ad-Nilpotent Ideals of a Borel Subalgebra

Paola Cellini

Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy
E-mail: cellini@math.unipd.it

and

Paolo Papi

Dipartimento di Matematica Istituto G. Castelnuovo, Università di Roma "La Sapienza," Piazzale Aldo Moro 5, 00185 Rome, Italy
E-mail: papi@mat.uniroma1.it

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INTRODUCTION

Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra. Let \( \mathfrak{h} \subset \mathfrak{g} \) be a fixed Cartan subalgebra, \( \Delta \) the corresponding root system of \( \mathfrak{g} \), and \( \mathcal{W} \) the Weyl group. Moreover let \( \hat{\Delta} \) and \( \hat{\mathcal{W}} \) be the affine real root system and the affine Weyl group associated to \( \Delta \) [3]. We fix a positive system \( \Delta^+ \) in \( \Delta \) and denote by \( \hat{\Delta}^+ \) the corresponding positive system in \( \hat{\Delta} \). For each \( \alpha \in \Delta^+ \) let \( L_\alpha \) be the root space of \( \mathfrak{g} \) relative to \( \alpha \), \( \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} L_\alpha \), and \( \mathfrak{b} \) be the Borel subalgebra \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \). An ideal \( \mathfrak{i} \) of \( \mathfrak{b} \) is ad-nilpotent (i.e., it consists of ad-nilpotent elements) if and only if it is included in \( \mathfrak{n} \); moreover, in such a case \( \mathfrak{i} \) is a direct sum of root spaces \( L_\alpha \), \( \alpha \in \Delta^+ \). In this paper we describe an encoding of the ad-nilpotent ideals of \( \mathfrak{b} \) by elements of \( \hat{\mathcal{W}} \); moreover we characterize the elements of \( \hat{\mathcal{W}} \) arising in this way. In fact, for \( w \in \hat{\mathcal{W}} \) we consider the set

\[
N(w) = \{ \alpha \in \hat{\Delta}^+ \mid w^{-1}(\alpha) \in \hat{\Delta}^- \},
\]

where \( \hat{\Delta}^- = -\hat{\Delta}^+ \). It is well known that \( N(w) \) determines \( w \) uniquely. We associate in a natural way to any ad-nilpotent ideal of \( \mathfrak{b} \) a set of roots in
which turns out to be of the form \( N(w) \); moreover such a set reflects the descending central series of the ideal. For the abelian ideals of \( b \), such a correspondence with subsets of type \( N(w) \), hence with elements in the affine Weyl group, has been established by Peterson (see [4]). As a consequence, he proves the following remarkable result: the number of abelian ideals of a Borel subalgebra of any simple Lie algebra of rank \( n \) is \( 2^n \), independently of the type of \( g \). The proofs of these results are unpublished but are clearly outlined in [4]. We generalize the idea of Peterson to all ad-nilpotent ideals of \( b \). We also give here a proof of his enumerative result.

The problem of enumerating ad-nilpotent ideals of fixed class of nilpotence seems to be quite complicated. On the other hand the set \( \mathcal{F} \) of all ad-nilpotent ideals is in bijection with the increasing set of roots studied by Shi [7] in a different context; using his results we determine the cardinality of \( \mathcal{F} \) for any \( g \). For classical Lie algebras the increasing sets are indeed encoded by suitable “subdiagrams” of a fixed (possibly shifted) staircase diagram. We give a simple characterization of the subdiagrams corresponding to abelian ideals.

1. PRELIMINARIES

\( \mathbb{Z}, \mathbb{N}, \mathbb{N}^+ \) will denote the sets of integer, non-negative integer and positive integer numbers, respectively. We set, for \( n \in \mathbb{N}, [n] = \{1, \ldots, n\} \).

We write \( \alpha > 0 \) (resp. \( \alpha < 0 \)) to mean that the root \( \alpha \) is positive (resp. negative).

We first recall the description of the affine Weyl group and root system given in [3] and the explicit relation between this description and the classical one [2]. We follow the approach of Kac’s book [3] (to which we refer for details and proofs). However, our interest focuses more on the geometric properties of \( \hat{\Delta}, \hat{W} \) rather than on the theory of Kac–Moody algebras: therefore we do not include imaginary roots in \( \hat{\Delta} \); in particular, our \( \hat{\Delta} \) is \( \Delta^{re} \) in Kac’s notation.

Let \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) be the simple roots of \( \Delta^+ \). We set \( V = \mathfrak{h}_\mathbb{R}^* = \bigoplus_{i=1}^{n} \mathbb{R} \alpha_i \) and denote by \( (, \) \) the positive symmetric bilinear form induced on \( V \) by the Killing form. We describe the affine root system associated to \( \Delta \) as follows [3, Chapt. 6]. We extend \( V \) and its inner product setting \( \hat{V} = V \oplus \mathbb{R} \delta \oplus \mathbb{R} \lambda, (\delta, \delta) = (\delta, V) = (\lambda, \lambda) = (\lambda, V) = 0, \) and \( (\delta, \lambda) = 1 \). We still denote by \( (, \) \) the resulting (non-degenerate) bilinear form. The affine root system associated to \( \Delta \) is \( \hat{\Delta} = \Delta + \mathbb{Z}\delta = \{\alpha + k\delta \mid \alpha \in \Delta, k \in \mathbb{Z}\} \); remark that the affine roots are non-isotropic vectors. The set of positive affine roots is \( \hat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta) \), where \( \Delta^- = -\Delta^+ \).

We denote by \( \theta \) the highest root of \( \Delta \) and set \( \alpha_0 = -\theta + \delta, \hat{\Pi} = \)
For each $\alpha \in \hat{\Delta}^+$ we denote by $s_\alpha$ the corresponding reflection of $\hat{V}$, $s_\alpha(x) = x - [2(\alpha, x)/(\alpha, \alpha)]x$ for $x \in \hat{V}$. The affine Weyl group associated to $\Delta$ is the group $\hat{W}$ generated by $\{s_\alpha \mid \alpha \in \hat{\Delta}^+\}$; the set $\{s_\alpha \mid \alpha \in \hat{\Pi}\}$ is a set of Coxeter generators for $\hat{W}$; we denote by $\ell$ the corresponding length function. Note that $w(\delta) = \delta$ for any $w \in \hat{W}$. Now consider the $\hat{W}$-invariant affine subspace $E = \{x \in V \mid (x, \delta) = 1\}$. Remark that $E = V \oplus \mathbb{R}\delta + \lambda$; let $\pi: E \to V$ be the natural projection. For $w \in \hat{W}$ we set $\bar{w} = \pi \circ w_{|E}$. We have that the map $w \mapsto \bar{w}$ gives an isomorphism of $\hat{W}$ onto a group $W_{af}$ of affine transformations of $V$ [3, 6.6]. $W_{af}$ is the usual affine representation of the affine Weyl group [2, VI, Sect. 2]. In fact, for $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$ set

$$H_{\alpha,k} = \{x \in V \mid (x, \alpha) = k\}$$

and let $s_{\alpha,k}$ denote the orthogonal reflection with respect to $H_{\alpha,k}$ in $V$. For $\beta \in V$, let $\hat{H}_\beta$ be the hyperplane orthogonal to $\beta$ in $\hat{V}$. Then it is easy to see that, for each $\alpha \in \Delta^+$, $k > 0$ and $h \geq 0$, we have

$$\pi(\hat{H}_{-\alpha+k\delta} \cap E) = H_{\alpha,k}, \quad s_{-\alpha+k\delta} = s_{\alpha,k},$$

$$\pi(\hat{H}_{\alpha+h\delta} \cap E) = H_{\alpha,-h}, \quad s_{\alpha+h\delta} = s_{\alpha,-h}.$$  

Moreover, if $C$ is the fundamental chamber of $\hat{W}$ and $A$ is the fundamental alcove of $W_{af}$,

$$C = \{x \in \hat{V} \mid (x, \alpha_i) > 0 \text{ for } i = 0, \ldots, n\},$$

$$A = \{x \in V \mid (x, \alpha) > 0 \forall \alpha \in \hat{\Pi}, (x, \theta) < 1\},$$

then $A = \pi(C \cap E)$. From this it easily follows that, for any $\alpha \in \Delta^+$, $k \in \mathbb{N}^+$, $h \in \mathbb{N}$

1.1.

$$w^{-1}(\alpha + k\delta) < 0 \text{ if and only if } H_{\alpha,k} \text{ separates } A \text{ and } w(A),$$

$$w^{-1}(\alpha + h\delta) < 0 \text{ if and only if } H_{\alpha,-h} \text{ separates } A \text{ and } w(A).$$

Now we recall the main properties of the sets $N(w), w \in \hat{W}$.

**Definition 1.2.** A subset $L \subset \hat{\Delta}^+$ is called compatible if it verifies the following conditions:

1. If $\lambda, \mu, \lambda + \mu \in L$ and $\lambda + \mu \in \hat{\Delta}^+$, then $\lambda + \mu \in L$.
2. If $\lambda + \mu \in L$, $\lambda, \mu \in \hat{\Delta}^+$ and $\mu \notin L$, then $\lambda \in L$.

If $L$ satisfies (1), we say that $L$ is closed.
THEOREM 1.3 [5]. Let $\hat{\Delta}^+$ be an affine root system not of type $\hat{A}_1$. Then a finite subset $L \subset \hat{\Delta}^+$ is compatible if and only if it is of the form $N(w)$, $w \in \hat{W}$. Moreover, $w$ is uniquely determined by $L$.

DEFINITION 1.4. Let $L$ be a compatible set. A total order $<$ on $L$ is called a compatible order if it verifies the following conditions:

(a) If $\lambda, \mu, \lambda + \mu \in L$ and $\lambda < \mu$, then $\lambda < \lambda + \mu < \mu$.

(b) If $\lambda + \mu \in L$, $\lambda, \mu \in \hat{\Delta}^+$ and $\mu \notin L$, then $\lambda \in L$ and $\lambda < \lambda + \mu$.

Remark. Fix a reduced expression $w = s_1 \cdots s_k$, with $s_i = s_{i+1}$, $i \in \hat{I}$; then $N(w) = \{s_1 \cdots s_{p-1}(\beta_p)\}$, $1 \leq p \leq k$. Moreover, the choice of a reduced expression for $w$ induces on $N(w)$ a total order, as follows: for $\alpha, \gamma \in N(w)$ set $\alpha < \gamma$ if $\alpha = s_1 \cdots s_{p-1}(\beta_p)$, $\gamma = s_1 \cdots s_{q-1}(\beta_q)$ with $1 \leq p < q \leq k$. Such an order is compatible; moreover, if $L$ is a finite compatible set, then any compatible order on $L$ is induced by a reduced expression of the unique element $w \in \hat{W}$ such that $L = N(w)$ [5, Sect. 2, Theorem].

2. ad-NILPOTENT IDEALS OF $\hat{b}$ AND AFFINE COMPATIBLE SETS

Assume that $i$ is an ideal of $\hat{b}$ and that $i \subseteq n$. Then it is clear that $i$ is a direct sum of root spaces $L_\alpha$, $\alpha \in \Delta^+$. We set

$$\mathcal{J} = \{i \subseteq n \mid i \text{ ideal of } \hat{b}\},$$

and, for $i \in \mathcal{J}$ we set

$$\Phi_i = \{\alpha \in \Delta^+ \mid L_\alpha \subseteq i\},$$

so that $i = \bigoplus_{\alpha \in \Phi_i} L_\alpha$. It is clear that if $\alpha \in \Phi_i$, $\beta \in \Delta^+$, and $\alpha + \beta \in \Delta^+$ then $\alpha + \beta \in \Phi_i$, and that, moreover, this property characterizes the sets $\Phi_i$, $i \in \mathcal{J}$. We set

$$\mathcal{F} = \{\Phi \subseteq \Delta^+ \mid \text{if } \alpha \in \Phi, \beta \in \Delta^+, \alpha + \beta \in \Delta^+ \text{ then } \alpha + \beta \in \Phi\}$$

and, for $\Phi \in \mathcal{F}$,

$$i_{\Phi} = \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Then $i_{\Phi} \in \mathcal{F}$ and $\Phi_{i_{\Phi}} = \Phi$, so that $i \leftrightarrow \Phi_i$ is a bijection between $\mathcal{J}$ and $\mathcal{F}$. For $i \in \mathcal{J}$ let

$$i^1 = i, \ldots, \quad i^k = [i^{k-1}, i], \ldots$$
be its descending central series, and \( n(i) \) its index of nilpotence (the number of nonzero terms of the descending central series). For \( \Phi \subseteq \Delta^+ \), set

\[
\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta.
\]

Then we have the following.

**Lemma 2.1.** If \( \phi \in \mathcal{F} \), then \((i_\phi)^k = i_{\phi^k} \in \mathcal{F} \). In particular \( \Phi^k \in \mathcal{F} \) \( \forall k \geq 1 \); moreover

\[
\Phi = \Phi^1 \supseteq \Phi^2 \supseteq \cdots \Phi^k \supseteq \cdots
\]

**Proof.** We first prove that if \( x = [x_1, x_2, \ldots, x_k] \), with \( x_i \in L_{\beta_i} \), \( \beta_i \in \Phi \), then \( x \in i_{\phi^k} \); this clearly implies the inclusion \((i_\phi)^k \subseteq i_{\phi^k} \). If \( x = 0 \) we are done; otherwise we have that \( \beta_1 + \cdots + \beta_k \in \Delta^+ \) for \( i \leq k \) so that \( \beta_1 + \cdots + \beta_k \in \Phi^k \) and \( x \in i_{\phi^k} \).

Conversely, assume \( x \in i_{\phi^k} \), say \( x \in L_{\beta} \) with \( \beta \in \Phi^k \). By definition \( \beta = \beta_1 + \cdots + \beta_k \) and \( \beta_1 + \cdots + \beta_i \in \Delta^+ \) for \( i \leq k \), so that \( L_{\beta} = [L_{\beta_1}, L_{\beta_2}, \ldots, L_{\beta_k}] \subseteq (i_\phi)^k \).

**Definition 2.2.** For \( \Phi \in \mathcal{F} \) we define

\[
L_{\phi} = \bigcup_{k \in \mathbb{N}^+} (-\Phi^k + k\delta).
\]

We are going to prove that the set \( L_{\phi} \) in the above definition is compatible and is, indeed, the minimal compatible set including \( -\alpha + \delta \) for each \( \alpha \in \Phi \).

**Lemma 2.3.** Let \( \Phi \in \mathcal{F}, \beta, \gamma \in \Delta^+ \), and \( k_1, k_2 \in \mathbb{N}^+ \). If

\[
\beta + \gamma \in \Phi^{k_1+k_2}
\]

then either \( \beta \in \Phi^{k_1} \) or \( \gamma \in \Phi^{k_2} \).

**Proof.** Set \( \alpha = \beta + \gamma \) and \( k = k_1 + k_2 \). We proceed by induction on \( k \geq 2 \). By the inductive definition of \( \Phi^k \) there exist \( \eta \in \Phi^{k-1} \) and \( \xi \in \Phi \) such that \( \eta + \xi = \beta + \gamma = \alpha \). We have

\[
(\alpha, \alpha) = (\eta, \beta) + (\eta, \gamma) + (\xi, \beta) + (\xi, \gamma) > 0,
\]

therefore one of the summands in the right-hand side of the previous relation is positive. Since the difference of two roots having positive scalar product is a root, and since \( \eta - \beta = \gamma - \xi \) and \( \eta - \gamma = \beta - \xi \), we have that either \( \eta - \beta \in \Delta \) or \( \eta - \gamma \in \Delta \). It suffices to consider the case \( \eta - \beta \in \Delta \). Suppose \( \beta - \eta \in \Delta^+ \). We have \( \beta = (\beta - \eta) + \eta \), \( \eta \in \Phi^{k-1} \), and \( \Phi^{k-1} \in \mathcal{F} \), therefore we obtain \( \beta \in \Phi^{k-1} \subseteq \Phi^k \). Assume
First note that if $k_2 = 1$ then, since $\gamma = \xi + (\eta - \xi)$, $\gamma - \xi = \eta - \beta \in \Delta^+$, and $\xi \in \Phi$, we obtain that $\gamma \in \Phi = \Phi^{k_2}$, so let $k_2 > 1$. Now we assume that $\beta \notin \Phi^{k_1}$ and show that $\gamma \in \Phi^{k_2}$, which concludes the proof. We have $\eta = (\eta - \beta) + \beta \in \Phi^{k_2-1}$, so, by the inductive assumption, we obtain $\eta - \beta \in \Phi^{k_2-1}$. Then $\gamma - \xi = \eta - \beta \in \Phi^{k_2-1}$, and therefore $\gamma = (\gamma - \xi) + \xi \in \Phi^{k_2}$.

**Proposition 2.4.** Let $\Phi \subseteq \mathcal{F}$. Then $L_\Phi$ is compatible.

**Proof.** We prove that properties (1) and (2) of Definition 1.2 hold for $L_\Phi$. Indeed, (1) is immediate from the definition of $L_\Phi$, so let us prove (2). Let $\alpha \in L_\Phi$, $\alpha = \beta + \gamma$, $\beta, \gamma \in \Delta^+$. Then $\alpha = -\alpha_0 + k\delta$ for some $k \in \mathbb{N}^+$ and $\alpha_0 \in \Phi^k$. We have $\beta = \beta_0 + k_1 \delta$ and $\gamma = \gamma_0 + k_2 \delta$ for some $\beta_0, \gamma_0 \in \Delta$ and $k_1, k_2$ in $\mathbb{N}$ such that $k_1 + k_2 = k$. Moreover at least one of $\beta_0$ and $\gamma_0$ is a negative root, since $\beta_0 + \gamma_0 = -\alpha_0$ is negative. Assume $\beta_0 < 0$ and $\gamma_0 > 0$. Then $-\beta_0 = \alpha_0 + \gamma_0$ and since $\alpha_0 \in \Phi^k \subseteq \Phi^{k_1}$ we obtain $-\beta_0 \in \Phi^{k_1}$ and hence $\beta_0 + k_1 \delta \in L_\Phi$. So we may assume that both $\beta_0$ and $\gamma_0$ are negative; remark that in such a case we have $k_1, k_2 > 0$. We have $(-\beta_0) + (-\gamma_0) = \alpha_0 \in \Phi^k$, hence, by Lemma 2.3 either $-\beta_0 \in \Phi^{k_1}$ or $-\gamma_0 \in \Phi^{k_2}$; hence either $\beta = \beta_0 + k_1 \delta \in L_\Phi$, or $\gamma = \gamma_0 + k_2 \delta \in L_\Phi$, as desired.

**Corollary 2.5.** If $\Phi \in \mathcal{F}$, then $L_\Phi$ is the minimal compatible set including $-\Phi + \delta$.

**Proof.** By definition each element in $L_\Phi$ is a sum of elements in $-\Phi + \delta$. By Definition 1.2 any compatible set is closed, hence if it includes $-\Phi + \delta$, it also includes $L_\Phi$.

**Remark.** A compatible set should contain at least one simple root. Since by definition $L_\Phi$ does not contain any of the simple roots of the finite system $\Delta$, we have that $-\theta + \delta$ belongs to $L_\Phi$.

For any ad-nilpotent ideal $i$ we set $L_i = L_{\Phi_i}$. Using 1.3 we obtain an injective map from the set $\mathcal{F}$ of ad-nilpotent ideals of $\mathfrak{b}$ to $\hat{W}$.

**Theorem 2.6.** Consider $i \in \mathcal{F}$. Then there exists a unique $w_i \in \hat{W}$ such that $L_i = N(w_i)$.

In 2.11 we shall characterize the set $\{w_i \mid i \in \mathcal{F}\}$. By the definition and Lemma 2.1 the compatible set $L_i$ naturally reflects the descending central series of $i$; in particular we have the following result.

**Corollary 2.7.** Suppose $i \in \mathcal{F}$. We have $n(i) = k$ if and only if $L_i$ contains some root of the form $-\alpha_0 + k\delta$ and no root of the form $-\beta_0 + (k + 1)\delta$ with $\alpha_0, \beta_0 \in \Delta^+$.

Restricting to the class of abelian ideals, we obtain the following result, due to Peterson.
PROPOSITION 2.8. The following statements are equivalent for \( \Phi \subset \Delta^+ \):

(i) \( \iota_\Phi \) is an abelian ideal.

(ii) The set \( \{ \delta - \beta \mid \beta \in \Phi \} \) is compatible.

(iii) \( \Phi \in \mathcal{S}_a \equiv \{ \Phi \in \mathcal{F} \mid \alpha, \beta \in \Phi \Rightarrow \alpha + \beta \not\in \Delta \} \).

The enumeration of abelian ideals follows now easily.

THEOREM 2.9 (Peterson). If \( g \) has rank \( n \), then it has exactly \( 2^n \) abelian ideals.

Proof. Let \( w \in \hat{W} \). We prove that \( N(w) = L_1 \) for some abelian ideal \( i \) of \( \beta \) if and only if \( w(A) \subseteq 2A \), where \( A \) is the fundamental alcove. Since the volume of \( 2A \) is \( 2^n \) times that of \( A \), and since \( 2A \) is a union of \( w(A) \) with \( w \in \hat{W} \), up to a null subset, we obtain the result. We use 1.1. First it is obvious that \( w(A) \subseteq 2A \), otherwise \( H_{\alpha, 2} \) separates \( A \) and \( w(A) \), and \( -\theta + 2\delta \in N(w) \) against the assumption. Conversely, if \( w(A) \subseteq 2A \), then each hyperplane which separates \( A \) and \( w(A) \) intersects \( 2A \). But for each \( x \in 2A \) and for each \( \alpha \in \Delta^+ \) we have \( 0 < (x, \alpha) < (x, \theta) < 2 \), since, \( (x, \alpha) > 0 \) for \( i = 1, \ldots, n \) and \( \theta \) is the highest root. Therefore if \( H_{\alpha, k} \cap 2A \neq \emptyset \), for some \( \alpha \in \Delta^+ \) and \( k \in \mathbb{Z} \), then \( 0 < k < 2 \).

Now we are going to characterize the compatible sets of type \( L_\Phi \), \( \Phi \in \mathcal{S} \).

LEMMA 2.10. Let \( L \subseteq \Delta^- + \mathbb{N}^+ \delta \) be compatible. Then

\[ \{ \alpha \in \Delta^+ \mid -\alpha + \delta \in L \} \in \mathcal{F}. \]

Proof. Set \( D = \{ \alpha \in \Delta^+ \mid -\alpha + \delta \in L \} \) and let \( \alpha \in D \) and \( \beta \in \Delta^+ \) be such that \( \alpha + \beta \in \Delta^+ \). Then \( -\alpha - \beta + \delta \in \hat{\Delta}^+ \), \( \beta \in \hat{\Delta}^+ \) and \( -\alpha - \beta + \delta = -\alpha + \delta \in L \). Then, by property (2) of Definition 1.2, either \( \beta \in L \), or \( -\alpha - \beta + \delta \in L \). Since \( L \subseteq \Delta^- + \mathbb{N}^+ \delta \), we obtain \( -\alpha - \beta + \delta \in L \), hence \( \alpha + \beta \in D \).

PROPOSITION 2.11. Let \( L \subseteq \Delta^- + \mathbb{N}^+ \delta \) be compatible. Then there exists \( \Phi \in \mathcal{S} \) such that \( L = L_\Phi \) if and only if, for any compatible order

\[ L = \{ \gamma_1 < \cdots < \gamma_k \}, \]

we have \( \gamma_k \in -\Delta^+ + \delta \).

Proof. Let \( \Phi = \{ \alpha \in \Delta^+ \mid -\alpha + \delta \in L \} \). By Lemma 2.10 \( \Phi \in \mathcal{F} \) and by Corollary 2.5 \( L_\Phi \subseteq L \). Moreover it is clear that \( L \neq L_\Phi \), for \( \Phi' \neq \Phi \). Recall from [1, Sect. 3] that \( N(w_1) \subseteq N(w_2) \) if and only if there exists \( u \in \hat{W} \) such that \( w_2 = w_1 u, \zeta'(w_2) = \zeta'(w_1) + \zeta'(u) \). Using this property, it is not difficult to deduce the following fact: if \( N \) is any compatible set and \( M \subset N \), then \( M \) is compatible if and only if there exists a compatible order
of $N$ having $M$ as an initial section, i.e., such that $x < y$ for each $x \in M$ and $y \in M \setminus N$. It follows that if $L = L_\Phi$ and $<$ is any compatible order on $L$, then the last element $\alpha$ of $<$ belongs to $-\Phi + \delta$, otherwise $L \setminus \{\alpha\}$ would be a compatible set including $-\Phi + \delta$, contradicting Corollary 2.5. Conversely, if $L_\Phi \subsetneq L$, then if we take a compatible order $<$ on $L$ having $L_\Phi$ as initial section, we obtain that the last element of $L$ for $<$ does not belong to $-\Phi + \delta$.

**Proposition 2.11.** Let $w \in \hat{W}$. Then $w = w_i$ for some $i \in \mathcal{I}$ if and only if the following conditions hold:

(i) $w^{-1}(\alpha) > 0$ for each $\alpha \in \Pi$;

(ii) if $w(\alpha) < 0$ for some $\alpha \in \hat{\Pi}$, then $w(\alpha) = \beta - \delta$ for some $\beta \in \Delta^+$. 

**Proof.** We have $w = w_i$ for some $i \in \mathcal{I}$ in and only if $N(w) = L_\Phi$ for some $\Phi \in \mathcal{F}$. By the definition of $L_\Phi$, (i) is a necessary condition for $N(w)$ to equal some $L_\Phi$ with $\Phi \in \mathcal{F}$. Moreover, if (i) holds, then we have $N(w) \subseteq \Delta^+ + \mathbb{N}^+ \delta$. We observe that if $\alpha \in \hat{\Pi}$, we have $w(\alpha) < 0$ if and only if there exists a reduced expression of $w^{-1}$ starting with $s_\alpha$, say $w^{-1} = s_\alpha s_{\beta_1} \cdots s_{\beta_k}$. In such a case $s_{\beta_k} \cdots s_{\beta_1} s_\alpha$ is a reduced expression of $w$; therefore $s_{\beta_k} \cdots s_{\beta_1}(\alpha) = -w(\alpha)$ belongs to $N(w)$ and is the last element of $N(w)$ for the order induced by $s_{\beta_k} \cdots s_{\beta_1} s_\alpha$. Now, if $N(w) = L_\Phi$ for some $\Phi \in \mathcal{F}$, by Proposition 2.11 we obtain that $-w(\alpha) = \beta - \delta$ for some $\beta \in \Delta^+$, as claimed. Conversely, if $\gamma$ is the last element of $N(w)$ for some compatible order, then $\alpha = -w^{-1}(\gamma)$ is a positive simple root and we have $w(\alpha) = -\gamma < 0$. Therefore, if condition (ii) holds, we obtain that $\gamma = -\beta + \delta$ for some $\beta \in \Delta^+$. If, moreover, condition (i) holds too, then we can apply Proposition 2.11, obtaining that $N(w) = L_\Phi$ for some $\Phi \in \mathcal{F}$.

**Proposition 2.12.** Let $w \in \hat{W}$. Then $w = w_i$ for some $i \in \mathcal{I}$ if and only if the following conditions hold:

(i) $w^{-1}(\alpha) > 0$ for each $\alpha \in \Pi$;

(ii) if $w(\alpha) < 0$ for some $\alpha \in \hat{\Pi}$, then $w(\alpha) = \beta - \delta$ for some $\beta \in \Delta^+$. 

**Proof.** We have $w = w_i$ for some $i \in \mathcal{I}$ in and only if $N(w) = L_\Phi$ for some $\Phi \in \mathcal{F}$. By the definition of $L_\Phi$, (i) is a necessary condition for $N(w)$ to equal some $L_\Phi$ with $\Phi \in \mathcal{F}$. Moreover, if (i) holds, then we have $N(w) \subseteq \Delta^+ + \mathbb{N}^+ \delta$. We observe that if $\alpha \in \hat{\Pi}$, we have $w(\alpha) < 0$ if and only if there exists a reduced expression of $w^{-1}$ starting with $s_\alpha$, say $w^{-1} = s_\alpha s_{\beta_1} \cdots s_{\beta_k}$. In such a case $s_{\beta_k} \cdots s_{\beta_1} s_\alpha$ is a reduced expression of $w$; therefore $s_{\beta_k} \cdots s_{\beta_1}(\alpha) = -w(\alpha)$ belongs to $N(w)$ and is the last element of $N(w)$ for the order induced by $s_{\beta_k} \cdots s_{\beta_1} s_\alpha$. Now, if $N(w) = L_\Phi$ for some $\Phi \in \mathcal{F}$, by Proposition 2.11 we obtain that $-w(\alpha) = \beta - \delta$ for some $\beta \in \Delta^+$, as claimed. Conversely, if $\gamma$ is the last element of $N(w)$ for some compatible order, then $\alpha = -w^{-1}(\gamma)$ is a positive simple root and we have $w(\alpha) = -\gamma < 0$. Therefore, if condition (ii) holds, we obtain that $\gamma = -\beta + \delta$ for some $\beta \in \Delta^+$. If, moreover, condition (i) holds too, then we can apply Proposition 2.11, obtaining that $N(w) = L_\Phi$ for some $\Phi \in \mathcal{F}$.

**3. Enumeration of ad-nilpotent ideals**

Endow $\Delta^+$ with the following partial order: $\alpha \leq \beta$ if $\beta - \alpha$ is a sum of positive roots. Then it is easy to see that $\Phi \in \mathcal{F}$ if and only if $\Phi$ is increasing w.r.t. $\leq$, i.e., if $\alpha \in \Phi$, $\beta \in \Delta^+$, $\alpha \leq \beta$ then $\beta \in \Phi$. Therefore, enumerating $ad$-nilpotent ideals is equivalent to enumerate the increasing subset of $\Delta^+$: this problem has been solved, for another purpose, by Shi in [6, Sect. 2-3]. He enumerates the increasing sets for $\mathfrak{g}$ of any type: for classical Lie algebras the proof is obtained by establishing a bijection between increasing sets and a suitable set of Young diagrams (possibly shifted). We shall give a simple characterization of the diagrams corresponding to the abelian ideals through this bijection. Moreover we provide an alternative proof of the enumeration for the classical types.
We first recall the bijection. Display the positive roots in a (possibly shifted) diagram $T$ of suitable shape; hence we can assign to a box $(i, j)$ in $T$ a positive root $t_{ij}$. We then consider the natural map $t: 2^\Delta^+ \to 2^T$ which associates to a subset $\Phi$ of positive roots the corresponding boxes $t(\Phi) = T_\Phi \subseteq T$. The shape and the filling of $T$ are chosen in such a way that $t$ restricts to a bijection between $\mathcal{F}$ and the set of subdiagrams $T' \subseteq T$.

For type $A_n$, $T$ is the (unshifted) diagram of shape $(n, n - 1, \ldots, 1)$; it is filled with the positive roots according to the assignment $t_{ij} = \alpha_i + \cdots + \alpha_{n-j+1}$, $1 \leq i \leq j \leq n$. For example, for $A_3$

$$\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 & \quad \alpha_1 + \alpha_2 + \alpha_1 \\
\alpha_2 + \alpha_3 & \quad \alpha_2 \\
\alpha_3 & 
\end{align*}$$

For type $B_n$, $T$ is the shifted diagram of shape $(2n - 1, 2n - 3, \ldots, 1)$ and the filling is defined as

$$t_{ij} = \begin{cases} 
\alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_n), & 1 \leq j \leq n - 1 \\
\alpha_i + \cdots + \alpha_{2n-j}, & n \leq j \leq 2n - 1.
\end{cases}$$

For example, for $B_3$

$$\begin{align*}
\alpha_1 + 2\alpha_2 + 2\alpha_3 & \quad \alpha_1 + \alpha_2 + 2\alpha_3 \\
\alpha_2 + 2\alpha_3 & \quad \alpha_2 + \alpha_3 \\
\alpha_3 & 
\end{align*}$$

The same shape $(2n - 1, 2n - 3, \ldots, 1)$ can be used to deal with case $C_n$; now

$$t_{ij} = \begin{cases} 
\alpha_i + \cdots + 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n, & 1 \leq j \leq n - 1 \\
\alpha_i + \cdots + \alpha_{2n-j}, & n \leq j \leq 2n - 1.
\end{cases}$$

For example, for $C_3$

$$\begin{align*}
2\alpha_1 + 2\alpha_2 + \alpha_3 & \quad \alpha_1 + 2\alpha_2 + \alpha_3 \\
2\alpha_2 + \alpha_3 & \quad \alpha_2 + \alpha_3 \\
\alpha_3 & 
\end{align*}$$

Recall that the long roots of a root system of type $B_n$ form a subsystem of type $D_n$; this motivates the choice for $T$ in case $D_n$. Let $T$ be the shifted diagram of shape $(2n - 2, 2n - 4, \ldots, 2)$, it is filled with the positive roots according to the assignment

$$t_{ij} = \begin{cases} 
\alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n, & 1 \leq j \leq n - 2 \\
\alpha_i + \cdots + \alpha_{n-2} + \alpha_n, & j = n - 1 \\
\alpha_i + \cdots + \alpha_{2n-j}, & n \leq j \leq 2n - 1.
\end{cases}$$
For example, for $D_4$, setting $\theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$

\[
\begin{array}{cccc}
\theta & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_1 + \alpha_2 + \alpha_4 & \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 \\
\alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_4 & \alpha_2 + \alpha_3 & \alpha_2 \\
\alpha_4 & \alpha_3 & & \\
\end{array}
\]

In case $D$ we need the following *ad hoc* definition. For $B \subseteq T$, denote by $B^*$ the set of boxes obtained from $B$ by exchanging columns $n - 1, n$. This definition mirrors the involution of the Dynkin diagram of $D_n$ exchanging $\alpha_{n-1}, \alpha_n$.

In the following we fix $T$ to be the diagram of shape $(n, n-1, \ldots, 1)$ (resp. shifted shape $(2n - 1, 2n - 3, \ldots, 1), (2n - 2, 2n - 4, \ldots, 2)$) in case $A_n$ (resp. $B_n$ or $C_n, D_n$). The filling of $T$ is the one specified above in each case.

The North-West corner of an element $t_{ij} \in T$ is defined as the set $\{h \in T | h < i, k < j\}$. We say that a set of boxes $A$ of $T$ is a subdiagram of $T$ if $A$ contains the North-West corner of its elements.

Let $\mathcal{Y}$ denote the class of subdiagrams of $T$.

Let finally $\mathcal{C}_n = \frac{1}{n+1}(\mathcal{C}_n)$ be the $n$th Catalan number.

**Theorem 3.1** [7, Th. 3.2]. The map $t$ is a bijection between $\mathcal{I}$ and $\mathcal{Y}$ when $\mathfrak{g}$ is of type $A$, $B$ or $C$. In case $D$, $t$ is a bijection between $\mathcal{I}$ and the sets of boxes $B \subseteq T$ such that either $B$ or $B^*$ belongs to $\mathcal{Y}$.

Let $a_n$ denote the cardinality of $\mathcal{I}$ for $\mathfrak{g}$ of rank $n$.

If $\mathfrak{g}$ is of type $A_n$, then $a_n = \mathcal{C}_{n+1}$.

If $\mathfrak{g}$ is of type $B_n$ or $C_n$, then $a_n = \binom{2n}{n} = (n + 1)\mathcal{C}_n$.

If $\mathfrak{g}$ is of type $D_n$, then $a_n = (n + 1)\mathcal{C}_n - n\mathcal{C}_{n-1}$.

**Remarks.** (1) The previous analysis can be completed with [7, Th. 3.6]: if $\mathfrak{g}$ is of type $E_6, E_7, E_8, F_4, G_2$, then $a_n = 832, 4160, 25080, 105, 8$, respectively.

(2) We provide an alternative approach to the second statement of the previous theorem. Let $\mathfrak{g}$ be of type $A_n$; from [8, 6.19 vv.] it follows that $a_n = \mathcal{C}_{n+1}$.

Let $\mathfrak{g}$ be of type $B_n$ or $C_n$; we have to count the number of diagrams contained in $T$: this can be recovered from [6], Corollary to Th. 1. There it is proved that the number of shifted tableaux $S = (S_{ij}), 1 \leq i \leq q, i \leq j \leq p + q - i$ of shifted shape $(p + q - 1, p + q - 3, \ldots, p - q + 1)$ with $0 \leq S_{ij} \leq m$ and $S_{ij} \geq S_{(i+1)j}, S_{ij} \geq S_{(i+1)(j+1)}$ is given by

\[
\prod_{i=1}^{p} \prod_{j=1}^{q} \prod_{k=1}^{m} \frac{i + j + k - 1}{i + j + k - 2}.
\]
If \( m = 1 \), the set of boxes filled with 1 in a tableau determines a subdiagram, and any subdiagram can be obtained in this way. Setting \( p = q = n, m = 1 \) in the previous formula, we obtain

\[
a_n = \prod_{i,j=1}^{n} \frac{i+j}{i+j-1} = \binom{2n}{n} = (n+1)\mathcal{S}_n.
\]

In the case of \( D_n \), we note that the sets of boxes \( B \subseteq T \) such that \( B \in \mathcal{Y} \) are counted by Proctor's formula with \( p = n, q = n - 1, m = 1 \); we have therefore a contribution

\[
\prod_{i=1}^{n} \prod_{j=1}^{n-1} \frac{i+j}{i+j-1} = \frac{n+1}{2} \mathcal{S}_n.
\]

On the other hand, the number of \( B \subseteq T \) such that \( B \notin \mathcal{Y} \) and \( B^* \in \mathcal{Y} \) is \( |\mathcal{D}|-|\mathcal{D}| \), where \( \mathcal{D} = \{B \subseteq T \mid B \in \mathcal{Y}\} \) and \( \mathcal{D} \) is the set of diagrams having columns \( n-1, n \) of the same length. Clearly, the elements of \( \mathcal{D} \) are as many as the subdiagrams of the shifted diagram obtained from \( T \) by removing the \( n \)th column, whose shape is \((2n-3, 2n-5, \ldots, 1)\). By the previous calculation in case \( B \) we get \( |\mathcal{D}| = n\mathcal{S}_{n-1} \). The claim follows.

Recall that the hook length of a box in a Young diagram is the number of boxes directly below and directly to the right of the box, including the box once. We need one more notation; for a set of boxes \( T_\Phi \subseteq T, \Phi \in \mathcal{F} \), we set \( r_i^\Phi = \max\{j \mid t_{ij} \in T_\Phi\} \).

**Proposition 3.2.** Consider \( \Phi \in \mathcal{F} \). We have \( \Phi \in \mathcal{F}_a \) if and only if

(a) the hook length of \( t_{11} = \theta \) in \( T_\Phi \) does not exceed \( n \) when \( \Phi \) of type \( A_n \);

(b) \( r_1^\Phi + r_2^\Phi \leq 2n - 1 \) in \( T_\Phi \) when \( \Phi \) of type \( B_n \);

(c) \( r_1^\Phi \leq n \) in \( T_\Phi \) when \( \Phi \) of type \( C_n \);

(d) \( r_1^\Phi + r_2^\Phi \leq 2n - 2 \) in \( T_\Phi \) when \( \Phi \) of type \( D_n \).

**Proof.** Let \( h \) denote the hook length of \( t_{11} = \theta \) in \( T_\Phi \). Remark that, by the definition of \( T_\Phi \) there exist two roots in \( \Phi \) which sum up to \( \theta \) if and only if \( h \leq n \); in particular \( h > n \) implies \( \Phi \notin \mathcal{F}_a \). To conclude the proof we show that if \( \Phi \notin \mathcal{F}_a \), then \( h > n \). In fact we show that if there exists a triple \( \alpha, \beta, \alpha + \beta \in \Phi \), then there exist \( \alpha', \beta' \in \Phi \) such that \( \alpha' + \beta' = \theta \). It suffices to consider the case \( \alpha = \alpha_i + \cdots + \alpha_j, \beta = \alpha_{j+1} + \cdots + \alpha_k, 1 \leq i \leq j \leq h \leq n \); indeed \( \alpha \in \Phi \) implies \( \alpha' = \alpha_1 + \cdots + \alpha_i \in \Phi \), whereas \( \beta \in \Phi \) implies \( \alpha_{j+1} + \cdots + \alpha_n \in \Phi \); clearly, \( \alpha' + \beta' = \theta \).

Type \( B \) is dealt with a similar argument; the details are as follows. As above, we remark that there exist two roots in \( \Phi \) which sum up to \( \theta \) if and only if \( \lambda_1 + \lambda_2 > 2n - 1 \): this can be checked directly, observing that if
$\xi + \eta = \theta$, then necessarily $\xi$ appears on the first row of $T$ and $\eta$ on the second row (or vice versa).

Then we show that a decomposition of any root $\gamma \in \Phi$ as a sum of two roots in $\Phi$ can be modified to a decomposition $\theta$ as a sum of two roots in $\Phi$. We have to consider two cases:

\[ \gamma = (\alpha_i + \cdots + \alpha_j) + (\alpha_{j+1} + \cdots + \alpha_h), \quad 1 \leq i \leq j \leq h \leq n \]

\[ \gamma = (\alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n)) + (\alpha_r + \cdots + \alpha_j), \quad 1 \leq r \leq j \leq n \]

\[ 1 \leq r \leq j, r \neq i. \]

The first case is dealt with an argument similar to the one used in type A. For the second case, we argue as follows. If $r > 1$ we note that $\alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n) \in \Phi$ implies $\xi = \alpha_1 + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n) \in \Phi$. On the other hand $\alpha_r + \cdots + \alpha_j \in \Phi$ implies $\eta = \alpha_2 + \cdots + \alpha_j \in \Phi$: therefore we obtain $\xi + \eta = \theta$, as desired. If instead $r = 1$ we may take $\xi = \alpha_1 + \cdots + \alpha_j, \eta = \alpha_2 + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n)$ and conclude as above.

Type $D$ is similar to the previous one. In type $C$ we remark that any decomposition of the highest root involves elements of the first row of $T$: by inspection one checks that there exists $\xi, \eta \in \Phi$ with $\xi + \eta = \theta$ if and only if $\tau_1^\Phi > n$; the rest of the proof is analogous to the previous cases. 

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