JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 26, 82–87 (1969)

Some Complements of Hölder's Inequality

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1. INTRODUCTION

Let f and g be nonnegative functions on the interval $[0, a]$. If $0 < \|f\|_p$, $\|g\|_q < \infty$ then the Hölder Inequality may be written

$$H_{p,q}(f,g) = \frac{(f,g)}{\|f\|_p \|g\|_q} = \frac{\int_0^a fg\,dX}{\left[\int_0^a f^p\,dX\right]^{1/p}\left[\int_0^a g^q\,dX\right]^{1/q}} \leqslant 1, \tag{1}$$

where $1/p + 1/q = 1$, $p, q \geqslant 1$. The quantity $H_{p,q}(f,g)$ defined by (1) will be called the Hölder quotient. A complement of the Hölder inequality gives a lower bound on the Hölder quotient for $p, q \geqslant 1$ and an upper bound for $p, q \leqslant 1$. It is known for example that if $\ell \leqslant f^{-p}g^q \leqslant \theta\ell$ for some constants ℓ, θ then (see [1], [2]),

$$H_{p,q}(f,g) \geqslant \frac{q^{1/q}p^{1/p}\theta^{1/pq}[\theta^{1/q} - 1]^{1/q}\,[\theta^{1/p} - 1]^{1/p}}{\theta - 1},$$

where $1/p + 1/q = 1$, $p, q \geqslant 1$. If however it is known that f, g are concave functions then (see [3] p. 41)

$$H_{p,q}(f,g) \geqslant \tfrac{1}{6}\,[1 + p]^{1/p}\,[1 + q]^{1/q},$$

where $1/p + 1/q = 1$, $p, q \geqslant 1$.

Nehari [4] has given some very general results along these lines using the idea of the convex hull of a class of functions. This paper gives a different method of obtaining complementing inequalities using a certain partial order "$\prec$" defined on a class of functions. In some cases these results are less general than Nehari's and in other cases they are more general. We shall deal only with the case of two functions f, g but we remove the restriction that $1/p + 1/q = 1$.

2. The General Method

We shall say $f_0 \prec f$ or equivalently $f \succ f_0$ (see [3] p. 33) provided

$$\int_0^X f(t)\,dt \geqslant \int_0^X f_0(t)\,dt \qquad \text{for all} \qquad X \in [0, a] \tag{2}$$

and

$$\int_0^a f(t)\,dt = \int_0^a f_0(t)\,dt. \tag{3}$$

Hardy, Littlewood and Pólya ([5] Theorem 250) have shown that if $\theta(U)$ is a convex function of U and if f_0, f are increasing functions of X with $f_0 \prec f$ then

$$\int_0^a \theta(f)\,dX \leqslant \int_0^a \theta(f_0)\,dX. \tag{4}$$

Clearly this same inequality will hold if f_0, f are decreasing functions with $f_0 \succ f$. The inequality will be reversed if $\theta(U)$ is a concave function of U. We denote by f^- the rearrangement of f into decreasing order and define $f^+(X) = f^-(a - X)$ the rearrangment of f into increasing order (see [5]). It follows from (4) that if $f_0 \prec f^+$ then

$$\|f\|_p = \|f^+\|_p \leqslant \|f_0\|_p \qquad \text{for} \qquad p \geqslant 1 \tag{5}$$

and

$$\|f\|_p = \|f^+\|_p \geqslant \|f_0\|_p \qquad \text{for} \qquad p \leqslant 1. \tag{6}$$

These same two inequalities will hold if $f_0 \succ f^-$. We shall also need the following result (see [5])

$$(f^+, g^+) \geqslant (f, g) \geqslant (f^+, g^-). \tag{7}$$

Our main result is the following

Theorem 1. *Let f, g be nonnegative functions on $[0, a]$ with $0 < \|f\|_p$, $\|g\|_q < \infty$. Then*

(a) *If f_0 is increasing with $f_0 \prec f^+$ and g_0 is decreasing with $g_0 \succ g^-$ then $H_{p,q}(f, g) \geqslant H_{p,q}(f_0, g_0)$ if $p, q \geqslant 1$.*

(b) *If f_0 and g_0 are both increasing functions with $f_0 \prec f^+$ and $g_0 \prec g^+$ then $H_{p,q}(f, g) \leqslant H_{p,q}(f_0, g_0)$ if $p, q \leqslant 1$.*

Note that we do not assume $1/p + 1/q = 1$ in this theorem. We first prove part (a). In view of (5) it will be sufficient to show that $(f, g) \geqslant (f_0, g_0)$. To this end we consider the elementary integration by parts formula

$$\int_0^a (f^+g^- - f_0g^-) = -\int_0^a \int_0^X [f^+(t) - f_0(t)]\,dt\,dg^-(X), \tag{8}$$

where we have used (3). Now (2) implies that the above integrand is nonnegative. Since $g^-(X)$ is decreasing we see that the right-hand side of (8) is nonnegative. Thus we obtain $(f^+, g^-) \geqslant (f_0, g^-)$. A similar formula implies $(f_0, g^-) \geqslant (f_0, g_0)$. Putting these inequalities together with (7) gives $(f, g) \geqslant (f_0, g_0)$. This proves part (a). The proof of part (b) is similar. We know from (7) that $(f, g) \leqslant (f^+, g^+)$. It is easy enough to construct the required integration by parts formulas to show in addition that

$$(f, g) \leqslant (f^+, g^+) \leqslant (f^+, g_0) \leqslant (f_0, g_0).$$

This inequality together with (6) will complete the proof of (b).

3. Applications of Theorem 1

Theorem 2. *Let f, g be nonnegative concave functions on* $[0, a]$ *with* $0 < \|f\|_p, \|g\|_q < \infty$. *Then*

(a) $H_{p,q}(f, g) \geqslant \frac{1}{6}[1 + p]^{1/p}[1 + q]^{1/q} a^{1-(1/p)-(1/q)}$ *for* $p, q \geqslant 1$. *Equality holds in case one of the functions f, g is αX and the other one is $\beta(a - X)$ with α, β positive constants.*

(b) $H_{p,q}(f, g) \leqslant \frac{1}{3}[1 + p]^{1/p}[1 + q]^{1/q} a^{1-(1/p)-(1/q)}$ *for* $-1 < p, q < 1$. *Equality holds in case* $f(X) = \alpha X$, $g(X) = \beta X$ *or else* $f(X) = \alpha(a - X)$, $g(X) = \beta(a - X)$.

Proof. We first suppose $p, q \geqslant 1$. Given functions f, g on $[0, a]$ define

$$f_0(X) = \frac{2M_1}{a^2} X, \qquad g_0(X) = \frac{2M_2}{a^2}(a - X), \tag{9}$$

where

$$M_1 = \int_0^a f(X)\, dX, \qquad M_2 = \int_0^a g(X)\, dX. \tag{10}$$

It is not difficult to show that if f, g are concave then the rearrangements f^+, g^- are also concave. It is then easy enough to see that $f_0 \prec f^+$ and $g_0 \succ g^-$. Theorem 1 implies

$$H_{p,q}(f, g) \geqslant H_{p,q}(f_0, g_0).$$

A simple calculation using (9) in the above inequality proves Theorem 2(a). The proof of Theorem 2(b) is much the same. In this case we take

$$f_0(X) = \frac{2M_1}{a^2} X, \qquad g_0(X) = \frac{2M_2}{a^2} X,$$

with M_1, M_2 as in (10). Part (b) of Theorem 1 then implies

$$H_{p,q}(f, g) \leqslant H_{p,q}(f_0, g_0).$$

Another very simple calculation completes the proof. If we let $p, q \to +\infty$ in Theorem 2(a) we obtain

$$\frac{1}{a}\int_0^a fg\, dX \geqslant \tfrac{1}{6} \sup_X f(X) \sup_X g(X).$$

Various other results of this type may be obtained from part (b) by letting p, q tend to zero, etc. Nehari [4] p. 418 has shown that this theorem has no generalization to higher dimensions (see [3] p. 42).

It is easy enough to generalize the methods used in Theorem 2 to cover a number of other cases. Given a nonnegative function $f(X)$ on $[0, a]$ we define the normalized function $F(X)$ by

$$F(X) = \frac{f(X)}{\int_0^a f(t)\, dt}.$$

The function $f(X)$ will be said to be of class

$C_1 = C_1(h, H, a)$ provided $0 < h \leqslant F(X) \leqslant H$,

$C_2 = C_2(h, H, a)$ provided $f(X)$ is convex and $0 \leqslant h \leqslant F(X) \leqslant H$,

$C_3 = C_3(L, a)$ provided $F(X)$ satisfies a lipschitz condition with constant L, $|F(X) - F(Y)| \leqslant L\,|X - Y|$,

$C_4 = C_4(a)$ provided $f(X)$ is concave.

Assuming that $f(X)$ belongs to Ci and $g(X)$ belongs to Cj for $i, j = 1, 2, 3, 4$, then Theorem 1 allows one to calculate the extreme values of the Hölder quotient

$$H_{p,q}(f, g) \geqslant K_{i,j} \qquad \text{for} \qquad p, q \geqslant 1 \tag{11}$$

$$H_{p,q}(f, g) \leqslant K_{i,j} \qquad \text{for} \qquad p, q \leqslant 1. \tag{12}$$

In many cases the above inequalities will be nontrivial. The minimum value in (11) will be attained whenever one of the functions f, g is a minimal element of Ci in the partial order $\prec$ and the other one is a maximal element of Cj. The maximum value in (12) will be attained whenever f and g are both maximal elements or else both are minimal elements of Ci, Cj in the partial order $\prec$.

4. Orlicz Spaces $L\phi$

The author would like to thank Professor J. A. Goldstein for pointing out this generalization. Let $\theta(U)$ be an increasing convex function of U with $\theta(0) = 0$. Define a norm $\|\cdot\|_\phi$ by

$$\|f\|_\phi = \inf\left\{K > 0 : \int_0^a \phi\left(\frac{|f|}{K}\right) dX \leqslant 1\right\}.$$

Then $L_\phi = \{f \mid \|f\|_\phi < \infty\}$ is a Banach space and $\|\cdot\|_\phi$ is a norm. If L_ϕ, L_ψ are complementary orlicz spaces and $f \in L_\phi$, $g \in L_\psi$ then

$$\int_0^a fg\, dX \leqslant \|f\|_\phi \|g\|_\psi .$$

For properties of Orlicz spaces see [6]. Equation (4) above implies

$$\|f\|_\phi = \|f^+\|_\phi \leqslant \|f_0\|_\phi .$$

Thus Theorem 1 generalizes to Orlicz spaces.

A direct imitation of the proof of Theorem 2 gives

THEOREM 3. *Let f, g be nonnegative concave functions on* $[0, a]$. *Then*

$$\frac{\int_\phi^a fg\, dX}{\|f\|_\phi \|g\|_\psi} \geqslant \frac{\frac{1}{6} a^3}{\|X\|_\phi \|a - X\|_\psi}.$$

Clearly there are many other generalizations of the above results. We shall not investigate them here.

5. Finite Sums

Given vectors $\mathbf{a} = (a_1, a_2, ..., a_n)$, $\mathbf{b} = (b_1, b_2, ..., b_n)$ we define $\mathbf{a} \prec \mathbf{b}$ (see [3] p. 45) if

$$a_1 + a_2 + \cdots + a_K \leqslant b_1 + b_2 + \cdots + b_K, \qquad K = 1, 2, ..., n-1$$

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n .$$

By analogy we define

$$H_{p,q}(\mathbf{a}, \mathbf{b}) = \frac{(\mathbf{a}, \mathbf{b})}{\|\mathbf{a}\|_p \|\mathbf{b}\|_q} = \frac{\sum_1^n a_i b_i}{\left[\sum_1^n a_i^p\right]^{1/p} \left[\sum_1^n b_i^q\right]^{1/q}}.$$

In view of [3] Section 28 the following theorem is obvious

THEOREM 4. *Let* $\mathbf{a}$, $\mathbf{b}$ *be vectors having nonnegative components with* $\|\mathbf{a}\|_p$, $\|\mathbf{b}\|_q > 0$. *Then*

(i) *If* $\mathbf{a}_0 = (a_{10}, a_{20}, \ldots, a_{n0})$, $\mathbf{b}_0 = (b_{10}, b_{20}, \ldots, b_{n0})$ *with* a_{i0} *increasing and* b_{i0} *decreasing and* $\mathbf{a}_0 \prec \mathbf{a}^+$, $\mathbf{b}_0 \succ \mathbf{b}^-$ *then* $H_{p,q}(\mathbf{a}, \mathbf{b}) \geqslant H_{p,q}(\mathbf{a}_0, \mathbf{b}_0)$ *for* $p, q \geqslant 1$.

(ii) *If* a_{i0} *and* b_{i0} *are increasing sequences with* $\mathbf{a}^+ \succ \mathbf{a}_0$, $\mathbf{b}^+ \succ \mathbf{b}_0$ *then* $H_{p,q}(\mathbf{a}, \mathbf{b}) \leqslant H_{p,q}(\mathbf{a}_0, \mathbf{b}_0)$ *for* $p, q \leqslant 1$.

One can easily use the above theorem to show for example that

$$\sum_{i=1}^{n} a_i b_i \geqslant \frac{n-2}{2n-1} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

if the sequences a_i, b_i are concave,

$$a_i \geqslant \frac{a_{i+1} + a_{i-1}}{2}, \qquad b_i \geqslant \frac{b_{i+1} + b_{i-1}}{2}.$$

One can now consider the finite sum analogue of (11), (12). The remarks following them remain equally valid in the finite case with the proper interpretation of $\prec$, Ci and Cj.

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