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Translation-invariant maps and applications

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ABSTRACT

In this paper we study translation-invariant maps on a linear space. A method for estimating quantities induced by such maps is presented. Applications for differences of operators, commutators, G-contractive maps and Grüss type inequalities are also given.

1. Introduction and summary

The purpose of the present paper is to give a unified framework for deriving upper bounds for some quantities induced by translation-invariant maps. To this end we employ vectorial intervals induced by two preorders on a linear space, and radii of families of maps at a point (to be defined in Section 2).

In Section 2 we begin with some relevant notation and terminology. In Theorem 2.2 we present a general framework that will be used throughout the paper. This result contains a set of assumptions leading to the required estimate. A particular case of Theorem 2.2 related to a compact group of linear operators is demonstrated in Theorem 2.5. In the second part of Section 2 we deal with so-called GIC preorders. In Theorem 2.9 we extend a result of Li and Mathias [20, Theorem 2] from the matrix case to a general setting of an inner product space equipped with a GIC preorder.

In Section 3 we focus on the differences of two operators, e.g., on commutators. Here we generalize recent results of Wang and Du [35, Corollary 4] and of Bhatia and Kittaneh [4, Theorem 4].

Section 4 is devoted to Grüss type inequalities. In Theorem 4.2 we apply vectorial intervals induced by cone preorders to get a generalization of some results of Dragomir (see [5, Theorem 1] and [7, Theorem 2.5]). Applications for L^2-functions are given in Corollary 4.5. The discrete case is presented in Corollary 4.7. Finally, Corollary 4.8 extends a result of Renaud [29, Theorem 2.1].

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2. Upper bounds for translation-invariant maps

Unless stated otherwise, throughout the paper X, Y and Z are linear spaces, and

$$\Phi : Y \to \mathbb{R} \quad \text{and} \quad \Psi : Z \to \mathbb{R} \quad \text{are real maps.}$$

It is also assumed that Z is equipped with two preorders $\prec_1$ and $\prec_2$.

Given two vectors $\alpha, \beta \in Z$, the *vectorial interval* $[\alpha, \beta]_{\prec_{1,2}}$ induced by $\prec_1$ and $\prec_2$ (in short, $\prec_{1,2}$*-interval*) is defined by (cf. [12, pp. 120–124])

$$[\alpha, \beta]_{\prec_{1,2}} = [\alpha, \beta] = \{z \in Z\colon\ \alpha \prec_1 z \prec_2 \beta\}.$$

The subscript is sometimes omitted, if not relevant.

The preorder $\prec_k$ $(k = 1, 2)$ is said to be *translation-invariant* with respect to (w.r.t.) the vector $z_0 \in Z$, if $z \prec_k v$ implies $z - z_0 \prec_k v - z_0$ for all $z, v \in Z$.

Given a point $x \in X$ and a (nonempty) subset M in X, a map $L : X \to Y$ is said to be *translation-invariant* at x w.r.t. M if $L(x - x_0) = Lx$ for all $x_0 \in M$.

Let $\mathcal{L} = \{L_i\colon\ i \in \mathcal{I}\}$ be a family of maps $L_i : X \to Z$. The $\Psi\mathcal{L}$*-radius* is the map $r_{\Psi\mathcal{L}} : X \to \mathbb{R}$ defined by

$$r_{\Psi\mathcal{L}}(x) = \sup_{i \in \mathcal{I}} \Psi L_i(x) < \infty \quad \text{for } x \in X.$$

The set

$$\mathcal{L}(x) = \big\{L_i(x)\colon\ i \in \mathcal{I}\big\}$$

is called the $\mathcal{L}$*-range* of $x \in X$.

Example 2.1 *(Numerical radius and numerical range).* Let $X = Y = \mathbb{M}_n$ be the linear space of $n \times n$ complex matrices. Consider $\Psi = |\cdot|$ = the modulus on $\mathbb{C}$, and

$$\mathcal{I} = \big\{a \in \mathbb{C}^n\colon\ a^*a = 1\big\} \quad \text{and} \quad \mathcal{L} = \big\{L_a(\cdot) = a^*(\cdot)a\colon\ a \in \mathcal{I}\big\},$$

where $(\cdot)^*$ stands for the conjugate transpose. It is easily seen that

$$r_{\Psi\mathcal{L}}(x) = \max\big\{|a^*xa|\colon\ a \in \mathbb{C}^n,\ a^*a = 1\big\}$$

is the *numerical radius* $w(x)$ of a matrix $x \in \mathbb{M}_n$ (see [14, p. 7], [19, Section 14]). It is known that $w(\cdot)$ is a (weakly unitarily invariant) norm on $\mathbb{M}_n$, and

$$w(x) \leqslant \|x\|_\infty \leqslant 2w(x) \quad \text{for } x \in \mathbb{M}_n, \tag{1}$$

where $\|\cdot\|_\infty$ denotes the operator norm on $\mathbb{M}_n$ [2, p. 4]. If x is a normal matrix then $\|x\|_\infty = w(x)$ [29, p. 96].

Furthermore,

$$\mathcal{L}(x) = \big\{a^*xa\colon\ a \in \mathbb{C}^n,\ a^*a = 1\big\}$$

is the *numerical range* $W(x)$ of x [19, Section 1] (also known as the *field of values* of x [14, Chapter 1]).

Take $\Phi = \|\cdot\|_\infty$. For a linear map $L : \mathbb{M}_n \to \mathbb{M}_n$ and $\eta_0 = 2$, it follows from (1) that

$$\|Lx\|_\infty \leqslant \|L\|\|x\|_\infty \leqslant \|L\|\eta_0 w(x) \quad \text{for } x \in X = \mathbb{M}_n.$$

(If x is a normal matrix then $\eta_0 = 1$.) Putting $\eta = \eta_0\|L\|$ yields

$$\Phi Lx \leqslant \eta r_{\Psi\mathcal{L}}(x) \quad \text{for } x \in X. \tag{2}$$

In this section, we are interested in an upper bound for the expression ΦLx, where x is a point in X, and $\Phi : Y \to \mathbb{R}$ is a given map and $L : X \to Y$ is a translation-invariant map at x.

In our considerations, the crucial assumption is an inequality of type (2) (see (4), (10), (22), (40)). It strongly depends on Φ. In some cases, it is a consequence of a certain nontrivial estimate (see e.g., (1)). On the other hand, (2) holds automatically when $\Phi = r_{\Psi\mathcal{L}}$, $\eta = 1$ and L is a contractive map w.r.t. $r_{\Psi\mathcal{L}}$.

For the statement of our results we need some notation.

Given a family $\mathcal{L} = \{L_i\colon\ i \in \mathcal{I}\}$ consisting of maps $L_i : X \to Z$ and given a (nonempty) set $M \subset X$, a vector $z_0 \in Z$ is said to be $\mathcal{L}, M$*-admissible* at $x \in X$, if there exists $x_0 \in M$ such that

$$L_i(x) - z_0 = L_i(x - x_0) \quad \text{for } i \in \mathcal{I}.$$

We denote

$$\mathrm{Adm}(\mathcal{L}, M, x) = \{z_0 \in Z\colon\ z_0 \text{ is } \mathcal{L}, M\text{-admissible at } x \in X\}.$$

If $r_0 \in \mathbb{R}$, $x_0 \in X$ and $\Theta : X \to \mathbb{R}$ is a map, then the set

$$B_\Theta(x_0, r_0) = \{x \in X\colon\ \Theta(x - x_0) \leqslant r_0\}$$

is called Θ-*ball* of radius r_0 centered at x_0.

We are now in a position to give Theorem 2.2. This result provides general assumptions of type (i), (ii) and/or (iii) leading to estimate (iv). Interpretations and applications of the implications (i) ⇒ (iv) and (ii) ⇒ (iv) are presented in Sections 3 and 4.

Theorem 2.2. *With the above notation, let X, Y and Z be linear spaces and let $\Phi : Y \to \mathbb{R}$ and $\Psi : Z \to \mathbb{R}$ be maps. Assume that $\prec_1$ and $\prec_2$ are preorders on Z such that*

$$-v \prec_1 z \prec_2 v \quad \text{implies} \quad \Psi(z) \leqslant \Psi(v) \quad \text{for } z, v \in Z. \tag{3}$$

Let $x \in X$ and M be a subset of X. Suppose that $L : X \to Y$ is a translation-invariant map at x w.r.t. M, and $\mathcal{L} = \{L_i\colon\ i \in \mathcal{I}\}$ is a family of maps $L_i : X \to Z$, and $\eta > 0$ is a constant such that

$$\Phi(Lv) \leqslant \eta r_{\Psi\mathcal{L}}(v) \quad \text{for } v \in x - M. \tag{4}$$

Consider the following four statements (i)–(iv):

(i) *For some vectors $\alpha, \beta \in Z$, with notation $z_0 = \frac{1}{2}(\alpha + \beta)$ and $r_0 = \Psi(\frac{1}{2}(\beta - \alpha))$, the following conditions* (a)–(c) *hold true*:
  (a) *the $\mathcal{L}$-range of x is included in the $\prec_{1,2}$-interval $[\alpha, \beta] \subset Z$,*
  (b) *the vector z_0 is $\mathcal{L}, M$-admissible at x,*
  (c) *the preorders $\prec_1$ and $\prec_2$ are translation-invariant w.r.t. the vector z_0.*

(ii) *The $\mathcal{L}$-range of x is included in certain Ψ-ball of radius r_0 centered at a point in* $\mathrm{Adm}(\mathcal{L}, M, x)$.

(iii) *The vector x lies in certain $r_{\Psi\mathcal{L}}$-ball of radius r_0 centered at a point in M.*

(iv) $\Phi Lx \leqslant \eta r_0$.

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv)*.*

Remark 2.3. In Theorem 2.2, statement (i) simplifies if the $\prec_{1,2}$-interval $[\alpha, \beta]$ is symmetric, i.e., $\alpha = -\delta$ and $\beta = \delta$ for some $\delta \in Z$. In this event, $z_0 = 0$ and $r_0 = \Psi(\delta)$. Then conditions (b) and (c) holds automatically, and can be dropped.

Proof of Theorem 2.2. (i) ⇒ (ii). On account of (a) we have $\mathcal{L}(x) \subset [\alpha, \beta]_{\prec_{1,2}}$, so $\alpha \prec_1 L_i(x) \prec_2 \beta$ for $i \in \mathcal{I}$. Hence, by (c),

$$-\frac{1}{2}(\beta - \alpha) = \alpha - \frac{1}{2}(\alpha + \beta) \prec_1 L_i(x) - \frac{1}{2}(\alpha + \beta) \prec_2 \beta - \frac{1}{2}(\alpha + \beta) = \frac{1}{2}(\beta - \alpha) \quad \text{for } i \in \mathcal{I}.$$

Therefore, by (3) applied to $v = \frac{1}{2}(\beta - \alpha)$ and $z = L_i(x) - \frac{1}{2}(\alpha + \beta)$, we deduce that

$$\Psi\left(L_i(x) - \frac{1}{2}(\alpha + \beta)\right) \leqslant \Psi\left(\frac{1}{2}(\beta - \alpha)\right) \quad \text{for } i \in \mathcal{I},$$

that is $\Psi(L_i(x) - z_0) \leqslant r_0$ for $i \in \mathcal{I}$. This means $\mathcal{L}(x) \subset B_\Psi(z_0, r_0)$, and z_0 is $\mathcal{L}, M$-admissible at x (see (b)). Thus (ii) is proved.

(ii) ⇒ (iii). We have $\mathcal{L}(x) \subset B_\Psi(z_0, r_0)$ for some $z_0 \in \mathrm{Adm}(\mathcal{L}, M, x)$. Therefore $\Psi(L_i(x) - z_0) \leqslant r_0$ for $i \in \mathcal{I}$. Since z_0 is $\mathcal{L}, M$-admissible at x, there exists $x_0 \in M$ such that $L_i(x) - z_0 = L_i(x - x_0)$ for all $i \in \mathcal{I}$. Consequently, $\Psi L_i(x - x_0) \leqslant r_0$ for $i \in \mathcal{I}$. Hence

$$r_{\Psi\mathcal{L}}(x - x_0) = \sup_{i \in \mathcal{I}} \Psi L_i(x - x_0) \leqslant r_0.$$

This means $x \in B_{r_{\Psi\mathcal{L}}}(x_0, r_0)$, as desired.

(iii) ⇒ (iv). Because $x \in B_{r_{\Psi\mathcal{L}}}(x_0, r_0)$ for some $x_0 \in M$, we have $r_{\Psi\mathcal{L}}(x - x_0) \leqslant r_0$, and further

$$\eta r_{\Psi\mathcal{L}}(x - x_0) \leqslant \eta r_0. \tag{5}$$

Clearly, by $Lx = L(x - x_0)$, we get

$$\Phi Lx = \Phi L(x - x_0). \tag{6}$$

In addition, by (4),

$$\Phi L(x - x_0) \leqslant \eta r_{\Psi\mathcal{L}}(x - x_0). \tag{7}$$

Now, combining (6), (7) and (5) yields (iv), completing the proof. □

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let G be a compact subgroup of the orthogonal/unitary group on X according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. So, $\langle ga, b\rangle = \langle a, g^{-1}b\rangle$ for $a, b \in X$ and $g \in G$. Given vectors $c, x \in X$, we define

$$r_{G,c}(x) = \max_{g\in G}\big|\langle x, gc\rangle\big| = \max_{g\in G}\big|\langle gc, x\rangle\big|,$$
$$W_{G,c}(x) = \big\{\langle x, gc\rangle\colon\ g \in G\big\},$$
$$M_G(X) = \{e \in X\colon\ ge = e \text{ for } g \in G\}.$$

The quantity $r_{G,c}(x)$ is called *$G(c)$-radius* of x [21,27]. The set $W_{G,c}(x)$ is said to be *$G(c)$-range* of x. The subspace $M_G(X)$ consists of fixed points for all operators in G.

Example 2.4 *(C-numerical radius and C-numerical range).* Let $X = Y = \mathbb{M}_n$ be equipped with the trace inner product $\langle x, y\rangle = \operatorname{tr} xy^*$ for $x, y \in \mathbb{M}_n$, where $(\cdot)^*$ stands for the conjugate transpose. Take $G = \{g = u^*(\cdot)u\colon\ u \in \mathbb{U}_n\}$ with $\mathbb{U}_n$ denoting the group of $n \times n$ unitary matrices. Then $M_G(X) = \operatorname{span} I_n$, where I_n denotes the $n \times n$ identity matrix.

Fix arbitrarily $c \in \mathbb{M}_n$. It is not hard to check that

$$r_{G,c^*}(x) = \max\big\{\big|\operatorname{tr} u^*cux\big|\colon\ u \in \mathbb{U}_n\big\}$$

is the *C-numerical radius* $r_c(x)$ of a matrix $x \in \mathbb{M}_n$, and

$$W_{G,c^*}(x) = \big\{\operatorname{tr} u^*cux\colon\ u \in \mathbb{U}_n\big\}$$

is the *C-numerical range* $W_c(x)$ of x (see [14, p. 81], [19, Sections 2 and 15]).

We now interpret Theorem 2.2 in terms of the $G(c)$-radius and $G(c)$-range of $x \in X$. To do this, we consider the following specification:

$$Z = \mathbb{F} = \mathbb{R} \quad\text{or}\quad \mathbb{C}, \qquad M \subset M_G(X), \qquad \Psi = |\cdot| \quad \text{is the modulus on } \mathbb{F}, \qquad \mathcal{I} = G \quad\text{and}\quad \mathcal{L} = \big\{L_g = \langle\cdot, gc\rangle\colon\ g \in G\big\},$$
$$\text{for } \alpha, \beta \in \mathbb{F} \text{ and } k = 1, 2, \text{ we define} \quad \alpha \prec_k \beta \quad \text{iff} \quad \beta - \alpha \in \mathbb{R}_+. \tag{8}$$

Then the $\prec_{1,2}$-interval is given by

$$[\alpha, \beta] = \{z \in \mathbb{F}\colon\ \operatorname{Re}\alpha \leqslant \operatorname{Re} z \leqslant \operatorname{Re}\beta,\ \operatorname{Im}\alpha = \operatorname{Im} z = \operatorname{Im}\beta\}. \tag{9}$$

We denote $\operatorname{span} e = \{te\colon\ t \in \mathbb{F}\}$ and $\operatorname{span}_R e = \{te\colon\ t \in \mathbb{R}\}$.

Theorem 2.5. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let G be a compact subgroup of the orthogonal/unitary group on X. Let $c \in X$ and $e \in M_G(X)$ be such that $\langle e, c\rangle \neq 0$. Assume that $\prec_1$ and $\prec_2$ are preorders on $\mathbb{F}$ given by* (8)*.*

Let $x \in X$ and $M = \operatorname{span} e$. Suppose that $\Phi : Y \to \mathbb{R}$ is a real map on a linear space Y, and $L : X \to Y$ is a translation-invariant map at x w.r.t. M, and $\eta > 0$ is a constant such that

$$\Phi(Lv) \leqslant \eta r_{G,c}(v) \quad \text{for } v \in x - M. \tag{10}$$

Consider the following four statements (i)–(iv):

- (i) *For some $\alpha, \beta \in \mathbb{F}$, the $G(c)$-range of x is included in the $\prec_{1,2}$-interval $[\alpha, \beta] \subset \mathbb{F}$, with notation $z_0 = \frac{1}{2}(\alpha + \beta)$ and $r_0 = \frac{1}{2}(\beta - \alpha)$.*
- (ii) *The $G(c)$-range of x is included in certain $|\cdot|$-ball of radius $r_0 \geqslant 0$.*
- (iii) *The vector x lies in certain $r_{G,c}$-ball of radius $r_0 \geqslant 0$ centered at a point in $M = \operatorname{span} e$.*
- (iv) *$\Phi Lx \leqslant \eta r_0$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)*.*

Proof. It follows that each $z_0 \in Z = \mathbb{F}$ is $\mathcal{L}, M$-admissible at $x \in X$, i.e., $\operatorname{Adm}(\mathcal{L}, M, x) = \mathbb{F}$. To see this, we set $t = z_0/\langle e, c\rangle$ and $x_0 = te$, where $e \in M_G(X)$ and $\langle e, c\rangle \neq 0$. Then for any $g \in G$ we can write

$$\begin{aligned} L_g(x) - z_0 &= \langle x, gc\rangle - t\langle e, c\rangle = \langle x, gc\rangle - t\big\langle g^{-1}e, c\big\rangle \\ &= \langle x, gc\rangle - t\langle e, gc\rangle = \langle x, gc\rangle - \langle x_0, gc\rangle = \langle x - x_0, gc\rangle = L_g(x - x_0). \end{aligned}$$

Thus $z_0 \in \operatorname{Adm}(\mathcal{L}, M, x)$, as claimed.

Moreover, the preorders $\prec_1$ and $\prec_2$ are translation-invariant w.r.t. any vector in $\mathbb{F}$ (see (8)). In addition, for $z, v \in \mathbb{F}$ the condition $-v \prec_1 z \prec_2 v$ implies $z, v \in \mathbb{R}$, and therefore (3) is fulfilled for the modulus $\Psi = |\cdot|$.

Now, the validity of the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follows from Theorem 2.2 applied to the specification (8). $\square$

Remark 2.6. In Theorem 2.5, if in addition $\frac{\alpha+\beta}{2}/\langle c, e\rangle \in \mathbb{R}$ then $\operatorname{span} e$ can be replaced by $\operatorname{span}_R e$.

In the remainder of this section, we deal with so-called GIC preorders.

Suppose that $(X, \langle\cdot,\cdot\rangle)$ is a finite-dimensional *real* inner product space and G is a closed subgroup of the orthogonal group $O(X)$ on X. For a vector $x \in X$, by $\operatorname{conv} Gx$ we denote the convex hull of the G-orbit $Gx = \{gx\colon\ g \in G\}$. Given two vectors $x, y \in X$, we write

$$y \prec_G x \quad \text{iff} \quad y \in \operatorname{conv} Gx. \tag{11}$$

The relation $\prec_G$ is a G-invariant preorder on X called *G-majorization*.

It is known [12, Theorem 1] that for $x, y \in X$

$$y \prec_G x \quad \text{iff} \quad m_{G,c}(y) \leqslant m_{G,c}(x) \quad \text{for } c \in X, \tag{12}$$

where

$$m_{G,c}(v) = \max_{g\in G}\langle gc, v\rangle, \quad v \in X.$$

Since $G \subset O(X)$ we have $m_{G,c}(v) = m_{G,v}(c)$ for all $c, v \in X$ [12, p. 114]. The function $m_{G,v}(\cdot)$ is the *support function* of the set $\operatorname{conv} Gv$ [30].

Following [9,10], we say that the G-majorization $\prec_G$ is a *group induced cone* (*GIC*) *preorder* if there exists a closed convex cone $D \subset X$ such that

(A1) $D \cap Gv$ is not the empty set for each $v \in X$,
(A2) $\max_{g\in G}\langle gc, v\rangle = \langle c, v\rangle$ for $c, v \in D$.

See [9,10,17,18] for some important examples and [31,32, Examples 1 and 2] for an interpretation of (A1)–(A2) in Lie theory.

Under (A1)–(A2), for each $v \in X$ there exist elements $v_\downarrow, v_\uparrow \in X$ such that

$$\{v_\downarrow\} = D \cap Gv \quad \text{and} \quad \{v_\uparrow\} = -D \cap Gv$$

(see [23, p. 14]). Then $m_{G,c}(v) = \langle c_\downarrow, v_\downarrow\rangle$ for $c, v \in X$, and (12) simplifies to

$$\begin{aligned} y \prec_G x \quad &\text{iff} \quad \langle c, y_\downarrow\rangle \leqslant \langle c, x_\downarrow\rangle \quad \text{for } c \in D, \\ &\text{iff} \quad \langle gc, y\rangle \leqslant \langle c, x_\downarrow\rangle \quad \text{for } c \in D \text{ and } g \in G. \end{aligned} \tag{13}$$

Furthermore,

$$\langle c, v_\uparrow\rangle \leqslant \langle gc, v\rangle \leqslant \langle c, v_\downarrow\rangle \quad \text{for } v \in X,\ c \in D \text{ and } g \in G \tag{14}$$

(see [31]). In other words, for each $c \in D$ the $G(c)$-range of $v \in X$ is contained in $[\alpha, \beta]$, where $\alpha = \langle c, v_\uparrow\rangle$ and $\beta = \langle c, v_\downarrow\rangle$.

Example 2.7. (See [9].) Given $x, y \in \mathbb{R}^n$, x is said to *weakly majorize* y, written $y \prec_w x$, if the sum of k largest entries of y does not exceed the sum of k largest entries of x for each $k = 1, \dots, n$ [22, p. 10]. If, in addition, equality holds for $k = n$, x is said to *majorize* y, written $y \prec_m x$ [22, p. 7].

If $X = \mathbb{R}^n$ and $G = \mathbb{P}_n$ is the group of $n \times n$ permutation matrices, then the G-majorization $\prec_G$ becomes the usual majorization $\prec_m$, i.e., for $x, y \in \mathbb{R}^n$,

$$y \prec_G x \quad \text{iff} \quad y \prec_m x.$$

It is well known that $\prec_m$ is a GIC preorder with the cone $D = \{v = (v_1, \dots, v_n)^T \in \mathbb{R}^n\colon\ v_1 \geqslant \dots \geqslant v_n\}$. Here $v_\downarrow = (v_{[1]}, \dots, v_{[n]})^T$ and $v_\uparrow = (v_{(1)}, \dots, v_{(n)})^T$, where $v_{[1]} \geqslant \dots \geqslant v_{[n]}$ and $v_{(1)} \leqslant \dots \leqslant v_{(n)}$ are the entries of $v = (v_1, \dots, v_n)^T$ stated in decreasing order and in increasing order, respectively.

Example 2.8. (See [17, Example 7.4], also cf. [9, Example 2.4].) Take X to be the (real) space $\mathbb{H}_n$ of $n \times n$ Hermitian matrices with the inner product $\langle x, y\rangle = \operatorname{tr} xy$ for $x, y \in \mathbb{H}_n$. Let G be the group of all unitary similarities $u(\cdot)u^*$, where u runs over the group $\mathbb{U}_n$ of $n \times n$ unitary matrices. Then the preorder $\prec_G$ is characterized by: for $x, y \in \mathbb{H}_n$,

$$y \prec_G x \quad \text{iff} \quad \lambda(y) \prec_m \lambda(x),$$

where $\lambda(v) = (\lambda_1(v), \dots, \lambda_n(v))^T$ stands for the vector of the eigenvalues of $v \in \mathbb{H}_n$ arranged in decreasing order, i.e., $\lambda_1(v) \geqslant \dots \geqslant \lambda_n(v)$.

The G-majorization $\prec_G$ is a GIC preorder with $D = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n)\colon\ \lambda_1 \geqslant \dots \geqslant \lambda_n\}$ and

$$v_\downarrow = \operatorname{diag}\bigl(\lambda_1(v), \dots, \lambda_n(v)\bigr) \quad \text{and} \quad v_\uparrow = \operatorname{diag}\bigl(\lambda_n(v), \dots, \lambda_1(v)\bigr).$$

In fact, (A1) is the Spectral Theorem for Hermitian matrices, and (A2) is Fan–Theobald's trace inequality [11,33].

Let Y be a subspace of X. A linear operator $L : X \to Y$ is said to be G-*contractive*, if

$$Lx \prec_G x \quad \text{for } x \in X. \tag{15}$$

In light (12), (15) is equivalent to

$$m_{G,c}(Lx) \leqslant m_{G,c}(x) \quad \text{for } c, x \in X. \tag{16}$$

Therefore (4) holds for

$$\Phi = m_{G,c}(\cdot), \qquad \eta = 1, \qquad \Psi = \text{the identity on } \mathbb{R}, \qquad \mathcal{L} = \big\{\langle gc, \cdot\rangle\colon\ g \in G\big\} \quad \text{and} \quad r_{\Psi\mathcal{L}}(\cdot) = m_{G,c}(\cdot). \tag{17}$$

The next result is motivated by [20, Theorem 2].

Theorem 2.9. *Let $(X, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real inner product space. Let G be a compact subgroup of the orthogonal group on X and $D \subset X$ be a closed convex cone such that the G-majorization $\prec_G$ is a GIC preorder satisfying axioms* (A1)–(A2).

Assume that $g_0 D \subset -D$ for some $g_0 \in G$. Denote $Q = \frac{1}{2}(\mathrm{id} - g_0)$ and $D_0 = QD$. Let Y be a subspace in X such that $Y = \bigcup_{g\in G_0} gD_0$ for some subset $G_0 \subset G$.

If $L : X \to Y$ is a G-contractive operator, then

$$Lx \prec_G \frac{1}{2}(x_\downarrow - x_\uparrow) \quad \textit{for } x \in X.$$

The proof of Theorem 2.9 will be simplified if we first prove a lemma.

Lemma 2.10. *Under the assumptions of Theorem* 2.9 *for $(X, \langle\cdot,\cdot\rangle)$, G and D, assume that $g_0 D \subset -D$ for some $g_0 \in G$. Denote $Q = \frac{1}{2}(\mathrm{id} - g_0)$.*

Then

(i) *g_0 is an involution on* $\operatorname{span} D$,
(ii) *for $c \in D$ and $x \in X$ we have $\frac{1}{2}(\alpha + \beta) = 0$, where $\alpha = \langle Qc, x_\uparrow\rangle$ and $\beta = \langle Qc, x_\downarrow\rangle$.*

Proof. (i). Since $-g_0 D \subset D$, for any $x_0 \in \operatorname{ri} D$ (the relative interior of D), we find that $g_0^2 x_0 = -g_0(-g_0 x_0) \in D$. By [23, Lemma 2.1], we deduce that $g_0^2|_{\operatorname{span} D} = \mathrm{id}|_{\operatorname{span} D}$, as desired.

(ii). Let $c \in D$ and $x \in X$. Define $P = \frac{1}{2}(\mathrm{id} + g_0)$. Notice that $x_\uparrow = g_0 x_\downarrow$. For this reason $\frac{1}{2}(x_\downarrow - x_\uparrow) = Qx_\downarrow$ and $\frac{1}{2}(x_\downarrow + x_\uparrow) = Px_\downarrow$. Since $g_0 \in G$ is an orthogonal operator, we have $g_0^T = g_0^{-1}$, where g_0^T denotes the dual operator of g_0 w.r.t. the inner product $\langle\cdot,\cdot\rangle$. Therefore we can write

$$\begin{aligned}\frac{1}{2}(\alpha + \beta) &= \frac{1}{2}\langle Qc, x_\downarrow + x_\uparrow\rangle = \langle Qc, Px_\downarrow\rangle = \big\langle P^T Qc, x_\downarrow\big\rangle = \frac{1}{4}\big\langle(\mathrm{id} + g_0)^T(\mathrm{id} - g_0)c, x_\downarrow\big\rangle \\ &= \frac{1}{4}\big\langle\big(\mathrm{id} + g_0^{-1}\big)(\mathrm{id} - g_0)c, x_\downarrow\big\rangle = \frac{1}{4}\big\langle\big(g_0^{-1} - g_0\big)c, x_\downarrow\big\rangle = \frac{1}{4}\big\langle g_0^{-1}\big(\mathrm{id} - g_0^2\big)c, x_\downarrow\big\rangle = 0,\end{aligned}$$

the last equality being a consequence of the proved part (i) of Lemma 2.10. □

Proof of Theorem 2.9. Fix arbitrarily $x \in X$. Since $x_\downarrow \in D$ and $x_\uparrow \in -D$, we get $\frac{1}{2}(x_\downarrow - x_\uparrow) \in D$. So, by (13), we have to prove that

$$\big\langle c, (Lx)_\downarrow\big\rangle \leqslant \left\langle c, \frac{1}{2}(x_\downarrow - x_\uparrow)\right\rangle \quad \text{for } c \in D. \tag{18}$$

Observe that $D_0 = QD \subset D$. Making use of Lemma 2.10(i), it is not hard to verify that $Q^2|_D = Q|_D$. It can be shown in a similar way as in the proof of Lemma 2.10 that $\langle Pc, Qy\rangle = 0$ for $c, y \in D$. Moreover, a simple computation shows that $\langle Qc, y\rangle = \langle c, Qy\rangle$ for $c \in D$ and $y \in \operatorname{span} D$, because $g_0^T y = g_0^{-1} y = g_0 y$ by $g_0^2 y = y$.

Since $Lx \in Y = \bigcup_{g\in G_0} gD_0$, we obtain $Lx = gy_0$ for some $g \in G_0$ and $y_0 \in D_0$. Hence $(Lx)_\downarrow = y_0 = g^{-1}Lx$ by $D_0 \subset D$. For this reason $Q(Lx)_\downarrow = (Lx)_\downarrow$, because $Qy_0 = y_0$ by $Q^2|_D = Q|_D$ and $y_0 \in D_0 = QD$. Therefore $0 = \langle Pc, (Lx)_\downarrow\rangle = \langle(\mathrm{id} - Q)c, (Lx)_\downarrow\rangle$ and further

$$\big\langle c, (Lx)_\downarrow\big\rangle = \big\langle Qc, (Lx)_\downarrow\big\rangle = \big\langle Qc, g^{-1}Lx\big\rangle = \langle gQc, Lx\rangle.$$

So, in order to show (18), it is sufficient prove that

$$\langle gQc, Lx\rangle \leqslant \left\langle c, \frac{1}{2}(x_\downarrow - x_\uparrow)\right\rangle \quad \text{for } c \in D. \tag{19}$$

Since $QD \subset D$, it follows from (14) that

$$\langle Qc, x_\uparrow \rangle \leqslant \langle gQc, \tilde{g}x \rangle \leqslant \langle Qc, x_\downarrow \rangle \quad \text{for } c \in D \text{ and } \tilde{g} \in G. \tag{20}$$

But L is G-contractive, so we have $Lx \prec_G x$. That is, the vector Lx is a convex combination of some vectors of the form $\tilde{g}x$ with $\tilde{g} \in G$. Hence (20) gives

$$\langle Qc, x_\uparrow \rangle \leqslant \langle gQc, Lx \rangle \leqslant \langle Qc, x_\downarrow \rangle \quad \text{for } c \in D.$$

Denoting $\alpha = \langle Qc, x_\uparrow \rangle$ and $\beta = \langle Qc, x_\downarrow \rangle$, we find that

$$-\frac{1}{2}(\beta - \alpha) \leqslant \langle gQc, Lx \rangle - \frac{1}{2}(\alpha + \beta) \leqslant \frac{1}{2}(\beta - \alpha) \quad \text{for } c \in D. \tag{21}$$

According to Lemma 2.10 (ii), we get $\frac{1}{2}(\alpha + \beta) = 0$. On the other hand, $x_\uparrow = g_0 x_\downarrow$ and $g_0 x_\downarrow = g_0^{-1} x_\downarrow$. As a result we obtain

$$\frac{1}{2}(\beta - \alpha) = \left\langle Qc, \frac{1}{2}(x_\downarrow - x_\uparrow) \right\rangle = \langle Qc, Qx_\downarrow \rangle = \langle c, Q^T Q x_\downarrow \rangle = \langle c, Qx_\downarrow \rangle = \left\langle c, \frac{1}{2}(x_\downarrow - x_\uparrow) \right\rangle.$$

Therefore (21) implies (19). This completes the proof. □

Remark 2.11. The last part of the proof of Theorem 2.9 can be obtained from Theorem 2.2 applied to the case when $z_0 = \frac{1}{2}(\alpha + \beta) = 0$ and $M = \{0\}$, with (17) updated by $\mathcal{L} = \{\langle gQc, \cdot \rangle \colon g \in G_0\}$.

3. Applications for differences of operators

Bounds for norms of the difference of two operators have attracted a great research interest [3, Section VI]. A particular attention is paid to commutators [4,15,16,35]. In this section we generalize some recent results of Wang and Du [35] and of Bhatia and Kittaneh [4].

The following result holds.

Corollary 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two norm spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Assume $\langle \cdot, \cdot \rangle$ is an inner product on X. Let G be a compact subgroup of the orthogonal/unitary group on X. Let $c \in X$ and $e \in M_G(X)$ be such that $\langle c, e \rangle \neq 0$. Assume that $\prec_1$ and $\prec_2$ are preorders on $\mathbb{F}$ given by* (8)*.*

Let $x \in X$ and $M = \operatorname{span} e$. Suppose that $\eta_0 > 0$ is a constant such that

$$\|v\|_X \leqslant \eta_0 r_{G,c}(v) \quad \text{for } v \in x - M. \tag{22}$$

Assume that $L_1 : X \to Y$ and $L_2 : X \to Y$ are linear operators such that the map $L = L_1 - L_2$ is translation-invariant at x w.r.t. M.

If the $G(c)$-range of x is included in $\prec_{1,2}$-interval $[\alpha, \beta] \subset \mathbb{F}$, then

$$\|L_1 x - L_2 x\|_Y \leqslant \kappa \eta_0 \frac{\beta - \alpha}{2}, \tag{23}$$

where $\kappa = \|L_1\| + \|L_2\|$ with $\|L_k\| = \sup\{\|L_k v\|_Y \colon v \in X, \ \|v\|_X = 1\}$ for $k = 1, 2$.

If in addition $\frac{\alpha+\beta}{2} / \langle c, e \rangle \in \mathbb{R}$, then $\operatorname{span} e$ can be replaced by $\operatorname{span}_R e$.

Proof. By (22), for $v \in x - M$ we obtain

$$\|L_1 v - L_2 v\|_Y \leqslant \|L_1 v\|_Y + \|L_2 v\|_Y \leqslant \kappa \|v\|_X \leqslant \kappa \eta_0 r_{G,c}(v).$$

Utilizing the implication (i) ⇒ (iv) of Theorem 2.5 for $\Phi = \|\cdot\|_Y$ and $\eta = \kappa \eta_0$, we conclude that (23) holds.

The last part of the theorem follows from Remark 2.6. □

We now illustrate Corollary 3.1 in matrix setting. Set $X = Y = \mathbb{M}_n$ with the operator norm $\|\cdot\|_\infty$ on $\mathbb{M}_n$ and with the trace inner product. Let G be the group of all unitary similarities acting on $\mathbb{M}_n$. Take e to be the $n \times n$ identity matrix I_n and $M = \operatorname{span}_R e$. Choose $c = \operatorname{diag}(1, 0, \dots, 0)$. Let x be an $n \times n$ Hermitian matrix with the smallest and largest eigenvalues α and β, respectively. Then $W_{G,c}(x) = W(x) = [\alpha, \beta]$ and $r_{G,c}(v) = w(v) = \|v\|_\infty$ for $v \in x - M \subset \mathbb{H}_n$ [14, p. 12]. So, (22) is fulfilled for $\eta_0 = 1$.

For given matrix $a \in \mathbb{M}_n$, we define

$$L_1 x = xa \quad \text{and} \quad L_2 x = ax \quad \text{for } x \in \mathbb{M}_n.$$

Evidently, $L_1 x - L_2 x$ is the *commutator* $xa - ax$. Since $\|\cdot\|_\infty$ is submultiplicative, we can put $\kappa = 2\|a\|_\infty$ into (23).

As a consequence of Corollary 3.1, we obtain

Corollary 3.2. *(See Wang and Du [35, Corollary 4].) Let a be an $n \times n$ matrix and let x be an $n \times n$ Hermitian matrix. Denote $\alpha = \min\{\lambda\colon \lambda \in \sigma(x)\}$ and $\beta = \max\{\lambda\colon \lambda \in \sigma(x)\}$, where $\sigma(x)$ stands for the spectrum of x.*

Then we have the inequality

$$\|xa - ax\|_\infty \leqslant \|a\|_\infty(\beta - \alpha).$$

A related result to Corollary 3.2 is Corollary 3.4. A more general framework is demonstrated in Theorem 3.3.

To state this theorem, we need some notation. Let $(X, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real inner product space and G be a compact subgroup of the orthogonal group on X. Assume that Y is a subspace in X and H is a subgroup in G. It is not hard to check that the set of all G-contractive operators on X is G-invariant and convex.

In Theorem 3.3 we deal with some special pairs of G-contractive operators. Namely, we introduce

$$\mathcal{C}_0 = \left\{ \left(\sum_{i=1}^m \alpha_i g_i h_{1i}, \sum_{i=1}^m \alpha_i g_i h_{2i} \right)\colon \sum_{i=1}^m \alpha_i = 1,\ \alpha_i > 0,\ g_i \in G \text{ and } h_{1i}, h_{2i} \in H \right\}. \tag{24}$$

This definition is motivated by the construction described in [4, pp. 147–148].

For instance, let $H_1 = \{h_{11}, \dots, h_{1q}\}$ and $H_2 = \{h_{21}, \dots, h_{2m}\}$ be finite subgroups of H of order q and m, respectively. Assume H_1 is a subgroup of H_2. By Lagrange's theorem, $m = kq$ for some positive integer k. It is known that the operators

$$L_1 = \frac{1}{q}\sum_{i=1}^q h_{1i} = \frac{1}{m}\sum_{i=1}^q k h_{1i} \quad \text{and} \quad L_2 = \frac{1}{m}\sum_{i=1}^m h_{2i}$$

are the orthogonal projections onto the subspaces $M_{H_1}(X) = \{x \in X\colon hx = x,\ h \in H_1\}$ and $M_{H_2}(X) = \{x \in X\colon hx = x,\ h \in H_2\}$, respectively (see [1,24]).

Theorem 3.3. *Let $(X, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real inner product space. Let G be a compact subgroup of the orthogonal group on X, and $D \subset X$ be a closed convex cone such that the G-majorization $\prec_G$ is a GIC preorder satisfying axioms* (A1)–(A2)*.*

Assume that Y is a subspace in X with the inherited inner product, H is a closed subgroup of G and $F \subset Y$ is a closed convex cone satisfying axioms (A1)–(A2)*.*

If $L_1, L_2 : X \to X$ are G-contractive operators on X such that (L_1, L_2) is in the class $\mathcal{C}_0$ defined by (24)*, then*

$$L_1 x - L_2 y \prec_G x^{\downarrow} - y^{\uparrow} \quad \textit{for } x, y \in Y, \tag{25}$$

where for $z \in Y$ the symbols $z^{\downarrow}$ and $z^{\uparrow}$ stand for the unique elements of the sets $F \cap Hz$ and $-F \cap Hz$, respectively.

Proof. By [25, Corollary 2.5] applied to the triple (Y, H, F) we have

$$x - y \prec_H x^{\downarrow} - y^{\uparrow} \quad \text{for } x, y \in Y.$$

Hence

$$h_1 x - h_2 y \prec_H x^{\downarrow} - y^{\uparrow} \quad \text{for } x, y \in Y,\ h_1, h_2 \in H,$$

because $(hz)^{\downarrow} = z^{\downarrow}$ and $(hz)^{\uparrow} = z^{\uparrow}$ for $h \in H$ and $z \in Y$. In consequence, by $H \subset G$, we obtain

$$h_1 x - h_2 y \prec_G x^{\downarrow} - y^{\uparrow} \quad \text{for } x, y \in Y,\ h_1, h_2 \in H.$$

Because the preorder $\prec_G$ is G-invariant, we get

$$g h_1 x - g h_2 y \prec_G x^{\downarrow} - y^{\uparrow} \quad \text{for } x, y \in Y,\ g \in G,\ h_1, h_2 \in H. \tag{26}$$

Fix arbitrarily $x, y \in Y$. Since $(L_1, L_2) \in \mathcal{C}_0$, there exist $g_i \in G$, $h_{1i}, h_{2i} \in H$ and $\alpha_i > 0$, $i = 1, \dots, m > 0$, with $\sum_{i=1}^m \alpha_i = 1$ satisfying

$$L_1 x = \sum_{i=1}^m \alpha_i g_i h_{1i} x \quad \text{and} \quad L_2 y = \sum_{i=1}^m \alpha_i g_i h_{2i} y.$$

It follows from (26) and from the definition of G-majorization that

$$\sum_{i=1}^m \alpha_i g_i h_{1i} x - \sum_{i=1}^m \alpha_i g_i h_{2i} y \prec_G x^{\downarrow} - y^{\uparrow}$$

In other words, (25) holds, as required. □

In the remainder of this section we interpret Theorem 3.3 when

$X = \mathbb{M}_n =$ the vector space of $n \times n$ complex matrices endowed with the (real) inner product $\langle x, y\rangle$
$= \operatorname{Re}\operatorname{tr} xy^*$ for $x, y \in \mathbb{M}_n$,

$G =$ the group of unitary equivalences $u_1(\cdot)u_2$ with u_1 and u_2 running over the unitary group $\mathbb{U}_n$,

$D = \{\operatorname{diag}(s_1, \dots, s_n):\ s_1 \geqslant \dots \geqslant s_n \geqslant 0\}$,

$x_\downarrow = \operatorname{diag} s(x)$, where $s(x) = (s_1(x), \dots, s_n(x))$ denote the n-vector of the singular values of $x \in \mathbb{M}_n$ ordered so that $s_1(x) \geqslant \dots \geqslant s_n(x)$,

$Y = \mathbb{H}_n =$ the vector space of $n \times n$ Hermitian matrices,

$H =$ the group of unitary similarities $u(\cdot)u^*$ with u running over $\mathbb{U}_n$,

$F = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n):\ \lambda_1 \geqslant \dots \geqslant \lambda_n\}$,

$y^\downarrow = \operatorname{diag}\lambda^\downarrow(y)$ and $y^\uparrow = \operatorname{diag}\lambda^\uparrow(y)$, where $\lambda_1(y) \geqslant \dots \geqslant \lambda_n(y)$ denote the eigenvalues of $y \in \mathbb{H}_n$, and $\lambda^\downarrow(y) = (\lambda_1(y), \dots, \lambda_n(y))$ and $\lambda^\uparrow(y) = (\lambda_n(y), \dots, \lambda_1(y))$.

It is known that (X, G, D) and (Y, H, F) satisfy conditions (A1)–(A2) (see [22, pp. 498, 514], [17, pp. 943–945], Example 2.8, also cf. [9, pp. 17–18]), and

$$y \prec_G x \quad \text{iff} \quad s(y) \prec_w s(x) \quad \text{for } x, y \in \mathbb{M}_n, \tag{27}$$

$$y \prec_H x \quad \text{iff} \quad \lambda(y) \prec_m \lambda(x) \quad \text{for } x, y \in \mathbb{H}_n. \tag{28}$$

A direct application of Theorem 3.3 gives

Corollary 3.4. *Let $L_1, L_2 : \mathbb{M}_n \to \mathbb{M}_n$ be two G-contractive operators on $\mathbb{M}_n$ (i.e., $s(L_k v) \prec_w s(v)$ for $v \in \mathbb{M}_n$, $k = 1, 2$) belonging to the class $\mathcal{C}_0$ with the groups G and H as above. Let x and y be $n \times n$ Hermitian matrices.*

Then we have the inequality

$$s(L_1x - L_2y) \prec_w s(\lambda^\downarrow(x) - \lambda^\uparrow(y)) = (|\lambda_1(x) - \lambda_n(y)|, \dots, |\lambda_n(x) - \lambda_1(y)|), \tag{29}$$

or, equivalently, for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n$,

$$|||L_1x - L_2y||| \leqslant |||\operatorname{diag}(\lambda^\downarrow(x) - \lambda^\uparrow(y))|||. \tag{30}$$

Corollary 3.5. *(See Bhatia and Kittaneh [4, Theorem 4].) Let a be an $n \times n$ matrix and let x and y be $n \times n$ Hermitian matrices.*

Then we have the inequality

$$s(xa - ay) \prec_w \|a\|_\infty (|\lambda_1(x) - \lambda_n(y)|, \dots, |\lambda_n(x) - \lambda_1(y)|), \tag{31}$$

or, equivalently, for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n$,

$$|||xa - ay||| \leqslant \|a\|_\infty |||\operatorname{diag}(\lambda^\downarrow(x) - \lambda^\uparrow(y))|||. \tag{32}$$

Proof. The operators

$$L_1x = \frac{1}{\|a\|_\infty} xa \quad \text{and} \quad L_2x = \frac{1}{\|a\|_\infty} ax \quad \text{for } x \in \mathbb{M}_n$$

are G-contractive on $\mathbb{M}_n$. In fact, we have

$$s(ab) \prec_w s(a) \circ s(b) \quad \text{for } a, b \in \mathbb{M}_n,$$

where $\circ$ denotes the Hadamard (entrywise) product on $\mathbb{R}^n$ (see [14, Theorem 3.3.14]). Furthermore, $\|a\|_\infty = s_1(a)$ and

$$s(a) \circ s(b) \leqslant s_1(b)s(a) \quad \text{and} \quad s(a) \circ s(b) \leqslant s_1(a)s(b) \quad \text{for } a, b \in \mathbb{M}_n.$$

Therefore

$$s(xa) \prec_w \|a\|_\infty s(x) \quad \text{and} \quad s(ax) \prec_w \|a\|_\infty s(x) \quad \text{for } a, x \in \mathbb{M}_n.$$

Employing (27) one sees that L_1 and L_2 are G-contractive.

In addition, (L_1, L_2) belongs to the class $\mathcal{C}_0$. To see this, use the argument given in [4, pp. 147–148]. By Corollary 3.4 we deduce that (31) and (32) hold. □

4. Applications for Grüss type inequalities

The classical Grüss' inequality [13] states that

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt\right| \leqslant \frac{1}{4}(\beta_0-\alpha_0)(\delta_0-\gamma_0) \tag{33}$$

for two bounded integrable functions $f, g : [a,b] \to \mathbb{R}$ such that

$$\alpha_0 \leqslant f(t) \leqslant \beta_0 \quad \text{and} \quad \gamma_0 \leqslant g(t) \leqslant \delta_0 \quad \text{for all } t \in [a,b].$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right), \quad t \in [a,b].$$

Let X be a linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. A *Grüss type inequality* (cf. [5–8,28,29,34]) estimates from above the quantity

$$\bigl|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle\bigr|, \tag{34}$$

where $x, y \in X$ and $e \in X$ is a given vector such that $\langle e, e\rangle = 1$.

A standard initial step in the problem of estimating (34) is as follows [5, p. 75]. By Schwarz inequality, for any $x, y \in X$, we have

$$\bigl|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle\bigr| = \bigl|\langle x - \langle x, e\rangle e, y - \langle y, e\rangle e\rangle\bigr| \leqslant \bigl\|x - \langle x, e\rangle e\bigr\|\,\bigl\|y - \langle y, e\rangle e\bigr\| = \|Qx\|\,\|Qy\|, \tag{35}$$

where $Q = \mathrm{id}_X - \langle\cdot, e\rangle e$ is the orthoprojector from X onto $M^\perp$, the subspace in X orthogonal to $M = \operatorname{span} e$.

Taking $x \in X$ and $x_0 \in M$, we find that $Q(x - x_0) = Qx$, i.e. Q is translation-invariant w.r.t. M. This allows to apply a special case of Theorem 2.2 when

$$\begin{aligned} &X = Y \quad \text{is a linear space with an inner product } \langle\cdot,\cdot\rangle, \text{ and } \Phi = \|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}, \\ &L = Q \quad \text{is the orthoprojector from } X \text{ onto } M^\perp, \text{ where } M = \operatorname{span} e. \end{aligned} \tag{36}$$

These assumptions are valid throughout this section.

In the sequel, we study Grüss type inequalities in two cases. The first employs Theorem 2.2 for vectorial intervals induced by two cone preorders (see Section 4.1). The second uses Theorem 2.5 for $G(c)$-ranges (see Section 4.2).

4.1. Making use of cone preorders

Before giving results, we recall some definitions.

If $K \subset Z = X$ is a convex cone, then the *dual cone* of K is defined by

$$\operatorname{dual} K = \bigl\{z \in Z\colon \operatorname{Re}\langle z, v\rangle \geqslant 0 \quad \text{for all } v \in K\bigr\}.$$

We define the cone preorders $\prec_K$ and $\prec_{\operatorname{dual} K}$ on Z by

$$\begin{aligned} &y \prec_K x \quad \text{iff} \quad x - y \in K, \\ &y \prec_{\operatorname{dual} K} x \quad \text{iff} \quad x - y \in \operatorname{dual} K. \end{aligned}$$

The symbol $[\alpha, \beta]_K$ stands for $[\alpha, \beta]_{\prec_K, \prec_{\operatorname{dual} K}} = \{z \in Z\colon \alpha \prec_K z \prec_{\operatorname{dual} K} \beta\}$.

Lemma 4.1 provides an interpretation of vectorial interval induced by convex cones K and $\operatorname{dual} K$. The equivalence (b) $\Leftrightarrow$ (c) is due to Dragomir [7, Lemma 2.1].

Lemma 4.1. *(See [7, Lemma 2.1], [26, Lemma 2.1].) Assume Z is a linear space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. For any vectors $\alpha, \beta, z \in Z$, the following statements are mutually equivalent*:

(a) *There exists a convex cone $K \subset Z$ such that $z \in [\alpha, \beta]_K$.*
(b) $\operatorname{Re}\langle \beta - z, z - \alpha\rangle \geqslant 0$.
(c) $\|z - \frac{1}{2}(\alpha + \beta)\| \leqslant \|\frac{1}{2}(\beta - \alpha)\|$.

In addition to the specification (36), in this section we also assume that

$$Z = X, \qquad \Psi = \|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}, \quad \text{and} \quad \mathcal{L} = \{\mathrm{id}_X\}, \tag{37}$$

where id_X denotes the identity on X. In this case,

$$\mathcal{L}(x) = \{x\} \quad \text{and} \quad r_{\Psi\mathcal{L}}(x) = \|x\| \quad \text{for } x \in X.$$

Notice that any $z_0 \in M = \operatorname{span} e$ is $\mathcal{L}, M$-admissible at any $x \in X$. Evidently, by (36)–(37), inequality (4) holds for $\eta = 1$.

We let $\prec_1$ and $\prec_2$ to be the cone preorders $\prec_K$ and $\prec_{\operatorname{dual} K}$, respectively, for some convex cone $K \subset Z$. Observe that Lemma 4.1 ensures that property (3) is valid for $\Psi = \|\cdot\|$.

We return to Grüss type inequalities.

Theorem 4.2. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $e \in X$ with $\|e\| = 1$, where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. Assume that $x, y, \alpha, \beta, \gamma, \delta \in X$ are vectors such that*

(a) *$\alpha \prec_{K_1} x \prec_{\operatorname{dual} K_1} \beta$ and $\gamma \prec_{K_2} y \prec_{\operatorname{dual} K_2} \delta$ for some convex cones $K_1, K_2 \subset X$,*
(b) *$\alpha + \beta \in \operatorname{span} e$ and $\gamma + \delta \in \operatorname{span} e$.*

Then we have the inequality

$$\left|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle\right| \leqslant \frac{1}{4}\|\beta - \alpha\|\|\delta - \gamma\|. \tag{38}$$

Proof. With the specifications (36) and (37), conditions (3)–(4) are fulfilled (see Lemma 4.1). By virtue of (a)–(b), one sees that condition (i) of Theorem 2.2 is satisfied.

Applying implication (i) $\Rightarrow$ (iv) of Theorem 2.2, we get

$$\|Qx\| \leqslant \left\|\frac{1}{2}(\beta - \alpha)\right\| \quad \text{and} \quad \|Qy\| \leqslant \left\|\frac{1}{2}(\delta - \gamma)\right\|.$$

Using the initial bound given at (35), we conclude that (38) follows. $\square$

With the additional restriction that the end-points of the vectorial intervals $[\alpha, \beta]$ and $[\gamma, \delta]$ are proportional to the vector e, the last theorem becomes

Corollary 4.3. *(See Dragomir [5, Theorem 1], [7, Theorem 2.5].) Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $e \in X$ with $\|e\| = 1$, where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. Assume that $x, y \in X$ and $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathbb{F}$.*

If

$$\alpha_0 e \prec_{K_1} x \prec_{\operatorname{dual} K_1} \beta_0 e \quad \text{and} \quad \gamma_0 e \prec_{K_2} y \prec_{\operatorname{dual} K_2} \delta_0 e$$

for some convex cones $K_1, K_2 \subset X$, or, equivalently,

$$\left\|x - \frac{1}{2}(\alpha_0 + \beta_0)e\right\| \leqslant \frac{1}{2}|\beta_0 - \alpha_0| \quad \text{and} \quad \left\|y - \frac{1}{2}(\gamma_0 + \delta_0)e\right\| \leqslant \frac{1}{2}|\delta_0 - \gamma_0|,$$

then we have the inequality

$$\left|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle\right| \leqslant \frac{1}{4}|\beta_0 - \alpha_0||\delta_0 - \gamma_0|.$$

A direct application of Theorem 4.2 for the space $\mathbb{R}^n$ with the inner product $\langle x, y\rangle = \frac{1}{n}\sum_{i=1}^n x_i y_i$ gives

Corollary 4.4. *Assume that $x, y, \alpha, \beta, \gamma, \delta \in \mathbb{R}^n$ are vectors such that*

(a) *$\alpha_i \leqslant x_i \leqslant \beta_i$ and $\gamma_i \leqslant y_i \leqslant \delta_i$ for all $i = 1, \ldots, n$, or more generally*

$$\sum_{i=1}^n (\beta_i - x_i)(x_i - \alpha_i) \geqslant 0 \quad \text{and} \quad \sum_{i=1}^n (\delta_i - y_i)(y_i - \gamma_i) \geqslant 0,$$

(b) *$\alpha + \beta \in \operatorname{span} e$ and $\gamma + \delta \in \operatorname{span} e$, where $e = (1, \ldots, 1)^T$.*

Then we have the inequality

$$\left|\frac{1}{n}\sum_{i=1}^n x_i y_i - \frac{1}{n}\sum_{i=1}^n x_i \cdot \frac{1}{n}\sum_{i=1}^n y_i\right| \leqslant \frac{1}{4n}\left(\sum_{i=1}^n (\beta_i - \alpha_i)^2\right)^{1/2}\left(\sum_{i=1}^n (\delta_i - \gamma_i)^2\right)^{1/2}.$$

The constant $\frac{1}{4}$ is best possible and is achieved (if n even) for

$$x_i = y_i = \operatorname{sgn}\left(i - \frac{n+1}{2}\right), \qquad \alpha_i = \gamma_i = -1 \quad \textit{and} \quad \beta_i = \delta_i = 1, \quad i = 1, \dots, n.$$

Using Theorem 4.2 for the space of real L^2-functions on a real interval $[a,b]$ with the inner product $\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(t)g(t)\,dt$, we obtain the following result.

Corollary 4.5. *Let $f, g, \alpha, \beta, \gamma, \delta \in L^2_{[a,b]}$ be functions such that*

(a) *$\alpha(t) \leqslant f(t) \leqslant \beta(t)$ and $\gamma(t) \leqslant g(t) \leqslant \delta(t)$ for all $t \in [a,b]$, or more generally*

$$\int_a^b \big(\beta(t) - f(t)\big)\big(f(t) - \alpha(t)\big)\,dt \geqslant 0 \quad \textit{and} \quad \int_a^b \big(\delta(t) - g(t)\big)\big(g(t) - \gamma(t)\big)\,dt \geqslant 0,$$

(b) *$\alpha + \beta \in \operatorname{span} e$ and $\gamma + \delta \in \operatorname{span} e$, where $e(t) = 1$ for $t \in [a,b]$.*

Then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt\right|$$
$$\leqslant \frac{1}{4(b-a)}\left(\int_a^b \big(\beta(t) - \alpha(t)\big)^2\,dt\right)^{1/2}\left(\int_a^b \big(\delta(t) - \gamma(t)\big)^2\,dt\right)^{1/2}. \tag{39}$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right), \qquad \alpha(t) = \gamma(t) = -1 \quad \textit{and} \quad \beta(t) = \delta(t) = 1, \quad t \in [a,b].$$

Example 4.6. Put

$$f(t) = g(t) = \sin t, \qquad \alpha(t) = \gamma(t) = -|t|, \qquad \beta(t) = \delta(t) = |t| \quad \text{for } t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then the left-hand side of (33) and of (39) equals $\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin^2 t\,dt = \frac{1}{2}$, while the right-hand side of (39) equals $\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}(2|t|)^2\,dt = \frac{\pi^2}{12} < \frac{5}{6}$. Taking $\alpha_0 = \gamma_0 = -1$ and $\beta_0 = \delta_0 = 1$, we find that the right-hand side of (33) equals 1.

This shows that (39) provides a more precise estimate than (33) does.

A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur-convex*, if $\psi(\alpha) \leqslant \psi(\beta)$ whenever $\alpha \prec_m \beta$ for $\alpha, \beta \in \mathbb{R}^n$. It is well known [22] that any convex permutation-invariant function $\psi : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex. For instance, $\alpha \prec_m \beta$ implies $\|\alpha\| \leqslant \|\beta\|$, where $\psi(z) = \|z\| = \frac{1}{\sqrt{n}}(\sum_{i=1}^n z_i^2)^{1/2}$ for $z = (z_1, \dots, z_n)^T \in \mathbb{R}^n$.

Referring to Example 2.7, it is known that $\alpha \prec_w \beta$ iff $\alpha \leqslant_1 x \prec_m \beta$ for some $x \in \mathbb{R}^n$, where $\leqslant_1$ is the usual componentwise order after ordering the entries of α and x, that is $\alpha_\downarrow \leqslant x_\downarrow$ [12, p. 120], and $z_\downarrow = (z_{[1]}, \dots, z_{[n]})^T$ and $z_{[1]} \geqslant \cdots \geqslant z_{[n]}$ are the entries of $z \in \mathbb{R}^n$ in decreasing order.

In the corollary below we combine the map $(\cdot)_\downarrow$ and the cone preorder $\prec_D$ induced by $D = \{z = (z_1, \dots, z_n)^T \in \mathbb{R}^n\colon z_1 \geqslant \cdots \geqslant z_n\}$. Namely, we write $\alpha \prec_1 x$ iff $\alpha_\downarrow \prec_D x_\downarrow$.

Two vectors $\alpha, x \in \mathbb{R}^n$ are said to be *similarly ordered (synchronous)* if $\alpha = p\alpha_\downarrow$ and $x = px_\downarrow$ for some permutation $p \in \mathbb{P}_n$ (see Example 2.7).

A majorization counterpart of Corollary 4.4 is as follows.

Corollary 4.7. *Assume that $x, y, \alpha, \beta, \gamma, \delta \in \mathbb{R}^n$ are vectors such that α and x are similarly ordered and γ and y are similarly ordered and*

(a) *$\alpha \prec_1 x \prec_m \beta$ and $\gamma \prec_1 y \prec_m \delta$.*
(b) *$\alpha_\downarrow + \beta_\downarrow \in \operatorname{span} e$ and $\gamma_\downarrow + \delta_\downarrow \in \operatorname{span} e$, where $e \in \mathbb{R}^n$ with $\frac{1}{n}\sum_{i=1}^n e_i^2 = 1$.*

Then we have the inequality

$$\left|\frac{1}{n}\sum_{i=1}^{n}x_{[i]}y_{[i]}-\frac{1}{n}\sum_{i=1}^{n}x_{[i]}e_i\cdot\frac{1}{n}\sum_{i=1}^{n}y_{[i]}e_i\right|\leqslant\frac{1}{4n}\left(\sum_{i=1}^{n}(\beta_{[i]}-\alpha_{[i]})^2\right)^{1/2}\left(\sum_{i=1}^{n}(\delta_{[i]}-\gamma_{[i]})^2\right)^{1/2}.$$

Proof. To see this result, remind that $x\prec_m\beta$ iff $x_\downarrow\prec_{\mathrm{dual}\,D}\beta_\downarrow$ (see (13) and Example 2.7). Similarly, $y\prec_m\delta$ iff $y_\downarrow\prec_{\mathrm{dual}\,D}\delta_\downarrow$. Next, use Corollary 4.4 for the vectors $x_\downarrow$, $y_\downarrow$, $\alpha_\downarrow$, $\beta_\downarrow$, $\gamma_\downarrow$ and $\delta_\downarrow$. □

4.2. Applying G(c)-ranges

In this section we utilize Theorem 2.5 (with (8) and (9)) for the spaces X, Y and M and for the maps L and Φ defined in the specification (36).

The following result holds.

Corollary 4.8. *Let $(X,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and let G be a compact subgroup of the orthogonal/unitary group on X. Let $c\in X$ and $e\in M_G(X)$ be such that $\langle c,e\rangle\neq 0$ and $\|e\|=1$, where $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$. Assume that $\prec_1$ and $\prec_2$ are preorders on $\mathbb{F}$ given by* (8)*.*

Denote $M=\operatorname{span}e$. For vectors $x,y\in X$, let $\eta_1,\eta_2>0$ be numbers such that

$$\|v\|\leqslant\eta_1 r_{G,c}(v)\quad\text{for } v\in x-M\quad\text{and}\quad\|v\|\leqslant\eta_2 r_{G,c}(v)\quad\text{for } v\in y-M. \tag{40}$$

(i) *If $G(c)$-ranges of $x\in X$ and of $y\in X$ are included in $\prec_{1,2}$-intervals $[\alpha,\beta]$ and $[\gamma,\delta]$, respectively, then*

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leqslant\eta_1\eta_2\frac{1}{4}|\beta-\alpha||\delta-\gamma|. \tag{41}$$

(ii) *If $G(c)$-ranges of $x\in X$ and of $y\in X$ are included in $|\cdot|$-balls of radii r_0 and s_0, respectively, then*

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leqslant\eta_1\eta_2 r_0 s_0. \tag{42}$$

Proof. Since $\|Qv\|\leqslant\|v\|$ for $v\in X$, it follows from (40) that (10) is satisfied for the vectors x and y and maps $\Phi=\|\cdot\|$ and $L=Q$.

(i). By virtue of the implication (i) ⇒ (iv) of Theorem 2.5 applied to the specification (36), we have

$$\|Qx\|\leqslant\eta_1\frac{1}{2}(\beta-\alpha)\quad\text{and}\quad\|Qy\|\leqslant\eta_2\frac{1}{2}(\delta-\gamma).$$

Using the initial bounds given at (35), we conclude that (41) holds.

(ii). Likewise, the implication (ii) ⇒ (iv) of Theorem 2.5 gives

$$\|Qx\|\leqslant\eta_1 r_0\quad\text{and}\quad\|Qy\|\leqslant\eta_2 s_0.$$

The assertion (42) now follows by (35). This completes the proof. □

To derive a result of Renaud [29] from Corollary 4.8, take $X=\mathbb{M}_n$ endowed with the inner product $\langle x,y\rangle=\operatorname{tr}Txy^*$ for $x,y\in\mathbb{M}_n$, where $T\in\mathbb{M}_n$ is positive definite with $\operatorname{tr}T=1$. Let G be the group of unitary similarities acting on $\mathbb{M}_n$. Putting $c=\operatorname{diag}(1,0,\dots,0)$ gives $r_{G,c}(x)=w(x)$, the numerical radius of $x\in\mathbb{M}_n$ (see Examples 2.1 and 2.4).

Let $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$ and $\|\cdot\|_\infty$ be the operator norm on $\mathbb{M}_n$. Since

$$\|x\|\leqslant\|x\|_\infty\leqslant 2w(x)\quad\text{for } x\in\mathbb{M}_n$$

(see [29, p. 97] and [2, p. 4]), condition (40) holds for $\eta_1=\eta_2=2$, $e=I_n$ and $M=\operatorname{span}e$. The constant 2 can be replaced by 1 if x and y are normal matrices.

Remind that $W(y^*)=\overline{W(y)}$ and $w(y^*)=w(y)$ for $y\in\mathbb{M}_n$ [19, pp. 52 and 71]. Finally, by Corollary 4.8(ii) applied for x and y^*, we obtain

Corollary 4.9. *(See Renaud [29, Theorem 2.1].) For matrices $x,y\in\mathbb{M}_n$, assume that the numerical ranges $W(x)$ and $W(y)$ are contained in disks of radii r_0 and s_0, respectively. Let $T\in\mathbb{M}_n$ be positive definite with* $\operatorname{tr}T=1$.

Then the following inequality holds

$$\left|\operatorname{tr}(Txy)-\operatorname{tr}(Tx)\operatorname{tr}(Ty)\right|\leqslant 4r_0 s_0.$$

If x and y are normal, the constant 4 *can be replaced by* 1.

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