{-2hu}}{2}\left(v - \int \frac{f'}{2u}\,du\right)^2\right). \tag{3.27}$$

Remark 3.3. In Theorem 3.4, for any positive constant c, the rotation surface S_r can be obtained by applying a rotation σ around spacelike axis e_2 on S_{r_0}, where σ is given by (3.9).

Corollary 3.2. *Locally, a minimal CSM-helicoidal surface of type II with pitch $h < 0$*

$$r(u, v) = \left(hv, u + \frac{hv^2}{2}, c_2 u^{3/2} - uv - \frac{hv^3}{6}\right), \tag{3.28}$$

is isometric to the minimal rotation surface

$$r(u, v) = \left(\sqrt{-2hu}/c, \sqrt{-2hu}\left(v - \frac{3}{2}c_2 u^{1/2} + c_3\right), c\left[g_0(u) - \frac{\sqrt{-2hu}}{2}\left(v - \frac{3}{2}c_2 u^{1/2} + c_3\right)^2\right]\right), \tag{3.29}$$

where

$$g_0(u) = \left(\frac{1}{6h^2} + \frac{3c_2{}^2}{16h}\right)(\sqrt{-2hu}\,)^3,$$

c_2 and c_3 are integration constants and $c_2{}^2 h \neq -8/9$.

Now we discuss the mean curvatures and Gauss maps of a pair of isometric helicoidal surface S and rotation surface S_r in Theorem 3.4. By differentiating the third equation of (3.24) with respect to $\bar{u}$ twice, we get

$$g_{\bar{u}} = c^2\left(-\frac{u}{h} - \frac{f'^2}{2}\right) \tag{3.30}$$

and

$$g_{\bar{u}\bar{u}} = c^3\sqrt{-2hu}\left(\frac{1}{h} + f'f''\right)/h. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.12), we have

$$H_r = \frac{h^3 f'}{\sqrt{-2hu}} H, \tag{3.32}$$

where H and H_r are mean curvatures of S and S_r in Theorem 3.4 respectively. And H is given by (2.6). This follows that $H_r = H$ is equivalent with $(h^3 f' - \sqrt{-2hu})H = 0$. From Proposition 2.2, we can see that $h^3 f' - \sqrt{-2hu} = 0$ implies $H = 0$. Thus we have $H_r = H$ is equivalent with $H_r = H = 0$. So we prove the following

Theorem 3.5. *Locally, a pair of isometric CSM-helicoidal surface of type II S with pitch $h < 0$ and rotation surface S_r in Theorem* 3.4 *have same mean curvature if and only if they are minimal.*

Remark 3.4. From (3.32), we can see that the mean curvature of S_r in Theorem 3.4 is independent of parameter c. Moreover, by a direct computation, we can see that the coefficients of the first and second fundamental form of S_r are independent of parameter c.

Let n (resp. n_r) be the normal vector field on the surface S and S_r given by (3.25) (resp. (3.26)). An easy computation leads to

$$n = |D|^{-1/2}\left(-h, h\big(f' - v\big), -u - hf'v + \frac{h}{2}v^2\right) \tag{3.33}$$

and

$$n_r = |D|^{-1/2}\left(-h/c, h\left(\int \frac{f'}{2u}\,du - v\right), -c\left[-u - \frac{h}{2}f'^2 + \frac{h}{2}\left(v - \int \frac{f'}{2u}\,du\right)^2\right]\right), \tag{3.34}$$

where $D = EG - F^2 = -2hu - h^2 f'^2$.

Comparing (3.33) with (3.34), we can see that $n = n_r$ if and only if

$$\begin{cases} c = 1, \\ f' = \displaystyle\int \frac{f'}{2u}\,du. \end{cases} \tag{3.35}$$

The general solution of (3.35) is

$$c = 1, \qquad f = c_2 u^{3/2} + c_4, \quad \text{and} \quad \int \frac{f'}{2u}\,du = \frac{3}{2}c_2 u^{1/2}, \tag{3.36}$$

where c_2 and c_4 are integration constants.

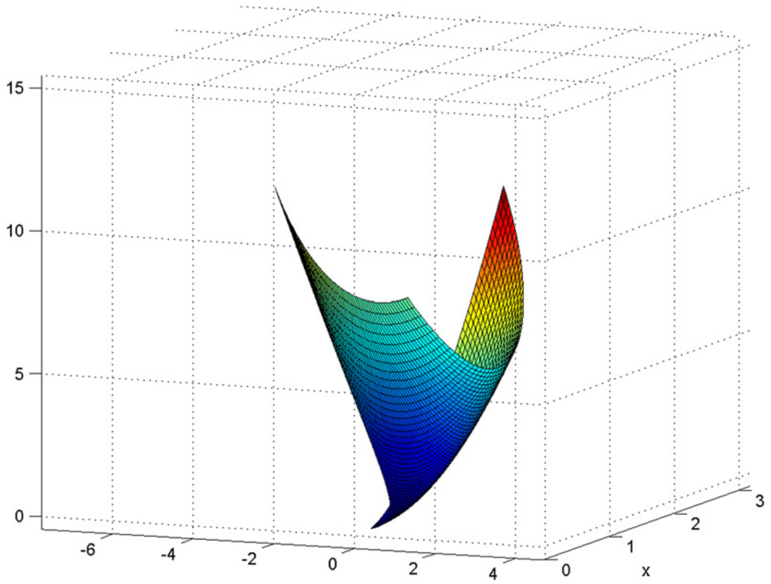

Fig. 3.

This implies that S and S_r have same Gauss map if and only if S and S_r are minimal surfaces given by (3.28) and (3.29) with $c = 1$ and $c_3 = 0$. In general, a pair of minimal isometric helicoidal and rotation surfaces in Corollary 3.21 have different Gauss maps.

Denote by S'_{r_0} the minimal rotation surface given by (3.29) with $c = 1$ and $c_3 = 0$ and S'_r the general case given by (3.29), with similar discussion in Case 1, S'_{r_0} can be obtained by applying a rotation σ_1 around spacelike axis e_2 together with a rotation σ_2 around null axis e_3 on S'_r, where σ_1 and σ_2 are given by (3.20) and (3.21) respectively.

Thus we prove the following

Theorem 3.6. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Locally, a pair of isometric helicoidal surface and rotation surface in Theorem* 3.4 *have same Gauss map, regardless of a rotation σ_1 around e_2 together with a rotation σ_2 around e_3 on the rotation surface, if and only if they are minimal.*

Example 3.2. A pair of isometric *CSM*-helicoidal surface of type *II* ($h < 0$) and rotation surface (see Figs. 3, 4) with same mean curvature ($H = 0$) and Gauss map

$$r(u, v) = \left(-\frac{v}{2}, u - \frac{v^2}{4}, 2u^{3/2} - uv + \frac{v^3}{12}\right)$$

and

$$r(\bar{u}, \bar{v}) = \left(\bar{u}, \bar{u}\bar{v}, -\frac{5}{6}\bar{u}^3 - \frac{\bar{u}\bar{v}^2}{2}\right), \quad \bar{u} = \sqrt{u},\ \bar{v} = v - 3\sqrt{u}.$$

Here the Gauss map is given by

$$n = \left|\frac{5u}{4}\right|^{-1/2}\left(\frac{1}{2}, -\frac{3}{2}u^{1/2} + \frac{v}{2}, -u + \frac{3}{2}u^{1/2}v - \frac{v^2}{4}\right).$$

Case 3. S is of type *II* with the pitch $h > 0$.

Comparing (3.22) with (3.3), we can see that $-2hu < 0$ and $\bar{u}^2 > 0$. Therefore, we cannot find an isometry condition as (3.23) between S and S_r, where S_r is given by (3.1). We have to consider rotation surface with non-null axis.

Let $[O; \eta_1, \eta_2, \eta_3]$ be the considered orthogonal frame, where $\{\eta_1, \eta_2, \eta_3\}$ is an orthonormal basis given by (2.1). Then S can be written as

$$r(u, v) = \left(hv, u + \frac{hv^2}{2}, f(u) - uv - \frac{hv^3}{6}\right)X^\top$$

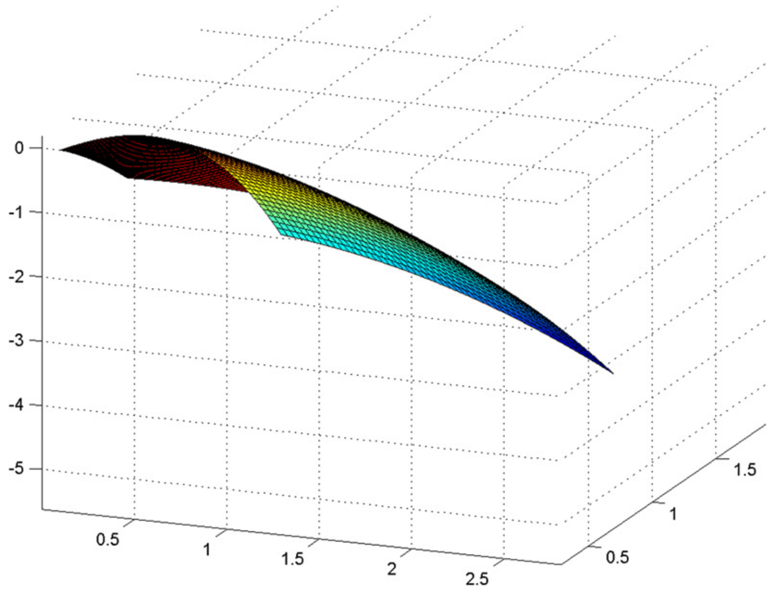

Fig. 4.

i.e.,

$$r(u,v)=\frac{\sqrt{2}}{2}\left(f(u)-uv-hv-\frac{hv^3}{6},\,f(u)-uv+hv-\frac{hv^3}{6},\,\sqrt{2}\left(u+\frac{hv^2}{2}\right)\right). \tag{3.37}$$

Similarly, the Gauss map of S is written as

$$n=\frac{\sqrt{2}}{2}|D|^{-1/2}\left(-u-hf'v+\frac{h}{2}v^2+h,\,-u-hf'v+\frac{h}{2}v^2-h,\,\sqrt{2}h\big(f'-v\big)\right). \tag{3.38}$$

Let $\gamma(\bar{u})=(0,\bar{u},g(\bar{u}))$, $\bar{u}>0$ be a curve in the $O\eta_2\eta_3$ plane. Suppose that $\bar{S}_r$ is the rotation surface generated by $\gamma(\bar{u})$ under a rotation around the axis η_3, the position vector r of which has the form

$$r(\bar{u},\bar{v})=\big(\bar{u}\sinh\bar{v},\bar{u}\cosh\bar{v},g(\bar{u})\big),\quad \bar{u}>0,\ \bar{v}\in R. \tag{3.39}$$

An easy computation leads to the first fundamental form, the mean curvature and Gauss map respectively being

$$I=\big(1+{g_{\bar{u}}}^2\big)\,d\bar{u}^2-\bar{u}^2\,d\bar{v}^2, \tag{3.40}$$

$$\bar{H}_r=\frac{\bar{u}^3g_{\bar{u}\bar{u}}+\bar{u}^2g_{\bar{u}}(1+{g_{\bar{u}}}^2)}{2\bar{D}|\bar{D}|^{1/2}} \tag{3.41}$$

and

$$\bar{n}_r=|\bar{D}|^{-1/2}(\bar{u}g_{\bar{u}}\sinh\bar{v},\bar{u}g_{\bar{u}}\cosh\bar{v},\bar{u}), \tag{3.42}$$

where $\bar{D}=-\bar{u}^2(1+{g_{\bar{u}}}^2)$.

Comparing (3.40) with (3.22), if

$$\big(1+{g_{\bar{u}}}^2\big)\,d\bar{u}^2=\left(1+\frac{hf'^2}{2u}\right)du^2,\qquad \bar{u}^2=2hu/c^2\quad\text{and}\quad d\bar{v}=c\left[dv-\frac{f'}{2u}\,du\right], \tag{3.43}$$

where c is a positive constant, then we have an isometry between S and $\bar{S}_r$.

(3.43) implies that

$$\bar{u}=\sqrt{2hu}/c,\qquad \bar{v}=c\left(v-\int\frac{f'}{2u}\,du\right)\quad\text{and}\quad g(\bar{u})=\int\sqrt{1+\frac{h(c^2f'^2-1)}{2c^2u}}\,du. \tag{3.44}$$

Substituting (3.44) into (3.39), we prove the following

Theorem 3.7. *Let $[O; \eta_1, \eta_2, \eta_3]$ be the considered orthogonal frame. Locally, a CSM-helicoidal surface of type II S with pitch $h > 0$*

$$r(u,v) = \frac{\sqrt{2}}{2}\left(f(u) - uv - hv - \frac{hv^3}{6}, f(u) - uv + hv - \frac{hv^3}{6}, \sqrt{2}\left(u + \frac{hv^2}{2}\right)\right), \tag{3.45}$$

is isometric to the one-parametric rotation surface $\bar{S}_r$

$$r(u,v) = \left(\frac{\sqrt{2hu}}{c}\sinh\left[c\left(v - \int \frac{f'}{2u}\,du\right)\right], \frac{\sqrt{2hu}}{c}\cosh\left[c\left(v - \int \frac{f'}{2u}\,du\right)\right], \int\sqrt{1 + \frac{h(c^2f'^2 - 1)}{2c^2u}}\,du\right) \tag{3.46}$$

where c is a positive constant, so that helices on the helicoidal surface correspond to parallel circles on the rotation surface.

Now we discuss the Gauss maps and mean curvatures of a pair of isometric helicoidal surface S and rotation surface $\bar{S}_r$ in Theorem 3.7. (3.44) implies

$$\frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} = \frac{\sqrt{2h}}{2\sqrt{u}} > 0.$$

Therefore, the Gauss map of $\bar{S}_r$ can be expressed as

$$\bar{n}_r = |\bar{D}|^{-1/2}(\bar{u}g_{\bar{u}}\sinh\bar{v}, \bar{u}g_{\bar{u}}\cosh\bar{v}, \bar{u}), \tag{3.47}$$

where $\bar{D} = -\bar{u}^2(1 + g_{\bar{u}}{}^2)$, $\bar{u}$, $\bar{v}$ and $g(\bar{u})$ are given by (3.44).

Comparing (3.47) with (3.38), we get the third coordinates of n and n_r are different. In fact, $|D|^{-1/2}h(f' - v)$ depends on parameters u and v but $|\bar{D}|^{-1/2}\bar{u}$ is independent of parameter v.

Thus we prove

Theorem 3.8. *Each pair of isometric CSM-helicoidal surface of type II with $h > 0$ and rotation surface with spacelike axis in Theorem 3.7 have different Gauss maps.*

By differentiating the third equation of (3.4) with respect to $\bar{u}$, we get

$$g_{\bar{u}} = \sqrt{c^2\left(f'^2 + 2u/h\right) - 1} \tag{3.48}$$

i.e.,

$$g_{\bar{u}}{}^2 = c^2\left(f'^2 + 2u/h\right) - 1. \tag{3.49}$$

Taking derivative with respect to $\bar{u}$ on (3.49), we get

$$g_{\bar{u}}g_{\bar{u}\bar{u}} = c^3\sqrt{\frac{2u}{h}}\left(1/h + f'f''\right). \tag{3.50}$$

Substituting (3.48)–(3.50) into (3.41), we have

Proposition 3.1. *Let H and $\bar{H}_r$ be the mean curvatures of a pair of isometric CSM-helicoidal surface and rotation surface with spacelike axis in Theorem 3.7 respectively. Then we have*

$$\bar{H}_r = \lambda_1(u)H + \lambda_2(u), \tag{3.51}$$

where

$$\lambda_1(u) = \frac{h^3f'}{[2hu(c^2f'^2 - 1) + 4u^2]^{1/2}} \tag{3.52}$$

and

$$\lambda_2(u) = \frac{c^2h(2u + hf'^2)^2}{-2(2hu + h^2f'^2)|2hu + h^2f'^2|^{1/2}[2hu(c^2f'^2 - 1) + 4u^2]^{1/2}}. \tag{3.53}$$

Remark 3.5. Unlike Cases 1 and 2, $\bar{H}_r$ depends on parameter c.

Remark 3.6. From Theorem 3.8 and Proposition 3.1, it's naturally to ask a question: *Is there a pair of isometric CSM-helicoidal surface of type II with $h > 0$ and rotation surface with same mean curvature*? That's an open problem.

Acknowledgments

The authors are grateful to the referee(s) for their valuable comments and suggestions.

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