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Rough paths analysis of general Banach space-valued Wiener processes

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Abstract

In this article, we carry out a rough paths analysis for Banach space-valued Wiener processes. We show that most of the features of the classical Wiener process pertain to its rough path analog. To be more precise, the enhanced process has the same scaling properties and it satisfies a Fernique type theorem, a support theorem and a large deviation principle in the same Hölder topologies as the classical Wiener process does. Moreover, the canonical rough paths of finite dimensional approximating Wiener processes converge to the enhanced Wiener process. Finally, a new criterion for the existence of the enhanced Wiener process is provided which is based on compact embeddings. This criterion is particularly handy when analyzing Kunita flows by means of rough paths analysis which is the topic of a forthcoming article.

Keywords: Rough paths; Wiener process; Support theorem; Large deviation principle

1. Introduction

The notion *rough path* was coined by Terry Lyons in 1994 [20]. The corresponding theory provides an extension of Young integrals to less regular driving signals. In the context of probability theory, it allows an alternative representation for solutions to Stratonovich differential equations as solutions to *rough path differential equations* (RDE). The power of the approach is that once the driving signal has been associated to a rough path, the solution can be written

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as continuous function (*Itô map*) of the rough path signal by Terry Lyons' universal limit theorem [22]. In general irregular controls admit several extensions to rough paths. Nonetheless, in the context of stochastic analysis there is one canonical choice which is uniquely defined up to a null set. Since the null set does not depend on the choice of the RDE this allows one to pick a random RDE depending on the path itself and, in particular, the concept of filtrations becomes obsolete for the existence of solutions.

As eluded by Ledoux, Qian, and Zhang [18] the theory of rough paths leads to natural proofs of *support theorems* (ST) and *large deviation principles* (LDP) since both properties behave nicely under an application of the continuous Itô map. Consequently, it suffices to prove a support theorem and a large deviation principle for the canonical rough path of the driving signal and then to infer the corresponding results for the solution of the SDE. This approach has been firstly carried out in [18] for the multi-dimensional Wiener process under the p-variation topology for $p > 2$. Later on, analogous results were proved under fine Hölder topologies by Friz and Victoir [10] (see also [12]). General Banach space-valued Wiener processes were embedded into the theory of rough paths by Ledoux et al. [17] in 2002. A series of articles by Inahama and Kawabi [14–16] followed which was mainly motivated by its applicability to heat kernel measures on loop spaces. Nowadays the theory of rough paths is well established and we refer the reader to the monographs [21,23], and [11] for a general account on the topic.

Our results are manifold. First we establish a representation of the enhanced Wiener process as limit of finite dimensional enhanced Wiener processes. This Itô–Nisio type theorem implies that the enhanced Wiener process has the same scaling properties as the classical Wiener process. For finite dimensional Wiener processes there are various ways to define the canonical rough path (either as solution to a Stratonovich stochastic differential equation or via the limit of certain smooth approximations, see for instance [11]) and it is thus conceived as a universal object. Since we can freely approximate the enhanced infinite dimensional Wiener process by finite dimensional approximations, also the infinite dimensional canonical rough path can be seen as a universal object.

We derive a support theorem and a large deviation principle in fine Hölder topologies similar as the one known for finite dimensional processes [10]. By doing so we extend results of [14] who analyzed the problem under p-variation topology.

In general, the existence of the canonical rough path is not trivial (at least for projective tensor products), and Ledoux et al. provide a sufficient criterion in [17]. We relate their concept of *finite dimensional approximation* to entropy numbers of compact embeddings. Since these are known for various embeddings [7,8], we obtain a new sufficient criterion which can be easily verified in many cases. We phrase its implications in the case where the state space of the Wiener process is a Hölder–Zygmund space. Our main interest in the theory developed here is its applicability to stochastic flows generated by Kunita type SDEs. Indeed, we will establish a support theorem and a large deviation principle for Brownian flows in a forthcoming article [5].

We start with summarizing the results of the article. Here we introduce some notation rather in an informal way in order to enhance readability. All notation will be introduced in great detail at the end of this section.

Results

Let $(V, |\cdot|_V)$ be a *separable* Banach space, and let $X = (X_t)_{t\in[0,1]}$ denotes a V-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More explicitly, X is measurable with respect to the Borel sets of $C([0,1], V)$ and satisfies, for $0 \leqslant s < t \leqslant 1$,

- $X_t - X_s$ is independent of $\sigma(X_w\colon\ w \leqslant s)$ and
- $\mathcal{L}((t-s)^{-1/2}(X_t - X_s)) = \mathcal{L}(X_1)$ is a centered Gaussian distribution.

We denote by $(H_1, |\cdot|_{H_1})$ and $(H, |\cdot|_H)$ the reproducing kernel Hilbert spaces of X_1 and $X = (X_t)_{t\in[0,1]}$. Note that H can be expressed in terms of H_1 as

$$H = \left\{ \int_0^{\cdot} f_t \, \mathrm{d}t\colon\ f \in L^2\big([0,1], H_1\big) \right\},$$

where the integral is to be understood as Bochner integral. For a general account on Gaussian distributions we refer the reader to the books by Lifshits [19] and Bogachev [2].

We let $\varphi : (0,1] \to (0,\infty)$ denote an increasing function with

$$\lim_{\delta\downarrow 0} \frac{\varphi(\delta)}{\sqrt{-\delta \log \delta}} = \infty \tag{1}$$

and consider the *geometric φ-Hölder rough path space $G\Omega_\varphi(\mathbf{V})$*, which will be rigorously introduced below. Moreover, X is assumed to possess a canonical rough path $\mathbf{X}$ in the sense of assumption (E), see Section 2.

Theorem 1.1. $\mathbf{X}$ *is almost surely an element of $G\Omega_\varphi(\mathbf{V})$ and its range in that space is the closure of the lift of the reproducing kernel Hilbert space H of X into $G\Omega_\varphi(\mathbf{V})$.*

Theorem 1.2. *The family $\{\mathbf{X}^\varepsilon\colon\ \varepsilon > 0\}$ with $\mathbf{X}^\varepsilon$ being the canonical rough path of $(\varepsilon \cdot X_t)_{t\in[0,1]}$ satisfies a LDP in $G\Omega_\varphi(\mathbf{V})$ with good rate function*

$$J(\mathbf{h}) = \begin{cases} \frac{1}{2}|h|_H^2 & \text{if } \exists h \in H \text{ with } \mathbf{h} = S(h), \\ \infty & \text{else}, \end{cases}$$

where S denotes the canonical lift of H into $G\Omega_\varphi(\mathbf{V})$.

Suppose now that the reproducing kernel Hilbert space H_1 of X_1 is infinite dimensional and fix a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ of H_1. We represent X as the in $C([0,1], V)$ almost sure limit

$$X_t = \lim_{n\to\infty} \sum_{i=1}^n \xi_t^{(i)} e_i, \tag{2}$$

with $\{(\xi_t^{(i)})_{t\in[0,1]}\colon\ i \in \mathbb{N}\}$ being an appropriate family of independent standard Wiener processes.

Theorem 1.3. *Each $X^{(n)} = (\sum_{i=1}^n \xi_t^{(i)} e_i)_{t\in[0,1]}$ possesses a canonical rough path $\mathbf{X}^{(n)}$, and one has almost sure convergence*

$$\lim_{n\to\infty} \mathbf{X}^{(n)} = \mathbf{X} \quad \text{in } G\Omega_\varphi(\mathbf{V}).$$

Remark 1.4. The latter theorem can be proved in a more general setting. One can replace the assumption that (e_i) is a complete orthonormal system of the reproducing kernel Hilbert space by the assumption that, for any $h \in H_1$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \langle h, e_i \rangle_{H_1} e_i = h.$$

We proceed with a sufficient criterion for the existence property (E) for Hölder–Zygmund spaces. For an open and bounded set D, for $n \in \mathbb{N}_0$, $\eta \in (0,1]$, and $\gamma = n + \eta$, we denote by $C_0^\gamma(D, \mathbb{R}^d)$ the set of n-times differentiable functions $f : \mathbb{R}^d \to \mathbb{R}^d$ whose derivatives are η-Hölder continuous and satisfy the zero boundary condition

$$f|_{D^c} = 0.$$

The space is endowed with a canonical norm $\|\cdot\|_{C^\gamma}$, see (14).

Theorem 1.5. *Let $0 < \gamma < \bar{\gamma}$ and $D \subset D'$ be bounded and open subsets of $\mathbb{R}^d$ with $\bar{D} \subset D'$. Let μ be a centered Gaussian measure on $C_0^{\bar{\gamma}}(D, \mathbb{R}^d)$. Then there exists a separable and closed subset $V \subset C_0^\gamma(D', \mathbb{R}^d)$ and a V-valued Wiener process $X = (X_t)_{t\in[0,1]}$ satisfying the existence property* (E) *and $\mathcal{L}(X_1) = \mu$. For instance, one may choose V as the closure of H_1.*

Remark 1.6. In the theorem, the two Banach spaces $C_0^{\bar{\gamma}}(D, \mathbb{R}^d)$ and $C_0^\gamma(D', \mathbb{R}^d)$ can be replaced by arbitrary Banach spaces V_1 and V_2 for which V_1 is compactly embedded into V_2 and for which the entropy numbers of the embedding decay at least at a polynomial order, see Section 5 for the details. In particular, one can use the results of Edmunds and Triebel [7,8] on embeddings of Sobolev and Besov spaces.

Let us summarize the implications in a language that does not incorporate rough paths. Let W denote a further Banach space, let $f : W \to L(V, W)$ be a Lip(γ)-function for a $\gamma > 2$ in the sense of [23, Def. 1.21]. For a fixed absolutely continuous path $g : [0,1] \to V$ with differential in $L^2([0,1], V)$, we consider the Young differential equation

$$\mathrm{d}y_t = f(y_t)\,\mathrm{d}[x+g]_t, \quad y_0 = \xi. \tag{3}$$

By Picard's theorem, the differential equation possesses a unique solution $I_g(x)$ for any $x \in \mathrm{BV}(V)$ (actually unique solutions exist under less restrictive assumptions).

For a V-valued Wiener process X and $n \in \mathbb{N}$, we denote by $X(n)$ the dyadic interpolation of X with breakpoints $D_n = [0,1] \cap (2^{-n}\mathbb{Z})$. We call Y the *Wong–Zakai solution* of (3) for the control X, if $\{I_g(X(n))\colon n \in \mathbb{N}\}$ is a convergent sequence in $C_\varphi([0,1], W)$ with limit Y.

As is well known Wong–Zakai solutions are tightly related to stochastic differential equations in the Stratonovich sense: If the state space W is an M-type 2 Banach space and if $\mathrm{d}g_t = v\,\mathrm{d}t$ for some $v \in V$, then the Wong–Zakai solution solves the corresponding Stratonovich stochastic differential equation, see [3].

Theorem 1.7. *Let X be a V-valued Wiener process with reproducing kernel Hilbert space H. If property* (E) *is valid, then the following is true*:

(I) *X admits a Wong–Zakai solution Y of* (3).
(II) *For $n \in \mathbb{N}$, let $X^{(n)}$ be as in Theorem* 1.3 *and denote by $Y^{(n)}$ the corresponding Wong–Zakai solution of* (3). *Then $\{Y^{(n)}\colon n \in \mathbb{N}\}$ converges almost surely to Y in $C_\varphi([0,1], W)$.*
(III) *The range of Y in $C_\varphi([0,1], W)$ is the closure of $I_g(H)$.*
(IV) *For $\varepsilon > 0$, we let Y^ε denote the Wong–Zakai solution of* (3) *for the control $\varepsilon \cdot X$. Then $\{Y^\varepsilon\colon \varepsilon > 0\}$ satisfies a large deviation principle in $C_\varphi([0,1], W)$ with speed $(\varepsilon^2)_{\varepsilon>0}$ and good rate function*

$$J(h) = \inf\left\{\frac{1}{2}\|x\|_H\colon x \in H,\ I_g(x) = y\right\}.$$

Here and elsewhere, the infimum of the empty set is defined as ∞.

Agenda

The article is organized as follows. Sections 3 and 4 are concerned with the derivation of Theorems 1.1 and 1.2, respectively. Section 2 has rather preliminary character. Here, we prove a preliminary version of the representation provided by Theorem 1.3 (see Remark 3.3 for the extension to the stronger statement). Moreover, we derive a Fernique type theorem together with Lévy's modulus of continuity. Section 5 is concerned with the proof of Theorem 1.5. Finally, we explain the implications of the theory of rough paths (Theorem 1.7) in Section 6.

Notation

We start with introducing the (for us) relevant notation of the theory of *rough paths*. For a profound background on the topic we refer the reader to the textbooks [23] and [11]. For two Banach spaces V_1 and V_2 we denote by $V_1 \otimes_a V_2$ the algebraic tensor product of V_1 and V_2. Moreover, we let $V_1 \otimes V_2$ denote the *projective tensor product*, that is the completion of $V_1 \otimes_a V_2$ under the *projective tensor norm* $|\cdot|_{V_1\otimes V_2}$ given by

$$|v|_{V_1\otimes V_2} = \inf \sum_{i=1}^n |f_i|_{V_1} |g_i|_{V_2},$$

where the infimum is taken over all representations

$$v = \sum_{i=1}^n f_i \otimes g_i \quad (n \in \mathbb{N},\ f_i \in V_1,\ g_i \in V_2 \text{ for } i = 1, \dots, n).$$

From now on V denotes a separable Banach space. For a (continuous) piecewise linear function $x : [0,1] \to V$, we may compute its iterated (Young) integrals

$$\mathbf{x}^1_{s,t} = \int_s^t \mathrm{d}x_u \in V,$$
$$\mathbf{x}^2_{s,t} = \int_{s<u_1<u_2<t} \mathrm{d}x_{u_1} \otimes \mathrm{d}x_{u_2} \in V^{\otimes 2}, \tag{4}$$

and set $\mathbf{x}_{s,t} = \mathbf{x}^1_{s,t} + \mathbf{x}^2_{s,t} \in V \oplus V^{\otimes 2}$, where $(s,t) \in \Delta := \{(s',t'):\ 0 \leqslant s' \leqslant t' \leqslant 1\}$. As is well known the iterated integrals satisfy the Chen condition:

$$\mathbf{x}^1_{s,t} = \mathbf{x}^1_{s,u} + \mathbf{x}^1_{u,t} \quad \text{and} \quad \mathbf{x}^2_{s,t} = \mathbf{x}^2_{s,u} + \mathbf{x}^2_{u,t} + \mathbf{x}^1_{s,u} \otimes \mathbf{x}^1_{u,t}, \quad \text{for } 0 \leqslant s \leqslant u \leqslant t \leqslant 1.$$

We need to consider the truncated tensor algebra. For ease of notation, we omit the real component and set

$$\mathbf{V} = V \oplus V^{\otimes 2}.$$

It is endowed with the standard addition and the multiplication

$$\mathbf{u} * \mathbf{v} = \mathbf{u}^1 + \mathbf{v}^1 + \mathbf{u}^2 + \mathbf{v}^2 + \mathbf{u}^1 \otimes \mathbf{v}^1 = \mathbf{u} + \mathbf{v} + \mathbf{u}^1 \otimes \mathbf{v}^1.$$

Using the convention $\mathbf{u} \otimes \mathbf{v} := \mathbf{u}^1 \otimes \mathbf{v}^1$ we also write

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \mathbf{u} \otimes \mathbf{v}.$$

In terms of $*$ the Chen condition can be rewritten as

$$\mathbf{x}_{s,t} = \mathbf{x}_{s,u} * \mathbf{x}_{u,t}.$$

We endow $\mathbf{V}$ and $C(\Delta, \mathbf{V})$ with the norms

$$|\mathbf{u}|_{\mathbf{V}} := \left|\mathbf{u}^1\right|_V + \left|\mathbf{u}^2\right|_{V \otimes V} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \sup_{(s,t)\in\Delta} |\mathbf{x}_{s,t}|_{\mathbf{V}},$$

and call

$$\Omega(\mathbf{V}) = \left\{(\mathbf{x}_{s,t})_{(s,t)\in\Delta} \in C(\Delta, \mathbf{V}):\ \mathbf{x} \text{ satisfies the Chen condition}\right\}$$

the set of *multiplicative functionals* on $\mathbf{V}$. It is a closed subset of $C(\Delta, \mathbf{V})$. In the following, we shall reserve the notation $\mathbf{x} = (\mathbf{x}_{s,t})_{(s,t)\in\Delta}$ for $\mathbf{V}$-valued multiplicative functionals. In the context of rough path theory we mainly work with the following homogeneous norm on $\mathbf{V}$,

$$\|\mathbf{u}\| = \left|\mathbf{u}^1\right|_V + \sqrt{\left|\mathbf{u}^2\right|_{V^{\otimes 2}}}.$$

It enjoys the following properties, for $\mathbf{u}, \mathbf{v} \in \mathbf{V}$,

(i) $\|\mathbf{u}\| = 0 \ \Leftrightarrow\ \mathbf{u} = 0$;
(ii) $\|\delta_t \mathbf{u}\| = |t|\|\mathbf{u}\|$ for the dilation operator $\delta_t \mathbf{u} = t\mathbf{u}^1 + t^2\mathbf{u}^2$ $(t \in \mathbb{R})$;
(iii) $\|\mathbf{u}\| = \|-\mathbf{u}\|$;
(iv) $\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$.

We consider rough path spaces with Hölder norm topologies: Let $\varphi : (0,1] \to (0,\infty)$ be an increasing function, and set, for $\mathbf{x} \in C(\Delta, \mathbf{V})$,

$$\|\mathbf{x}\|_\varphi = \sup_{0 \leqslant s < t \leqslant 1} \frac{\|\mathbf{x}_{s,t}\|}{\varphi(t-s)}.$$

We denote by

$$\Omega_\varphi(\mathbf{V}) = \bigl\{\mathbf{x} \in \Omega(\mathbf{V}):\ \|\mathbf{x}\|_\varphi < \infty\bigr\}$$

the set of *φ-Hölder rough paths* in $\mathbf{V}$. It is equipped with the metric $(\mathbf{x}, \mathbf{y}) \mapsto \|\mathbf{x} - \mathbf{y}\|_\varphi$. The set is nontrivial whenever $\liminf_{\delta \downarrow 0} \varphi(\delta)/\delta > 0$ which we assume in the following without further mentioning. Moreover, the set of *geometric φ-Hölder rough paths* is given by

$$G\Omega_\varphi(\mathbf{V}) = \overline{\bigl\{S(x):\ x \in C\bigl([0,1], V\bigr) \text{ piecewise linear}\bigr\}} \subset \Omega_\varphi(\mathbf{V}),$$

where $S(x) = (\mathbf{x}_{s,t})_{(s,t)\in\Delta}$ with $\mathbf{x}_{s,t}$ as in (4), and the closure is taken with respect to $\|\cdot\|_\varphi$.

Analogously, we denote by $C_\varphi([0,1], V)$ the Hölder space induced by φ that is the space of all functions $x : [0,1] \to V$ with finite Hölder norm

$$\|x\|_\varphi = \|x\|_\infty + \sup_{0 \leqslant s < t \leqslant 1} \frac{|x_{s,t}|_V}{\varphi(t-s)},$$

where we – as usual – denote $x_{s,t} = x_t - x_s$ and $\|x\|_\infty = \sup_{t\in[0,1]} |x_t|_V$ for functions $x \in C([0,1], V)$. Moreover, we denote by $\mathrm{BV}(V)$ the set of all functions $x \in C([0,1], V)$ with finite *bounded variation norm*

$$\|x\|_{\mathrm{BV}(V)} = \|x\|_\infty + \sup_{0 \leqslant t_0 < \cdots < t_n \leqslant 1} \sum_{l=1}^n |x_{t_{l-1}, t_l}|_V.$$

Interpreting the integrals in (4) as Young integrals allows us to assign each path $x \in \mathrm{BV}(V)$ to a path $S(x) := \mathbf{x} \in \Omega(\mathbf{V})$. Furthermore, we call $x \in \mathrm{BV}(V)$ *absolutely continuous* if it admits $\dot{x} \in L^1([0,1], V)$ with $x_{0,t} = \int_0^t \dot{x}_s \,\mathrm{d}s$.

We will make use of the following three operators: The *dilation operator*, which appeared already above, is given by

$$\delta_t : \mathbf{V} \to \mathbf{V}, \qquad \delta_t(\mathbf{u}) = t\mathbf{u}^1 + t^2\mathbf{u}^2, \quad \text{for } t \in \mathbb{R},$$

and the *logarithm* which is defined by

$$\log : \mathbf{V} \to \mathbf{V}, \quad \mathbf{v} \mapsto \mathbf{v} - \frac{1}{2}\mathbf{v} \otimes \mathbf{v}.$$

Both operators naturally extend to continuous functions that map $C(\Delta, \mathbf{V})$ into itself via $\delta_t(\mathbf{x}) = (\delta_t(\mathbf{x}_{s,u}))_{(s,u)\in\Delta}$ and $\log \mathbf{x} = (\log \mathbf{x}_{s,t})_{(s,t)\in\Delta}$. Additionally, we consider the *translation operator* on the set of multiplicative functionals which is defined for $f \in \mathrm{BV}(V)$ by

$$T_f : \Omega(\mathbf{V}) \to \Omega(\mathbf{V}),$$

$$T_f(\mathbf{x})_{s,t} = \mathbf{x}_{s,t} + f_{s,t} + \int_s^t x_{s,u} \otimes \mathrm{d}f_u + \int_s^t f_{s,u} \otimes \mathrm{d}x_u + \int_s^t f_{s,u} \otimes \mathrm{d}f_u.$$

All integrals are well defined Young integrals and restricting the translation operator to more regular paths allows to relax the assumption on f. Note that the definition of T_f is motivated by the following property: for $x \in C([0,1], V)$ for which $\Gamma(x)$ exists (in the sense that the limit converges in $\Omega(\mathbf{V})$) and $f \in \mathrm{BV}(V)$ one has

$$T_f\big(\Gamma(x)\big) = \Gamma(x + f). \tag{5}$$

Finally, we denote by $\pi_V : \mathbf{V} \to V$ and $\pi_{V^{\otimes 2}} : \mathbf{V} \to V^{\otimes 2}$ the projections onto the V- and $V^{\otimes 2}$-component, respectively. In general, we use analogous notation for W-valued paths.

2. The canonical rough path

In this section, we introduce the canonical rough path (called *enhanced Wiener process*) associated to a Banach space-valued Wiener process. The main task will be to establish Lévy's modulus of continuity together with a Fernique type theorem.

As before we let X denote a Wiener process attaining values in a separable Banach space V. Let $D_n = (2^{-n}\mathbb{Z}) \cap [0,1]$ ($n \in \mathbb{N}_0$) and denote by $X(n)$ the linear interpolation of X with dyadic breakpoints D_n. Moreover, let $\mathbf{X}(n) = S(X(n))$. The definition of the *enhanced Wiener process* relies on the following assumption:

(E) There exists an $\Omega(\mathbf{V})$-valued random element $\mathbf{X}$ such that

$$\lim_{n\to\infty} \mathbf{X}(n) = \mathbf{X} \quad \text{in } L^1\big(\mathbb{P}; C(\Delta, \mathbf{V})\big).$$

A sufficient criterion for property (E) is provided in [17]:

Definition 2.1. Let μ be a centered Gaussian measure on the Borel sets of the separable Banach space V. For a fixed tensor product norm $|\cdot|_{V\otimes V}$ the measure μ is called *exact*, if for independent μ-distributed random elements $G_l, \tilde{G}_l$ ($l \in \mathbb{N}$) and some constants C and $\alpha < 1$ one has

$$\mathbb{E}\left|\sum_{l=1}^N G_l \otimes \tilde{G}_l\right|_{V\otimes V} \leqslant CN^\alpha$$

for all $N \in \mathbb{N}$.

By [17, formula on p. 566] property (E) is satisfied if the measure μ is *exact* with respect to the projective tensor norm. From now on we assume that property (E) is satisfied without further mentioning.

In general, we denote, for a path $x \in C([0,1], V)$,

$$\Gamma(x) = \lim_{n\to\infty} S\big(x(n)\big), \tag{6}$$

provided that the limit exists in $C(\Delta, \mathbf{V})$. Here and elsewhere, we denote by $x(n)$ the interpolation of x with dyadic breakpoints D_n. The proposition below allows us to choose $\mathbf{X} = \Gamma(X)$ as the *canonical rough path* (*enhanced Wiener process*) of X.

Proposition 2.2. *The family* $(\log \mathbf{X}(n))_{n\in\mathbb{N}}$ *is a* $C(\Delta, \mathbf{V})$*-valued martingale and one has*

$$\log \mathbf{X}(n) = \mathbb{E}[\log \mathbf{X}|\mathcal{G}_n],$$

where $\mathcal{G}_n = \sigma(X_t\colon t \in D_n) = \sigma(X(n))$. *In particular,*

$$\lim_{n\to\infty} \mathbf{X}(n) = \mathbf{X}, \quad \textit{almost surely}.$$

The proof is based on the following lemma.

Lemma 2.3. *Let* $f = (f_t)_{t\in[0,1]}$ *be a* $\mathrm{BV}(V)$*-valued random element such that almost surely* f *is a one-dimensional excursion, that is* $\dim(\mathrm{span}(\mathrm{im}(f))) = 1$, *almost surely. If the distribution of* f *is symmetric in the sense that* $f \stackrel{\mathcal{L}}{=} -f$ *and if* $\mathbb{E}\|f\|^2_{\mathrm{BV}(V)} < \infty$, *then for an arbitrary path* $\mathbf{x} \in \Omega(\mathbf{V})$ *we have*

$$\mathbb{E}\big[\log T_f(\mathbf{x})\big] = \log \mathbf{x}.$$

Proof. According to the definition of log and the shift operator we have (with $\mathbf{f} = S(f)$)

$$\log T_f(\mathbf{x})_{s,t} = \log \mathbf{x}_{s,t} + \log \mathbf{f}_{s,t} + \frac{1}{2}\Bigg[\int_s^t x_{s,u} \otimes \mathrm{d}f_u + \int_s^t f_{s,u} \otimes \mathrm{d}x_u - \int_s^t \mathrm{d}x_u \otimes f_{s,u} - \int_s^t \mathrm{d}f_u \otimes x_{s,u}\Bigg].$$

Due to the symmetry of the distribution of f we have

$$\mathbb{E}\Bigg[\int_s^t x_{s,u} \otimes \mathrm{d}f_u + \int_s^t f_{s,u} \otimes \mathrm{d}x_u - \int_s^t \mathrm{d}x_u \otimes f_{s,u} - \int_s^t \mathrm{d}f_u \otimes x_{s,u}\Bigg] = 0.$$

The expectation of the first component $\pi_V(\log \mathbf{f}_{s,t})$ vanishes for the same reason. The second component of $\log \mathbf{f}_{s,t}$ is identically zero since f takes values in a one-dimensional subspace of V. Indeed, we can write $f_t = \xi_t \cdot v$ for a V-valued random vector v and a real valued process (ξ_t). Correspondingly, $f_{s,t} = \xi_{s,t} \cdot v$ and therefore

$$\begin{aligned}\pi_{V\otimes V}(\log \mathbf{f}_{s,t}) &= \frac{1}{2}\Bigg[\int_s^t f_{s,u} \otimes \mathrm{d}f_u - \int_s^t \mathrm{d}f_u \otimes f_{s,u}\Bigg] \\ &= \frac{1}{2}\Bigg[\int_s^t \xi_{s,u}\,\mathrm{d}\xi_u - \int_s^t \mathrm{d}\xi_u\,\xi_{s,u}\Bigg] v \otimes v = 0. \qquad \square\end{aligned}$$

Proof of Proposition 2.2. Note that $\Delta X(n) := X(n+1) - X(n)$ can be written as the sum of 2^n independent symmetric one-dimensional excursions that are also independent of the σ-algebra $\mathcal{G}_n$. Applying Lemma 2.3 2^n-times then gives that

$$\mathbb{E}\big[\log \mathbf{X}(n+1)_{s,t} | \mathcal{G}_n\big] = \log \mathbf{X}(n)_{s,t}. \tag{7}$$

Hence, for general $k \in \mathbb{N}$, we have

$$\mathbb{E}\big[\log \mathbf{X}(n+k)_{s,t} | \mathcal{G}_n\big] = \log \mathbf{X}(n)_{s,t}$$

so that by assumption (E)

$$\mathbb{E}[\log \mathbf{X} | \mathcal{G}_n] = \log \mathbf{X}(n).$$

In particular, $\log \mathbf{X}(n)$ converges almost surely to $\log \mathbf{X}$ in $C(\Delta, \mathbf{V})$ (see for instance [6, Proposition 5.3.20]). The inverse of log (that is $\mathbf{x} \mapsto (\mathbf{x} + \frac{1}{2}\mathbf{x} \otimes \mathbf{x})_{(s,t)\in\Delta}$) is continuous on $C(\Delta, \mathbf{V})$, and we also get that $(\mathbf{X}(n))_{n\in\mathbb{N}}$ converges to $\mathbf{X}$. □

Finite dimensional approximation

Based on a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ of the reproducing kernel Hilbert space H_1, we establish a limit theorem for certain finite dimensional approximations to $\mathbf{X}$. We represent X as the in $C([0,1], V)$ almost sure limit

$$X_t = \sum_{i=1}^{\infty} \xi_t^{(i)} e_i,$$

where $\xi^{(i)} = (\xi_t^{(i)})_{t\in[0,1]}$ ($i \in \mathbb{N}$) are independent Wiener processes. For $m \in \mathbb{N}$ denote

$$X^*(m)_t = \sum_{i=1}^{m} \xi_t^{(i)} e_i.$$

Proposition 2.4. *For* $\mathbf{X}^*(m) = \Gamma(X^*(m))$ $(m \in \mathbb{N})$ *one has*

$$\lim_{m\to\infty} \mathbf{X}^*(m) = \mathbf{X}, \quad \textit{almost surely in } \Omega(\mathbf{V}). \tag{8}$$

Remark 2.5. The theorem yields that, for the enhanced Wiener process $\mathbf{X}$,

$$\mathbf{X}_{s,t} \stackrel{\mathcal{L}}{=} \delta_{\sqrt{t-s}}(\mathbf{X}_{0,1}), \quad \text{for } (s,t) \in \Delta,$$

where δ denotes the dilation operator. Indeed, this statement is true for $\mathbb{R}^d$-valued Wiener processes and it can be easily extended via (8).

Proof. We denote the n-th dyadic interpolation of $X^*(m)$ by $X^*(n,m)$ and let $\mathbf{X}^*(n,m) = S(X^*(n,m))$ and $\mathcal{G}_{n,m} = \sigma(X^*(n,m))$. By Lemma A.3, one has, for $n, m \in \mathbb{N}$,

$$\begin{aligned}&\big|\log \mathbf{X}^*(n,m)_{s,t} - \log \mathbf{X}(n)_{s,t}\big|_{\mathbf{V}} \\ &\quad \leqslant 2^{n+2}\big\|X^*(n,m) - X(n)\big\|_\infty\big(1 + \big\|X^*(n,m)\big\|_\infty + \big\|X(n)\big\|_\infty\big).\end{aligned}$$

Thus the Cauchy–Schwarz inequality implies together with the equivalence of moments of Gaussian measures that

$$\lim_{m\to\infty} \log \mathbf{X}^*(n,m) = \log \mathbf{X}(n), \quad \text{in } L^1\big(\mathbb{P}, C(\Delta, \mathbf{V})\big).$$

As in the proof of Proposition 2.2 one verifies that $(\log \mathbf{X}^*(n,m))_{m\in\mathbb{N}}$ is a $(\mathcal{G}_{n,m})_{m\in\mathbb{N}}$-martingale for any fixed $n \in \mathbb{N}$. Hence,

$$\log \mathbf{X}^*(n,m) = \mathbb{E}\big[\log \mathbf{X}(n)|\mathcal{G}_{n,m}\big] = \mathbb{E}[\log \mathbf{X}|\mathcal{G}_{n,m}].$$

Recall that $X^*(m)$ is a Wiener process for which

$$\log \mathbf{X}^*(m) = \lim_{n\to\infty} \log \mathbf{X}^*(n,m) = \lim_{n\to\infty} \mathbb{E}[\log \mathbf{X}|\mathcal{G}_{n,m}] = \mathbb{E}[\log \mathbf{X}|\mathcal{G}_{\infty,m}], \quad \text{in } C(\Delta, \mathbf{V}),$$

where $\mathcal{G}_{\infty,m} = \sigma(\bigcup_{n\in\mathbb{N}} \mathcal{G}_{n,m})$. □

Lévy's modulus of continuity

In the rest of this section, we derive Lévy's modulus of continuity for the enhanced Wiener process, that is $\mathbf{X}$ is almost surely an element of $\Omega_\phi(\mathbf{V})$, where $\phi : (0,1] \to (0,\infty)$ is a fixed increasing function with

$$\lim_{\delta\downarrow 0} \frac{\phi(\delta)}{\sqrt{-\delta \ln \delta}} = 1.$$

Theorem 2.6. *The canonical rough path $\mathbf{X}$ of the Wiener process X is almost surely an element of $\Omega_\phi(\mathbf{V})$ and one has*

$$\mathbb{E}e^{\alpha\|\mathbf{X}\|_\phi^2} < \infty$$

for some $\alpha > 0$.

Remark 2.7. Fernique's theorem is also proven in [14] for p-variation topology. Moreover, finite dimensional analogs can be found in [9].

We remark that a Fernique type result is already known to hold for p-variation norm under the exactness assumption [14]. The proof of this assertion parallels a variant of the proof of the classical statement. Basically our approach relies on the isoperimetric inequality and the Garsia–Rodemich–Rumsey inequality and it is similar to the approach taken in [10].

Lemma 2.8. *For a multiplicative functional* $\mathbf{x} \in \Omega(\mathbf{V})$ *and a function* $f \in \mathrm{BV}(V)$*, one has*

$$\left\| T_f(\mathbf{x})_{s,t} \right\| \leqslant 2\Big[\sup_{s \leqslant u \leqslant v \leqslant t} \|\mathbf{x}_{u,v}\| + \|f\|_{\mathrm{BV}([s,t],V)} \Big], \quad \textit{for } (s,t) \in \Delta,$$

where $\|f\|_{\mathrm{BV}([s,t],V)} := \sup_{s \leqslant t_0 < \cdots < t_n \leqslant t} |x_{t_{l-1},t_l}|_V$.

Proof. We set $\kappa(s,t) = \sup_{s \leqslant u \leqslant v \leqslant t} \|\mathbf{x}_{u,v}\|$ and observe that $\pi_V(T_f(\mathbf{x})) = \mathbf{x}^1 + f$ so that

$$\left|\pi_V\big(T_f(\mathbf{x})_{s,t}\big)\right|_V \leqslant \left|\mathbf{x}^1_{s,t}\right|_V + |f_{s,t}|_V \leqslant \left|\mathbf{x}^1_{s,t}\right|_V + \|f\|_{\mathrm{BV}([s,t],V)}.$$

It remains to analyze the terms in $V^{\otimes 2}$ (the second level):

$$\mathbf{x}^2_{s,t} + \int_s^t f_{s,u} \otimes \mathrm{d}x_u + \int_s^t x_{s,u} \otimes \mathrm{d}f_u + \int_{s,t} f_{s,u} \otimes \mathrm{d}f_u.$$

Note that

$$\left| \int_s^t x_{s,u} \otimes \mathrm{d}f_u \right|_{V\otimes V} \leqslant \kappa(s,t)\|f\|_{\mathrm{BV}([s,t],V)}.$$

The same estimate is valid for $|\int_s^t f_u \otimes \mathrm{d}x_{s,u}|_{V\otimes V}$. Moreover, one has

$$\left| \int_{s,t} f_{s,u} \otimes \mathrm{d}f_u \right|_{V\otimes V} \leqslant \|f\|^2_{\mathrm{BV}([s,t],V)}$$

so that

$$\left|\pi_{V^{\otimes 2}}\big(T_f(\mathbf{x})_{s,t}\big)\right|_{V\otimes V} \leqslant \left|\mathbf{x}^2_{s,t}\right|_{V\otimes V} + \big(\kappa(s,t) + \|f\|_{\mathrm{BV}([s,t],V)}\big)^2.$$

Finally, combining all estimates gives

$$\left\| T_f(\mathbf{x})_{s,t} \right\| \leqslant \|\mathbf{x}_{s,t}\| + \kappa(s,t) + 2\|f\|_{\mathrm{BV}([s,t],V)}. \qquad \square$$

Corollary 2.9. *For an increasing function* $\varphi : (0,1] \to (0,\infty)$ *with* $\frac{1}{\alpha} := \inf_{\delta \in (0,1]} \varphi(\delta)/\sqrt{\delta} > 0$*, one has for all* $\mathbf{x} \in \Omega_\varphi(\mathbf{V})$ *and* f *absolutely continuous with* $\dot f \in L^2([0,1],V)$*,*

$$\left\| T_f(\mathbf{x}) \right\|_\varphi \leqslant 2\big[\alpha\|\mathbf{x}\|_\varphi + \|\dot f\|_{L^2([0,1],V)}\big]$$

and

$$\left\| T_f(\mathbf{x})_{0,1} \right\| \leqslant 2\Big[\sup_{0 \leqslant s \leqslant t \leqslant 1} \|\mathbf{x}_{s,t}\| + \|\dot f\|_{L^2([0,1],V)} \Big].$$

Proof. The proof is an immediate consequence of Lemma 2.8 and the estimate

$$\|f\|_{\mathrm{BV}([s,t],V)} \leqslant \sqrt{t-s}\,\|\dot f\|_{L^2([0,1],V)} \leqslant \alpha\varphi(t-s)\|\dot f\|_{L^2([0,1],V)}. \qquad \square$$

Next, we apply the isoperimetric inequality for Gaussian measures together with the above estimates to infer the following lemma.

Lemma 2.10. *There exists* $\alpha > 0$ *with*

$$\mathbb{E}e^{\alpha\|\mathbf{X}_{0,1}\|^2} < \infty.$$

Moreover, if $\mathbf{X}$ *is in* $\Omega_\phi(\mathbf{V})$ *with positive probability, then there exists* $\beta > 0$ *such that*

$$\mathbb{E}e^{\beta\|\mathbf{X}\|_\phi^2} < \infty.$$

Proof. Fix $\delta > 0$ sufficiently large such that

$$\mathbb{P}(X \in A) \geqslant 1/2$$

for

$$A = \{x \in C([0,1],V):\ \Gamma(x) \text{ exists and } \|\mathbf{x}\|_\infty \leqslant \delta\}.$$

Recall that H is the reproducing kernel Hilbert space of X, and thus we get with the isoperimetric inequality that for $r \geqslant 0$

$$\mathbb{P}\big(X \in \underbrace{A + B_H(0,r)}_{=:A_r}\big) \geqslant \Phi(r),$$

where Φ is the standard normal distribution function. By (5) and Corollary 2.9, we have for $z = x + h \in A_r$ with $x \in A$ and $h \in B_H(0,r)$,

$$\big\|\Gamma(z)_{0,1}\big\| = \big\|T_h\big(\Gamma(x)\big)_{0,1}\big\| \leqslant 2[\delta + \sigma r],$$

where σ is the norm of the canonical embedding of H_1 into V. Hence, we can couple $\|\mathbf{X}_{0,1}\|$ with a standard normal random variable N such that $\|\mathbf{X}_{0,1}\| \leqslant 2[\delta + \sigma(N \vee 0)]$ and Fernique's theorem implies the first assertion. The second assertion is proved analogously. $\square$

Proposition 2.11. *One has* $\mathbf{X} \in \Omega_\phi(\mathbf{V})$*, almost surely.*

This proposition implies together with Lemma 2.10 the assertion of Theorem 2.6.

Proof. The proof is based on the Garsia–Rodemich–Rumsey Lemma [13] and since it is classical we only focus on the crucial facts that allow us to apply the argument.

By Lemma A.1 in Appendix A, $\|\cdot\|$ possesses an equivalent norm $|||\cdot|||$ which satisfies the triangle inequality with respect to the group operation $*$. Moreover, one infers from the scaling property of the Wiener process (Remark 2.5) and Lemma 2.10 that

$$\mathbb{E}e^{\alpha(t-s)^{-1}|||\mathbf{X}_{s,t}|||^2} = \mathbb{E}e^{\alpha|||\mathbf{X}_{0,1}|||^2} < \infty$$

for an $\alpha > 0$. The rest of the proof can be literally translated from classical proofs of that result (see for instance [10]). □

3. Support theorem

Again we assume the validity of property (E), and let $\varphi : (0, 1] \to (0, \infty)$ be an increasing function satisfying (1). The objective of this section is to prove the following support theorem for the enhanced Wiener process:

Theorem 3.1. *If X is a V-valued Wiener process satisfying assumption* (E)*, then* $\mathbf{X}$ *is almost surely an element of $G\Omega_\varphi(\mathbf{V})$ and its range* (*in $G\Omega_\varphi(\mathbf{V})$*) *is the closure of $S(H)$, where H is again the RKHS of X.*

Again we denote by $\varphi, \phi : (0, 1] \to (0, \infty)$ an increasing functions with $\lim_{\delta\downarrow 0} \frac{\phi(\delta)}{\sqrt{-\delta\log\delta}} = 1$. For $\mathbf{x} \in C(\Delta, \mathbf{V})$ we set

$$|||\mathbf{x}|||_\varphi = \sup_{0\leqslant s<t\leqslant 1} \left[\frac{|\mathbf{x}^1_{s,t}|_V}{\varphi(t-s)} + \frac{|\mathbf{x}^2_{s,t}|_{V^{\otimes 2}}}{\varphi(t-s)^2} \right]$$

and we consider the space

$$C_\varphi(\Delta, \mathbf{V}) = \left\{ (\mathbf{x}_{s,t})_{(s,t)\in\Delta} \in C(\Delta, \mathbf{V}) \colon |||\mathbf{x}|||_\varphi < \infty \right\}$$

endowed with the norm $|||\cdot|||_\varphi$. It is a (non-separable) Banach space. Clearly, the distances $\|\cdot\|_\varphi$ and $|||\cdot|||_\varphi$ generate the same topology on $G\Omega_\varphi(\mathbf{V}) \subset C_\varphi(\Delta, \mathbf{V})$. The proof of Theorem 3.1 relies on the following theorem.

Theorem 3.2. *One has almost sure convergence*

$$\lim_{n\to\infty} \mathbf{X}(n) = \mathbf{X} \quad \text{in } G\Omega_\varphi(\mathbf{V}). \tag{9}$$

Remark 3.3. Theorem 3.2 is an extension of Proposition 2.2. With the same techniques we can strengthen the statement of Proposition 2.4 in order to get Theorem 1.3.

The following criterion allows us to verify convergence in $G\Omega_\varphi(\mathbf{V})$ (see [10] for a similar criterion in the finite dimensional setting).

Lemma 3.4. *Let $\mathbf{x}(n)$ $(n \in \mathbb{N})$ and $\mathbf{x}$ be elements of $C_\phi(\Delta, \mathbf{V})$. If*

- $\sup_{n\in\mathbb{N}} |||\mathbf{x}(n)|||_\phi < \infty$ *and*
- $(\mathbf{x}(n))_{n\in\mathbb{N}}$ *converges to* $\mathbf{x}$ *in* $C(\Delta, \mathbf{V})$,

then $(\mathbf{x}(n))_{n\in\mathbb{N}}$ converges in $C_\varphi(\Delta, \mathbf{V})$ to $\mathbf{x}$.

Proof. Set $\eta(t) := \phi(t)/\varphi(t)$ $(t \in [0,1])$ and note that for $M = |||\mathbf{x}|||_\phi \vee \sup_{n\in\mathbb{N}} |||\mathbf{x}(n)|||_\phi$,

$$\left|\mathbf{x}(n)^1_{s,t} - \mathbf{x}^1_{s,t}\right|_V \leqslant 2M\phi(t-s) = 2M\eta(t-s)\varphi(t-s).$$

On the other hand,

$$\left|\mathbf{x}(n)^1_{s,t} - \mathbf{x}^1_{s,t}\right|_V \leqslant \frac{|||\mathbf{x}(n) - \mathbf{x}|||_\infty}{\varphi(t-s)}\varphi(t-s),$$

where $|||\cdot|||_\infty$ denotes the supremum norm $|||\mathbf{x}|||_\infty = \sup_{(s,t)\in\Delta} |\mathbf{x}_{s,t}|_\mathbf{V}$. Recall that $\eta(t)$ converges to 0 for $t \downarrow 0$ so that for arbitrary $\varepsilon > 0$ there exists $\zeta \in (0,1)$ with $\eta(\zeta) \leqslant \varepsilon/(2M)$. We apply the first estimate in the case where $t - s \leqslant \zeta$ and apply the second estimate otherwise:

$$\left|\mathbf{x}(n)^1_{s,t} - \mathbf{x}^1_{s,t}\right| \leqslant \left[\varepsilon \vee \frac{|||\mathbf{x}(n) - \mathbf{x}|||_\infty}{\varphi(\zeta)}\right]\varphi(t-s).$$

Thus

$$\limsup_{n\to\infty} \left|\left|\left|\mathbf{x}(n)^1 - \mathbf{x}^1\right|\right|\right|_\varphi \leqslant \varepsilon.$$

Analogously, one shows that $\limsup_{n\to\infty} |||\mathbf{x}(n)^2 - \mathbf{x}^2|||_\varphi \leqslant \varepsilon$ so that the assertion follows by the triangle inequality and by noticing that $\varepsilon > 0$ is arbitrary. □

From now on we denote for $n \in \mathbb{N}$ by Υ_n the canonical embedding of $C(D_n, V)$ into $C([0,1], V)$ which linearly interpolates the values at the breakpoints $D_n = (2^{-n}\mathbb{Z}) \cap [0,1]$. With slight misuse of notation we sometimes also apply Υ_n on functions in $C([0,1], V)$.

Proof of Theorem 3.2. By Proposition 2.2, one has

$$\mathbb{E}[\log \mathbf{X}|\mathcal{G}_n] = \log \mathbf{X}(n)$$

and we apply Jensen's inequality [6, Thm. 5.1.15] to get

$$\begin{aligned}\left|\left|\left|\log \mathbf{X}(n)\right|\right|\right|_\phi &= \left|\left|\left|\mathbb{E}\left[\log \mathbf{X}(n+1)|\mathcal{G}_n\right]\right|\right|\right|_\phi \leqslant \mathbb{E}\left[\left|\left|\left|\log \mathbf{X}(n+1)\right|\right|\right|_\phi |\mathcal{G}_n\right] \\ &\leqslant \mathbb{E}\left[|||\log \mathbf{X}|||_\phi |\mathcal{G}_n\right]. \end{aligned} \tag{10}$$

Hence, $(|||\log \mathbf{X}(n)|||_\phi)_{n\in\mathbb{N}\cup\{\infty\}}$ is a submartingale with

$$\mathbb{E}\left|\left|\left|\log \mathbf{X}(n)\right|\right|\right|_\phi \leqslant \mathbb{E}|||\log \mathbf{X}|||_\phi \leqslant \mathbb{E}|||\mathbf{X}|||_\phi + 2\mathbb{E}|||\mathbf{X}|||_\phi^2 < \infty.$$

Here, we used the general inequality $|||\log \mathbf{x}|||_\phi \leqslant |||\mathbf{x}|||_\phi + 2|||\mathbf{x}|||_\phi^2$ and Theorem 2.6. Consequently, the submartingale converges almost surely to a finite value so that $\sup_{n\in\mathbb{N}} |||\mathbf{X}(n)|||_\phi$ is almost surely finite, and the statement is an immediate consequence of Lemma 3.4. □

Next, we consider $\bar{X}(-n) = X - X(n)$ for $n \in \mathbb{N}_0$ together with its enhanced process

$$\bar{\mathbf{X}}(-n) = T_{-X(n)}\mathbf{X} = \Gamma\big(\bar{X}(n)\big).$$

Lemma 3.5. *The sequence* $(\log \bar{\mathbf{X}}(k))_{k\in-\mathbb{N}_0}$ *is a* $C_\varphi(\Delta,\mathbf{V})$*-valued* $(\bar{\mathcal{G}}_k)$*-martingale, where* $\bar{\mathcal{G}}_k = \sigma(\bar{X}(k))$ $(k\in-\mathbb{N}_0)$. *Moreover, one has*

$$\lim_{k\to-\infty} \bar{\mathbf{X}}(k) = 0, \quad \textit{almost surely.}$$

Proof. Set $\Delta X_k = X(k+1) - X(k)$ and $\bar{\mathcal{G}}_k = \sigma(\bar{X}_k)$. Notice that as in the proof of Lemma 3.2 the processes ΔX_k are independent of $\bar{\mathcal{G}}_k$ and they can be written as a finite sum of one-dimensional symmetric excursions so that

$$\log \bar{\mathbf{X}}(k) = \mathbb{E}\big[\log \bar{\mathbf{X}}(k+1)|\bar{\mathcal{G}}_k\big].$$

Hence, $(\log \bar{\mathbf{X}}(k))_{k\in-\mathbb{N}_0}$ defines a $C_\varphi(\Delta,\mathbf{V})$-valued martingale. Applying a convergence theorem for reversed martingales (see for instance [6, p. 213]) we obtain almost sure convergence

$$\lim_{k\to-\infty} \log \bar{\mathbf{X}}(k) = \mathbb{E}[\log \mathbf{X}|\bar{\mathcal{G}}_{-\infty}],$$

in $C_\varphi(\Delta,\mathbf{V})$, where $\bar{\mathcal{G}}_{-\infty} = \bigcap_{k\in-\mathbb{N}} \bar{\mathcal{G}}_k$. Since $\bar{\mathcal{G}}_{-\infty}$ is a tail σ-field it only contains 0-1-events. Thus using the symmetry of $\log \mathbf{X}$ the limit has to be zero. □

We are now in a position to prove the support theorem.

Proof of Theorem 3.1. Let $X(n)$ and $\bar{X}(n)$ be as above. We use the standard notation for their enhanced processes and abridge $\Omega_\varphi = \Omega_\varphi(V)$. As a consequence of the standard support theorem and Lemma A.3 one has

$$\mathrm{range}_{\Omega_\varphi}\big(\mathbf{X}(n)\big) = \mathrm{range}_{\Omega_\varphi}\big(S\circ\Upsilon_n(X)\big) = \overline{S\circ\Upsilon_n(H)}^{\Omega_\varphi} \subset \overline{S(H)}^{\Omega_\varphi}.$$

By Theorem 3.2, the family $(\mathbf{X}(n))_{n\in\mathbb{N}}$ converges almost surely in $\Omega_\varphi(\mathbf{V})$ to $\mathbf{X}$ so that

$$\mathrm{range}_{\Omega_\varphi}(\mathbf{X}) \subset \overline{S(H)}^{\Omega_\varphi}.$$

For the converse statement fix $f\in H$ and let $\mathbf{f} = S(f)$. Due to the continuity of T_f and T_{-f} (see Lemma A.2) one has

$$\forall\varepsilon>0: \quad P\big(\mathbf{X}\in B_{\Omega_\varphi}(\mathbf{f},\varepsilon)\big) > 0$$

if and only if

$$\forall\varepsilon>0: \quad P\big(T_{-f}\mathbf{X}\in B_{\Omega_\varphi}(0,\varepsilon)\big) > 0.$$

Due to the Cameron Martin Theorem the latter statement is equivalent to

$$\forall\varepsilon>0: \quad P\big(\mathbf{X}\in B_{\Omega_\varphi}(0,\varepsilon)\big) > 0.$$

On the other hand, we have $\mathbf{X} = T_{X(n)}(\bar{\mathbf{X}}(n))$, and by Corollary 2.9 there exists a constant $c = c(\varphi)$ such that for any $\varepsilon > 0$:

$$\begin{aligned}\mathbb{P}(\|\mathbf{X}\|_\varphi \leqslant 2c\varepsilon) &\geqslant \mathbb{P}\big(\|\bar{\mathbf{X}}(n)\|_\varphi \leqslant \varepsilon, \|\dot{X}(n)\|_{L^2([0,1],V)} \leqslant \varepsilon\big)\\ &\geqslant \underbrace{\mathbb{P}\big(\|\bar{\mathbf{X}}(n)\|_\varphi \leqslant \varepsilon\big)}_{\to 1\ (n\to\infty)} \underbrace{\mathbb{P}\big(\|\dot{X}(n)\|_{L^2([0,1],V)} \leqslant \varepsilon\big)}_{>0}.\end{aligned}$$

In the last step we have used the independence of $X(n)$ and $\bar{X}(n)$. □

4. Large deviations

Let again X denote a V-valued Wiener process satisfying assumption (E) with enhanced process $\mathbf{X}$. For $\varepsilon > 0$ let $X^\varepsilon = (X_t^\varepsilon)_{t\in[0,1]} = (\varepsilon X_t)_{t\in[0,1]}$ and recall that the family $\{X^\varepsilon\colon\ \varepsilon > 0\}$ satisfies a large deviation principle in $C([0,1],V)$ with good rate function

$$J(h) = \begin{cases} \frac{1}{2}\|h\|_H^2 & \text{if } h \in H,\\ \infty & \text{else.}\end{cases}$$

The aim of this section is to prove:

Theorem 4.1. *The family $\{\mathbf{X}^\varepsilon\colon\ \varepsilon > 0\}$ given by $\mathbf{X}^\varepsilon = (\delta_\varepsilon(\mathbf{X}_t))_{t\in[0,1]} = \Gamma(X^\varepsilon)$ satisfies a LDP in $G\Omega_\varphi(\mathbf{V})$ with good rate function*

$$J(\mathbf{h}) = \begin{cases} \frac{1}{2}\|h\|_H^2 & \text{if } \exists h \in H \text{ with } \mathbf{h} = S(h),\\ \infty & \text{else}.\end{cases}$$

Similarly as in [17] and [10], the proof uses the concept of exponentially good approximations. It mainly relies on the isoperimetric inequality and the following estimate.

Lemma 4.2. *Let $n \in \mathbb{N}$, and denote by $h : [0,1] \to V$ an absolutely continuous function with $\dot{h} \in L^2([0,1],V)$. Set $f = \Upsilon_n(h)$ and $g = h - f$. There exists a universal constant C such that for $\mathbf{x},\mathbf{y} \in \Omega_\varphi(\mathbf{V})$ and $\kappa = \|\mathbf{x}\|_\varphi + \|\mathbf{y}\|_\varphi$, one has*

$$\big\|T_h(\mathbf{x}) - T_f(\mathbf{y})\big\|_\varphi \leqslant \left(1 + 2\frac{\beta}{\beta_n}\right)\big(\|\mathbf{x}\|_\varphi + \|\mathbf{y}\|_\varphi\big) + 4\beta_n\|\dot{h}\|_{L^2([0,1],V)},$$

where $\beta = \sup_{\delta\in(0,1]} \frac{\sqrt{\delta}}{\varphi(\delta)}$ and $\beta_n = \sup_{\delta\in(0,1\wedge 2^{1-n/2}]} \frac{\sqrt{\delta}}{\varphi(\delta)} \to 0$ as $n \to \infty$.

Proof. We denote $x = \mathbf{x}^1$, $y = \mathbf{y}^1$, $f = \Upsilon_n(h)$ and $g = h - f$. By Jensen's inequality one has $\|\dot{f}\|_{L^2([0,1],V)} \leqslant \|\dot{h}\|_{L^2([0,1],V)}$ and thus the triangle inequality gives that $\|\dot{g}\|_{L^2([0,1],V)} \leqslant 2\|\dot{h}\|_{L^2([0,1],V)}$. Moreover,

$$\begin{aligned}T_h(\mathbf{x})_{s,t} - T_f(\mathbf{y})_{s,t} = \mathbf{x}_{s,t} + S(f+g)_{s,t} + \int_s^t (f+g)_{s,u}\otimes \mathrm{d}x_u + \int_s^t x_{s,u}\otimes \mathrm{d}(f+g)_u\\ - \mathbf{y}_{s,t} - S(f)_{s,t} - \int_s^t f_{s,u}\otimes \mathrm{d}y_u - \int_s^t y_{s,t}\otimes \mathrm{d}f_u\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{x}_{s,t} - \mathbf{y}_{s,t} + g_{s,t} + \int_s^t h_{s,u} \otimes \mathrm{d}g_u + \int_s^t g_{s,u} \otimes \mathrm{d}f_u \\
&\quad + \int_s^t f_{s,u} \otimes \mathrm{d}(x-y)_u + \int_s^t g_{s,u} \otimes \mathrm{d}x_u \\
&\quad + \int_s^t (x-y)_{s,u} \otimes \mathrm{d}f_u + \int_s^t x_{s,u} \otimes \mathrm{d}g_u. \qquad (11)
\end{aligned}
$$

We need to control the norms of each single term above. We start with $|\int_s^t h_{s,u} \otimes \mathrm{d}g_u|_{V^{\otimes 2}}$. Clearly,

$$
\left| \int_s^t h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} \leqslant \|h_{s,\cdot}\|_{L^\infty([s,t],V)} \|\dot{g}\|_{L^1([s,t],V)} \leqslant 2(t-s)\|\dot{h}\|^2_{L^2([s,t],V)}.
$$

For $t - s \geqslant 2^{-n}$ one can refine this estimate as follows. Let $s \leqslant t_0 \leqslant \cdots \leqslant t_N \leqslant t$ with $\{t_0, \ldots, t_N\} = D_n \cap [s,t]$, and observe that

$$
\left| \int_s^t h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} \leqslant \left| \int_s^{t_0} h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} + \sum_{i=0}^{N-1} \left| \int_{t_i}^{t_{i+1}} h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} + \left| \int_{t_N}^t h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}}.
$$

Since $t_0 - s \leqslant 2^{-n}$ the first term is bounded by $2^{-n+1}\|\dot{h}\|^2_{L^2([s,t_0],V)}$. For $v \in V$ one has $\int_{t_i}^{t_{i+1}} v \otimes \mathrm{d}g_u = 0$ so that

$$
\left| \int_{t_i}^{t_{i+1}} h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} = \left| \int_{t_i}^{t_{i+1}} h_{t_i,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} \leqslant 2^{-n+1}\|\dot{h}\|^2_{L^2([t_i,t_{i+1}],V)}.
$$

Moreover, the remaining term is bounded by

$$
\left| \int_{t_N}^t h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} \leqslant \|h_{s,\cdot}\|_{L^\infty([s,t],V)} \|\dot{g}\|_{L^1([t_N,t],E)} \leqslant 2^{-n/2+1}\|\dot{h}\|^2_{L^2([0,1],V)}.
$$

Combining the above estimates yields

$$
\left| \int_s^t h_{s,u} \otimes \mathrm{d}g_u \right|_{V^{\otimes 2}} \leqslant 2\big(2^{-n} + 2^{-n/2}\big)\|\dot{h}\|^2_{L^2([0,1],V)} \leqslant 2^{-n/2+2}\|\dot{h}\|^2_{L^2([0,1],V)}
$$

so that in general

$$\left|\int_s^t h_{s,u}\otimes \mathrm{d}g_u\right|_{V^{\otimes 2}} \leqslant 2\big[(t-s)\wedge 2^{1-n/2}\big]\|\dot h\|^2_{L^2([0,1],V)}.$$

Similarly, one finds the same estimate for $\int_s^t g_{s,u}\otimes \mathrm{d}f_u$ and

$$|g_{s,t}|_V \leqslant 2\sqrt{(t-s)\wedge 2^{1-n}}\,\|\dot h\|_{L^2([0,1],V)}.$$

We proceed with the next term in (11):

$$\begin{aligned}\left|\int_s^t f_{s,u}\otimes \mathrm{d}(x-y)_u\right|_{V^{\otimes 2}} &\leqslant \varphi(t-s)\sqrt{t-s}\,\|x-y\|_\varphi\|\dot h\|_{L^2([0,1],V)}\\ &\leqslant 2\kappa\varphi(t-s)\sqrt{t-s}\,\|\dot h\|_{L^2([0,1],V)},\end{aligned}$$

where $\kappa := \|\mathbf{x}\|_\varphi + \|\mathbf{y}\|_\varphi$. Analogously one finds that also the remaining three terms from (11) have norm smaller or equal to $2\kappa\varphi(t-s)\sqrt{t-s}\|\dot h\|_{L^2([0,1],V)}$. We now combine the above estimates:

$$\begin{aligned}&\big\|T_h(\mathbf{x})_{s,t}-T_f(\mathbf{y})_{s,t}\big\|\\ &\quad\leqslant \kappa\varphi(t-s)+2\sqrt{(t-s)\wedge 2^{1-n}}\,\|\dot h\|_{L^2([0,1],V)}\\ &\qquad+2\sqrt{\big[(t-s)\wedge 2^{1-n/2}\big]\|\dot h\|^2_{L^2([0,1],V)}+2\kappa\varphi(t-s)\sqrt{t-s}\,\|\dot h\|_{L^2([0,1],V)}}.\end{aligned}$$

Next, we use that $\sqrt{(t-s)\wedge 2^{1-n/2}} \leqslant \beta_n\varphi(t-s)$ to conclude that

$$\begin{aligned}&\big\|T_h(\mathbf{x})_{s,t}-T_f(\mathbf{y})_{s,t}\big\|\\ &\quad\leqslant \Big(\kappa+2\beta_n\|\dot h\|_{L^2([0,1],V)}+2\sqrt{\beta_n^2\|\dot h\|^2_{L^2([0,1],V)}+2\kappa\beta\|\dot h\|_{L^2([0,1],V)}}\Big)\varphi(t-s)\\ &\quad\leqslant \left[\left(1+2\frac{\beta}{\beta_n}\right)\kappa+4\beta_n\|\dot h\|_{L^2([0,1],V)}\right]\varphi(t-s). \qquad \square\end{aligned}$$

Proof of Theorem 4.1. We will use the concept of exponentially good approximations. Recall that by Lemma A.3, the map $S\circ\Upsilon_n : C([0,1],E)\to G\Omega_\varphi(E)$ is continuous. Therefore, the processes $\mathbf{X}^\varepsilon(n) = S(X^\varepsilon(n))$ $(\varepsilon>0)$ satisfy a large deviation principle with good rate function

$$J_n(\mathbf{h}) = \begin{cases}\frac12\|h\|_H^2 & \text{if } \exists h\in H\colon \mathbf{h}=S(h) \text{ and } h=\Upsilon_n(h),\\ \infty & \text{else.}\end{cases}$$

It remains to show that the approximation is exponentially good (see [4, Thm. 4.2.23]), in the sense that for every $\delta>0$,

$$\lim_{n\to\infty}\limsup_{\varepsilon\downarrow 0}\varepsilon^2\log\mathbb{P}\big(d_\varphi\big(\mathbf{X}^\varepsilon,\mathbf{X}^\varepsilon(n)\big)>\delta\big)=-\infty \tag{12}$$

and that for every $\alpha>0$,

$$\lim_{n\to\infty}\sup\big\{d_\varphi\big(S(x),S\big(\Upsilon_n(x)\big)\big)\colon\ x\in H,\ \|x\|_H\leqslant\alpha\big\}=0. \tag{13}$$

We start with verifying (12). Recall that we can fix $\kappa>0$ such that

$$\mathbb{P}(X\in A)\geqslant\frac{1}{2}$$

for

$$A=\Big\{x\in C\big([0,1],V\big)\colon\ x_0=0,\ \Gamma(x)\text{ exists},\ \big\|\Gamma(x)\big\|_\varphi+\sup_{n\in\mathbb{N}}\big\|\Upsilon_n(x)\big\|_\varphi\leqslant\kappa\Big\}.$$

With the isoperimetric inequality we conclude that $\mathbb{P}(X\in A_r)\geqslant\Phi(r)$ for the sets

$$A_r=A+B_H(0,r)\quad(r\geqslant 0).$$

Here Φ denotes again the standard normal distribution function. By Lemma 4.2 it follows that for $z=x+h\in A_r$ (with $x\in A$ and $h\in B_H(0,r)$):

$$\big\|\mathbf{z}-\mathbf{z}(n)\big\|_\varphi=\big\|T_h(\mathbf{x})-T_f\big(\mathbf{x}(n)\big)\big\|_\varphi\leqslant C\big[\beta_n\sigma\|h\|_H+\kappa(1+\beta/\beta_n)\big],$$

where β and $\beta_n\to 0$ are as in the lemma and σ is the norm of the canonical embedding of H_1 into V. Hence, we can find a standard normal random variable N (on a possibly larger probability space) such that for any $n\in\mathbb{N}$:

$$\big\|\mathbf{X}-\mathbf{X}(n)\big\|_\varphi\leqslant C\big[\beta_n\sigma N_+ +\kappa(1+\beta/\beta_n)\big],$$

where $N_+=N\vee 0$. Now choose $\alpha_n=\frac{1}{3}(C^2\sigma^2\beta_n^2)^{-1}$. Then $\lim_{n\to\infty}\alpha_n=\infty$ and

$$\begin{aligned}C_n:=\mathbb{E}e^{\alpha_n\|\mathbf{X}-\mathbf{X}(n)\|_\varphi^2}&\leqslant\mathbb{E}\big[\exp\big\{\alpha_n C^2\big(\beta_n\sigma N_+ +\kappa(1+\beta/\beta_n)\big)^2\big\}\big]\\&=\mathbb{E}\bigg[\exp\bigg\{\frac{1}{3}N_+^2+\mathcal{O}(N_+)\bigg\}\bigg]<\infty.\end{aligned}$$

By Chebychev's inequality we get for $\eta\geqslant 0$:

$$\mathbb{P}\big(\big\|\mathbf{X}-\mathbf{X}(n)\big\|_\varphi\geqslant\eta\big)\leqslant C_n e^{-\alpha_n\eta^2}$$

and hence

$$\limsup_{\eta\to\infty}\frac{1}{\eta^2}\log\mathbb{P}\big(\big\|\mathbf{X}-\mathbf{X}(n)\big\|_\varphi\geqslant\eta\big)\leqslant-\alpha_n\to-\infty\quad\text{as } n\to\infty.$$

Now assertion (12) is a consequence of

$$\|\mathbf{X}^\varepsilon - \mathbf{X}^\varepsilon(n)\|_\varphi = \|\delta_\varepsilon(\mathbf{X}) - \delta_\varepsilon(\mathbf{X}(n))\|_\varphi = \|\delta_\varepsilon(\mathbf{X} - \mathbf{X}(n))\|_\varphi = \varepsilon\|\mathbf{X} - \mathbf{X}(n)\|_\varphi.$$

Finally note that (13) is an immediate consequence of Lemma 4.2. Indeed, for $h \in H$ and $f = \Upsilon_n(h)$ one gets

$$\|S(h) - S(f)\|_\varphi = \|T_h(0) - T_f(0)\|_\varphi \leqslant 3\beta_n \|\dot{h}\|_{L^2([0,1],E)} \leqslant 3\sigma\beta_n \|\dot{h}\|_H$$

with $\lim_{n\to\infty} \beta_n = 0$. □

5. A sufficient criterion for exactness

In this section we introduce a new sufficient criterion for the exactness of a Gaussian random vector attaining values in a Hölder–Zygmund space.

For $n \in \mathbb{N}$ and for an n-times continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}^d$, we set

$$\|f\|_{\mathcal{C}^n} = \sum_{|\alpha| \leqslant n} \sup_{x \in \mathbb{R}^d} |\partial^\alpha f(x)|,$$

where the sum is taken over all multiindices α with entries in $\{1, \dots, d\}$ of length smaller or equal to n. Moreover, for $\gamma = n + \eta$ with $n \in \mathbb{N}_0$ and $\eta \in (0, 1]$, and an n-times continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}^d$, we consider the *Hölder–Zygmund norm* of order γ

$$\|f\|_{C^\gamma} = \|f\|_{\mathcal{C}^n} + \sum_{|\alpha| = n} \sup_{x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\eta} \tag{14}$$

and we denote by $C_0^\gamma(D, \mathbb{R}^d)$ ($D \subset \mathbb{R}^d$ open) the set of n-times continuously differentiable functions satisfying

$$f|_{D^c} \equiv 0 \quad \text{and} \quad \|f\|_{C^\gamma} < \infty.$$

It is endowed with the norm $\|\cdot\|_{C^\gamma}$.

Theorem 5.1. *Let $0 \leqslant \gamma < \bar{\gamma}$ and let $D, D' \subset \mathbb{R}^d$ denote bounded open sets with $\bar{D} \subset D'$. Every centered Gaussian measure μ on $C_0^{\bar{\gamma}}(D, \mathbb{R}^d)$ is exact and Bochner measurable when viewed as Gaussian measure on $C_0^\gamma(D', \mathbb{R}^d)$.*

The proof is based on the concept of finite dimensional approximation introduced in [17]. For a Banach space V (not necessarily separable) and a V-valued random vector Y, we denote the *linear average Kolmogorov widths* of Y by

$$\ell_n(Y) = \ell_n^V(Y) = \inf\{\mathbb{E}|Y - T_n(Y)|_V:\ T_n : V \to V \text{ linear},\ \mathrm{rk}(T_n) \leqslant n\} \quad (n \in \mathbb{N}).$$

Lemma 5.2. *Let Y be a V-valued Gaussian random vector and suppose that $\ell_n(Y) \lesssim n^{-\varepsilon}$ for some $\varepsilon > 0$. Then Y is exact and Bochner measurable in V.*

Proof. Since $\ell_n(Y)$ decays to zero Y is Bochner measurable. Without loss of generality we assume that V is infinite dimensional. Let G_1 be a μ-distributed r.v. For fixed $n \in \mathbb{N}$ there exists a bounded operator $T_n : V \to V$ with n-dimensional range and

$$\mathbb{E}\left|G_1 - T_n(G_1)\right| \leqslant 2\ell_n(Y) =: \varepsilon(n).$$

Set $F_1 := T_n(G_1)$ and observe that there are n independent standard normals $\xi_1^1, \ldots, \xi_n^1$ and n vectors $e_1, \ldots, e_n \in V$ such that

$$F_1 = \sum_{i=1}^n \xi_i^1 e_i.$$

Moreover, set $H_1 = G_1 - F_1$.

Then for each $i = 1, \ldots, n$ one has

$$\sqrt{2/\pi}\,|e_i|_V = \mathbb{E}\left|\xi^1 e_i\right| \leqslant \mathbb{E}|F_1| \leqslant \mathbb{E}|G_1| + \mathbb{E}|H_1| \leqslant C,$$

where $C := C(Y) := 2\ell_1(Y) + \mathbb{E}|Y|$ does only depend on the distribution of Y.

Let now $(G_l, F_l, H_l, (\xi_i^l)_{i=1,\ldots,n})_{l\geqslant 2}$ and $(\tilde{G}_l, \tilde{F}_l, \tilde{H}_l, (\tilde{\xi}_i^l)_{i=1,\ldots,n})_{l\in\mathbb{N}}$ denote independent copies of $(G_1, F_1, H_1, (\xi_i^1)_{i=1,\ldots,n})$. Then

$$\sum_{l=1}^N G_l \otimes \tilde{G}_l = \sum_{l=1}^N F_l \otimes \tilde{F}_l + \sum_{l=1}^N F_l \otimes \tilde{H}_l + \sum_{l=1}^N H_l \otimes \tilde{F}_l + \sum_{l=1}^N H_l \otimes \tilde{H}_l \tag{15}$$

and

$$\mathbb{E}\left|\sum_{l=1}^N F_l \otimes \tilde{F}_l\right|_{V^{\otimes 2}} \leqslant \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\left|\sum_{l=1}^N \xi_i^l \tilde{\xi}_j^l\right| |e_i \otimes e_j|_{V^{\otimes 2}}.$$

Using that $\mathbb{E}|\sum_{l=1}^N \xi_i^l \tilde{\xi}_j^l| \leqslant \sqrt{N}$ we arrive at

$$\mathbb{E}\left|\sum_{l=1}^N F_l \otimes \tilde{F}_l\right|_{V^{\otimes 2}} \leqslant d^2\sqrt{N} \max_{i=1,\ldots,n} |e_i|_V^2 \leqslant \frac{\pi}{2} C^2 d^2 \sqrt{N}. \tag{16}$$

On the other hand,

$$\mathbb{E}\left|\sum_{l=1}^N F_l \otimes \tilde{H}_l\right|_{V^{\otimes 2}} \leqslant N\mathbb{E}|F_1|\mathbb{E}|\tilde{H}_1| \leqslant N\varepsilon(n)\mathbb{E}|F_1| \leqslant CN\varepsilon(n). \tag{17}$$

The same estimate holds for $\mathbb{E}|\sum_{l=1}^N H_l \otimes \tilde{F}_l|_{V^{\otimes 2}}$. Finally, the last term in (15) is bounded by

$$\mathbb{E}\left|\sum_{l=1}^N H_l \otimes \tilde{H}_l\right|_{V^{\otimes 2}} \leqslant N\big(\mathbb{E}|H_1|\big)^2 \leqslant N\varepsilon(n)^2. \tag{18}$$

When choosing $n = n(N) = \lfloor N^{1/(4+2\varepsilon)} \rfloor$ and letting n tend to infinity one obtains with (16), (17) and (18) that

$$\mathbb{E}\left|\sum_{l=1}^{N} G_l \otimes \tilde{G}_l\right| \precsim N^{(4+\varepsilon)/(4+2\varepsilon)}, \quad \text{as } N \to \infty,$$

which implies exactness. □

In the forthcoming proof of Theorem 5.1, we use a result by Pisier [25] that provides an estimate for the average Kolmogorov width against entropy numbers of generating operators. For two Banach spaces E and V, and a compact operator $\rho : E \to V$ we define the *n-th entropy number* as

$$e_n(\rho) = \inf\left\{\varepsilon > 0\colon\ u\big(B_E(0,1)\big) \subset \bigcup_{j=1}^{2^{n-1}} B_V(b_j, \varepsilon) \text{ for some } b_1, \ldots, b_{2^{n-1}} \in V\right\}.$$

Proof of Theorem 5.1. We denote by ρ the canonical embedding of the reproducing kernel Hilbert space of Y into $C_0^{\bar{\gamma}}(D, \mathbb{R}^d)$. By Pajor and Tomczak-Jaegermann [24], one has

$$e_n(\rho) \precsim n^{-1/2}.$$

The asymptotic behavior of the entropy numbers for general Besov embeddings were studied by Edmunds and Triebel [7,8]. In particular, one has for the canonical embedding $\varrho : C_0^{\bar{\gamma}}(D, \mathbb{R}^d) \to C_0^{\gamma}(D', \mathbb{R}^d)$ that

$$e_n(\varrho) \asymp n^{-\frac{\bar{\gamma}-\gamma}{d}}.$$

Combining the above estimates with the general estimate $e_{k+l-1}(\varrho \circ \rho) \leqslant e_k(\varrho) e_l(\rho)$ $(k, l \in \mathbb{N}_0)$ gives

$$e_n(\varrho \circ \rho) \precsim n^{-\frac{1}{2}-\frac{\bar{\gamma}-\gamma}{d}}.$$

Now note that $\varrho \circ \rho$ generates the Gaussian random element Y on $C_0^{\gamma}(D', \mathbb{R}^d)$. In order to control $\ell(Y)$ we use a result of Pisier (see [25, Thm. 9.1, p. 141]) combined with the duality of metric entropy found in [1]:

$$\ell_n(Y) \leqslant C_1 \sum_{k \geqslant C_2 n} k^{-1/2} (\log k) e_k(\varrho \circ \rho)$$

for two universal constants $C_1, C_2 > 0$. Combining this estimate with the above result on the entropy numbers gives

$$\ell_n(Y) \precsim n^{-\frac{\bar{\gamma}-\gamma}{d}} \log n. \qquad \square$$

6. Consequences of Lyons' universal limit theorem

In this section, we use Lyons' universal limit theorem together with our findings to derive Theorem 1.7. Let V and W denote two Banach spaces and let $f : W \to L(V, W)$ a Lip(γ)-function for a $\gamma > 2$ in the sense of [23, Def. 1.21]. As in the introduction, I_g denotes the solution operator for controls $x \in \mathrm{BV}(V)$ to the differential equation

$$\mathrm{d}y_t = f(y_t)\,\mathrm{d}[x+g]_t, \quad y_0 = \xi,$$

where $g \in C([0,1], V)$ is an absolutely continuous function with $\dot{g} \in L^2([0,1], V)$.

Next, fix a real $p \in (2, 3 \wedge \gamma)$ and an increasing function $\bar{\varphi} : (0,1] \to (0,\infty)$ that is dominated by φ and satisfies $\lim_{\delta \downarrow 0} \frac{-\delta \log \delta}{\bar{\varphi}(\delta)} = 0$ such that $\bar{\varphi}^p$ is convex. By Lemma A.4, there always exists an appropriate $\bar{\varphi}$ and since $C_{\bar{\varphi}}([0,1], W)$ is continuously embedded into $C_{\varphi}([0,1], W)$, it suffices to prove Theorem 1.7 in $C_{\bar{\varphi}}([0,1], W)$.

By choice of $\bar{\varphi}$ the space $G\Omega_{\bar{\varphi}}(\mathbf{V})$ is continuously embedded into $G\Omega_p(\mathbf{V})$, where $G\Omega_p(\mathbf{V})$ denotes the space of all geometric rough paths induced by the p-variation norm. Hence, the universal limit theorem (see for instance [23, Thm. 5.3]) implies that the rough path differential equation

$$\mathrm{d}\mathbf{y}_t = f(\mathbf{y}_t)\,\mathrm{d}\mathbf{x}_t, \quad \mathbf{y}_0 = \mathrm{id}_D$$

induces a continuous solution operator $\mathbf{I} : G\Omega_{\bar{\varphi}}(\mathbf{V}) \to G\Omega_{\bar{\varphi}}(\mathbf{W})$ (Itô map) and the following diagram commutes for any piecewise linear V-valued path x:

$$\begin{array}{ccccc}
\mathbf{x} & \xrightarrow{T_g} & \mathbf{x}_g & \xrightarrow{\mathbf{I}_0} & \mathbf{y} \\
{\scriptstyle S}\uparrow & & & & \downarrow{\scriptstyle \xi+\pi_W(\cdot)} \\
x & & \xrightarrow{\quad I \quad} & & y
\end{array}$$

Assuming assumption (E), the processes $\{\mathbf{X}(n)\colon n \in \mathbb{N}\}$ converge in $G\Omega_{\bar{\varphi}}(V)$ to $\mathbf{X}$. By the continuity of $T_g : G\Omega_{\bar{\varphi}}(V) \to G\Omega_{\bar{\varphi}}(V)$ (Lemma A.2), $\mathbf{I} : G\Omega_{\bar{\varphi}}(V) \to G\Omega_{\bar{\varphi}}(W)$ (Lyons' universal limit theorem), and $\xi + \pi_W(\cdot) : G\Omega_{\bar{\varphi}}(W) \to C_{\bar{\varphi}}([0,1], W)$, we conclude that $\{Y(n)\colon n \in \mathbb{N}\}$ converges to $\xi + \pi_W \circ \mathbf{I}_0 \circ T_g(\mathbf{X})$ which is statement (I) of the theorem. Assertions (II)–(IV) are now immediate consequences of Theorems 1.3, 1.1, and 1.2, respectively.

Appendix A. Preliminary results

For $\mathbf{u} \in \mathbf{V}$ we set

$$|||\mathbf{u}||| = \inf \sum_{i=1}^{n} \|\mathbf{u}_i\|, \tag{19}$$

where the infimum is taken over all representations $\mathbf{u} = \prod_{i=1}^{n} \mathbf{u}_i$ with $n \in \mathbb{N}$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbf{V}$ arbitrary.

Lemma A.1.

- $|||\cdot|||$ *satisfies the triangle inequality with respect to* $*$.
- *For* $t \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbf{V}$, *one has* $|||\delta_t \mathbf{u}||| = |t|\,|||\mathbf{u}|||$.
- *Moreover,* $\|\cdot\|$ *and* $|||\cdot|||$ *are equivalent*:

$$|||\mathbf{u}||| \leqslant \|\mathbf{u}\| \leqslant 2|||\mathbf{x}|||.$$

Proof. The proof of the first two statements is straight forward and we only present the proof of the third statement. Let $\mathbf{u} = \prod_{i=1}^n \mathbf{u}_i$. Then $\mathbf{u}^1 = \sum_{i=1}^n \mathbf{u}_i^1$ and $|\mathbf{u}^1|_V \leqslant \sum_{i=1}^n |\mathbf{u}_i^1|_V$. Moreover, since $\mathbf{u}^2 = \sum_{i=1}^n \mathbf{u}_i^2 + \sum_{i<j} \mathbf{u}_i^1 \otimes \mathbf{u}_j^1$ we get

$$\begin{aligned}\sqrt{|\mathbf{u}^2|_{V\otimes V}} &= \sqrt{\left|\sum_{i=1}^n \mathbf{u}_i^2 + \sum_{i<j} \mathbf{u}_i^1 \otimes \mathbf{u}_j^1\right|_{V\otimes V}} \\ &\leqslant \sqrt{\sum_{i=1}^n |\mathbf{u}_i^2|_{V\otimes V} + \left(\sum_{i=1}^n |\mathbf{u}_i^1|_V\right)^2} \\ &\leqslant \sum_{i=1}^n \left(|\mathbf{u}_i^1|_V + \sqrt{|\mathbf{u}_i^2|_{V\otimes V}}\right) = \sum_{i=1}^n \|\mathbf{u}_i\|\end{aligned}$$

so that $\|\mathbf{u}\| \leqslant 2|||\mathbf{u}|||$. □

Lemma A.2. *Let* $f \in C([0,1], V)$ *be an absolutely continuous with* $\dot f \in L^2([0,1], V)$ *and let* $\varphi : (0,1] \to (0,\infty)$ *be an increasing function with* $\inf_{\delta\in(0,1]} \frac{\varphi(\delta)}{\sqrt{\delta}} > 0$. *Then the shift operator*

$$T_f : \Omega_\varphi(\mathbf{V}) \to \Omega_\varphi(\mathbf{V})$$

is continuous.

Proof. For $\mathbf{x}, \mathbf{y} \in \Omega_\varphi(\mathbf{V})$, one has

$$T_f(\mathbf{x})_{s,t} - T_f(\mathbf{y})_{s,t} = \mathbf{x}_{s,t} - \mathbf{y}_{s,t} + \int_s^t f_{s,u} \otimes \mathrm{d}(x-y)_u + \int_s^t (x-y)_{s,u} \otimes \mathrm{d}f_u,$$

where $x = \pi_V(\mathbf{x})$ and $y = \pi_V(\mathbf{y})$. The process f is of bounded variation, and one has

$$\left|\int_s^t f_{s,u} \otimes \mathrm{d}(x-y)_u\right|_{V\otimes V} \leqslant \|\dot f\|_{L^1([0,1],V)} \sup_{s\leqslant u\leqslant v\leqslant t} |x_{u,v} - y_{u,v}|_V.$$

Next, recall that $\|\dot f\|_{L^1([s,t],V)} \leqslant \sqrt{t-s}\,\|\dot f\|_{L^2([0,1],V)}$ and that, by assumption, there exists a constant $c = c(\varphi)$ with $\sqrt{t-s} \leqslant c\,\varphi(t-s)$. Consequently,

$$\left| \int_s^t f_{s,u} \otimes \mathrm{d}(x-y)_u \right|_{V\otimes V} \leqslant c\, \|\dot f\|_{L^2([0,1],V)}\, \|\mathbf{x}-\mathbf{y}\|_\varphi\, \varphi(t-s)^2,$$

and

$$\left\| \left(\int_s^t f_{s,u} \otimes \mathrm{d}(x-y)_u \right)_{(s,t)\in\Delta} \right\|_\varphi \leqslant \sqrt{c\, \|\dot f\|_{L^2([0,1],V)}\, \|\mathbf{x}-\mathbf{y}\|_\varphi}.$$

Analogously one finds the same estimate for $\|(\int_s^t (x-y)_{s,u} \otimes \mathrm{d} f_u)_{(s,t)\in\Delta}\|_\varphi$ so that

$$\left\| T_f(\mathbf{x}) - T_f(\mathbf{y}) \right\|_\varphi \leqslant \|\mathbf{x}-\mathbf{y}\|_\varphi + 2\sqrt{c\|\dot f\|_{L^2([0,1],V)}\, \|\mathbf{x}-\mathbf{y}\|_\varphi}. \qquad \square$$

Lemma A.3. *For $n \in \mathbb{N}$, the maps*

$$S \circ \Upsilon_n : C(D_n, V) \to G\Omega_\varphi(\mathbf{V}) \quad \textit{and} \quad \log \circ S \circ \Upsilon_n : C(D_n, V) \to C_\varphi(\Delta, \mathbf{V})$$

are continuous. Moreover, for $x, y \in C([0,1], V)$, $\mathbf{x} = S \circ \Upsilon_n(x)$, and $\mathbf{y} = S \circ \Upsilon_n(y)$, one has

$$|\log \mathbf{x}_{s,t} - \log \mathbf{y}_{s,t}|_{\mathbf{V}} \leqslant 2^{n+1} \|x-y\|_\infty \big(1 + 2\|x\|_\infty + 2\|y\|_\infty\big)(t-s), \quad \textit{for } (s,t) \in \Delta.$$

The proof is straight-forward and therefore omitted.

Lemma A.4. *For any $p > 2$ and $\varphi : (0,1] \to (0,\infty)$ increasing with $\sqrt{-\delta \log \delta} \ll \varphi(\delta)$ there exists a function $\bar\varphi : (0,1] \to (0,\infty)$ such that $\bar\varphi^p$ is convex and*

$$\sqrt{-\delta \log \delta} \ll \bar\varphi(\delta) \leqslant \varphi(\delta), \quad \textit{for } \delta \in (0,1].$$

Proof. Set $\phi(\delta) = \sqrt{-\delta \log \delta}$ and define $\phi_{m,u} : (0,1] \to [0,\infty)$ for $m \in \mathbb{N}$ and $u > 0$ via

$$\phi_{m,u}^p(\delta) = \begin{cases} m\phi^p(\delta) & \text{if } \delta \leqslant u, \\ m\phi^p(u) + (m\phi^p)'(u)(\delta - u) & \text{otherwise.} \end{cases}$$

As one easily verifies by taking derivatives, ϕ^p is convex on an appropriate set $(0,\varepsilon)$. Hence, $\phi_{m,u}^p$ is convex provided that $u \leqslant \varepsilon$. Moreover, one can check that, for every $m \in \mathbb{N}$, there exists $u(m) \in (0,\varepsilon)$ such that $\phi_{m,u(m)} \leqslant \varphi$. Consequently, taking $\bar\varphi = \sup_{m\in\mathbb{N}} \phi_{m,u(m)}$ finishes the proof. $\square$

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