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Quasi 1D Bose–Einstein condensate flow past a nonlinear barrier

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ABSTRACT

The problem of a quasi 1D repulsive BEC flow past wide and narrow nonlinear barriers is investigated. It is shown that in contrast to the linear barrier case, for a wide nonlinear barrier an interval of velocities $0 < v < v_-$ always exists, where the flow is superfluid regardless of the barrier potential strength. In the case of a short range barrier stable and unstable steady solutions exist below some critical velocity. An unstable solution is shown to decay into a gray soliton moving downstream and a stable solution located at the barrier position.

1. Introduction

The problem of the transcritical flow of BEC through the penetrable barriers has been under recent active investigations [1–4]. Damping processes for a superfluid flow moving through the barrier are of a fundamental interest. In multidimensional case above some critical velocity of the obstacle motion the damping accompanied by the radiation emission [2] is observed. Thus in the region when the motion is still superfluid, the velocity is bounded above. The damping is associated with the Landau type damping and related to the emission of the elementary excitations. Landau damping can be described in the framework of the mean field theory and is not associated with thermalization processes [5]. The critical velocity value at which the damping is observed, differs essentially from the values predicted by the Landau theory. As it was shown firstly by Feynman [6], the reason is in the nonlinearity of the system. In the case of a quasi 1D Bose–Einstein condensate flow passing through a penetrable barrier, some interval of velocities $v_- < v < v_+$ exists, where trains of dark solitons are generated, that leads to deviation from predictions based on the matching with the spectrum of elementary linear excitations [1,7]. In addition in this range of velocities, generation of dispersive shock waves occurs. Experimental proof of the existence of the velocities interval was given in the work [3]. Haddad and Hakim [8] have indicated that for supersonic velocities (including ones above supercritical velocity v_+) some radiation is still nonzero and its amplitude rapidly decreases at the ratio of the potential variation length to the GPE coherence length. Amplitude of the wake can be characterized by the Fourier transform of the obstacle potential [9]. Thus, wide and smooth potentials can be considered as radiationless at velocities above *supercritical*. Seemingly in the one dimensional case only stable dark solitons can exist. Peculiarity of one dimension is in the fact that generation of the solitons is possible till some *supercritical* velocity, v_+. Above this velocity the emission is strongly damped and the quasi-superfluidity is restored. The radiation exists, but exponentially small-decay rate is proportional to l_{pot}/l_h, where l_h is the healing length of the order of the dark soliton width.

In this work we consider the phenomena occurring in the flow of a quasi 1D BEC past *a nonlinear* barrier which is a localized space inhomogeneity of the nonlinearity coefficient in the Gross–Pitaevskii equation. Such a type of barriers can be formed by some area of BEC where the effective value of the atomic scattering length is varied in the *space*. It can be achieved both by the Feshbach resonance techniques [10], and by the local variation of the transverse frequency of the trap potential. In the former case, varying external magnetic field in space near the resonance, one can vary the value of the atomic scattering length a_s. Another way is to use optically induced Feshbach resonances [11]. In this case the variation can be achieved by local change in the intensity of a laser field. Variation of a_s in a half space recently has been suggested in generation vortices in BEC by a nonlinear quantum piston [12].

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The present Letter is motivated by the works [1,4] where flow of a BEC past an obstacle in one dimension was investigated. We consider two cases, wide obstacle potential and short range one.

2. The model

Let us consider a nonlinear penetrable barrier moving through an elongated BEC. A quasi 1D BEC can be described by the Gross–Pitaevskii (GP) equation with standard dimensionless variables

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = V(x+vt)|\psi|^2\psi, \tag{1}$$

where

$$t = T\omega_\perp, \qquad x = X/l_\perp, \qquad \psi(x,t) = \sqrt{2|a_{s0}|}\Psi(x,t),$$
$$l_\perp = \sqrt{\hbar/m\omega_\perp}, \tag{2}$$

a_s is the atomic scattering length, $\omega_\perp$ is the transverse frequency of the trap, $V \to \frac{a_s}{a_{s0}}$, a_{s0} is the background value of the scattering length a_s. For the further study of the flow problem it is useful to pass to the reference frame moving with the barrier $x' = x + vt$, $t = t$. So we come to the equation

$$i\psi_t + iv\psi_{x'} + \frac{1}{2}\psi_{x'x'} - |\psi|^2\psi = V(x')|\psi|^2\psi. \tag{3}$$

The scattering length can be manipulated with a laser field tuned near a photo association transition, e.g., close to the resonance of one of the bound p levels of the excited molecules. Virtual radiative transitions of a pair of interacting atoms to this level can change the value and even reverse the sign of the scattering length [11]. Recently spatial modulations of the atomic scattering length by the optical Feshbach resonance method was realized experimentally in BEC [13]. Such approach implies some spontaneous emission loss which is inherent in the optical Feshbach resonance technique. Here we assume that such dissipative effects can be ignored, since they become possible if one uses laser fields of sufficiently high intensity detuned from the resonance. Thus the repulsive nonlinear barrier can be formed by a focused external laser beam with the parameters lying near the optically induced Feshbach resonance.

2.1. Wide obstacle potential

We analyze this case following the method developed in [1,4] for the linear barrier case. Let us pass to the hydrodynamical form for the GP equation (1). It can be obtained by the following transformation

$$\psi(x',t) = \sqrt{\rho(x',t)}e^{i\int^{x'} u(x,t)\,dx}. \tag{4}$$

Substituting it into (1) and introducing $u' = u + v$ we obtain the system

$$\rho_t + (\rho u')_{x'} = 0, \tag{5}$$
$$u'_t + u'u'_{x'} + \left(\frac{\rho_{x'}^2}{8\rho^2} - \frac{\rho_{x'x'}}{4\rho}\right)_{x'} + \rho_{x'} + \left(V(x')\rho\right)_{x'} = 0. \tag{6}$$

For a wide smooth obstacle potential we can neglect the terms in the bracket in the second equation that corresponds to the hydrodynamical approximation. Omitting also primes, for stationary solutions we can put $\rho_t = 0$ and $u_t = 0$, and obtain the following system of equations

$$(\rho u)_x = 0, \tag{7}$$
$$uu_x + \rho_x + (V\rho)_x = 0 \tag{8}$$

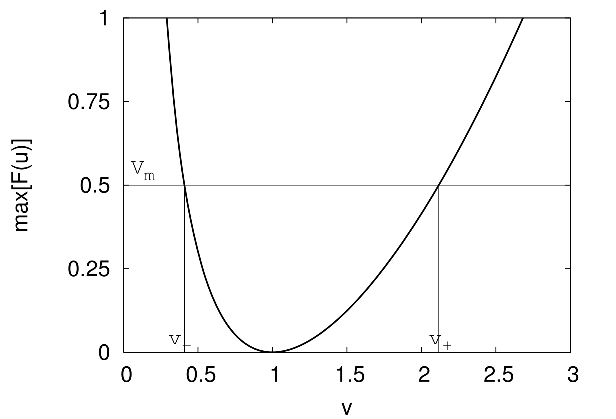

Fig. 1. Maximum of the function $F(u)$ (see Eq. (13)) versus x. For given obstacle potential maximum $V_m = 0.5$, critical values of the velocity $v_- = 0.409$, $v_+ = 2.117$.

with the boundary conditions

$$\rho \to 1, \qquad u \to v, \qquad V(x) \to 0, \quad \text{when } |x| \to \infty. \tag{9}$$

Integrating over x we find

$$\rho u = v, \tag{10}$$
$$\frac{1}{2}u^2 + \rho + V(x)\rho = \frac{1}{2}v^2 + 1. \tag{11}$$

Eliminating the function ρ from these equations, we get

$$V(x) = \frac{1}{2v}(u-v)\left[2 - u(u+v)\right] \equiv F(u). \tag{12}$$

Since we consider repulsive obstacle potential $V(x) > 0$ we have the condition $F(u) > 0$. Maximum of $F(u)$ is realized at $u_m = \sqrt{(v^2+2)/3}$. Thus the maximum of the function $F(u)$ is

$$\max\left[F(u)\right] = \mu(v) = \frac{1}{v}\sqrt{\left(\frac{v^2+2}{3}\right)^3} - 1. \tag{13}$$

Stationary solution $u(x)$ is obtained by solving Eq. (12) with respect to u. This equation has a real solution defined for all x provided that

$$V_m \equiv \max\left[V(x)\right] \leqslant \max\left[F(u)\right], \tag{14}$$

i.e. the range of values of $V(x)$, which is $[0, V_m]$, lies within the range of values of the function $F(u)$ [1].

Maximum of the function $F(u)$ versus the obstacle velocity v of BEC is presented in Fig. 1. As seen for any value of V_m two critical values of the velocity exist, v_- and v_+, determined by algebraic equation $V_m = \mu(v)$. In transcritical regime, in the interval $v_- < v < v_+$, the condition of the stationary flow (14) does not hold. Out of this region, in subcritical ($v < v_-$) and supercritical ($v > v_+$) regimes the radiation phenomena are negligible and the motion of the system can be considered as superfluid.

Analyzing expression (13) and Fig. 1 it should be noted that unlike the case of a wide linear barrier, considered in [1], the velocity v_- does not vanish and there always exists an interval $0 < v < v_-$ where the flow is superfluid.

Eq. (12) can be rewritten as

$$u^3 - (v^2+2)u + 2v\left(V(x)+1\right) = 0, \tag{15}$$

which is a cubic equation with respect to $u(x)$. Solving it we obtain the following solutions for $u(x)$ satisfying the boundary conditions

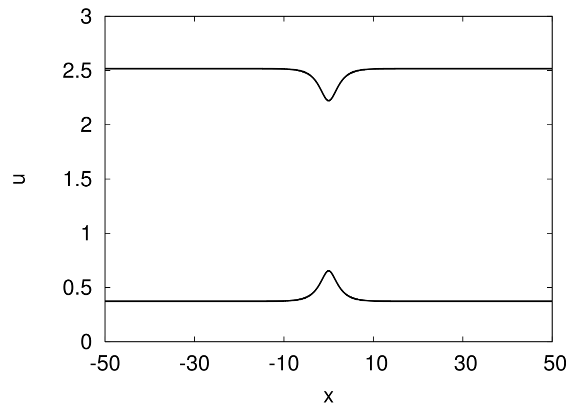

Fig. 2. Spatial profiles of the local velocity $u(x)$. The barrier velocities are equal to $v = 0.373$ and $v = 2.517$ for lower and upper lines respectively.

$$u(x) = -2\sqrt{q}\cos\left(s(x) - \frac{2\pi}{3}\right) \quad \text{for } v < v_-, \tag{16}$$

$$u(x) = -2\sqrt{q}\cos\left(s(x) + \frac{2\pi}{3}\right) \quad \text{for } v_+ < v, \tag{17}$$

where

$$q = \frac{v^2+2}{3}, \qquad s(x) = \frac{1}{3}\arccos\left(\frac{v(V(x)+1)}{\sqrt{q^3}}\right).$$

Spatial profiles of the local velocity u for subcritical $v = 0.373$ ($v < v_-$) and supercritical $v = 2.517$ ($v > v_+$) regimes are depicted in Fig. 2. The NL obstacle potential is taken in the form $V(x) = V_m/\cosh(x/2)$ with its maximum value $V_m = 0.5$.

Fig. 3 depicts time evolution of a BEC flow through a repulsive nonlinear potential $V(x) = V_m/\cosh(x/2)$ with $V_m = 0.5$ in (a) subcritical ($v = 0.373 < v_-$) and (b) supercritical ($v = 2.517 > v_+$) regimes, respectively. Initial form of the condensate density $\rho(x)$ is determined by Eq. (10) as $\rho(x) = v/u(x)$, where initial distribution of local velocities $u(x)$ is given by Eqs. (16), (17). One can see that in these regimes the flow through the barrier is steady. Existence of small amplitude waves, spreading from the hump in the beginning is a result of neglecting small terms in the course of derivation of Eqs. (7) and (8). In Fig. 3(b) one can see that in supercritical regime the solution at the center has the hump form. The numerical simulations show stability of this kind of steady flows.

In order to carry out numerical simulations of the behavior of a BEC at transcritical velocities ($v_- < v < v_+$), we cannot use Eqs. (16), (17) as initial wave packets, because they have been derived for a steady flow.

In numerical simulations it is more convenient to increase adiabatically the strength of NL potential V_m. In Fig. 4 we show time evolution of BEC flow through a NL potential barrier in the transcritical regime with $v = 0.47$ ($v > v_-$). The NL potential is taken in the form $V(x) = V_m/\cosh(x/2)$. V_m is increasing from 0 to 0.5 in the time interval $0 < t < 1000$ and then is kept constant. One can see that in the transcritical regime the flow becomes unsteady and a train of dark solitons emerges from the NL barrier at the barrier potential strength $V_m = 0.5$.

2.2. Short range nonlinear obstacle (δ-function potential)

In this section we follow the approach used in the work [4]. Let us suppose the condensate to have a chemical potential $\mu = 1$. Then in the frame of the moving obstacle with the velocity v Eq. (1) takes the form

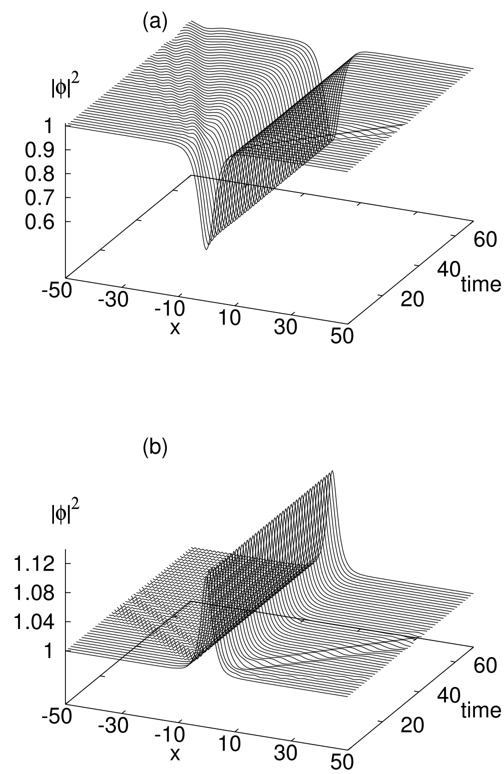

Fig. 3. Time evolution of a BEC flow in (a) subcritical regime, $v = 0.373$ and (b) supercritical regime, $v = 2.517$ through a nonlinear repulsive potential barrier $V_m/\cosh(x/2)$ with $V_m = 0.5$. The initial wave packet and distribution of the BEC local velocities $u(x)$ are taken in the form determined by formulas (16), (17) and Eq. (10).

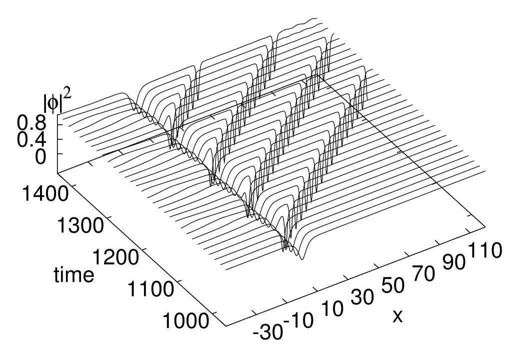

Fig. 4. Time evolution of a BEC flow in the transcritical regime when the NL barrier velocity $v = 0.47$ ($v_- < v < v_+$). The NL barrier is taken in the form of $V_m/\cosh(x/2)$ with $V_m = 0.5$. During the time period from $t = 0$ to $t = 1000$ (that is not presented in the figure) the value of V_m is adiabatically being increased from 0 to 0.5. Further evolution is given at $V_m = 0.5$.

$$i\psi_t + iv\psi_x + \frac{1}{2}\psi_{xx} - \psi - |\psi|^2\psi = V(x)|\psi|^2\psi \tag{18}$$

with uniform boundary conditions $|\psi(x)|^2 = 1$ at $x \to \pm\infty$.

Looking for time independent solution in the form $\psi(x) = R(x)\exp(i\phi(x))$ we get equations for amplitude $R(x)$ and phase $\phi(x)$

$$\phi_x = v\left(1 - \frac{1}{R^2}\right), \tag{19}$$

$$R_{xx} = v^2\left(-R + \frac{1}{R^3}\right) + R^3 + V(x)R^3 - R. \tag{20}$$

In the case of the δ-function barrier potential (a sharp jump in the nonlinearity) $V(x) = \gamma\delta(x)$ the solution $R(x)$ has the form

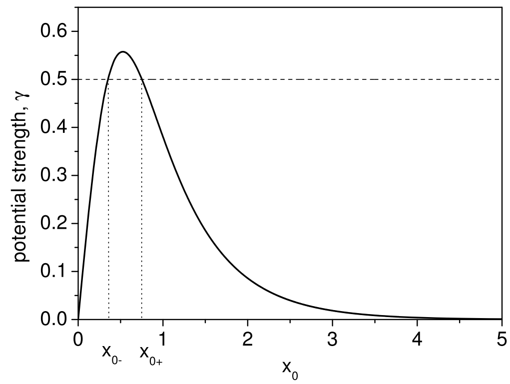

Fig. 5. Dependence of the parameter x_0 on the nonlinear potential strength γ for $v = 0.65$.

$$R^2(x) = 1 - \frac{1 - v^2}{\cosh^2[\sqrt{1 - v^2}(x \mp x_0)]} \quad \text{at } x \lessgtr 0. \tag{21}$$

Substituting obtained $R(x)$ into Eq. (19) and solving it we obtain phase $\phi(x)$ as

$$\phi(x) = f(x) = \arctan\left(\frac{2v\sqrt{1 - v^2}}{\exp(\sqrt{1 - v^2}(x + x_0)) + 2v^2 - 1}\right)$$

$$\text{at } x > 0,$$

$$\text{and} \quad \phi(x) = 2f(0) - f(-x) \quad \text{at } x < 0, \tag{22}$$

where unknown parameter x_0 depending on the potential strength γ is determined by the relation

$$\gamma = \frac{(1 - v^2)^{3/2} \cosh(\sqrt{1 - v^2}x_0) \sinh(\sqrt{1 - v^2}x_0)}{(v^2 + \sinh^2(\sqrt{1 - v^2}x_0))^2} \tag{23}$$

obtained from matching condition for derivatives $R_x(x)$ at $x = 0$

$$R_x(+0) - R_x(-0) = \gamma R^3(0).$$

Fig. 5 depicts a typical relation between the potential strength γ and parameter x_0 at $v = 0.65$. As seen for given strength γ there are two values of the parameter x_0 (or not a single) corresponding to a pair of steady solutions. One of the solutions ($x_0 = x_{0-}$) is unstable and another ($x_0 = x_{0+}$) is stable [4,14].

Time evolution of stable and unstable steady solutions corresponding to $x_{0+} = 0.752048$ and $x_{0-} = 0.350966$ are shown in Fig. 6. As seen the unstable solution decays into a gray soliton moving downstream with the velocity less than v and a stable solution localized at the barrier position. The decay is accompanied by a dispersive wave emitted upstream in front of the barrier.

The BEC density profile of a decaying unstable steady state at $t = 180$ is depicted in Fig. 7. Unlike the case of a wide barrier, in the case of the δ-function nonlinear barrier potential, localized steady states exist only at $v < v_c < v_s$ where v_s is the sound velocity. In our case $v_s = 1$. Critical velocity v_c is determined by the potential strength γ

$$\gamma = \frac{16(1 - v_c^2)^2}{(6v_c^2 - 3 + \alpha(v_c))^2} \frac{(2v_c^2 - 3 + \alpha(v_c))^{1/2}}{(-2v_c^2 - 1 + \alpha(v_c))^{1/2}}, \tag{24}$$

where $\alpha(v_c) = \sqrt{9 - 4v_c^2 + 4v_c^4}$.

The plot of the function $v_c(\gamma)$ is presented in Fig. 8. The particular values of the critical velocity are: $v_c = 0.8$ for $\gamma = 0.2$, $v_c = 0.476$ for $\gamma = 3$ and $v_c = 0.3$ for $\gamma = 10$. We can obtain asymptotic formulas for small and large γ.

For $\gamma \ll 1$ we find

$$v_c \approx 1 - \left(\frac{3\sqrt{3}}{8}\gamma\right)^{2/3} \tag{25}$$

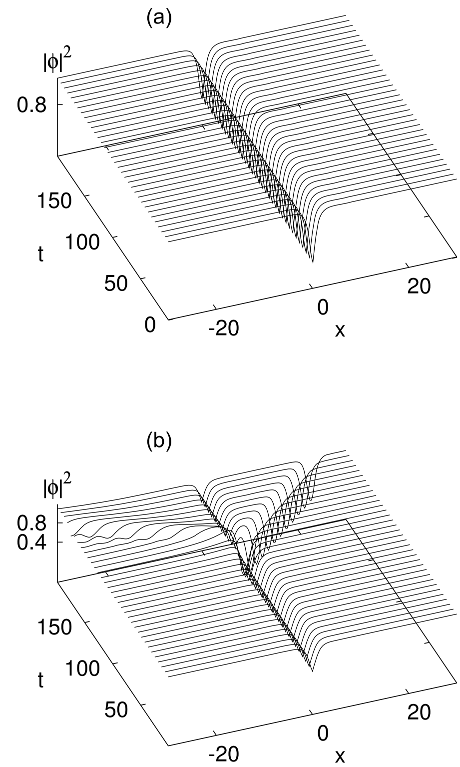

Fig. 6. Time evolution of steady solutions with the parameter $x_0 = x_{0+} = 0.752048$ and $x_0 = x_{0-} = 0.350966$ corresponding to (a) stable and (b) unstable BEC flows past a nonlinear repulsive δ-potential barrier in subcritical regime. The other parameters $v = 0.65$ and $\gamma = 0.5$. For this value of γ the critical velocity $v_c = 0.663946$.

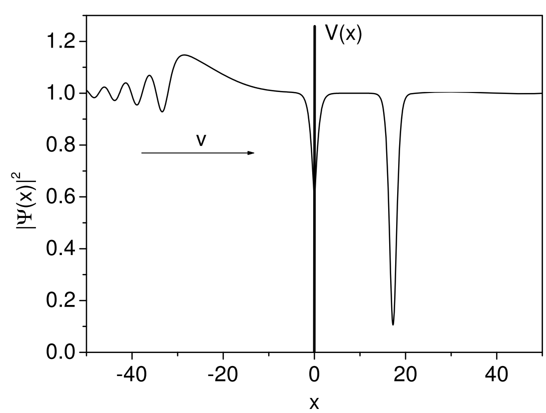

Fig. 7. The BEC density profile of a decaying unstable steady state at $t = 180$. The flow velocity v and strength of the barrier γ are the same as in Fig. 6.

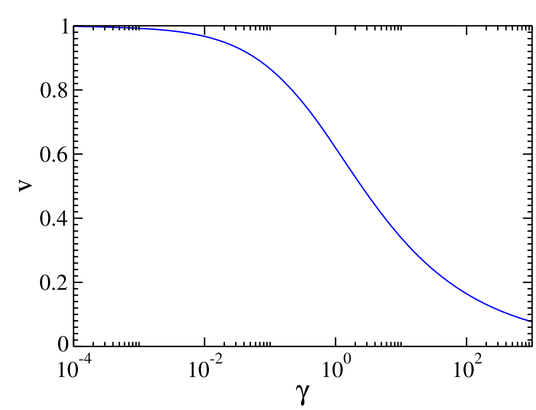

Fig. 8. The critical velocity v_c versus the nonlinear obstacle strength γ.

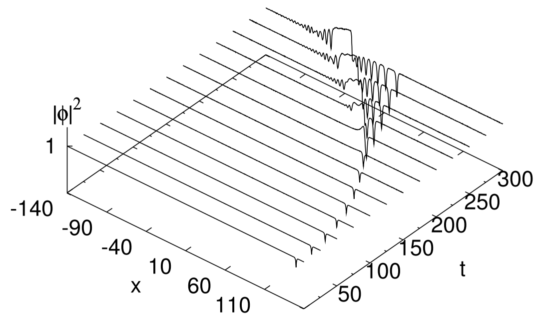

Fig. 9. Time evolution of a BEC flow through the δ-potential nonlinear barrier moving with the acceleration equal to 0.004. The barrier potential strength $\gamma = 0.5$, initial velocity of the flow $v_0 = 0$. The initial wave packet corresponds to a stable steady state of the BEC. The image corresponds to the coordinate system where the BEC is immobile and the barrier is moving.

and for large $\gamma \gg 1$ we have

$$v_c \approx \left(\frac{3\sqrt{3}}{8\sqrt{2}\gamma}\right)^{1/3}. \tag{26}$$

Comparing with the corresponding formulas for the short range linear barriers, obtained in [4,15,16], we observe that for small γ we have a similar dependence. In the case of large γ the decay law for the critical velocity v_c is essentially slower ($\sim \gamma^{-1/3}$), than in the linear barrier case ($\sim \gamma^{-1}$).

In order to cover a wide range of velocities we have carried out numerical simulations of the flow of a BEC through the δ-potential nonlinear barrier moving with small acceleration beginning from zero velocity. Fig. 9 depicts the time evolution of a BEC flow when the acceleration is equal to 0.004. The image corresponds to the coordinate system where the BEC is immobile and the barrier is moving. The barrier potential strength $\gamma = 0.5$. The initial wave packet is taken in the form of Eq. (21) and corresponds to a stable steady state of the BEC. Time interval $0 < t < 165$ ($0 < v < v_{cr}$) corresponds to a superfluid flow. At times $170 < t < 250$ the flow velocity becomes greater than the critical one (but still less than the sound velocity $v_{cr} < v < v_s$) and generation of a chain of gray solitons becomes possible. Time interval $250 < t$ where $v_s < v$ corresponds to transcritical flow of BEC at supersonic velocities. One can observe a dispersive shock wave propagating upstream with generation of soliton-like waves propagating downstream. As seen emerging wave pattern is qualitatively the same as in the work [1].

3. Conclusion

In conclusion, we studied steady flows in a defocusing quasi 1D BEC moving through a nonlinear *repulsive* barrier. Such a kind of barriers can be formed by variation of the atomic scattering length of BEC in *space*. For the case of a wide nonlinear barrier we found critical velocities of a steady flow. Within the interval of velocities $v_- < v < v_+$, in the transcritical regime we observed generation of a slowly moving train of dark solitons that disappears at velocities above supercritical.

At the same time in this regime one can observe formation of a hump localized at the place of *the barrier*. For the case of a δ-function nonlinear barrier potential the dependence of the steady solution parameters and the critical velocity v_c on the potential strength γ was obtained in an analytical form. As numerical simulations show, in subcritical regime $v < v_c$ an unstable solution decays into a gray soliton moving downstream and a stable solution localized at the barrier position. The decay is accompanied by a dispersive wave propagating upstream in front of the barrier.

The dynamics of flows past through a linear and nonlinear barriers are qualitatively similar except the following. In the case of a wide *linear* barrier, the superfluidity is broken at any small velocities if the barrier potential strength greater than some threshold value (see Fig. 2 in [1]). For a wide *nonlinear* barrier an interval of velocities $0 < v < v_-$ *always* exists, where the flow is superfluid regardless of the barrier potential strength.

When using the optically induced Feshbach resonance technique to generate a repulsive nonlinear barrier by focused laser beam, one should in general take into account the losses, induced by spontaneous emission of atoms. Phenomenologically it can be described by adding a nonlinear loss term $-i\gamma|u|^2u$ in the GP equation. Atom feeding can be described by introducing a linear gain term $i\alpha u$. This case requires a separate investigation. It should be noted that this problem relates to one considered in the recent work [17], where the flow of polariton condensate [18] past a linear barrier was studied taking into account linear amplification and nonlinear damping.

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