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Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$-modules, II: Higher-level case

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Abstract

We give an a priori proof of the known presentations of (that is, completeness of families of relations for) the principal subspaces of all the standard $A_1^{(1)}$-modules. These presentations had been used by Capparelli, Lepowsky and Milas for the purpose of obtaining the classical Rogers–Selberg recursions for the graded dimensions of the principal subspaces. This paper generalizes our previous paper.

MSC: 17B69; 17B65

1. Introduction

The affine Kac–Moody algebra $A_1^{(1)} = \widehat{\mathfrak{sl}(2)}$ is the simplest infinite-dimensional Kac–Moody Lie algebra, and in some sense the most prominent one. Not only does $\widehat{\mathfrak{sl}(2)}$ give insight into the higher-rank affine Lie algebras, but in fact, considerations of standard (= integrable highest weight) $\widehat{\mathfrak{sl}(2)}$-modules have frequently led to new ideas. For instance, explicit constructions of the standard $\widehat{\mathfrak{sl}(2)}$-modules have been used to obtain vertex-operator-theoretic derivations of the classical Rogers–Ramanujan identities and related q-series identities (cf. [21–24,19,20,26,27]). Another important use of standard $\widehat{\mathfrak{sl}(2)}$-modules is in the "coset" construction of unitary Virasoro-algebra minimal models [14]. These developments are deeply related to two-dimensional conformal field theory.

More recently, to each standard $\widehat{\mathfrak{sl}(n)}$-module $L(\Lambda)$, Feigin and Stoyanovsky associated a distinguished subspace $W(\Lambda)$, which they called the "principal subspace" of $L(\Lambda)$ ([7,8]), and interestingly, the graded dimensions of the principal subspaces of the standard $\widehat{\mathfrak{sl}(2)}$-modules are essentially the Gordon–Andrews q-series ([7,13]; cf. [1]). These q-series had previously appeared as the graded dimensions of the "vacuum" subspaces, with respect to a

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certain twisted Heisenberg subalgebra, of the odd-level standard $\widehat{\mathfrak{sl}(2)}$-modules ([22–24,26]). Since each standard $\widehat{\mathfrak{sl}(n)}$-module $L(\Lambda)$ of level k, $k \geq 1$, is a module for a certain vertex-operator algebra ([11]; cf. [6,18]), it is natural to employ ideas from vertex-operator-algebra theory to gain a better insight into the structure of principal subspaces. In [4,5], for the case $\widehat{\mathfrak{sl}(2)}$, the theory of vertex algebras (cf. [10,18]) and related algebraic structures, including intertwining operators [9], has been used to do this, via the construction of certain exact sequences, which led to a vertex-algebra-theoretic interpretation of the classical Rogers–Ramanujan and Rogers–Selberg recursions. This in turn explained the appearance of the Gordon–Andrews q-series, and these q-series can be implemented by means of "combinatorial bases" of the principal subspaces, revealing a fundamental "difference-two condition" that had already arisen in the setting of [22–24].

An important technical result used in [4,5] was a certain presentation of (that is, the completeness of a certain family of relations for) the principal subspaces of the standard $\widehat{\mathfrak{sl}(2)}$-modules (cf. Theorem 2.1 in [5]). This result had been stated as Theorem 2.2.1$'$ in [7]. However, the proofs of this result that we are aware of all turn out to require either a priori knowledge of a combinatorial basis of the principal subspace $W(\Lambda)$ (see (2.6) below) or information closely related to such knowledge. But what one ideally wants is rather an a priori of the presentation, which could then be used to construct the exact sequences mentioned above, and thereby to produce the bases. Thus it is an important problem to try to find an a priori proof of the presentation of $W(\Lambda)$, and we were able to achieve this for the level one standard $\widehat{\mathfrak{sl}(2)}$-modules in [2]. Our proof in [2] was obtained in two steps. We first argued that the presentation of $W(\Lambda_1)$ follows from the presentation of $W(\Lambda_0)$, and then we proved the presentation of $W(\Lambda_0)$. (These two steps are in fact interchangeable, so we could have placed the proof of the presentation of $W(\Lambda_0)$ first.)

In the present paper we give an a priori proof of the presentation of the principal subspaces more generally for all the standard $\widehat{\mathfrak{sl}(2)}$-modules. The higher-level case brings additional subtleties, and our approach is different from that in [2]. Instead of trying to reduce the problem of proving the presentation of principal subspaces to a "preferred" principal subspace (e.g., $W(k\Lambda_0)$), we found it more convenient and more elegant to prove the presentation of all the principal subspaces of a given level at once. This is done in the proofs of Theorems 3.1 and 3.2 through a (necessarily) rather delicate argument, which uses various properties of principal subspaces and intertwining operators among standard modules. In our new approach all the principal subspaces are on more-or-less equal footing. Thus we not only generalize the main result in [2] to all the standard $\widehat{\mathfrak{sl}(2)}$-modules, but we also give a new proof of the presentation of the principal subspaces in the level one case, different from the one in [2]. This is why we write the proof of the $k = 1$ case separately and in full detail below; we will also be generalizing this $k = 1$ proof in a different direction elsewhere.

This paper brings our in-depth analysis of the principal subspaces of the standard $\widehat{\mathfrak{sl}(2)}$-modules to an end. Even though the study of the principal subspaces of the $\widehat{\mathfrak{sl}(2)}$-modules is facilitated by the commutativity of the underlying nilpotent Lie algebra used to define these subspaces, many methods in this paper can be applied to more general affine Lie algebras, both untwisted and twisted. In a sequel [3] we will shift our attention to standard modules for affine Lie algebras of types A, D, E, in which case the relevant nilpotent Lie algebras are nonabelian.

This paper is organized as follows. Section 2 gives the setting. In Sections 3 and 4 we state and prove our main result, Theorem 3.1, which we also reformulate as Theorem 3.2. As in [2], finding a further reformulation of the presentation of the principal subspaces in terms of ideals of vertex (operator) algebras is a natural problem. This is achieved in Section 5 (Theorem 5.1), at least for the principal subspaces stemming from the "vacuum" higher-level $\widehat{\mathfrak{sl}(2)}$-modules.

2. The setting

We start by recalling some background from [2], for the reader's convenience. Set

$$\mathfrak{g} = \mathfrak{sl}(2) = \mathbb{C}x_{-\alpha} \oplus \mathbb{C}h \oplus \mathbb{C}x_{\alpha},$$

with bracket relations

$$[h, x_{\alpha}] = 2x_{\alpha}, \qquad [h, x_{-\alpha}] = -2x_{-\alpha}, \qquad [x_{\alpha}, x_{-\alpha}] = h.$$

The symmetric invariant bilinear form $\langle a, b\rangle = \mathrm{tr}\,(ab)$ $(a, b \in \mathfrak{g})$ allows us to identify the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$ with its dual $\mathfrak{h}^*$. The simple root $\alpha \in \mathfrak{h}^*$ corresponding to the root vector x_{α} identifies with $h \in \mathfrak{h}$, that is, $h = \alpha$, and $\langle \alpha, \alpha\rangle = 2$. Set $\mathfrak{n} = \mathbb{C}x_{\alpha}$.

We shall use the affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}, \tag{2.1}$$

with brackets

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n}\mathbf{k} \tag{2.2}$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, with $\mathbf{k}$ central, and its subalgebras

$$\bar{\mathfrak{n}} = \mathbb{C}x_\alpha \otimes \mathbb{C}[t, t^{-1}],$$
$$\bar{\mathfrak{n}}_- = \mathbb{C}x_\alpha \otimes t^{-1}\mathbb{C}[t^{-1}],$$
$$\bar{\mathfrak{n}}_{\leq -2} = \mathbb{C}x_\alpha \otimes t^{-2}\mathbb{C}[t^{-1}].$$

The Lie algebra $\widehat{\mathfrak{g}}$ has the triangular decompositions

$$\widehat{\mathfrak{g}} = (\mathbb{C}x_{-\alpha} \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{h} \oplus \mathbb{C}\mathbf{k}) \oplus (\mathbb{C}x_\alpha \oplus \mathfrak{g} \otimes t\mathbb{C}[t]) \tag{2.3}$$

and

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{<0} \oplus \widehat{\mathfrak{g}}_{\geq 0}, \tag{2.4}$$

where

$$\widehat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$$

and

$$\widehat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k}.$$

Let $k \geq 1$. We consider the level k standard $\widehat{\mathfrak{g}}$-modules $L((k-i)\Lambda_0 + i\Lambda_1)$, where $\Lambda_0, \Lambda_1 \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{k})^*$ are the fundamental weights of $\widehat{\mathfrak{g}}$ ($\Lambda_j(\mathbf{k}) = 1$, $\Lambda_j(h) = \delta_{j,1}$ for $j = 0, 1$) and $0 \leq i \leq k$ (cf. [15]), so that $L(\Lambda_0)$ and $L(\Lambda_1)$ are the level 1 standard $\widehat{\mathfrak{g}}$-modules used in [2]. For such i, we set

$$\Lambda_{k,i} = (k-i)\Lambda_0 + i\Lambda_1. \tag{2.5}$$

Denote by $v_{\Lambda_{k,i}}$ a highest weight vector of $L(\Lambda_{k,i})$. (These highest weight vectors will be normalized in Section 4 below.)

Throughout this paper we will write $x(m)$ for the action of $x \otimes t^m \in \widehat{\mathfrak{g}}$ on any $\widehat{\mathfrak{g}}$-module, where $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$. In particular, we have the operator $x_\alpha(m)$, the image of $x_\alpha \otimes t^m$. Sometimes we will simply write $x(m)$ for the Lie algebra element $x \otimes t^m$. It will be clear from the context whether $x(m)$ is an operator or a Lie algebra element.

We generalize the definition of the principal subspace of a standard module for an untwisted affine Lie algebra of type A given in [7,8]:

Definition 2.1. Consider any finite-dimensional semisimple Lie algebra and its associated affine Lie algebra. The *principal subspace* of a highest weight module V for the affine Lie algebra is $U(\bar{\mathfrak{n}}) \cdot v \subset V$, where $\mathfrak{n}$ is the nilradical of a fixed Borel subalgebra of the finite-dimensional Lie algebra, $\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$ and v is a highest weight vector of V.

In particular, the principal subspace $W(\Lambda_{k,i})$ of $L(\Lambda_{k,i})$ is

$$W(\Lambda_{k,i}) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda_{k,i}} \tag{2.6}$$

for $i = 0, \ldots k$, as in [7].

We have

$$W(\Lambda_{k,i}) = U(\bar{\mathfrak{n}}_-) \cdot v_{\Lambda_{k,i}}. \tag{2.7}$$

Set

$$W(\Lambda_{k,k})' = U(\bar{\mathfrak{n}}_{\leq -2}) \cdot v_{\Lambda_{k,k}}, \tag{2.8}$$

generalizing (2.8) in [2]. Since $x_\alpha(-1)\cdot v_{\Lambda_{k,k}} = 0$, we have

$$W(\Lambda_{k,k})' = W(\Lambda_{k,k}), \tag{2.9}$$

generalizing (2.9) in [2].

For $i = 0, \ldots k$, consider the surjective maps

$$\begin{aligned} F_{\Lambda_{k,i}} : U(\widehat{\mathfrak{g}}) &\longrightarrow L(\Lambda_{k,i}) \\ a &\mapsto a\cdot v_{\Lambda_{k,i}}. \end{aligned} \tag{2.10}$$

Restrict $F_{\Lambda_{k,i}}$ to $U(\bar{\mathfrak{n}}_-)$ and $F_{\Lambda_{k,k}}$ to $U(\bar{\mathfrak{n}}_{\le -2})$ and denote these (surjective) restrictions by $f_{\Lambda_{k,i}}$ and $f'_{\Lambda_{k,k}}$:

$$\begin{aligned} f_{\Lambda_{k,i}} : U(\bar{\mathfrak{n}}_-) &\longrightarrow W(\Lambda_{k,i}) \\ a &\mapsto a\cdot v_{\Lambda_{k,i}}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} f'_{\Lambda_{k,k}} : U(\bar{\mathfrak{n}}_{\le -2}) &\longrightarrow W(\Lambda_{k,k})' \\ a &\mapsto a\cdot v_{\Lambda_{k,k}}, \end{aligned} \tag{2.12}$$

generalizing (2.11) and (2.12) in [2]. Our main goal is to give a precise description of the kernels Ker $f_{\Lambda_{k,i}}$ and Ker $f'_{\Lambda_{k,k}}$.

For every $t \in \mathbb{Z}$ we consider the following formal infinite sums:

$$R_{k,t} = \sum_{m_1+\cdots+m_{k+1}=-t} x_\alpha(m_1)\cdots x_\alpha(m_{k+1}). \tag{2.13}$$

For each t, $R_{k,t}$ acts naturally on any highest weight $\widehat{\mathfrak{g}}$-module and, in particular, on each $L(\Lambda_{k,i})$ for $0 \le i \le k$. For $k = 1$ these are the formal sums R_t introduced in [2].

Continuing to generalize the corresponding objects in [2], in order to describe Ker $f_{\Lambda_{k,i}}$ and Ker $f'_{\Lambda_{k,k}}$ we shall truncate each $R_{k,t}$ as follows:

$$R^0_{k,t} = \sum_{\substack{m_1,\ldots,m_{k+1}\le -1,\\ m_1+\cdots+m_{k+1}=-t}} x_\alpha(m_1)\cdots x_\alpha(m_{k+1}), \quad t \ge k+1. \tag{2.14}$$

Just as in [2], we shall often be viewing $R^0_{k,t}$ as an element of $U(\bar{\mathfrak{n}})$, and in fact of $U(\bar{\mathfrak{n}}_-)$, rather than as an endomorphism of a $\widehat{\mathfrak{g}}$-module. In order to describe Ker $f'_{\Lambda_{k,k}}$ it will also be convenient to take $m_1, \ldots, m_{k+1} \le -2$ in (2.13), to obtain other elements of $U(\bar{\mathfrak{n}})$, which we denote by $R^1_{k,t}$:

$$R^1_{k,t} = \sum_{\substack{m_1,\ldots,m_{k+1}\le -2,\\ m_1+\cdots+m_{k+1}=-t}} x_\alpha(m_1)\cdots x_\alpha(m_{k+1}), \quad t \ge 2(k+1). \tag{2.15}$$

Again as in [2], one can view $U(\bar{\mathfrak{n}}_-)$ and $U(\bar{\mathfrak{n}}_{\le -2})$ as the polynomial algebras

$$U(\bar{\mathfrak{n}}_-) = \mathbb{C}[x_\alpha(-1), x_\alpha(-2), \ldots] \tag{2.16}$$

and

$$U(\bar{\mathfrak{n}}_{\le -2}) = \mathbb{C}[x_\alpha(-2), x_\alpha(-3), \ldots], \tag{2.17}$$

so that

$$U(\bar{\mathfrak{n}}_-) = U(\bar{\mathfrak{n}}_{\le -2}) \oplus U(\bar{\mathfrak{n}}_-)x_\alpha(-1) \tag{2.18}$$

and we have the corresponding projection

$$\rho : U(\bar{\mathfrak{n}}_-) \longrightarrow U(\bar{\mathfrak{n}}_{\le -2}). \tag{2.19}$$

From (2.14) and (2.15) we have

$$R^1_{k,t} = \rho(R^0_{k,t}). \tag{2.20}$$

(For $t < 2(k+1)$, $R^1_{k,t} = 0$.)

Generalizing the corresponding constructions in [2], we set

$$I_{\Lambda_{k,0}} = \sum_{t \geq k+1} U(\bar{\mathfrak{n}}_-) R^0_{k,t} \subset U(\bar{\mathfrak{n}}_-), \tag{2.21}$$

$$I_{\Lambda_{k,i}} = \sum_{t \geq k+1} U(\bar{\mathfrak{n}}_-) R^0_{k,t} + U(\bar{\mathfrak{n}}_-) x_\alpha(-1)^{k-i+1} \subset U(\bar{\mathfrak{n}}_-) \quad \text{for } i \geq 0 \tag{2.22}$$

(note that (2.22) indeed agrees with (2.21) for $i = 0$, since $R^0_{k,k+1} = x_\alpha(-1)^{k+1}$) and

$$I'_{\Lambda_{k,k}} = \sum_{t \geq 2(k+1)} U(\bar{\mathfrak{n}}_{\leq -2}) R^1_{k,t} \subset U(\bar{\mathfrak{n}}_{\leq -2}). \tag{2.23}$$

Remark 2.1. We have the inclusions

$$I_{\Lambda_{k,0}} \subset I_{\Lambda_{k,1}} \subset \cdots \subset I_{\Lambda_{k,k-1}} \subset I_{\Lambda_{k,k}} \tag{2.24}$$

among the $U(\bar{\mathfrak{n}}_-)$-ideals $I_{\Lambda_{k,i}}$. We also have

$$I_{\Lambda_{k,i}} = I_{\Lambda_{k,0}} + U(\bar{\mathfrak{n}}_-) x_\alpha(-1)^{k-i+1} \quad \text{for every } i \geq 1, \tag{2.25}$$

and this holds for $i = 0$ as well. In addition,

$$\rho(I_{\Lambda_{k,k}}) = I'_{\Lambda_{k,k}}, \tag{2.26}$$

and in fact,

$$I_{\Lambda_{k,k}} = I'_{\Lambda_{k,k}} \oplus U(\bar{\mathfrak{n}}_-) x_\alpha(-1). \tag{2.27}$$

These relations generalize the corresponding ones in [2].

3. Formulations of the main result

It is well known that the level k standard $\widehat{\mathfrak{sl}(2)}$-module $L(\Lambda_{k,0})$ has a natural vertex-operator algebra structure with $v_{\Lambda_{k,0}}$ as vacuum vector; the vertex-operator map

$$\begin{aligned} &Y(\cdot, x) : L(\Lambda_{k,0}) \longrightarrow \operatorname{End} L(\Lambda_{k,0})\,[[x, x^{-1}]] \\ &v \mapsto Y(v, x) = \sum_{m \in \mathbb{Z}} v_m x^{-m-1} \end{aligned} \tag{3.1}$$

has the property

$$Y(x_\alpha(-1) \cdot v_{\Lambda_{k,0}}, x) = \sum_{m \in \mathbb{Z}} x_\alpha(m) x^{-m-1}. \tag{3.2}$$

It is also well known that each $L(\Lambda_{k,i})$, $0 \leq i \leq k$, has a natural $L(\Lambda_{k,0})$-module structure, with (3.2) remaining valid for the module action. (See [11,6,25,18].)

The standard action of the Virasoro algebra operator $L(0)$ (not to be confused with the trivial $\widehat{\mathfrak{g}}$-module) provides the usual grading by *conformal weights* on the spaces $L(\Lambda_{k,i})$. We have

$$\operatorname{wt} x_\alpha(m) = -m \tag{3.3}$$

for $m \in \mathbb{Z}$, where $x_\alpha(m)$ is viewed as either an operator or as an element of $U(\bar{\mathfrak{n}})$. For any i with $0 \leq i \leq k$,

$$\operatorname{wt} v_{\Lambda_{k,i}} = \frac{\langle i\alpha/2, i\alpha/2 + \alpha \rangle}{2(k+2)} = \frac{i^2 + 2i}{4(k+2)} \tag{3.4}$$

(cf. [15,6,18]).

There is also a grading by *charge*, given by the eigenvalues of the operator $\frac{1}{2}\alpha(0) = \frac{1}{2}h(0)$, on $L(\Lambda_{k,i})$. The weight and charge gradings are compatible. For any $m \in \mathbb{Z}$, $x_\alpha(m)$, viewed as either an operator or as an element of $U(\bar{\mathfrak{n}})$, has charge 1. Also, $v_{\Lambda_{k,i}}$ has charge $\langle i\alpha/2, \alpha/2\rangle = i/2$. The principal subspaces $W(\Lambda_{k,i})$ are graded by weight and charge. For any $m_1, \ldots, m_r \in \mathbb{Z}$,

$$x_\alpha(m_1) \cdots x_\alpha(m_r) \cdot v_{\Lambda_{k,i}} \in W(\Lambda_{k,i}) \tag{3.5}$$

has weight $-m_1 - \cdots - m_r + \frac{i^2+2i}{4(k+2)}$ and charge $r + \frac{i}{2}$. See [4,5,2] for further details, background and notation.

Remark 3.1. As in [2], we have

$$L(0) \text{ Ker } f_{\Lambda_{k,i}} \subset \text{Ker } f_{\Lambda_{k,i}} \quad \text{for all } 0 \leq i \leq k$$

and

$$L(0) \text{ Ker } f'_{\Lambda_{k,k}} \subset \text{Ker } f'_{\Lambda_{k,k}}.$$

Also, $R^0_{k,t}$ and $R^1_{k,t}$ have conformal weight t:

$$L(0) R^0_{k,t} = t R^0_{k,t} \quad \text{for all } t \geq k+1$$

and

$$L(0) R^1_{k,t} = t R^1_{k,t} \quad \text{for all } t \geq 2(k+1),$$

so that in particular, the subspaces $I_{\Lambda_{k,i}}$ and $I'_{\Lambda_{k,k}}$ are $L(0)$-stable. Also, $R^0_{k,t}$ and $R^1_{k,t}$ have charge $k+1$, and the spaces Ker $f_{\Lambda_{k,i}}$, Ker $f'_{\Lambda_{k,k}}$, $I_{\Lambda_{k,i}}$ and $I'_{\Lambda_{k,k}}$ are graded by charge. Thus these spaces are graded by both weight and charge, and the two gradings are compatible.

We will prove the following description of the kernels Ker $f_{\Lambda_{k,i}}$ and Ker $f'_{\Lambda_{k,k}}$ (recall (2.18) and (2.27)):

Theorem 3.1. *For any $i = 0, \ldots, k$, we have*

$$\text{Ker } f_{\Lambda_{k,i}} = I_{\Lambda_{k,i}}. \tag{3.6}$$

In particular,

$$\text{Ker } f'_{\Lambda_{k,k}} = I'_{\Lambda_{k,k}}.$$

As in [2], we will actually prove a restatement of this assertion (see Theorem 3.2 below) that uses generalized Verma modules, in the sense of [16,12,17], for $\widehat{\mathfrak{g}}$, and the principal subspaces of these generalized Verma modules.

The generalized Verma module $N(\Lambda_{k,0})$ is defined as the induced $\widehat{\mathfrak{g}}$-module

$$N(\Lambda_{k,0}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} \mathbb{C}v^N_{\Lambda_{k,0}}, \tag{3.7}$$

where $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially and $\mathbf{k}$ acts as the scalar k on $\mathbb{C}v^N_{\Lambda_{k,0}}$; $v^N_{\Lambda_{k,0}}$ is a highest weight vector. From the Poincaré–Birkhoff–Witt theorem we have

$$N(\Lambda_{k,0}) = U(\widehat{\mathfrak{g}}_{<0}) \otimes_{\mathbb{C}} U(\widehat{\mathfrak{g}}_{\geq 0}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} \mathbb{C}v^N_{\Lambda_{k,0}} = U(\widehat{\mathfrak{g}}_{<0}) \otimes_{\mathbb{C}} \mathbb{C}v^N_{k\Lambda_0} = U(\widehat{\mathfrak{g}}_{<0}), \tag{3.8}$$

with the natural identifications. We similarly define the generalized Verma module

$$N(\Lambda_{k,i}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} U_i$$

for $i = 1, \ldots k$, where U_i is an $(i+1)$-dimensional irreducible $\mathfrak{g}$-module and where $\mathfrak{g} \otimes t\mathbb{C}[t]$ acts trivially and $\mathbf{k}$ acts by k. By the Poincaré–Birkhoff–Witt theorem we have the identifications

$$N(\Lambda_{k,i}) = U(\widehat{\mathfrak{g}}_{<0}) \otimes_{\mathbb{C}} U_i .$$

For $0 \leq i \leq k$ we have the natural surjective $\widehat{\mathfrak{g}}$-module maps

$$F^N_{\Lambda_{k,i}} : U(\widehat{\mathfrak{g}}) \longrightarrow N(\Lambda_{k,i}) \qquad (3.9)$$
$$a \mapsto a \cdot v^N_{\Lambda_{k,i}},$$

where $v^N_{\Lambda_{k,i}}$ is a highest weight vector of U_i (cf. (2.10)).

Remark 3.2. The restriction of (3.9) to $U(\widehat{\mathfrak{g}}_{<0})$ is a $U(\widehat{\mathfrak{g}}_{<0})$-module isomorphism for $i = 0$ and a $U(\widehat{\mathfrak{g}}_{<0})$-module injection for $i \geq 1$.

From Definition 2.1, the $\bar{\mathfrak{n}}$-submodule

$$W^N(\Lambda_{k,i}) = U(\bar{\mathfrak{n}}) \cdot v^N_{\Lambda_{k,i}} \qquad (3.10)$$

of $N(\Lambda_{k,i})$ is the principal subspace of the generalized Verma module $N(\Lambda_{k,i})$, generalizing the corresponding structure in [2]. We have

$$W^N(\Lambda_{k,i}) = U(\bar{\mathfrak{n}}_-) \cdot v^N_{\Lambda_{k,i}}. \qquad (3.11)$$

We also consider the subspace

$$W^N(\Lambda_{k,k})' = U(\bar{\mathfrak{n}}_{\leq -2}) \cdot v^N_{\Lambda_{k,k}} \qquad (3.12)$$

of $W^N(\Lambda_{k,k})$.

Remark 3.3. In view of Remark 3.2, the restrictions of $F^N_{\Lambda_{k,i}}$ to $U(\bar{\mathfrak{n}}_-)$,

$$U(\bar{\mathfrak{n}}_-) \longrightarrow W^N(\Lambda_{k,i}) \qquad (3.13)$$
$$a \mapsto a \cdot v^N_{\Lambda_{k,i}},$$

are $\bar{\mathfrak{n}}_-$-module isomorphisms and the restriction of $F^N_{\Lambda_{k,k}}$ to $U(\bar{\mathfrak{n}}_{\leq -2})$,

$$U(\bar{\mathfrak{n}}_{\leq -2}) \longrightarrow W^N(\Lambda_{k,k})' \qquad (3.14)$$
$$a \mapsto a \cdot v^N_{\Lambda_{k,k}},$$

is an $\bar{\mathfrak{n}}_{\leq -2}$-module isomorphism.

In particular, by using (2.18) we have the natural identifications

$$W^N(\Lambda_{k,k})' \simeq W^N(\Lambda_{k,k})/U(\bar{\mathfrak{n}}_-)x_\alpha(-1) \cdot v^N_{\Lambda_{k,k}} \simeq U(\bar{\mathfrak{n}}_{\leq -2}). \qquad (3.15)$$

Consider the natural surjective $\widehat{\mathfrak{g}}$-module maps

$$\Pi_{\Lambda_{k,i}} : N(\Lambda_{k,i}) \longrightarrow L(\Lambda_{k,i}) \qquad (3.16)$$
$$a \cdot v^N_{\Lambda_{k,i}} \mapsto a \cdot v_{\Lambda_{k,i}}$$

for $a \in U(\widehat{\mathfrak{g}})$ and set

$$N^1(\Lambda_{k,i}) = \mathrm{Ker}\ \Pi_{\Lambda_{k,i}}. \qquad (3.17)$$

The restrictions of $\Pi_{\Lambda_{k,i}}$ to $W^N(\Lambda_{k,i})$ (respectively, $W^N(\Lambda_{k,k})'$) are $\bar{\mathfrak{n}}$-module (respectively, $\bar{\mathfrak{n}}_{\leq -2}$-module) surjections:

$$\pi_{\Lambda_{k,i}} : W^N(\Lambda_{k,i}) \longrightarrow W(\Lambda_{k,i}) \qquad (3.18)$$

for $0 \leq i \leq k$ and

$$\pi'_{\Lambda_{k,k}} : W^N(\Lambda_{k,k})' \longrightarrow W(\Lambda_{k,k}) \tag{3.19}$$

(recall (2.9)).

As in the case of $L(\Lambda_{k,i})$, the generalized Verma modules $N(\Lambda_{k,i})$ are compatibly graded by conformal weight and by charge. We shall restrict these gradings to the principal subspaces $W^N(\Lambda_{k,i})$. The elements of $W^N(\Lambda_{k,i})$ given by (3.5) with $v_{\Lambda_{k,i}}$ replaced by $v^N_{\Lambda_{k,i}}$ have the same weights and charges as in those cases.

Remark 3.4. Since the maps $\pi_{\Lambda_{k,i}}$, and $\pi'_{\Lambda_{k,k}}$ commute with the actions of $L(0)$, the kernels Ker $\pi_{\Lambda_{k,i}}$ and Ker $\pi'_{\Lambda_{k,k}}$ are $L(0)$-stable. These maps also preserve charge, so that Ker $\pi_{\Lambda_{k,i}}$ and Ker $\pi'_{\Lambda_{k,k}}$ are also graded by charge.

Using Remark 3.2, we see that Theorem 3.1 can be reformulated as follows:

Theorem 3.2. *For $i = 0, \ldots, k$, we have*

$$\text{Ker}\, \pi_{\Lambda_{k,i}} = I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}} \ (\subset N^1(\Lambda_{k,i})). \tag{3.20}$$

In particular,

$$\text{Ker}\, \pi'_{\Lambda_{k,k}} = I'_{\Lambda_{k,k}} \cdot v^N_{\Lambda_{k,k}} \ (\subset N^1(\Lambda_{k,k})).$$

4. Proof of the main result

Using the setting of [4,5], with $P = \frac{1}{2}\mathbb{Z}\alpha$ the weight lattice of $\mathfrak{sl}(2)$, we have the space

$$V_P = L(\Lambda_0) \oplus L(\Lambda_1) \tag{4.1}$$

and its vertex-operator structure. We shall use the identifications

$$v_{\Lambda_{1,0}} = 1 \in L(\Lambda_0) \quad \text{and} \quad v_{\Lambda_{1,1}} = e^{\alpha/2} \cdot v_{\Lambda_{1,0}} \in L(\Lambda_1) \tag{4.2}$$

as in formula (2.5) in [2] and Section 2 of [4] (with $v_{\Lambda_0} = v_{\Lambda_{1,0}}$ and $v_{\Lambda_1} = v_{\Lambda_{1,1}}$, using our current notation for highest weight vectors). We consider

$$V_P^{\otimes k} = V_P \otimes \cdots \otimes V_P \tag{4.3}$$

(k times). For any k-tuple $(j_1, \ldots, j_k)$ with $j_1, \ldots, j_k \in \{0, 1\}$ we consider the element

$$v_{j_1,\ldots,j_k} = v_{\Lambda_{1,j_1}} \otimes \cdots \otimes v_{\Lambda_{1,j_k}} \in V_P^{\otimes k}, \tag{4.4}$$

where exactly $k - i$ indices j_l ($l = 1, \ldots, k$) are equal to 0 (and exactly i indices are equal to 1); recall (4.2). This vector is of course a highest weight vector for $\widehat{\mathfrak{sl}(2)}$, and

$$L(\Lambda_{k,i}) \simeq U(\widehat{\mathfrak{g}}) \cdot v_{j_1,\ldots,j_k} \subset V_P^{\otimes k} \tag{4.5}$$

(cf. [15,5]), using the natural extension to $U(\widehat{\mathfrak{g}})$ of the usual comultiplication

$$a \cdot v = (a \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes a)v \tag{4.6}$$

for $a \in \widehat{\mathfrak{g}}$ and $v \in V_P^{\otimes k}$. As in [5] we will use the embeddings

$$\iota_{j_1,\ldots,j_k} : L(\Lambda_{k,i}) \hookrightarrow V_P^{\otimes k} \quad \text{for } 0 \leq i \leq k, \tag{4.7}$$

uniquely determined by the identifications

$$v_{\Lambda_{k,i}} = v_{j_1,\ldots,j_k}. \tag{4.8}$$

Of course, this element $v_{\Lambda_{k,i}}$ and the embedding (4.7) depend on $j_1, \ldots, j_k$.

Recall the linear isomorphism (3.20) in [4],

$$e^{\alpha/2} : V_P \longrightarrow V_P,$$

and consider the linear isomorphism

$$e^{\alpha/2}_{(k)} = \underbrace{e^{\alpha/2} \otimes \cdots \otimes e^{\alpha/2}}_{k \text{ times}} : V_P^{\otimes k} \longrightarrow V_P^{\otimes k}. \tag{4.9}$$

We denote by $e^{\alpha/2}_{(k,i)}$ the restriction of $e^{\alpha/2}_{(k)}$ to the principal subspace $W(\Lambda_{k,i})$ of $L(\Lambda_{k,i})$, using an embedding of the form (4.7). The action of $e^{\alpha/2}_{(k)}$ on $L(\Lambda_{k,i})$ and its action $e^{\alpha/2}_{(k,i)}$ on $W(\Lambda_{k,i})$ as well as other features of this map are given as follows (see Lemma 3.2 of [5] and its proof):

Lemma 4.1 *([5]). Fix $j_1, \ldots, j_k$ as in (4.4)–(4.8) and consider the standard module $L(\Lambda_{k,i})$ embedded in $V_P^{\otimes k}$ via $\iota_{j_1,\ldots,j_k}$. The image of the restriction of $e^{\alpha/2}_{(k)}$ to $L(\Lambda_{k,i})$ lies in $L(\Lambda_{k,k-i})$, embedded in $V_P^{\otimes k}$ via $\iota_{1-j_1,\ldots,1-j_k}$. For any i with $0 \le i \le k$, we have*

$$e^{\alpha/2}_{(k,i)} : W(\Lambda_{k,i}) \longrightarrow W(\Lambda_{k,k-i}). \tag{4.10}$$

When $i = 0$ the map (4.10) is a linear isomorphism. We also have

$$e^{\alpha/2}_{(k,i)} (x_\alpha(m_1) \cdots x_\alpha(m_r) \cdot v_{\Lambda_{k,i}}) = \frac{1}{i!} x_\alpha(m_1 - 1) \cdots x_\alpha(m_r - 1) x_\alpha(-1)^i \cdot v_{\Lambda_{k,k-i}} \tag{4.11}$$

for any $m_1, \ldots, m_r \in \mathbb{Z}$. □

We emphasize that according to our notation, the embeddings of the two spaces $W(\Lambda_{k,i})$ and $W(\Lambda_{k,k-i})$ in (4.10) are "opposite" even when i and $k - i$ happen to coincide.

We now generalize the lifting procedures in [2]. For each $i = 0, \ldots, k$ we construct a lifting

$$\widehat{e^{\alpha/2}_{(k,i)}} : W^N(\Lambda_{k,i}) \longrightarrow W^N(\Lambda_{k,k-i}) \tag{4.12}$$

of

$$e^{\alpha/2}_{(k,i)} : W(\Lambda_{k,i}) \longrightarrow W(\Lambda_{k,k-i}), \tag{4.13}$$

making the diagram

$$\begin{array}{ccc} W^N(\Lambda_{k,i}) & \xrightarrow{\widehat{e^{\alpha/2}_{(k,i)}}} & W^N(\Lambda_{k,k-i}) \\ \pi_{\Lambda_{k,i}} \downarrow & & \pi_{\Lambda_{k,k-i}} \downarrow \\ W(\Lambda_{k,i}) & \xrightarrow{e^{\alpha/2}_{(k,i)}} & W(\Lambda_{k,k-i}) \end{array}$$

commute; here, in (4.13) we continue to use the particular embeddings depending on $j_1, \ldots, j_k$ used in (4.10). In fact, for any i with $0 \le i \le k$ and any integers $m_1, \ldots, m_r < 0$ we set

$$\widehat{e^{\alpha/2}_{(k,i)}} (x_\alpha(m_1) \cdots x_\alpha(m_r) \cdot v^N_{\Lambda_{k,i}}) = \frac{1}{i!} x_\alpha(m_1 - 1) \cdots x_\alpha(m_r - 1) x_\alpha(-1)^i \cdot v^N_{\Lambda_{k,k-i}}, \tag{4.14}$$

which is well defined, since $U(\bar{\mathfrak{n}}_-)$, viewed as the polynomial algebra

$$\mathbb{C}[x_\alpha(-1), x_\alpha(-2), \ldots],$$

maps isomorphically onto $W^N(\Lambda_{k,i})$ under the map (3.13). This gives our desired lifting (4.12).

For the case $i = 0$, the map (4.12) is a linear isomorphism onto the subspace $W^N(\Lambda_{k,k})'$:

$$\widehat{e^{\alpha/2}_{(k,0)}} : W^N(\Lambda_{k,0}) \longrightarrow W^N(\Lambda_{k,k})', \tag{4.15}$$

a lifting of the linear isomorphism

$$e^{\alpha/2}_{(k,0)} : W(\Lambda_{k,0}) \longrightarrow W(\Lambda_{k,k}); \tag{4.16}$$

the diagram

$$\begin{array}{ccc} W^N(\Lambda_{k,0}) & \xrightarrow[\sim]{\widehat{e^{\alpha/2}_{(k,0)}}} & W^N(\Lambda_{k,k})' \\ \pi_{\Lambda_{k,0}} \big\downarrow & & \pi'_{\Lambda_{k,k}} \big\downarrow \\ W(\Lambda_{k,0}) & \xrightarrow[\sim]{e^{\alpha/2}_{(k,0)}} & W(\Lambda_{k,k}) \end{array}$$

commutes. Indeed, since

$$W^N(\Lambda_{k,k})' = U(\bar{\mathfrak{n}}_{\leq -2}) \cdot v^N_{\Lambda_{k,k}} = \widehat{e^{\alpha/2}_{(k,0)}}\,(U(\bar{\mathfrak{n}}_-) \cdot v^N_{\Lambda_{k,0}}),$$

the linear map (4.15) is surjective, and by Remark 3.3 it is also injective and thus a linear isomorphism. Denote by

$$(\widehat{e^{\alpha/2}_{(k,0)}})^{-1} = \widehat{\mathrm{e}^{-\alpha/2}_{(k,0)}} : W^N(\Lambda_{k,k})' \longrightarrow W^N(\Lambda_{k,0}) \tag{4.17}$$

its inverse; the map $\widehat{\mathrm{e}^{-\alpha/2}_{(k,0)}}$ is correspondingly a lifting of the inverse

$$\mathrm{e}^{-\alpha/2}_{(k,0)} : W(\Lambda_{k,k}) \longrightarrow W(\Lambda_{k,0}). \tag{4.18}$$

Remark 4.1. We have just noticed that, as in the $k = 1$ special case of [2], the image of $W^N(\Lambda_{k,0})$ under the map $\widehat{e^{\alpha/2}_{(k,0)}}$ is the subspace $W^N(\Lambda_{k,k})' \subset W^N(\Lambda_{k,k})$ and not the full space $W^N(\Lambda_{k,k})$. Both the maps (4.15) and (4.16) are isomorphisms, while the map (4.12) for $i = 0$, from $W^N(\Lambda_{k,0})$ to $W^N(\Lambda_{k,k})$, is only an injection.

Remark 4.2. The restriction

$$\widehat{\mathrm{e}^{\alpha/2}_{(k,k)}} : W^N(\Lambda_{k,k})' \longrightarrow W^N(\Lambda_{k,0}) \tag{4.19}$$

of (4.12) for $i = k$ to $W^N(\Lambda_{k,k})'$ is a lifting of

$$e^{\alpha/2}_{(k,k)} : W(\Lambda_{k,k}) \longrightarrow W(\Lambda_{k,0}), \tag{4.20}$$

making the diagram

$$\begin{array}{ccc} W^N(\Lambda_{k,k})' & \xrightarrow{\widehat{e^{\alpha/2}_{(k,k)}}} & W^N(\Lambda_{k,0}) \\ \pi'_{\Lambda_{k,k}} \big\downarrow & & \pi_{\Lambda_{k,0}} \big\downarrow \\ W(\Lambda_{k,k}) & \xrightarrow{e^{\alpha/2}_{(k,k)}} & W(\Lambda_{k,0}) \end{array}$$

commute; it is an injection and not a surjection. The maps (4.15) and (4.19) were used in [2] for $k = 1$.

Now we describe the actions of our liftings (4.12) on the spaces $I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}}$:

Lemma 4.2. *For any i with $0 \leq i \leq k$, we have*

$$\widehat{e^{\alpha/2}_{(k,i)}}\,(I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}}) \subset I_{\Lambda_{k,k-i}} \cdot v^N_{\Lambda_{k,k-i}}. \tag{4.21}$$

Proof. By (2.22) we have

$$I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}} = \sum_{t \geq k+1} U(\bar{\mathfrak{n}}_-) R^0_{k,t} \cdot v^N_{\Lambda_{k,i}} + U(\bar{\mathfrak{n}}_-) x_\alpha(-1)^{k-i+1} \cdot v^N_{\Lambda_{k,i}} \tag{4.22}$$

and

$$I_{\Lambda_{k,k-i}} \cdot v^N_{\Lambda_{k,k-i}} = \sum_{t \geq k+1} U(\bar{\mathfrak{n}}_-) R^0_{k,t} \cdot v^N_{\Lambda_{k,k-i}} + U(\bar{\mathfrak{n}}_-) x_\alpha(-1)^{i+1} \cdot v^N_{\Lambda_{k,k-i}}. \tag{4.23}$$

We use (4.14). For any $t \geq k+1$,

$$\begin{aligned} \widehat{e^{\alpha/2}_{(k,i)}} \, (R^0_{k,t} \cdot v^N_{\Lambda_{k,i}}) &= \frac{1}{i!} \sum_{\substack{m_1,\ldots,m_{k+1} \leq -1 \\ m_1+\cdots+m_{k+1} = -t}} x_\alpha(m_1 - 1) \cdots x_\alpha(m_{k+1} - 1) x_\alpha(-1)^i \cdot v^N_{\Lambda_{k,k-i}} \\ &= \frac{1}{i!} R^1_{k,t+k+1} x_\alpha(-1)^i \cdot v^N_{\Lambda_{k,k-i}} \\ &= \frac{1}{i!} R^0_{k,t+k+1} x_\alpha(-1)^i \cdot v^N_{\Lambda_{k,k-i}} + a x_\alpha(-1)^{i+1} \cdot v^N_{\Lambda_{k,k-i}} \in I_{\Lambda_{k,k-i}} \cdot v^N_{\Lambda_{k,k-i}}, \end{aligned}$$

where $a \in U(\bar{\mathfrak{n}}_-)$. We also have

$$\begin{aligned} \widehat{e^{\alpha/2}_{(k,i)}} (x_\alpha(-1)^{k-i+1} \cdot v^N_{\Lambda_{k,i}}) &= \frac{1}{i!} x_\alpha(-2)^{k-i+1} x_\alpha(-1)^i \cdot v^N_{\Lambda_{k,k-i}} \\ &= \gamma R^0_{k,2k-i+2} \cdot v^N_{\Lambda_{k,k-i}} + b x_\alpha(-1)^{i+1} \cdot v^N_{\Lambda_{k,k-i}} \in I_{\Lambda_{k,k-i}} \cdot v^N_{\Lambda_{k,k-i}}, \end{aligned} \tag{4.24}$$

where γ is a nonzero scalar and $b \in U(\bar{\mathfrak{n}}_-)$. Indeed, the expression $R^0_{k,2k-i+2}$ does not have any terms involving $x_\alpha(-1)^t$ with $0 \leq t < i$, since if there is such a term $c x_\alpha(-1)^t$, with $c \in U(\bar{\mathfrak{n}}_-)$ a product of $k+1-t$ elements $x_\alpha(m)$, where each $m \leq -2$, then

$$\text{wt}\,(c x_\alpha(-1)^t) \geq 2(k+1-t) + t = 2k+2-t > 2k+2-i = \text{wt}\,(R^0_{k,2k-i+2}),$$

and this contradicts the fact that $c x_\alpha(-1)^t$ is a summand of $R^0_{k,2k-i+2}$. We also observe that $x_\alpha(-2)^{k-i+1} x_\alpha(-1)^i$ is the only type of term in the sum R^0_{2k-i+2} involving $x_\alpha(-1)^i$. This proves (4.24), and hence (4.21). □

Remark 4.3. We have

$$\widehat{e^{\alpha/2}_{(k,0)}} \, (I_{\Lambda_{k,0}} \cdot v^N_{\Lambda_{k,0}}) = I'_{\Lambda_{k,k}} \cdot v^N_{\Lambda_{k,k}}. \tag{4.25}$$

Indeed, for any $t \geq k+1$,

$$\widehat{e^{\alpha/2}_{(k,0)}} \, (R^0_{k,t} \cdot v^N_{\Lambda_{k,0}}) = R^1_{k,t+k+1} \cdot v^N_{\Lambda_{k,k}},$$

and from the descriptions (2.21) and (2.23) of the ideals $I_{\Lambda_{k,0}}$ and $I'_{\Lambda_{k,k}}$ we see that (4.25) holds. The $k = 1$ case of (4.25) was used in [2].

For the reader's convenience we recall from [2] the shift, or translation, automorphism

$$\tau : U(\bar{\mathfrak{n}}) \longrightarrow U(\bar{\mathfrak{n}}) \tag{4.26}$$

given by

$$\tau(x_\alpha(m_1) \cdots x_\alpha(m_r)) = x_\alpha(m_1 - 1) \cdots x_\alpha(m_r - 1)$$

for any integers $m_1, \ldots, m_k$. For any integer s, the sth power

$$\tau^s : U(\bar{\mathfrak{n}}) \longrightarrow U(\bar{\mathfrak{n}}) \tag{4.27}$$

is given by

$$\tau^s(x_\alpha(m_1) \cdots x_\alpha(m_r)) = x_\alpha(m_1 - s) \cdots x_\alpha(m_r - s).$$

Recall from Remark 3.1 in [2] that for any nonzero element $a \in U(\bar{\mathfrak{n}})$ homogeneous with respect to both the weight and charge gradings such that a has positive charge, the element $\tau^s(a)$ has the same properties, and

$$\text{wt}\ \tau^s(a) > \text{wt}\ a \quad \text{for } s > 0 \tag{4.28}$$

and

$$\text{wt}\ \tau^s(a) < \text{wt}\ a \quad \text{for } s < 0. \tag{4.29}$$

If a is a constant, that is, a has charge zero, then

$$\tau^s(a) = a \tag{4.30}$$

and $\tau^s(a)$ and a have the same weight and charge.

Generalizing Remark 3.2 in [2], we now have:

Remark 4.4. Using the map τ we can re-express (4.10) and (4.12) as follows:

$$e^{\alpha/2}_{(k,i)}(a \cdot v_{\Lambda_{k,i}}) = \frac{1}{i!}\tau(a)x_\alpha(-1)^i \cdot v_{\Lambda_{k,k-i}}, \quad a \in U(\bar{\mathfrak{n}}),\ 0 \le i \le k \tag{4.31}$$

and

$$\widehat{e^{\alpha/2}_{(k,i)}}(a \cdot v^N_{\Lambda_{k,i}}) = \frac{1}{i!}\tau(a)x_\alpha(-1)^i \cdot v^N_{\Lambda_{k,k-i}}, \quad a \in U(\bar{\mathfrak{n}}_-),\ 0 \le i \le k. \tag{4.32}$$

(recall (4.11) and (4.14)).

Lemma 3.3 in [2] generalizes to:

Lemma 4.3. *We have*

$$\tau(I_{\Lambda_{k,0}}) \subset I_{\Lambda_{k,0}} + U(\bar{\mathfrak{n}}_-)x_\alpha(-1) = I_{\Lambda_{k,k}}.$$

Proof. Let $t \ge k+1$. Then

$$\tau(R^0_{k,t}) = R^1_{k,t+k+1} = R^0_{k,t+k+1} + a \cdot x_\alpha(-1)$$

for some $a \in U(\bar{\mathfrak{n}}_-)$. □

Intertwining vertex operators (in the sense of [9,6]) among triples of $L(\Lambda_{k,0})$-modules play an important role in this paper, as they did in [4,5] and [2]. The following theorem is well known:

Theorem 4.1 ([11]). *For integers i, j and m with $0 \le i, j, m \le k$, write*

$$I\begin{pmatrix} L(\Lambda_{k,m}) \\ L(\Lambda_{k,i})\ \ L(\Lambda_{k,j}) \end{pmatrix}$$

for the vector space of intertwining operators of type $\begin{pmatrix} L(\Lambda_{k,m}) \\ L(\Lambda_{k,i})\ \ L(\Lambda_{k,j}) \end{pmatrix}$. The dimensions of these spaces (the fusion rules) are given by:

$$\dim\ I\begin{pmatrix} L(\Lambda_{k,m}) \\ L(\Lambda_{k,i})\ \ L(\Lambda_{k,j}) \end{pmatrix} = 1$$

if and only if

$$i + j - 2\max\{0, i+j-k\} \ge m \ge i + j - 2\min\{i, j\}, \quad m \equiv i + j \bmod 2,$$

and otherwise, the fusion rule is zero. □

As a consequence, we see that $L(\Lambda_{k,0})$ and $L(\Lambda_{k,k})$ are "group-like" elements in the fusion ring at level k. Such modules are sometimes called "simple currents".

An intertwining operator $\mathcal{Y}(\cdot, x)$ of type $\binom{L(\Lambda_{k,m})}{L(\Lambda_{k,i})\ L(\Lambda_{k,j})}$ satisfies the condition

$$\mathcal{Y}(v_{\Lambda_{k,i}}, x) \in x^{\mathrm{wt}\, v_{\Lambda_{k,m}} - \mathrm{wt}\, v_{\Lambda_{k,i}} - \mathrm{wt}\, v_{\Lambda_{k,j}}} \mathrm{Hom}\, (L(\Lambda_{k,j}), L(\Lambda_{k,m}))[[x, x^{-1}]]$$

(cf. [9,11]). Denote by

$$\mathcal{Y}_c(v_{\Lambda_{k,i}}, x) \in \mathrm{Hom}\, (L(\Lambda_{k,j}), L(\Lambda_{k,m}))$$

the constant term of $x^{-\mathrm{wt}\, v_{\Lambda_{k,m}} + \mathrm{wt}\, v_{\Lambda_{k,i}} + \mathrm{wt}\, v_{\Lambda_{k,j}}} \mathcal{Y}(v_{\Lambda_{k,i}}, x)$. Then as in [5], we have

$$[x_\alpha(n), \mathcal{Y}(v_{\Lambda_{k,i}}, x)] = 0 \quad \text{for all } n \in \mathbb{Z}. \tag{4.33}$$

If $\mathcal{Y}_c(v_{\Lambda_{k,i}}, x)v_{\Lambda_{k,j}}$ is nonzero then it is a highest weight vector of $L(\Lambda_{k,m})$, so that

$$\mathcal{Y}_c(v_{\Lambda_{k,i}}, x)v_{\Lambda_{k,j}} = \gamma v_{\Lambda_{k,m}}, \tag{4.34}$$

where $\gamma \neq 0$; this will hold for our cases below. Using these remarks about intertwining operators and constant terms we prove the following:

Lemma 4.4. *For any i with $0 \leq i < k$ we have*

$$\mathrm{Ker}\, f_{\Lambda_{k,i}} \subset \mathrm{Ker}\, f_{\Lambda_{k,i+1}}, \tag{4.35}$$

so that

$$\mathrm{Ker}\, f_{\Lambda_{k,0}} \subset \mathrm{Ker}\, f_{\Lambda_{k,1}} \subset \cdots \subset \mathrm{Ker}\, f_{\Lambda_{k,k}}. \tag{4.36}$$

Proof. We consider a nonzero intertwining operator $\mathcal{Y}$ of type

$$\binom{L(\Lambda_{k,i+1})}{L(\Lambda_{k,1})\ L(\Lambda_{k,i})};$$

the corresponding fusion rule is one by Theorem 4.1. Consider $\mathcal{Y}_c(v_{\Lambda_{k,1}}, x)$, the constant term of the nonzero operator $x^{-\mathrm{wt}\, v_{\Lambda_{k,i+1}} + \mathrm{wt}\, v_{\Lambda_{k,1}} + \mathrm{wt}\, v_{\Lambda_{k,i}}} \mathcal{Y}(v_{\Lambda_{k,1}}, x)$.

Let $a \in U(\bar{\mathfrak{n}}_-)$ be such that $a \in \mathrm{Ker}\, f_{\Lambda_{k,i}}$, so that $a \cdot v_{\Lambda_{k,i}} = 0$. By applying the map $\mathcal{Y}_c(v_{\Lambda_{k,1}}, x)$ to $a \cdot v_{\Lambda_{k,i}}$ and using (4.33) and (4.34) we obtain

$$\gamma a \cdot v_{\Lambda_{k,i+1}} = 0 \quad \text{with } \gamma \neq 0,$$

so that

$$a \in \mathrm{Ker}\, f_{\Lambda_{k,i+1}},$$

as desired. □

Remark 4.5. The maps $\mathcal{Y}_c(v_{\Lambda_{k,1}}, x)$ $(0 \leq i < k)$ used here are exactly the same as the constant-term maps crucially used in Theorem 4.2 (formula (4.44)) of [5].

Remark 4.6. In order to construct $\mathcal{Y}_c$ and prove Lemma 4.4 we do not in fact need results from [11]. The construction of $\mathcal{Y}_c$ follows easily from results in Chapter 13 of [6], while Lemma 4.4 follows from the relation $\mathrm{Ker}\, f_{\Lambda_{1,0}} \subset \mathrm{Ker}\, f_{\Lambda_{1,1}}$ and (4.7) (cf. also Chapter 13 of [6]).

Our next goal is to prove the main result, Theorem 3.1, or equivalently, Theorem 3.2 (formula (3.20)), which is what we will in fact prove.

We notice first the inclusion

$$I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}} \subset \mathrm{Ker}\, \pi_{\Lambda_{k,i}}, \quad 0 \leq i \leq k. \tag{4.37}$$

Indeed, as is well known, the $(k+1)$th power of the vertex operator $Y(x_\alpha(-1)\cdot v_{\Lambda_{k,0}}, x)$ is well defined (the components $x_\alpha(m)$, $m \in \mathbb{Z}$, of this vertex operator commute) and equals zero on each $L(\Lambda_{k,i})$, and in particular on $W(\Lambda_{k,i})$. The expansion coefficients of $Y(x_\alpha(-1)\cdot v_{\Lambda_{k,0}}, x)^{k+1}$ are the operators $R_{k,-t}$, $t \in \mathbb{Z}$:

$$Y(x_\alpha(-1)\cdot v_{\Lambda_{k,0}}, x)^{k+1} = \sum_{t\in\mathbb{Z}} \left(\sum_{m_1+m_2+\cdots+m_{k+1}=t} x_\alpha(m_1)x_\alpha(m_2)\cdots x_\alpha(m_{k+1}) \right) x^{-t-k-1} \tag{4.38}$$

(recall (2.13) and (3.2)). Thus the operators (2.14) annihilate the highest weight vector $v_{\Lambda_{k,i}}$, and (4.37) follows.

Before we prove our main result for the general level $k \geq 1$ (Theorem 3.2) we first prove this result for $k = 1$, for the reasons mentioned in Remark 4.7 below. We have $i = 0, 1$, and we shall use the notation Λ_0 and Λ_1 instead of $\Lambda_{1,0}$ and $\Lambda_{1,1}$ (recall (2.5)).

Proof of the $k = 1$ case of Theorem 3.2. By (4.37) it is sufficient to show that

$$\text{Ker}\, \pi_{\Lambda_i} \subset I_{\Lambda_i} \cdot v^N_{\Lambda_i} \quad \text{for } i = 0, 1. \tag{4.39}$$

We will prove this by contradiction. Assume then that there exists $a \in U(\bar{\mathfrak{n}}_-)$ such that

$$a \cdot v^N_{\Lambda_i} \in \text{Ker}\, \pi_{\Lambda_i} \quad \text{but } a \cdot v^N_{\Lambda_i} \notin I_{\Lambda_i} \cdot v^N_{\Lambda_i} \quad \text{for } i = 0 \text{ or } 1. \tag{4.40}$$

By Remarks 3.1 and 3.4 we may and do assume that a is doubly homogeneous, that is, homogeneous with respect to the weight and charge gradings. By the second statement in (4.40), a is nonzero, and by the first statement in (4.40), a is in fact nonconstant, so that a has positive weight and positive charge. Let

$$L = \min\{\text{wt}\, d \mid d \in U(\bar{\mathfrak{n}}_-) \text{ doubly homogeneous such that (4.40) holds for } d\}. \tag{4.41}$$

Any such element d is nonzero and in fact nonconstant (just as for the chosen element a), so that any such d has positive weight and charge; thus $L > 0$. We further assume that wt $a = L$. Note that i might be 0 or 1 or both. We shall show that in fact i cannot be 1, and then we shall use this to show that i cannot be 0, giving our desired contradiction.

By (2.18) we have a unique decomposition

$$a = r_0 x_\alpha(-1) + s_0 \tag{4.42}$$

with $r_0 \in U(\bar{\mathfrak{n}}_-)$ and $s_0 \in U(\bar{\mathfrak{n}}_{\leq -2})$. The elements r_0 and s_0 are doubly homogeneous, and in fact,

$$\text{wt}\, r_0 = \text{wt}\, a - 1, \qquad \text{wt}\, s_0 = \text{wt}\, a; \tag{4.43}$$

similarly, the charge of r_0 is one less than that of a and the charges of s_0 and a are equal. Applying τ^{-1} to (4.42) gives

$$\tau^{-1}(a) = \tau^{-1}(r_0)x_\alpha(0) + \tau^{-1}(s_0), \tag{4.44}$$

and $\tau^{-1}(s_0)$ is doubly homogeneous,

$$\tau^{-1}(s_0) \in U(\bar{\mathfrak{n}}_-), \tag{4.45}$$

and

$$\text{wt}\, \tau^{-1}(s_0) < \text{wt}\, a, \tag{4.46}$$

from (4.29) and the fact that the charge of a and hence of s_0 is positive.

Suppose now that $i = 1$. Then we have

$$a \cdot v^N_{\Lambda_1} \in \text{Ker}\, \pi_{\Lambda_1} \quad \text{but } a \cdot v^N_{\Lambda_1} \notin I_{\Lambda_1} \cdot v^N_{\Lambda_1} \quad (\text{that is, } a \notin I_{\Lambda_1}), \tag{4.47}$$

where a is doubly homogeneous and wt $a = L$ (recall (4.41)). We are going to show that there exists a doubly homogeneous element of $U(\bar{\mathfrak{n}}_-)$, namely, $\tau^{-1}(s_0)$, whose weight is less than L and which satisfies (4.40). We note that $s_0 \neq 0$, because $a \notin U(\bar{\mathfrak{n}}_-)x_\alpha(-1)$, by (2.25) and (4.47). We have seen that $\tau^{-1}(s_0)$ is doubly homogeneous and that its weight is less than wt a. Since $s_0 \cdot v^N_{\Lambda_1} \in$ Ker π_{Λ_1} (by (4.42) and (4.47)), we have $s_0 \cdot v_{\Lambda_1} = 0$. We also have

$$e^{\alpha/2}_{(1,0)}(\tau^{-1}(s_0)\cdot v_{\Lambda_0}) = s_0 \cdot v_{\Lambda_1} = 0$$

(recall (4.31)), which together with the injectivity of $e^{\alpha/2}_{(1,0)}$ implies

$$\tau^{-1}(s_0)\cdot v^N_{\Lambda_0}\in \operatorname{Ker}\pi_{\Lambda_0}. \tag{4.48}$$

We also have

$$\tau^{-1}(s_0)\cdot v^N_{\Lambda_0}\notin I_{\Lambda_0}\cdot v^N_{\Lambda_0}. \tag{4.49}$$

Indeed, if (4.49) does not hold, then $\tau^{-1}(s_0)\in I_{\Lambda_0}$, and by Lemma 4.3 we get $s_0\in I_{\Lambda_1}$. Now (4.42) yields $a\in I_{\Lambda_1}$, and thus $a\cdot v^N_{\Lambda_1}\in I_{\Lambda_1}\cdot v^N_{\Lambda_1}$, contradicting (4.47). Hence (4.49) holds. Now (4.48) and (4.49) give a contradiction since $\tau^{-1}(s_0)$ is a doubly homogeneous element satisfying (4.40) but whose weight is less than wt $a=L$. We have shown that i cannot be 1.

Now we may and do assume that $i=0$, that is,

$$a\cdot v^N_{\Lambda_0}\in \operatorname{Ker}\pi_{\Lambda_0}\quad \text{but } a\cdot v^N_{\Lambda_0}\notin I_{\Lambda_0}\cdot v^N_{\Lambda_0}, \tag{4.50}$$

where a is doubly homogeneous of weight L (recall (4.41)). Since $a\cdot v^N_{\Lambda_0}\in \operatorname{Ker}\pi_{\Lambda_0}$,

$$a\cdot v_{\Lambda_0}=0\quad \text{in } W(\Lambda_0)$$

and by Lemma 4.4 we obtain

$$a\cdot v_{\Lambda_1}=0\quad \text{in } W(\Lambda_1). \tag{4.51}$$

Hence $a\cdot v^N_{\Lambda_1}\in \operatorname{Ker}\pi_{\Lambda_1}$, and so by what we have just proved (that i cannot be 1), we obtain

$$a\cdot v^N_{\Lambda_1}\in I_{\Lambda_1}\cdot v^N_{\Lambda_1},$$

and so

$$a\in I_{\Lambda_1}.$$

Our goal is to show that in fact $a\in I_{\Lambda_0}$, which will contradict (4.50).

From (2.25) we have

$$a=b_1x_\alpha(-1)+c_1 \tag{4.52}$$

with

$$b_1\in U(\bar{\mathfrak{n}}_-)\quad \text{and}\quad c_1\in I_{\Lambda_0}. \tag{4.53}$$

By Remark 3.1 we may and do assume that b_1 and c_1 are doubly homogeneous; then wt $b_1=$ wt $a-1$, wt $c_1=$ wt a, the charge of b_1 is one less than that of a, and c_1 and a have the same charge.

We now claim that

$$b_1x_\alpha(-1)\in I_{\Lambda_0}. \tag{4.54}$$

Assume then that

$$b_1x_\alpha(-1)\notin I_{\Lambda_0}. \tag{4.55}$$

Then

$$b_1\notin U(\bar{\mathfrak{n}}_-)x_\alpha(-1); \tag{4.56}$$

otherwise, $b_1x_\alpha(-1)\in U(\bar{\mathfrak{n}}_-)x_\alpha(-1)^2\subset I_{\Lambda_0}$. By (2.18) we have a unique decomposition

$$b_1=r_1x_\alpha(-1)+s_1,\qquad r_1\in U(\bar{\mathfrak{n}}_-),\ s_1\in U(\bar{\mathfrak{n}}_{\le -2}), \tag{4.57}$$

and r_1 and s_1 are doubly homogeneous, with wt $r_1=$ wt b_1-1, wt $s_1=$ wt b_1, and similarly for charge. We have $s_1\neq 0$ by (4.56). We will use the vector s_1 to produce a contradiction. We have

$$\tau^{-1}(b_1)=\tau^{-1}(r_1)x_\alpha(0)+\tau^{-1}(s_1)\quad \text{and}\quad \tau^{-1}(s_1)\in U(\bar{\mathfrak{n}}_-). \tag{4.58}$$

Since $b_1 x_\alpha(-1) \cdot v_{\Lambda_0}^N = a \cdot v_{\Lambda_0}^N - c_1 \cdot v_{\Lambda_0}^N \in \operatorname{Ker} \pi_{\Lambda_0}$,

$$b_1 x_\alpha(-1) \cdot v_{\Lambda_0} = 0,$$

and so by (4.31),

$$\tau^{-1}(b_1) \cdot v_{\Lambda_1} = 0.$$

Thus (4.58) gives

$$\tau^{-1}(s_1) \cdot v_{\Lambda_1} = 0,$$

so that

$$\tau^{-1}(s_1) \cdot v_{\Lambda_1}^N \in \operatorname{Ker} \pi_{\Lambda_1}. \tag{4.59}$$

By combining (4.32), (4.55) and (4.57) we also have

$$\widehat{e_{(1,1)}^{\alpha/2}}(\tau^{-1}(s_1) \cdot v_{\Lambda_1}^N) = s_1 x_\alpha(-1) \cdot v_{\Lambda_0}^N = b_1 x_\alpha(-1) \cdot v_{\Lambda_0}^N - r_1 x_\alpha(-1)^2 \cdot v_{\Lambda_0}^N \notin I_{\Lambda_0} \cdot v_{\Lambda_0}^N, \tag{4.60}$$

which by Lemma 4.2 implies

$$\tau^{-1}(s_1) \cdot v_{\Lambda_1}^N \notin I_{\Lambda_1} \cdot v_{\Lambda_1}^N. \tag{4.61}$$

Since s_1 is doubly homogeneous, so is $\tau^{-1}(s_1)$, and

$$\operatorname{wt} \tau^{-1}(s_1) \leq \operatorname{wt} s_1 = \operatorname{wt} b_1 < \operatorname{wt} a = L \tag{4.62}$$

(note that if s_1 has charge 0, that is, is a constant, then $\operatorname{wt} \tau^{-1}(s_1) = \operatorname{wt} s_1$). Now (4.59) and (4.61) together with the fact that $\tau^{-1}(s_1)$ is doubly homogeneous of weight less than L give us a contradiction. This proves our claim (4.54), and hence that

$$a = b_1 x_\alpha(-1) + c_1 \in I_{\Lambda_0}, \tag{4.63}$$

which contradicts (4.50). We have proved that i cannot be 0 and we have thus established (4.39), completing the proof of Theorem 3.2 for $k = 1$. □

Remark 4.7. We have just proved Theorem 3.2 (formula (3.20)) for $k = 1$, by contradiction, in such a way that the assertion to be contradicted, namely, (4.40), involves *both* $W^N(\Lambda_0)$ and $W^N(\Lambda_1)$. A different proof of this theorem was given in [2] (see the proof of Theorem 2.2), where our argument proved the result for $W^N(\Lambda_0)$ and used this result to prove the result for $W^N(\Lambda_1)$. Also, the proof given here does not use the space $W^N(\Lambda_1)'$ and related "primed" spaces, which played a crucial role in the proof of the corresponding result in [2]. We have, however, included information about such "primed" spaces in the present paper, including conclusions about them in Theorems 3.1 and 3.2, partly for reasons of comparison with the arguments in [2]. Our new argument for proving the $k = 1$ case of Theorem 3.2 naturally generalizes to $k \geq 1$ (see the proof below, which, while it certainly reduces to the proof above when $k = 1$, appears more complicated in the greater generality), and it will also be generalized in a different direction in subsequent work [3].

We now generalize the $k = 1$ proof to all $k \geq 1$.

Proof of Theorem 3.2. In view of (4.37) it is sufficient to prove that

$$\operatorname{Ker} \pi_{\Lambda_{k,i}} \subset I_{\Lambda_{k,i}} \cdot v_{\Lambda_{k,i}}^N \quad \text{for all } i = 0, \ldots, k. \tag{4.64}$$

Again we will prove this by contradiction. Suppose then that there exists $a \in U(\bar{\mathfrak{n}}_-)$ such that

$$a \cdot v_{\Lambda_{k,i}}^N \in \operatorname{Ker} \pi_{\Lambda_{k,i}} \quad \text{but } a \cdot v_{\Lambda_{k,i}}^N \notin I_{\Lambda_{k,i}} \cdot v_{\Lambda_{k,i}}^N \quad \text{for some } i = 0, \ldots, k. \tag{4.65}$$

By Remarks 3.1 and 3.4 we may and do assume that a is doubly homogeneous. Since a is nonzero and in fact nonconstant (as above), it has positive weight and charge. Let

$$L = \min\{\operatorname{wt} d \mid d \in U(\bar{\mathfrak{n}}_-) \text{ doubly homogeneous such that (4.65) holds for } d\} \quad (>0). \tag{4.66}$$

We further assume that wt $a = L$. Note that i might be any one or more of the indices from 0 to k. We shall show first that in fact i cannot be k.

Formulas (4.42)–(4.46) hold, exactly as in the $k = 1$ case.

Suppose that $i = k$, that is,

$$a \cdot v^N_{\Lambda_{k,k}} \in \operatorname{Ker} \pi_{\Lambda_{k,k}} \quad \text{but } a \cdot v^N_{\Lambda_{k,k}} \notin I_{\Lambda_{k,k}} \cdot v^N_{\Lambda_{k,k}} \quad (\text{that is, } a \notin I_{\Lambda_{k,k}}), \tag{4.67}$$

where a is doubly homogeneous and wt $a = L$ (recall (4.66)). We will show that $\tau^{-1}(s_0)$ (recall (4.42)) is a doubly homogeneous element of $U(\bar{\mathfrak{n}}_-)$ whose weight is less than L and which satisfies (4.65). We see that $s_0 \neq 0$, since $a \notin U(\bar{\mathfrak{n}}_-)x_\alpha(-1)$, by (2.25) and (4.67), and we know that wt $\tau^{-1}(s_0) <$ wt a. From (4.42) and (4.67) we obtain $s_0 \cdot v^N_{\Lambda_{k,k}} \in \operatorname{Ker} \pi_{\Lambda_{k,k}}$, which is equivalent to $s_0 \cdot v_{\Lambda_{k,k}} = 0$. Since

$$e^{\alpha/2}_{(k,0)}(\tau^{-1}(s_0) \cdot v_{\Lambda_{k,0}}) = s_0 \cdot v_{\Lambda_{k,k}} = 0$$

(from (4.31)) and since $e^{\alpha/2}_{(k,0)}$ is injective we obtain

$$\tau^{-1}(s_0) \cdot v^N_{\Lambda_{k,0}} \in \operatorname{Ker} \pi_{\Lambda_{k,0}}. \tag{4.68}$$

Just as in the proof of the case $k = 1$ we show that

$$\tau^{-1}(s_0) \cdot v^N_{\Lambda_{k,0}} \notin I_{\Lambda_{k,0}} \cdot v^N_{\Lambda_{k,0}}, \tag{4.69}$$

and we have constructed a doubly homogeneous element $\tau^{-1}(s_0)$ of $U(\bar{\mathfrak{n}}_-)$ satisfying (4.65) whose weight is less than wt $a = L$. This is a contradiction, and so i cannot be k.

Now we may and do assume that

$$a \cdot v^N_{\Lambda_{k,i}} \in \operatorname{Ker} \pi_{\Lambda_{k,i}} \qquad \text{but } a \cdot v^N_{\Lambda_{k,i}} \notin I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}} \quad \text{for some } i = 0, \ldots, k-1, \tag{4.70}$$

where a is doubly homogeneous of weight L (recall (4.66)). We now fix any one of the indices i for which (4.70) holds. Our next goal is to show that i cannot be $k-1$.

Since $a \cdot v_{\Lambda_{k,i}} \in \operatorname{Ker} \pi_{\Lambda_{k,i}}$ we have

$$a \cdot v_{\Lambda_{k,i}} = 0 \quad \text{in } W(\Lambda_{k,i}),$$

and thus by Lemma 4.4 we obtain

$$a \cdot v_{\Lambda_{k,k}} = 0 \quad \text{in } W(\Lambda_{k,k}), \tag{4.71}$$

that is, $a \cdot v^N_{\Lambda_{k,k}} \in \operatorname{Ker} \pi_{\Lambda_{k,k}}$. The case we just proved (that i cannot be k) thus gives us

$$a \cdot v^N_{\Lambda_{k,k}} \in I_{\Lambda_{k,k}} \cdot v^N_{\Lambda_{k,k}},$$

and so

$$a \in I_{\Lambda_{k,k}}.$$

Just as in (4.52) and (4.53), we use (2.25) to write

$$a = b_1 x_\alpha(-1) + c_1 \tag{4.72}$$

with

$$b_1 \in U(\bar{\mathfrak{n}}_-) \quad \text{and} \quad c_1 \in I_{\Lambda_{k,0}}, \tag{4.73}$$

and by Remark 3.1 we may and do assume that b_1 and c_1 are doubly homogeneous. Then in fact wt $b_1 =$ wt $a - 1$, wt $c_1 =$ wt a, the charge of b_1 is one less than that of a, and c_1 and a have the same charge.

We next claim that

$$b_1 x_\alpha(-1) \in I_{\Lambda_{k,k-1}}. \tag{4.74}$$

Suppose instead that

$$b_1 x_\alpha(-1) \notin I_{\Lambda_{k,k-1}}. \tag{4.75}$$

Then

$$b_1 \notin U(\bar{\mathfrak{n}}_-) x_\alpha(-1) \tag{4.76}$$

(otherwise, $b_1 x_\alpha(-1) \in U(\bar{\mathfrak{n}}_-) x_\alpha(-1)^2 \subset I_{\Lambda_{k,k-1}}$). We have a unique decomposition

$$b_1 = r_1 x_\alpha(-1) + s_1, \quad r_1 \in U(\bar{\mathfrak{n}}_-),\ s_1 \in U(\bar{\mathfrak{n}}_{\le -2}) \tag{4.77}$$

by (2.18), and r_1 and s_1 are doubly homogeneous, with wt $r_1 =$ wt $b_1 - 1$, wt $s_1 =$ wt b_1, and similarly for charge. Note that by (4.76) we have $s_1 \neq 0$. We also have

$$\tau^{-1}(b_1) = \tau^{-1}(r_1) x_\alpha(0) + \tau^{-1}(s_1) \quad \text{and} \quad \tau^{-1}(s_1) \in U(\bar{\mathfrak{n}}_-). \tag{4.78}$$

Remark 2.1 and (4.37) yield the inclusions

$$I_{\Lambda_{k,0}} \cdot v^N_{\Lambda_{k,i}} \subset I_{\Lambda_{k,i}} \cdot v^N_{\Lambda_{k,i}} \subset \operatorname{Ker} \pi_{\Lambda_{k,i}}, \tag{4.79}$$

so that

$$b_1 x_\alpha(-1) \cdot v^N_{\Lambda_{k,i}} = (a - c_1) \cdot v^N_{\Lambda_{k,i}} \in \operatorname{Ker} \pi_{\Lambda_{k,i}}$$

(recall (4.70), (4.72) and (4.73)), and this is equivalent to

$$b_1 x_\alpha(-1) \cdot v_{\Lambda_{k,i}} = 0.$$

Now by Lemma 4.4 we obtain

$$b_1 x_\alpha(-1) \cdot v_{\Lambda_{k,k-1}} = 0, \tag{4.80}$$

and so (4.31) yields

$$\tau^{-1}(b_1) \cdot v_{\Lambda_{k,1}} = 0.$$

Hence from (4.78) we get

$$\tau^{-1}(s_1) \cdot v^N_{\Lambda_{k,1}} = \tau^{-1}(b_1) \cdot v^N_{\Lambda_{k,1}} \in \operatorname{Ker} \pi_{\Lambda_{k,1}}. \tag{4.81}$$

On the other hand, by (4.32) (using the fact that $\tau^{-1}(s_1) \in U(\bar{\mathfrak{n}}_-)$), (4.75) and (4.77)) we also have

$$\widehat{e^{\alpha/2}_{(k,1)}}(\tau^{-1}(s_1) \cdot v^N_{\Lambda_{k,1}}) = s_1 x_\alpha(-1) \cdot v^N_{\Lambda_{k,k-1}} = b_1 x_\alpha(-1) \cdot v^N_{\Lambda_{k,k-1}} - r_1 x_\alpha(-1)^2 \cdot v^N_{\Lambda_{k,k-1}} \notin I_{\Lambda_{k,k-1}} \cdot v^N_{\Lambda_{k,k-1}}.$$

Thus by using Lemma 4.2 we obtain

$$\tau^{-1}(s_1) \cdot v^N_{\Lambda_{k,1}} \notin I_{\Lambda_{k,1}} \cdot v^N_{\Lambda_{k,1}}. \tag{4.82}$$

Just as in the proof of the case $k = 1$ (recall (4.62)) we see that $\tau^{-1}(s_1)$ is doubly homogeneous of weight less than wt $a = L$. We have obtained a contradiction by constructing the doubly homogeneous element $\tau^{-1}(s_1)$ satisfying (4.81) and (4.82), and hence (4.65), whose weight is less than L. (Note that if $k = 1$, we are *not* claiming that $\tau^{-1}(s_1)$ also satisfies (4.70).) This proves our claim (4.74).

Thus

$$a = b_1 x_\alpha(-1) + c_1 \in I_{\Lambda_{k,k-1}} \tag{4.83}$$

with $b_1 \in U(\bar{\mathfrak{n}}_-)$ and $c_1 \in I_{\Lambda_{k,0}}$, by (4.72) and (2.24), and we have shown that the index i in (4.65) and in (4.70) cannot be $k - 1$. In particular, if $k = 1$ we are done.

Suppose then that $k \geq 2$. Then we may and do choose the index i in (4.70) so that $0 \leq i \leq k - 2$. We shall next show that this index i cannot be $k - 2$. This argument will be similar to the previous one, and it will make the general pattern clear.

Since

$$a \cdot v_{\Lambda_{k,i}} = 0 \quad \text{for some } i = 0, \dots, k-2, \tag{4.84}$$

by Lemma 4.4 we get

$$a \cdot v_{\Lambda_{k,k-1}} = 0, \tag{4.85}$$

that is, $a \cdot v^N_{\Lambda_{k,k-1}} \in \operatorname{Ker} \pi_{\Lambda_{k,k-1}}$. By the previous case (that i cannot be $k-1$),

$$a \cdot v^N_{\Lambda_{k,k-1}} \in I_{\Lambda_{k,k-1}} \cdot v^N_{\Lambda_{k,k-1}}, \tag{4.86}$$

so that

$$a \in I_{\Lambda_{k,k-1}}. \tag{4.87}$$

Thus from (2.25) we obtain

$$a = b_2 x_\alpha(-1)^2 + c_2 \quad \text{with } b_2 \in U(\bar{\mathfrak{n}}_-) \text{ and } c_2 \in I_{\Lambda_{k,0}}, \tag{4.88}$$

and as usual, we may and do assume that b_2 and c_2 are doubly homogeneous (by Remark 3.1), so that wt b_2 = wt $a-2$, wt c_2 = wt a, the charge of b_2 is two less than that of a, and c_2 and a have the same charge.

We now prove by contradiction that

$$b_2 x_\alpha(-1)^2 \in I_{\Lambda_{k,k-2}} \tag{4.89}$$

(cf. (4.74)): If instead

$$b_2 x_\alpha(-1)^2 \notin I_{\Lambda_{k,k-2}}, \tag{4.90}$$

then $b_2 \notin U(\bar{\mathfrak{n}}_-) x_\alpha(-1)$ (cf. (4.76)), and thus we have a unique decomposition

$$b_2 = r_2 x_\alpha(-1) + s_2, \qquad r_2 \in U(\bar{\mathfrak{n}}_-), \qquad 0 \neq s_2 \in U(\bar{\mathfrak{n}}_{\leq -2}), \tag{4.91}$$

and r_2 and s_2 are doubly homogeneous, with wt r_2 = wt $b_2 - 1$, wt s_2 = wt b_2, and similarly for charge (as in (4.77)). We follow the argument of (4.78)–(4.82): We apply τ^{-1} to (4.91). Since (4.79) still holds, we obtain that

$$b_2 x_\alpha(-1)^2 \cdot v_{\Lambda_{k,i}} = (a - c_2) \cdot v_{\Lambda_{k,i}} = 0,$$

which gives

$$b_2 x_\alpha(-1)^2 \cdot v_{\Lambda_{k,k-2}} = 0$$

by Lemma 4.4. Thus by (4.31) we get

$$\tau^{-1}(b_2) \cdot v_{\Lambda_{k,2}} = 0,$$

and so

$$\tau^{-1}(s_2) \cdot v^N_{\Lambda_{k,2}} = \tau^{-1}(b_2) \cdot v^N_{\Lambda_{k,2}} \in \operatorname{Ker} \pi_{\Lambda_{k,2}}. \tag{4.92}$$

Using (4.32), the fact that $\tau^{-1}(s_2) \in U(\bar{\mathfrak{n}}_-)$, (4.90) and (4.91), we also obtain

$$(2!)\, \widehat{e^{\alpha/2}_{(k,2)}} (\tau^{-1}(s_2) \cdot v^N_{\Lambda_{k,2}}) = b_2 x_\alpha(-1)^2 \cdot v^N_{\Lambda_{k,k-2}} - r_2 x_\alpha(-1)^3 \cdot v^N_{\Lambda_{k,k-2}} \notin I_{\Lambda_{k,k-2}} \cdot v^N_{\Lambda_{k,k-2}},$$

and so by Lemma 4.2,

$$\tau^{-1}(s_2) \cdot v^N_{\Lambda_{k,2}} \notin I_{\Lambda_{k,2}} \cdot v^N_{\Lambda_{k,2}}. \tag{4.93}$$

Just as in the proof above, $\tau^{-1}(s_2)$ is a doubly homogeneous element satisfying (4.92) and (4.93) and hence (4.65) (but not necessarily (4.70)) and of weight less than L. This proves (4.89).

By (4.88), (4.89) and (2.24) we now have

$$a = b_2 x_\alpha(-1)^2 + c_2 \in I_{\Lambda_{k,k-2}} \tag{4.94}$$

with $b_2 \in U(\bar{\mathfrak{n}}_-)$ and $c_2 \in I_{\Lambda_{k,0}}$, and this proves that i cannot be $k-2$. In particular, we are done if $k = 2$.

Now we give the general inductive step. Fix $m \geq 1$ and assume that the assertion of Theorem 3.2 has been proved for $k = 1, 2, \ldots, m$ and that i in (4.65) (or in (4.70)) cannot be $k, k-1, \ldots, k-m$. We shall show that if $k \geq m+1$, then the index i cannot be $k-(m+1)$ either, and that in particular, the assertion of Theorem 3.2 thus holds for $k = m+1$. This will complete the proof of the theorem.

Suppose then that $k \geq m+1$ and that the index i in (4.70) is such that $0 \leq i \leq k-(m+1)$. To show that this index i in fact cannot be $k-(m+1)$, we first observe that exactly as in (4.84)–(4.87) we have

$$a \in I_{\Lambda_{k,k-m}},$$

and so from (2.25) we see that

$$a = b_{m+1} x_\alpha(-1)^{m+1} + c_{m+1} \quad \text{with } b_{m+1} \in U(\bar{\mathfrak{n}}_-) \text{ and } c_{m+1} \in I_{\Lambda_{k,0}}. \tag{4.95}$$

Again, as above, we may and do assume that b_{m+1} and c_{m+1} are doubly homogeneous (by Remark 3.1); then $\mathrm{wt}\, b_{m+1} = \mathrm{wt}\, a - (m+1)$, $\mathrm{wt}\, c_{m+1} = \mathrm{wt}\, a$, the charge of b_{m+1} is $m+1$ less than that of a, and c_{m+1} and a have the same charge.

Exactly as in (4.89)–(4.93), we obtain by contradiction that

$$b_{m+1} x_\alpha(-1)^{m+1} \in I_{\Lambda_{k,k-(m+1)}} : \tag{4.96}$$

Assume that

$$b_{m+1} x_\alpha(-1)^{m+1} \notin I_{\Lambda_{k,k-(m+1)}}.$$

In place of formula (4.91), we now have the unique decomposition

$$b_{m+1} = r_{m+1} x_\alpha(-1) + s_{m+1}, \qquad r_{m+1} \in U(\bar{\mathfrak{n}}_-), \quad 0 \neq s_{m+1} \in U(\bar{\mathfrak{n}}_{\leq -2}),$$

with r_{m+1} and s_{m+1} doubly homogeneous, $\mathrm{wt}\, r_{m+1} = \mathrm{wt}\, b_{m+1} - 1$, $\mathrm{wt}\, s_{m+1} = \mathrm{wt}\, b_{m+1}$, and similarly for charge. As in formula (4.78) we now have

$$\tau^{-1}(b_{m+1}) = \tau^{-1}(r_{m+1}) x_\alpha(0) + \tau^{-1}(s_{m+1}) \quad \text{and} \quad \tau^{-1}(s_{m+1}) \in U(\bar{\mathfrak{n}}_-).$$

By (4.79) we obtain

$$b_{m+1} x_\alpha(-1)^{m+1} \cdot v_{\Lambda_{k,i}} = (a - c_{m+1}) \cdot v_{\Lambda_{k,i}} = 0,$$

so that

$$b_{m+1} x_\alpha(-1)^{m+1} \cdot v_{\Lambda_{k,k-(m+1)}} = 0,$$

by Lemma 4.4, and so (4.31) gives

$$\tau^{-1}(b_{m+1}) \cdot v_{\Lambda_{k,m+1}} = 0.$$

Thus

$$\tau^{-1}(s_{m+1}) \cdot v^N_{\Lambda_{k,m+1}} = \tau^{-1}(b_{m+1}) \cdot v^N_{\Lambda_{k,m+1}} \in \mathrm{Ker}\, \pi_{\Lambda_{k,m+1}}. \tag{4.97}$$

Since $\tau^{-1}(s_{m+1}) \in U(\bar{\mathfrak{n}}_-)$, we can use (4.32), and exactly as above we find that

$$(m+1)!\, \widehat{e^{\alpha/2}_{(k,m+1)}} (\tau^{-1}(s_{m+1}) \cdot v^N_{\Lambda_{k,m+1}}) = b_{m+1} x_\alpha(-1)^{m+1} \cdot v^N_{\Lambda_{k,k-(m+1)}} - r_{m+1} x_\alpha(-1)^{m+2} \cdot v^N_{\Lambda_{k,k-(m+1)}},$$

so that

$$\widehat{e^{\alpha/2}_{(k,m+1)}} (\tau^{-1}(s_{m+1}) \cdot v^N_{\Lambda_{k,m+1}}) \notin I_{\Lambda_{k,k-(m+1)}} \cdot v^N_{\Lambda_{k,k-(m+1)}}.$$

Thus by Lemma 4.2,

$$\tau^{-1}(s_{m+1}) \cdot v^N_{\Lambda_{k,m+1}} \notin I_{\Lambda_{k,m+1}} \cdot v^N_{\Lambda_{k,m+1}}. \tag{4.98}$$

Since $\tau^{-1}(s_{m+1})$ is a doubly homogeneous element satisfying (4.97) and (4.98) and thus (4.65) (but not necessarily (4.70)) and of weight less than L, we have proved (4.96).

Hence from (4.95), (4.96) and (2.24) we finally obtain

$$a = b_{m+1}x_\alpha(-1)^{m+1} + c_{m+1} \in I_{\Lambda_{k,k-(m+1)}}, \tag{4.99}$$

proving that i cannot be $k-(m+1)$ and thus proving Theorem 3.2. □

Remark 4.8. The first part of the proof, in which we showed that i cannot be k, is actually essentially the same argument as the successive arguments showing that i cannot be $k-1$, $k-2$, and so on.

Remark 4.9. As an immediate consequence of Theorem 3.1, we see that any nonzero doubly homogeneous element $a \in U(\bar{\mathfrak{n}}_-)$ such that $a \in \operatorname{Ker} f_{\Lambda_{k,0}} = I_{\Lambda_{k,0}}$ has charge at least $k+1$; that is, no nonzero linear combination of monomials $x_\alpha(m_1)\cdots x_\alpha(m_r)$ with $r \le k$ and each $m_i < 0$ belongs to $\operatorname{Ker} f_{\Lambda_{k,0}}$. We observe similarly that any homogeneous element of charge $k+1$ that lies in $\operatorname{Ker} f_{\Lambda_0}$ is a multiple of R_t^0 for some $t \ge k+1$.

5. Another reformulation

Generalizing the last section of [2], we shall finally give a further reformulation of the $i=0$ case of Theorem 3.2, formula (3.20), in terms of principal ideals of vertex (operator) algebras. As in [2], we shall invoke [18] for material on ideals of vertex (operator) algebras and on vertex-operator algebra and module structure on generalized Verma modules.

The generalized Verma module $N(\Lambda_{k,0})$ has a natural structure of vertex-operator algebra, with vertex-operator map

$$Y(\cdot,x) : N(\Lambda_{k,0}) \longrightarrow \operatorname{End} N(\Lambda_{k,0})[[x,x^{-1}]]$$
$$v \mapsto Y(v,x) = \sum_{m\in\mathbb{Z}} v_m x^{-m-1}$$

satisfying the conditions given in Theorem 6.2.18 of [18], with $v^N_{\Lambda_{k,0}}$ as vacuum vector. The conformal vector gives rise to the Virasoro algebra operators $L(m)$, $m \in \mathbb{Z}$, including the operator $L(0)$ used above. Also, $N(\Lambda_{k,i})$ for $0 \le i \le k$ is naturally a module for the vertex-operator algebra $N(\Lambda_{k,0})$, as described in Theorem 6.2.21 of [18].

Just as in [2], $W^N(\Lambda_{k,0})$ is a vertex subalgebra of $N(\Lambda_{k,0})$ and $W^N(\Lambda_{k,i})$ is a $W^N(\Lambda_{k,0})$-submodule of $N(\Lambda_{k,i})$ for $0 \le i \le k$. Also, $L(0)$ preserves $W^N(\Lambda_{k,i})$ for $0 \le i \le k$ and $L(-1)$ preserves only $W^N(\Lambda_{k,0})$.

We recall from Section 3 the natural surjective $\widehat{\mathfrak{g}}$-module maps

$$\Pi_{\Lambda_{k,i}} : N(\Lambda_{k,i}) \longrightarrow L(\Lambda_{k,i}) \tag{5.1}$$
$$a\cdot v^N_{\Lambda_{k,i}} \mapsto a\cdot v_{\Lambda_{k,i}}, \quad a \in U(\widehat{\mathfrak{g}})$$

and their kernels

$$N^1(\Lambda_{k,i}) = \operatorname{Ker} \Pi_{\Lambda_{k,i}}, \tag{5.2}$$

for $0 \le i \le k$. Then $N^1(\Lambda_{k,i})$ is the unique maximal proper ($L(0)$-graded) $\widehat{\mathfrak{g}}$-submodule of $N(\Lambda_{k,i})$ and

$$N^1(\Lambda_{k,i}) = U(\widehat{\mathfrak{g}})x_\alpha(-1)^{k-i+1}\cdot v^N_{\Lambda_{k,i}} = U(\mathbb{C}x_{-\alpha} \oplus \mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])x_\alpha(-1)^{k-i+1}\cdot v^N_{\Lambda_{k,i}}$$

for $0 \le i \le k$ (cf. [15,18]).

As in [2], a *principal ideal* of a vertex (operator) algebra is an ideal generated by a single element. The following result, which generalizes Proposition 4.1 in [2] and which is proved the same way, says that $N^1(\Lambda_{k,0})$ is the principal ideal of $N(\Lambda_{k,0})$ generated by the "null vector" $x_\alpha(-1)^{k+1}\cdot v^N_{\Lambda_{k,0}}$:

Proposition 5.1. *The space $N^1(\Lambda_{k,0})$ is the ideal of the vertex-operator algebra $N(\Lambda_{k,0})$ generated by $x_\alpha(-1)^{k+1}\cdot v^N_{\Lambda_{k,0}}$.* □

The kernels of the restrictions $\pi_{\Lambda_{k,i}}$ of the maps (5.1) to the principal subspaces $W^N(\Lambda_{k,i})$ (recall (3.18)) are

$$\operatorname{Ker} \pi_{\Lambda_{k,i}} = N^1(\Lambda_{k,i}) \cap W^N(\Lambda_{k,i}) \tag{5.3}$$

for $0 \le i \le k$. As in Remark 4.2 in [2] we have that $\mathrm{Ker}\ \pi_{\Lambda_{k,0}} = N^1(\Lambda_{k,0}) \cap W^N(\Lambda_{k,0})$, which equals $I_{\Lambda_{k,0}} \cdot v^N_{\Lambda_{k,0}}$ by Theorem 3.2, is an ideal of the vertex algebra $W^N(\Lambda_{k,0})$. Moreover, generalizing Proposition 4.2 of [2] and using essentially the same proof, we have that this ideal is also a principal ideal, generated by the same null vector:

Proposition 5.2. *The space $I_{\Lambda_{k,0}} \cdot v^N_{\Lambda_{k,0}}$ is the ideal of the vertex algebra $W^N(\Lambda_{k,0})$ generated by $x_\alpha(-1)^{k+1} \cdot v^N_{\Lambda_{k,0}}$.* □

Again as in [2] we write $(v)_V$ for the ideal generated by an element v of a vertex (operator) algebra V. Combining Propositions 5.1 and 5.2 with Theorem 3.2, we have obtained a reformulation of the $i = 0$ case of Theorem 3.2, formula (3.20), generalizing Theorem 4.1 of [2]:

Theorem 5.1. *For every $k > 0$,*

$$\mathrm{Ker}\ \pi_{\Lambda_{k,0}} = (x_\alpha(-1)^{k+1} \cdot v^N_{\Lambda_{k,0}})_{N(\Lambda_{k,0})} \cap W^N(\Lambda_{k,0}) = (x_\alpha(-1)^{k+1} \cdot v^N_{\Lambda_{k,0}})_{W^N(\Lambda_{k,0})}. \tag{5.4}$$

In particular, the intersection with the vertex subalgebra $W^N(\Lambda_{k,0})$ of the principal ideal of $N(\Lambda_{k,0})$ generated by the null vector $x_\alpha(-1)^{k+1} \cdot v^N_{\Lambda_{k,0}}$ coincides with the principal ideal of the vertex subalgebra $W^N(\Lambda_{k,0})$ generated by the same null vector. □

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