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Shadowing orbits of ordinary differential equations

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Abstract

A new notion of shadowing of a pseudo orbit, an approximate solution, of an autonomous system of ordinary differential equations by an associated nearby true orbit is introduced. Then a general shadowing theorem for finite time, which guarantees the existence of shadowing in ordinary differential equations and provides error bounds for the distance between the true and the pseudo orbit in terms of computable quantities, is proved. The use of this theorem in numerical computations of orbits is outlined.

Keywords: Numerical orbits; Ordinary differential equations; Shadowing; Chaos

1. Introduction

Numerical computations are playing an increasingly more important role in the investigations of chaotic differential and difference equations. Indeed, some of the more noteworthy recent discoveries have largely been based on suggestive numerical experiments. The reliability of numerical computations in such computer experiments is a serious concern. This concern is well justified in light of the fact that chaotic systems tend to amplify small computational errors due to the sensitive dependence of solutions of chaotic systems on initial data. Because of this sensitivity, it is, in fact, unreasonable to expect to be able to approximate well numerically a particular solution of a chaotic system for any considerable length of time.

In computer experiments with chaotic systems one is mainly interested in knowing whether a solution one sees on the computer has anything to do at all with the chaotic system. Despite the inherent numerical difficulties, for certain "well-behaved" chaotic systems, it is indeed the case that near a computed solution there exists a true solution of the chaotic system albeit with slightly different initial data. This is the content of the famous *Shadowing Lemma* proved by Anosov [1]

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and Bowen [2] for hyperbolic sets of diffeomorphisms (see also [12]), and extended by Franke and Selgrade [7] for hyperbolic sets of vector fields.

The classical Shadowing Lemma is applicable only to uniformly hyperbolic systems, that is, the linearized flow about each solution possesses exponentially contracting and expanding directions. From the practical viewpoint, this hyperbolicity assumption is a serious constraint as it is almost impossible to verify in specific systems. Moreover, it appears that many well-studied chaotic systems are not uniformly hyperbolic. However, some of these chaotic systems, although not uniformly hyperbolic, possess enough hyperbolicity to ensure that computed solutions can be shadowed by true solutions for finite lengths of time. Indeed, Hammel et al. [8,9] demonstrated finite-time shadowing in the logistic and Hénon maps for certain parameter values. Later, Chow and Palmer [3,4] proved a general finite-time Shadowing Lemma for maps (cf. [11]).

The objective of the present paper is to prove a general Finite-time Shadowing Theorem for systems of autonomous ordinary differential equations. Since it is our aim to use this shadowing result in numerical simulations of specific differential equations, we will put no special requirements on our vector fields that are difficult to verify in practice. Our method of proof of this Shadowing Theorem leads to an algorithm to shadow orbits which are numerically computed using one-step methods. We include here a synopsis of our algorithm; the details will appear in [6]. In the case of uniformly hyperbolic vector fields, our methods yield a new proof of the Classical Shadowing Lemma, which is reported in [5].

2. Statement of the Finite-time Shadowing Theorem

In this section we first develop a precise notion of shadowing of a pseudo orbit by an associated nearby true orbit. Then we present the statement of our main theorem which guarantees the existence of shadowing in ordinary differential equations and provides error bounds for the distance between the true and the pseudo orbit in terms of computable quantities associated with the linearized flow along the pseudo orbit.

We proceed with a definition of a pseudo orbit of the autonomous system

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 vector field. We denote the associated flow of (1) by φ^t. Throughout this paper, unless otherwise stated, we use the Euclidean norm for vectors and the relevant operator norm for matrices and linear operators.

Definition 2.1. For a given positive number δ, a sequence of points $\{y_k\}_{k=0}^N$, with $f(y_k) \neq \mathbf{0}$ for all k, is said to be a *δ-pseudo orbit* of (1) if there is an associated sequence $\{h_k\}_{k=0}^{N-1}$ of positive times such that

$$\|y_{k+1} - \varphi^{h_k}(y_k)\| \leqslant \delta, \quad \text{for } k = 0, \ldots, N-1.$$

Next, we introduce the notion of shadowing a pseudo orbit by a true orbit.

Definition 2.2. For a given positive number ε, an orbit of (1) is said to *ε-shadow* a δ-pseudo orbit $\{y_k\}_{k=0}^N$ with associated times $\{h_k\}_{k=0}^{N-1}$ if there are points $\{x_k\}_{k=0}^N$ on the true orbit and times $\{t_k\}_{k=0}^{N-1}$

with $\varphi^{t_k}(\boldsymbol{x}_k) = \boldsymbol{x}_{k+1}$ such that

$$\|\boldsymbol{x}_k - \boldsymbol{y}_k\| \leqslant \varepsilon \quad \text{and} \quad |t_k - h_k| \leqslant \varepsilon.$$

To state our theorem we need to develop a bit of notation and introduce certain relevant mathematical constructs.

Let $\{y_k\}_{k=0}^{N}$ be a δ-pseudo orbit of (1) with associated times $\{h_k\}_{k=0}^{N-1}$. Also suppose that we have a sequence $\{Y_k\}_{k=0}^{N-1}$ of $n \times n$ matrices such that

$$\|Y_k - D\varphi^{h_k}(\boldsymbol{y}_k)\| \leqslant \delta, \quad \text{for } k = 0, \ldots, N-1. \tag{2}$$

We will define a sequence $\{A_k\}_{k=0}^{N-1}$ of $(n-1) \times (n-1)$ matrices in the following way. For $k = 0, \ldots, N$, we let S_k be an $n \times (n-1)$ matrix chosen so that its columns form an orthonormal basis for the subspace orthogonal to $f(\boldsymbol{y}_k)$. Now, we let

$$A_k = S_{k+1}^{*} Y_k S_k, \quad \text{for } k = 0, \ldots, N-1, \tag{3}$$

where $*$ denotes transpose. Geometrically, A_k is Y_k restricted to the subspace orthogonal to $f(\boldsymbol{y}_k)$ and then projected to the subspace orthogonal to $f(\boldsymbol{y}_{k+1})$.

Next, we define a linear operator $L : (\mathbb{R}^{(n-1)})^{(N+1)} \to (\mathbb{R}^{(n-1)})^{N}$ in the following way. If $\boldsymbol{\xi} = \{\boldsymbol{\xi}_k\}_{k=0}^{N}$ is in $(\mathbb{R}^{(n-1)})^{(N+1)}$, then we take $L\boldsymbol{\xi} = \{(L\boldsymbol{\xi})_k\}_{k=0}^{N-1}$ to be

$$(L\boldsymbol{\xi})_k = \boldsymbol{\xi}_{k+1} - A_k\boldsymbol{\xi}_k, \quad \text{for } k = 0, \ldots, N-1. \tag{4}$$

The operator L has right inverses and we choose one such right inverse L^{-1}.

As the last piece of our notational collection, we define seven constants. Let ε_0 be a positive number and let U be a convex open set containing $\{\boldsymbol{y}_k\}_{k=0}^{N}$ such that if $\boldsymbol{x}$ is in the ball about $\boldsymbol{y}_k$ with radius ε_0, then the solution $\varphi^t(\boldsymbol{x})$ is defined for $0 \leqslant t \leqslant 2h_k$ and is in U. For such a U, we define

$$M_0 = \sup_{\boldsymbol{x}\in U} \|f(\boldsymbol{x})\|, \qquad M_1 = \sup_{\boldsymbol{x}\in U} \|Df(\boldsymbol{x})\|, \qquad M_2 = \sup_{\boldsymbol{x}\in U} \|D^2 f(\boldsymbol{x})\|.$$

Finally, we define

$$\Delta = \inf_{0\leqslant k\leqslant N} \|f(\boldsymbol{y}_k)\|, \qquad \Theta = \sup_{0\leqslant k\leqslant N-1} \|Y_k\|,$$

and

$$h = \sup_{0\leqslant k\leqslant N-1} h_k.$$

Now, we can state our main theorem.

Finite-time Shadowing Theorem. *Let $\{\boldsymbol{y}_k\}_{k=0}^{N}$ be a δ-pseudo orbit of the autonomous system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, and let*

$$C = \max\left\{\Delta^{-1}(\Theta\|L^{-1}\| + 1), \|L^{-1}\|\right\}.$$

If δ satisfies the inequalities

(i) $C(M_1 + 1)\delta \leqslant \frac{1}{2}$,

(ii) $4C\delta < \min_{0\leqslant k\leqslant N-1} h_k$, $4C\delta < \varepsilon_0$,
(iii) $8(M_0M_1 + 2M_1 e^{2M_1 h} + 2M_2 h e^{4M_1 h})C^2\delta \leqslant 1$,
then the pseudo orbit $\{y_k\}_{k=0}^N$ *is ε-shadowed by a true orbit* $\{x_k\}_{k=0}^N$ *with*

$$\varepsilon \leqslant 4C\delta.$$

Before delving into the technical details of the proof of our theorem, we now summarize briefly its use in numerical computations.

3. The Finite-time Shadowing Theorem in numerics

As we have mentioned in the Introduction, the primary impetus for the development of our theorem is to establish the existence of a true solution near a carefully computed approximate solution of a system of ordinary differential equations. Full details of a numerical algorithm for this purpose and the particulars of its implementation on the computer will be forthcoming in a subsequent publication [6]. In this section, we offer a synopsis of our numerical algorithm.

We use a standard one-step method for numerically approximating the solution of the initial-value problem

$$\dot{x} = f(x), \quad x(0) = y_0,$$

and generate a sequence of points $\{y_k\}_{k=0}^N$ and a sequence of times $\{h_k\}_{k=0}^{N-1}$. We also compute each Y_k, which is the approximation to $D\varphi^{h_k}(y_k)$ obtained by applying the one-step method for a time step of h_k to the enlarged initial-value problem

$$\dot{x} = f(x), \quad \dot{X} = Df(x)X, \qquad x(0) = y_k, \quad X(0) = I.$$

With an appropriate choice of a numerical integration method and times $\{h_k\}_{k=0}^{N-1}$, we control the local errors to produce a suitable δ-pseudo orbit $\{y_k\}_{k=0}^N$ and matrices $\{Y_k\}_{k=0}^{N-1}$ satisfying (2).

Depending on the particulars of the differential equation under consideration, we select suitable U and ε_0. In the case of dissipative systems, for example, it is convenient to make the set U a set which is forward invariant under the flow. Then M_0, M_1 and M_2 can readily be determined once and for all.

The most troublesome of the other quantities to be calculated is $\|L^{-1}\|$. We prefer to postpone the details of the lengthy computation of $\|L^{-1}\|$ to the forthcoming longer exposition [6] where we will present an efficient algorithmic way involving the sequence of matrices $\{S_k\}_{k=0}^N$.

We calculate the quantities h, δ, Δ, Θ and $\|L^{-1}\|$ as we go along, updating them when we compute the next point on the pseudo orbit; simultaneously we check conditions (i)–(iii) of our theorem. If they are satisfied, we conclude that the pseudo orbit is shadowed by a true orbit for this duration of time. The computations are terminated when the conditions of our theorem can no longer be satisfied.

4. Proof of the Finite-time Shadowing Theorem

We begin with a δ-pseudo orbit $\{y_k\}_{k=0}^N$ of (1) and an associated sequence $\{Y_k\}_{k=0}^{N-1}$ of $n \times n$ matrices satisfying (2). We wish to show that $\{y_k\}_{k=0}^N$ shadows a true orbit containing $\{x_k\}_{k=0}^N$, with

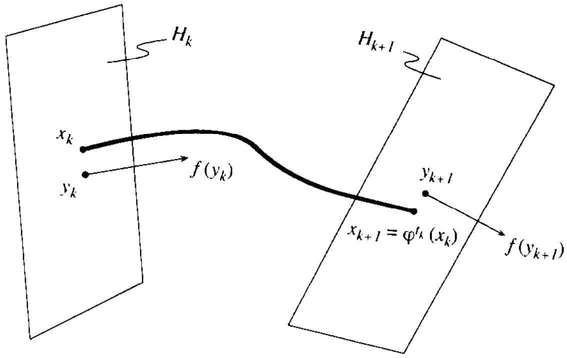

Fig. 1. Pseudo orbit y_k and the shadowing true orbit x_k.

x_k being contained in the hyperplane $\mathcal{H}_k$ through y_k and normal to $f(y_k)$ (see Fig. 1). In fact, we will find a sequence of times $\{t_k\}_{k=0}^{N-1}$ and a sequence of points $\{x_k\}_{k=0}^{N}$ with $x_k \in \mathcal{H}_k$ and near y_k such that $x_{k+1} = \phi^{t_k}(x_k)$ for $k = 0, \ldots, N-1$.

We first identify $\mathcal{H}_k$ with $\mathbb{R}^{n-1}$ via the map $z \mapsto y_k + S_k z$. The problem of finding appropriate sequences of t_k's and x_k's becomes that of finding a sequence of times $\{t_k\}_{k=0}^{N-1}$ and a sequence of points $\{z_k\}_{k=0}^{N}$ in $\mathbb{R}^{n-1}$ such that

$$y_{k+1} + S_{k+1} z_{k+1} = \varphi^{t_k}(y_k + S_k z_k), \quad k = 0, \ldots, N-1.$$

We next introduce the space $X = (\mathbb{R})^N \times (\mathbb{R}^{n-1})^{N+1}$ with norm

$$\|(\{s_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N})\| = \max\left\{\max_{0 \leqslant k \leqslant N-1} |s_k|, \max_{0 \leqslant k \leqslant N} \|w_k\|\right\}, \tag{5}$$

and the space $Y = (\mathbb{R}^n)^N$ with norm

$$\|\{g_k\}_{k=0}^{N-1}\| = \max_{0 \leqslant k \leqslant N-1} \|g_k\|. \tag{6}$$

Now, we let B be the open set in X consisting of those $v = (\{s_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N})$ with $0 < s_k < 2h_k$ and $\|w_k\| < \varepsilon_0$, and introduce the function $\mathcal{G} : B \to Y$ given by

$$[\mathcal{G}(v)]_k = y_{k+1} + S_{k+1} w_{k+1} - \varphi^{s_k}(y_k + S_k w_k), \quad k = 0, \ldots, N-1. \tag{7}$$

Thus the theorem will be proved if we can find a solution $\bar{v} = \left(\{t_k\}_{k=0}^{N-1}, \{z_k\}_{k=0}^{N}\right)$ of the equation

$$\mathcal{G}(\bar{v}) = \mathbf{0}, \tag{8}$$

in the closed ball of radius ε about $v_0 = \left(\{h_k\}_{k=0}^{N-1}, \mathbf{0}\right)$. In order to solve this equation, we use the following lemma.

Lemma 4.1. *Let X and Y be finite-dimensional vector spaces, let B be an open subset of X, and let $\mathcal{G} : B \to Y$ be a C^2 function satisfying the following properties.*

(i) *The derivative $D\mathcal{G}(v_0)$ at $v_0 \in B$ has a right inverse $\mathcal{K}$.*

(ii) *The closed ball about v_0 with radius ε, where*

$$\varepsilon = 2\|\mathcal{K}\| \, \|\mathcal{G}(v_0)\|,$$

is contained in B.

(iii) *The inequality*

$$2M\|\mathcal{K}\|^2\|\mathcal{G}(v_0)\| \leqslant 1$$

holds, where

$$M = \sup\{\|D^2\mathcal{G}(v)\|: \|v - v_0\| \leqslant \varepsilon\}.$$

Then there is a solution $\bar{v}$ *of the equation*

$$\mathcal{G}(\bar{v}) = \mathbf{0},$$

satisfying $\|\bar{v} - v_0\| \leqslant \varepsilon$.

Proof. For $\|v - v_0\| \leqslant \varepsilon$, a fixed point of the mapping

$$T(v) = v_0 - \mathcal{K}(\mathcal{G}(v) - D\mathcal{G}(v_0)(v - v_0))$$

corresponds to a zero of the function $\mathcal{G}$. To establish the existence of a fixed point of T, we show the invariance of the ball of radius ε under the mapping T. To wit, for v in such a ball, we have

$$\begin{aligned}\|T(v) - v_0\| &\leqslant \|\mathcal{K}\|\{\|\mathcal{G}(v_0)\| + \|\mathcal{G}(v) - \mathcal{G}(v_0) - D\mathcal{G}(v_0)(v - v_0)\|\}\\ &\leqslant \|\mathcal{K}\|\{\|\mathcal{G}(v_0)\| + \tfrac{1}{2}M\|v - v_0\|^2\} \leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{2}M\|\mathcal{K}\|\varepsilon^2\\ &= \tfrac{1}{2}\varepsilon + M\|\mathcal{K}\|^2\|\mathcal{G}(v_0)\|\varepsilon \leqslant \varepsilon.\end{aligned}$$

Then, the conclusion of the lemma follows from the Brouwer Fixed-Point Theorem. □

We now undertake the somewhat arduous task of verifying that the map $\mathcal{G}$ in (7), which is clearly C^2, does indeed satisfy hypotheses (i)–(iii) of Lemma 4.1.

Verification of (i). We begin with the most difficult part of the verification, that is, constructing a right inverse for $D\mathcal{G}(v_0)$. Notice that for $\boldsymbol{u} = (\{\tau_k\}_{k=0}^{N-1}, \{\xi_k\}_{k=0}^{N}) \in X$, the derivative of $\mathcal{G}$ at v_0 is given by

$$[D\mathcal{G}(v_0)\boldsymbol{u}]_k = -\tau_k f(\varphi^{h_k}(y_k)) + S_{k+1}\xi_{k+1} - D\varphi^{h_k}(y_k)S_k\xi_k, \quad k = 0, \ldots, N-1. \tag{9}$$

It is convenient to approximate $D\mathcal{G}(v_0)$ by the linear operator $T : X \to Y$ defined in terms of the pseudo orbit and its associated data:

$$[T\boldsymbol{u}]_k = -\tau_k f(y_{k+1}) + S_{k+1}\xi_{k+1} - Y_k S_k \xi_k, \quad k = 0, \ldots, N-1. \tag{10}$$

We will find a right inverse for T and then use standard operator theory to obtain a right inverse for $D\mathcal{G}(v_0)$.

In order to find a right inverse for T, for given $\boldsymbol{g} = \{g_k\}_{k=0}^{N-1}$ in Y, we must solve the set of equations

$$[T\boldsymbol{u}]_k = g_k, \quad k = 0, \ldots, N-1, \tag{11}$$

for $\boldsymbol{u}$. Since, for each k, the matrix $[f(y_{k+1})/\|f(y_{k+1})\| \,|\, S_{k+1}]$ is orthogonal, this set of equations is equivalent to the following two sets of equations, one set obtained by premultiplying the kth member in (11) by $f(y_{k+1})^*$, and the other set obtained by premultiplying the kth member in (11) by S_{k+1}^*:

$$-\|f(y_{k+1})\|^2\tau_k - f(y_{k+1})^* Y_k S_k \xi_k = f(y_{k+1})^* g_k, \quad k = 0, \ldots, N-1,$$

$$\xi_{k+1} - A_k\xi_k = S_{k+1}^* g_k, \quad k = 0, \ldots, N-1,$$

where A_k is as given in (3). If we write $\bar{g} = \{S_{k+1}^* g_k\}_{k=0}^{N-1}$, then, referring to (4), we see that a solution of the second set is

$$\xi_k = (L^{-1}\bar{g})_k, \quad k = 0, \ldots, N-1. \tag{12}$$

We substitute this into the first set and obtain

$$\tau_k = -\frac{f(y_{k+1})^*}{\|f(y_{k+1})\|^2}\{Y_k S_k (L^{-1}\bar{g})_k + g_k\}, \quad k = 0, \ldots, N-1. \tag{13}$$

Then we define our right inverse of T by

$$T^{-1}g = \left(\{\tau_k\}_{k=0}^{N-1}, \{\xi_k\}_{k=0}^{N}\right), \tag{14}$$

with ξ_k and τ_k given in (12) and (13), respectively. From its form, it is evident that T^{-1} is linear. Also, we note here for future use that (12)–(14) imply the inequality

$$\|T^{-1}\| \leqslant C, \tag{15}$$

where C is given in the statement of the Theorem.

We next turn to the construction of the right inverse of $D\mathcal{G}(v_0)$. First, we observe that the operator $\mathcal{K}$, defined by

$$\mathcal{K} = \left(I + T^{-1}(D\mathcal{G}(v_0) - T)\right)^{-1} T^{-1}, \tag{16}$$

is a right inverse of $D\mathcal{G}(v_0)$, provided that the operator $I + T^{-1}(D\mathcal{G}(v_0) - T)$ is invertible. To establish that this operator is indeed invertible, we appeal to a standard result from operator theory [10, p.253] which tells us that if

$$\|T^{-1}(D\mathcal{G}(v_0) - T)\| \leqslant \tfrac{1}{2}, \tag{17}$$

then the inverse $\left(I + T^{-1}(D\mathcal{G}(v_0) - T)\right)^{-1}$ exists and

$$\left\|\left(I + T^{-1}(D\mathcal{G}(v_0) - T)\right)^{-1}\right\| \leqslant 2. \tag{18}$$

To verify the condition in (17), we estimate that

$$\begin{aligned}
\|T^{-1}(D\mathcal{G}(v_0) - T)\| &\leqslant \|T^{-1}\|\, \|D\mathcal{G}(v_0) - T\| \\
&\leqslant \|T^{-1}\| \sup_{0 \leqslant k \leqslant N-1} \{\|f(\varphi^{h_k}(y_k)) - f(y_{k+1})\| + \|D\varphi^{h_k}(y_k) - Y_k\|\} \\
&\leqslant \|T^{-1}\| \sup_{0 \leqslant k \leqslant N-1} \{M_1\|\varphi^{h_k}(y_k) - y_{k+1}\| + \|D\varphi^{h_k}(y_k) - Y_k\|\} \\
&\leqslant \|T^{-1}\|(M_1 + 1)\delta \leqslant C(M_1 + 1)\delta.
\end{aligned}$$

Thus, it follows from hypothesis (i) of our theorem that

$$\|T^{-1}(D\mathcal{G}(v_0) - T)\| \leqslant \tfrac{1}{2}.$$

Hence, we conclude that the operator $\mathcal{K}$ given in (16) is a right inverse of $D\mathcal{G}(v_0)$.

Verification of (ii). In light of the inequalities

$$\|\mathcal{K}\| \leqslant 2C \tag{19}$$

derived from (15), (16) and (18), and

$$\|\mathcal{G}(v_0)\| = \sup_{0\leqslant k\leqslant N-1} \|y_{k+1} - \varphi^{h_k}(y_k)\| \leqslant \delta, \tag{20}$$

it follows from the expression for ε in the statement of Lemma 4.1 that

$$\varepsilon \leqslant 4C\delta.$$

Thus, hypothesis (ii) in the statement of the Theorem implies that the closed ball of radius ε around v_0 is contained in the open set B.

Verification of (iii). First, we find a bound for $\|D^2\mathcal{G}(v)\|$. If $v = (\{s_k\}_{k=0}^{N-1}, \{w_k\}_{k=0}^{N})$, $u = (\{\tau_k\}_{k=0}^{N-1}, \{\xi_k\}_{k=0}^{N})$ and $\bar{u} = (\{\sigma_k\}_{k=0}^{N-1}, \{\eta_k\}_{k=0}^{N})$, one calculates that

$$\begin{aligned}[D^2\mathcal{G}(v)u\bar{u}]_k = &-\tau_k\sigma_k Df(\varphi^{s_k}(y_k + S_k w_k))f(\varphi^{s_k}(y_k + S_k w_k)) \\ &- \tau_k Df(\varphi^{s_k}(y_k + S_k w_k))D\varphi^{s_k}(y_k + S_k w_k)S_k\eta_k \\ &- \sigma_k Df(\varphi^{s_k}(y_k + S_k w_k))D\varphi^{s_k}(y_k + S_k w_k)S_k\xi_k \\ &- D^2\varphi^{s_k}(y_k + S_k w_k)(S_k\xi_k)(S_k\eta_k).\end{aligned}$$

One can also produce the following estimates:

$$\|D\varphi^t(x)\| \leqslant e^{M_1 t}, \qquad \|D^2\varphi^t(x)\| \leqslant M_2 t e^{2M_1 t}, \tag{21}$$

when $0 \leqslant t \leqslant 2h_k$ and $\|x - y_k\| < \varepsilon_0$ for some k, whose verifications we defer to the end of this section. With these estimates, we have, for $v \in B$,

$$\|D^2\mathcal{G}(v)\| \leqslant M_0M_1 + 2M_1e^{2M_1h} + 2M_2he^{4M_1h}. \tag{22}$$

Then it follows from (19), (20) and (22), and hypothesis (iii) in the statement of the Theorem that

$$2M\|\mathcal{K}\|^2\|\mathcal{G}(v_0)\| \leqslant 2(M_0M_1 + 2M_1e^{2M_1h} + 2M_2he^{4M_1h})(2C)^2\delta \leqslant 1.$$

This completes the verification of the conditions of Lemma 4.1 and thus we may assert that (8) has a solution $\bar{v}$ which corresponds to a true orbit satisfying the desired properties. This concludes the proof of our Finite-time Shadowing Theorem less the verification of the estimates in (21) to which we now turn.

To verify the estimates in (21), we begin with the observation that $D\varphi^t(x)$ solves the initial-value problem

$$\frac{d}{dt}D\varphi^t(x) = Df(\varphi^t(x))D\varphi^t(x), \quad D\varphi^0(x) = I.$$

Notice that if $0 \leqslant t \leqslant 2h_k$ and $\|x - y_k\| < \varepsilon_0$, then $\varphi^t(x) \in U$ and thus $\|Df(\varphi^t(x))\| \leqslant M_1$. Now, the first estimate in (21) is an immediate consequence of Gronwall's Lemma.

Next, we notice that $D^2\varphi^t(x)$ solves the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}D^2\varphi^t(x) = Df(\varphi^t(x))D^2\varphi^t(x) + D^2f(\varphi^t(x))D\varphi^t(x)D\varphi^t(x), \quad D^2\varphi^0(x) = \mathbf{0}.$$

Then the variation of constants formula implies that

$$D^2\varphi^t(x) = \int_0^t D\varphi^{t-s}(\varphi^s(x))D^2f(\varphi^s(x))D\varphi^s(x)D\varphi^s(x)\,\mathrm{d}s.$$

Then, if $0 \leqslant t \leqslant 2h_k$ and $\|x - y_k\| < \varepsilon_0$,

$$\|D^2\varphi^t(x)\| \leqslant \int_0^t \mathrm{e}^{M_1(t-s)}M_2\mathrm{e}^{M_1 s}\mathrm{e}^{M_1 s}\,\mathrm{d}s = M_2\mathrm{e}^{M_1 t}\frac{\mathrm{e}^{M_1 t}-1}{M_1} \leqslant M_2\mathrm{e}^{M_1 t}t\mathrm{e}^{M_1 t},$$

from which the second estimate in (21) follows.

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