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Note

On endo-homology of complexes of graphs[1]

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Abstract

Let L be a subcomplex of a complex K. If the homomorphism from inclusion $i_* : H_q(L) \mapsto H_q(K)$ is an isomorphism for all $q \geqslant 0$, then we say that L and K are endo-homologous. The clique complex of a graph G, denoted by $C(G)$, is an abstract complex whose simplices are the cliques of G. The present paper is a generalization of Ivashchenko (1994) along several directions. For a graph G and a given subgraph F of G, some necessary and sufficient conditions for $C(G)$ to be endo-homologous to $C(F)$ are given. Similar theorems hold also for the independence complex $I(G)$ of G, where $I(G) = C(G^c)$, the clique complex of the complement of G. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

All graphs considered are finite and simple, all complexes considered are finite and simplicial. Undefined terms can be found in [1,2].

Given a n-complex K (n-complex means K is a n-dimensional complex) and a subcomplex L of K. For each q $(0 \leqslant q \leqslant n)$, we use $H_q(K, A)$ and $H_q(K, L; A)$ to denote the qth homology group of K and the qth relative homology group of K modulo L, respectively, where A is the coefficient group, which may be the group of integers or any other abelian group. Without confussion, we briefly denote $H_q(K, A)$ and $H_q(K, L; A)$ by $H_q(K)$ and $H_q(K, L)$, respectively.

The augmented complex [4] of complex K, denoted by K^+, is a complex obtained from K by adding a -1-dimensional simplex to K, which is the face of every simplex

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of K and we have

$$H_{-1}(K^+)=0, \qquad H_0(K)\cong A+H_0(K^+), \qquad H_q(K^+)=H_q(K) \quad (q>0)$$

when K is not empty and

$$H_{-1}(K^+)\cong A, \qquad H_q(K^+)=H_q(K) \quad (q\geqslant 0)$$

when K is empty [4].

A complex K is said to be acyclic, if $H_q(K^+)=0$ for each $q\geqslant -1$. Obviously, K is acyclic, if and only if $H_q(K)=0$ $(q\geqslant 1)$ and $H_0(K)\cong A$. Thus, an acyclic complex must be nonempty and connected.

Given a graph $G=(V(G),E(G))$, where $V(G)$ and $E(G)$ denote the vertex-set and edge-set of G, respectively. $G[S]$ denotes the subgraph of G induced by $S\subset V(G)$. $N_G(u)$ denotes the set of vertices adjacent to u in G. Without confusion, we still use $N_G(u)$ to denote the subgraph of G induced by $N_G(u)$. $S\subset V(G)$ is called an independent set of G if $E(G[S])=\emptyset$. A subgraph H of G is called a clique of G if H is a complete graph. We use G^c to denote the complement graph of G.

There are various abstract complexes [2] related to a simple graph [1] G. Notable are the following:

1. The neibourhood complex [5] of G, each of whose simplices is formed by the neighbour set of a vertex of G.
2. The independence complex of G, denoted by $I(G)$, whose simplices are the independent sets of G.
3. The clique complex of G, denoted by $C(G)$, whose simplices are the cliques of G.

We are interested in the last two kinds by obvious reason that isomorphic graphs have isomorphic clique complexes and vice versa [8]. Since $C(G)=I(G^c)$ and $I(G)=C(G^c)$, we need only to study the clique complexes of graphs. The results on independence complexes can be obtained analogously.

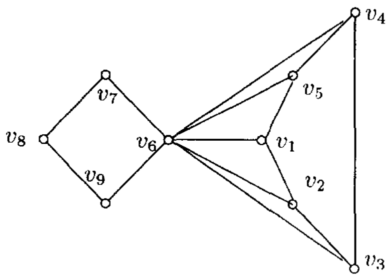
The present paper may be considered as a generalization of [3] along several directions. In [3], the homology groups of a graph G are defined, which, in fact, are the homology groups of the clique complex $C(G)$ of G. For convenience, we do not make distinction between an abstract complex and its geometrical realization as a simplicial complex K in *Euclidean* spaces, so we may apply the usual theorems about simplicial complex [2,4] to $I(G)$ and $C(G)$ freely.

Let L be a subcomplex of a complex K. If the homomorphism from inclusion

$$i_*: H_q(L)\mapsto H_q(K)$$

is an isomorphism for all q, then we say that L and K are *endo-homologous* to each other. In general, if the homology groups of K and L are isomorphic, i.e., $H_q(L)\cong H_q(K)$ $(q\geqslant 0)$, then we say that L and K are homologous. Obviously, if L and K are endo-homologous, they are surely homologous; but the converse may not be true. For example, let G be the graph shown in the following figure and cycle

$F = v_1v_2v_3v_4v_5v_1$ be a subgraph of G. Then $C(F)$ and $C(G)$ are homologous but not endo-homologous. In fact, i_* maps the 1-*cycle* of $C(F)$ to zero.



2. Main theorems

Theorem 1. *Let F be an induced subgraph of a given graph G. Graph $\tilde{G}$ is obtained from G by adding a new vertex v and joining it to each vertex of $V(F)$. Then, $C(F)$ and $C(G)$ are endo-homologous, if and only if $C(\tilde{G})$ is acyclic.*

Theorem 1′. *Let F be an induced subgraph of a given graph G. Graph $\tilde{G}$ is obtained from G by adding a new vertex v and joining it to each vertex of $V(G)\setminus V(F)$. Then, $I(F)$ and $I(G)$ are endo-homologous, if and only if $I(\tilde{G})$ is acylic.*

Theorem 2. *Let G be a graph and v be an arbitrary vertex of G. The clique complexes $C(G-v)$ and $C(G)$ are endo-homologous, if and only if $C(N_G(v))$ is acyclic, where $N_G(v)$ denotes the subgraph of G induced by the neighbour set of v in G.*

Theorem 2′. *Let G be a graph, v be an arbitrary vertex of G and $L'(v) = G - v - N_G(v)$. The independence complexes $I(G-v)$ and $I(G)$ are endo-homologous, if and only if $I(L'(v))$ is acyclic.*

Theorem 3. *Let uv be an edge of a given graph G. The clique complexes $C(G-uv)$ and $C(G)$ are endo-homologous, if and only if $C(N_G(v)\cap N_G(u))$ is acyclic, where $N_G(v)\cap N_G(u)$ denotes the subgraph of G induced by the common neighbour set of v and u in G.*

Theorem 3′. *Let u and v be two non-adjacent vertices of a given graph G. The independence complexes $I(G+uv)$ and $I(G)$ are endo-homologous, if and only if $I(L'_G(v)\cap L'_G(u))$ is acyclic.*

3. Some useful lemmas

To prove our main theorems, we need the following lemmas.

Lemma 1 [4,6,7]. *Let L be a subcomplex of complex K. Then L and K are endo-homologous if and only if*

$$H_q(K,L) \cong 0, \quad q \geqslant 0.$$

Proof. This is a direct consequence of the exactness of the homology sequence of K and L. □

Lemma 2 [2,4]. *Let L be a subcomplex of a complex K, v be a single 0-simplex which is not in K. $v \circ L$ represents the cone on L with vertex v, $\tilde{K} = K \cup v \circ L$. Then*

$$H_q(\tilde{K}^+) \cong H_q(K,L) \quad q \geqslant 0.$$

Lemma 3 [2,4]. *Let L be a complex and v be a single 0-simplex which is not in L. Then*

$$H_q(v \circ L, L) \cong H_{q-1}(L^+) \quad q \geqslant 0.$$

Lemma 4. *Let L be a complex, u and v be distinct 0-simplices which are not in L. Then*

$$H_q(u \circ L, L) \cong H_q((u \circ L \cup v \circ L)^+), \quad q \geqslant 0.$$

Proof. Let $K = u \circ L$, $\tilde{K} = K \cup v \circ L$. By Lemma 2, for $q \geqslant 0$,

$$H_q((u \circ L \cup v \circ L)^+) = H_q(\tilde{K}^+) \cong H_q(K,L) = H_q(u \circ L, L).$$

Lemma 5. *Let K be a cone of dimension n and L be a nonempty subcomplex of K. Then*

$$\partial_* : H_q(K,L) \mapsto H_{q-1}(L^+), \quad q \geqslant 0$$

are all isomorphisms.

Proof. We know that the following homology sequence is exact provided L is non-empty.

$$0 \mapsto H_n(L) \overset{i_*}{\mapsto} H_n(K) \overset{j_*}{\mapsto} H_n(K,L) \overset{\partial_*}{\mapsto} H_{n-1}(L) \overset{i_*}{\mapsto} H_{n-1}(K)$$

$$\overset{j_*}{\mapsto} \cdots \overset{\partial_*}{\mapsto} H_q(L) \overset{i_*}{\mapsto} H_q(K) \overset{j_*}{\mapsto} H_q(K,L) \overset{\partial_*}{\mapsto} H_{q-1}(L) \overset{i_*}{\mapsto} \cdots$$

$$\overset{j_*}{\mapsto} H_1(K,L) \overset{\partial_*}{\mapsto} H_0(L^+) \overset{i_*^+}{\mapsto} H_0(K^+) \overset{j_*^+}{\mapsto} H_0(K,L) \mapsto 0, \quad (1)$$

where i_q* are homomorphisms from inclusion; j_q*, homomorphisms from j; ∂_q*, homomorphisms from ∂; and i_*^+ and j_*^+ are reduced homomorphisms.

Since K is a cone, we have for each $q \geqslant 0$, $H_q(K^+) \cong 0$. Since L is nonempty, we have $H_{-1}(L^+) = 0$.

From these facts and the exactness of the homology sequence (1), it is easy to see that our lemma holds. □

4. Proofs of main theorems

Proof of Theorem 1. Let $K = C(G)$, $L = C(F)$ and $\tilde{K} = C(\tilde{G})$. Because $\tilde{G} = G + v$, where $N_{\tilde{G}}(v) = V(F)$, then

$$\tilde{K} = C(G) \cup v \circ C(F) = K \cup v \circ L. \tag{2}$$

By Lemma 2,

$$H_q(\tilde{K}^+) \cong H_q(K, L), \quad q \geqslant 0,$$

i.e.,

$$H_q(C(\tilde{G})^+) \cong H_q(C(G), C(F)), \quad q \geqslant 0. \tag{3}$$

By Lemma 1 and (3), $C(F)$ and $C(G)$ are endo-homologous if and only if

$$0 \cong H_q(C(G), C(F)) \cong H_q(C(\tilde{G}^+)), \quad q \geqslant 0.$$

The theorem immediately follows because that $C(\tilde{G})$ is a nonempty complex. □

Proof of Theorem 2. By Lemma 1, $C(G - v)$ and $C(G)$ are endo-homologous if and only if

$$H_q(C(G), C(G - v)) \cong 0, \quad q \geqslant 0. \tag{4}$$

Let $\overline{N_G(v)} = N_G(v) + v$. Then, $C(\overline{N_G(v)}) = v \circ C(N_G(v))$, $C(G) = C(G - v) \cup C(\overline{N_G(v)})$. In other words, we have

$$C(G) = C(G - v) \cup v \circ C(N_G(v))$$

and

$$C(G - v) \cap v \circ C(N_G(v)) = C(N_G(v)).$$

According to *Excision-Theorem* and Lemma 3,

$$H_q(C(G), C(G - v)) = H_q(v \circ C(N_G(v)), C(N_G(v)))$$
$$\cong H_{q-1}(C(N_G(v))^+), \quad q \geqslant 0. \tag{5}$$

Combining (4) and (5), $C(G-v)$ and $C(G)$ are endo-homologous if and only if

$$H_q(C(N_G(v))^+)\cong 0, \quad q\geqslant -1,$$

i.e., if and only if $C(N_G(v))$ is an acyclic complex. □

Proof of Theorem 3. Let $F=C(N_G(u)\cap N_G(v))$. It is obvious that

$$C(G)=C(G-uv)\cup C(\overline{N_G(u)}\cap\overline{N_G(v)}), \tag{6}$$

$$C(G-uv)\cap C(\overline{N_G(u)}\cap\overline{N_G(v)})=u\circ F\cup v\circ F \tag{7}$$

and

$$C(\overline{N_G(u)}\cap\overline{N_G(v)})=u\circ(v\circ F)=v\circ(u\circ F). \tag{8}$$

According to Excision-Theorem, Lemma 5 and (6)–(8),

$$H_q(C(G),C(G-uv))\cong H_q(u\circ v\circ F,u\circ F\cup v\circ F)$$

$$\cong H_{q-1}((u\circ F\cup v\circ F)^+), \quad q\geqslant 0 \tag{9}$$

By Lemmas 3 and 4,

$$H_q((u\circ F\cup v\circ F)^+)\cong H_{q-1}(F^+), \quad q\geqslant 0. \tag{10}$$

Combining (9) and (10), we have

$$H_q(C(G),C(G-uv))\cong H_{q-2}(F^+), \quad q\geqslant 1. \tag{11}$$

By Lemma 1, $C(G-uv)$ and $C(G)$ are endo-homologous if and only if

$$H_q(C(G),C(G-uv))\cong 0, \quad q\geqslant 0. \tag{12}$$

Noticing that, when F is nonempty, $u\circ F\cup v\circ F$ is connected and $H_0(C(G), C(G-uv))\cong 0$ by (9), our theorem immediately follows from (11) and (12). □

Note: The clique complexes of contractible graphs defined in [3] are all proved to be acyclic. Among the four kinds of contractible transformations in [3] which do not change the homology groups of graphs, the first two transformations, i.e., deleting or gluing of a vertex v, are to transform G to $G-v$ or, conversely, transform $G-v$ to G, when the induced subgraph $N_G(v)$ in G is contractible. The other two transformations are applied to G and $G-uv$, when $N_G(u)\cap N_G(v)$ is a contractible graph. Thus, the main theorems in [3] are special cases of the sufficient parts of Theorems 2 and 3 above.

Finally, we propose the following.

Conjecture. The clique complexes of graphs G and H are homologous, if and only if there exists a finite number of graphs,

$$G=F_0,F_1,\ldots,F_n=H \quad (n\geqslant 1),$$

such that, for each i $(0 \leqslant i \leqslant n-1)$, either F_i is isomorphic to F_{i+1}, or one of the two graphs F_i and F_{i+1} is a subgraph of the other with $C(F_i)$ being endo-homologous to $C(F_{i+1})$. Similar conjecture can also be made for the independence complexes of G and H.

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