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Best Simple Octagonal Distances in Digital Geometry

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A subclass of general octagonal distances defined by neighbourhood sequences [2] have been characterized here which have a strikingly simple closed functional form. These are called *simple distances*. Minimization of the average absolute (normalized) and average relative errors of these simple distances with regard to the euclidean norm have been carried out to identify the *best approximate digital distances* in 2-D digital geometry. The direct errors have also been analyzed and the effect of finite domain sizes on the approximation has been highlighted. It is shown that the neighbourhood sequences $\{2\}$, $\{1, 2\}$, $\{1, 1, 2\}$, and $\{1, 1, 2, 1, 2\}$ have special significance in distance measurement in digital geometry.

1. INTRODUCTION

Integral approximations of the true euclidean distance e in the digital plane have long been attempted, particularly for the purpose of digital picture processing. Though pictures seemingly exist in the continuous domain, their fast processing using a computer has often been envisaged in the quantized space in which digital computers operate. In particular, a number of distance propagation and transformation algorithms have been worked out which necessarily operate with integer values. Thus the pertinent question which has obtained frequent attention is the issue of close approximation of the true euclidean norm using integer valued metrics. The first obvious choices were e^2, $\lfloor e \rfloor$, $\text{round}(e) = \lfloor e + 0.5 \rfloor$, and $\lceil e \rceil$, where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are floor and ceiling functions [3], respectively. Though all these four are integer valued, the first three of them fail to satisfy the metric properties [4]. The fourth approximation $\lceil e \rceil$ is a metric and provides a workable solution, but unfortunately it has received little attention, possibly due to its limitation in the definitions of suitable point neighbourhoods and minimal paths. Consequently most of the research efforts in digital distance approximation have been diverted to the search of proper digital distances and the class of *octagonal distances* [2] have

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emerged as a viable solution in the digital plane. In this paper we have analyzed the octagonal distances from the point of view of approximation and identified a few very simple integer metrics which can be widely used for the above tasks. Most importantly we prove that in the framework of octagonal distances we can hardly expect to achieve a better approximation.

The *octagonal distance* for digital pictures was introduced in digital geometry by Pfaltz and Rosenfeld in [4] when they proved that an alternating use of cityblock and chessboard motions defines a new integer-valued metric which can approximate the true euclidean norm better, than the conventional cityblock or chessboard distances. Recently Das and Chatterji [2] have extended their definition to allow for arbitrarily long cyclic sequences of cityblock and chessboard motions called *neighbourhood sequences*. This general definition has been shown to be "octagonal" still, since it always corresponds to constant radius "disks" which are digital octagons (see Fig. 1(a, b)). Detailed analysis of such octagons with respect to the area and perimeter errors for a euclidean circle shows that in every such neighbourhood sequence the actual order in which the two motions are arranged is of little consequence in an asymptotic sense so long as the length of the sequence and the number of cityblock/chessboard motions remain constant. This fact is reflected in the *characteristic value* [2] of every sequence which is invariant under the reordering of motions. A general closed form expression for such distances has also been derived in [2] and it is proved that a neighbourhood sequence defines a *metric* in the topological sense if and only if the sequence is *well-behaved.*

Unfortunately the functional form of the class of octagonal distances is mathematically fairly complex and involves a long chain of integer functions (floor operations) in the computation. In practical use this functional complexity not only leads to unnecessary programming difficulty but at the same time hinders the physical understanding of the properties of the metric. So the simplification of the distance function needed special attention for effective usage. We show in this paper that out of the class of neighbourhood sequences which have the same characteristic value (and hence identical error behaviour) there exists exactly one metric which has a strikingly simple functional form (involving only one ceiling function) and incidentally satisfies the metricity conditions too. So after a revision of the available results on octagonal distances in Section 2, we derive a characterization for such simple octagonal distances in Section 3. In Section 4 we introduce new error analyses involving these simple metrics. In these analyses the error between the octagonal and the true euclidean distances has been estimated in the asymptotic order by using a continuous (and hence asymptotic) approximation of the octagonal metric. Finally we have attempted to minimize the maxima of the absolute (normalized) difference, the relative difference, the average absolute (normalized) difference, and the average relative difference through the selection of the proper characteristic value and the corresponding neighbourhood distance. Interestingly most of the errors minimize for some special metrics. We have analyzed these in detail and recommended, in Section 5, four different simple metrics for practical use in digital approximation.

2. Octagonal Distances—A Revision

For the sake of completeness we highlight in this section the relevant results on octagonal distances from [2].

Rosenfeld and Pfaltz [4] identified two types of motions in the two dimensional digital plane Z^2, where Z is the set of integers. The first type of motion (cityblock motion) restricts movements to the horizontal or vertical directions, while the second kind (chessboard motion) also allows diagonal movements. The length of the shortest path between any two points restricted by a particular type of motion defines a distance function between two points. Thus the two types of motions in two dimensions determine two distances, cityblock distance and chessboard distance.

Cityblock movement as such involves a unit change in at most one coordinate at every step, whereas chessboard motion allows a unit change in both coordinates. The first kind of motion will be said to involve type 1-neighbours, while the latter will use type 2-neighbours. Any distance which is obtained by combining these two motions is determined by a *Neighbourhood Sequence* (N-sequence, for short) which defines the type of motion to be used at every step. Here a distance function between any two points (u_1, u_2) and (v_1, v_2) using the N-sequence $B = \{b(1), b(2), ..., b(p)\}$ (where $b(i)$ is a particular type of neighbourhood, $1 \leqslant b(i) \leqslant 2$, and $p = |B|$ is the length of the sequence beyond which B repeats itself) is denoted by $d((u_1, u_2), (v_1, v_2); B)$ or $d(B)$ for short. For example, the octagonal distance d_{oct} [4] is defined by an N-sequence $B = \{1, 2\}$ which corresponds to a cycle of neighbourhood relationships $\{1, 2, 1, 2, ...\}$. Any N-sequence B defines a unique distance function $d(B)$. However, any distance function may be associated with an infinite number of N-sequences; e.g., $B = \{1\}$, $\{1, 1\}$, $\{1, 1, 1\}$, ... all define the same cityblock distance.

The functional form of the octagonal distance is given in the following theorem.

Theorem 1 [Theorem 3.1 of [2]]. *Let x_1 and x_2 be the lengths of the sides of a digital rectangle. The minimal length of the diagonal $d((x_1, x_2); B)$ of the rectangle as determined by B is* $d((x_1, x_2); B) = \max(x_1, x_2,$

$\sum_{j=1}^{p} \lfloor ((x_1+x_2)+g(j))/f(p) \rfloor)$, *where* $f(i)=\sum_{j=1}^{i} b(j)$, $1 \leqslant i \leqslant p$, *and* $f(0)=0$, $g(i)=f(p)-f(i-1)-1$, $1 \leqslant i \leqslant p$.

Clearly the distance between two arbitrary points (u_1, u_2) and (v_1, v_2) in the digital plane becomes $d((u_1, u_2), (v_1, v_2); B) = d((|u_1 - v_1|, |u_2 - v_2|); B)$.

Unfortunately not all B's define *metric* (*positive definite*, *symmetric*, and *triangular*) $d(B)$'s. The following theorem states the necessary and sufficient condition for metric $d(B)$'s.

THEOREM 2 [Theorem 4.1 of [2]]. *$d(B)$ is a metric if and only if B is well-behaved, that is,*

$$\begin{aligned} f(i)+f(j) &\leqslant f(i+j) & i+j \leqslant p \\ &\leqslant f(p)+f(i+j-p) & i+j \geqslant p. \end{aligned}$$

Finally, these distances are octagonal in the sense that for every integral radius the corresponding disk $H(r, B) = \{(x_1, x_2) \mid (x_1, x_2) \in Z^2, d((x_1, x_2); B) \leqslant r\}$, $r \geqslant 0$ is a digital octagon having vertices at $(\pm r, \pm h(r))$ and $(\pm h(r), \pm r)$, where $h(r)$ is a function of B and r as given in the next lemma. For example, we illustrate the first quadrants of $H(6, \{1, 2\})$ and $H(6, \{1, 1, 2, 1, 2\})$ in Fig. 1(a, b).

LEMMA 1 [Lemma 6.2 of [2]]. *For any B and r, the corner function $h(r)$ is* $h(r) = \lfloor r/p \rfloor (f(p)-p) + f(r \bmod p) - (r \bmod p)$.

Note that in Fig. 1(a), for $B = \{1, 2\}$, $h(6) = \lfloor 6/2 \rfloor (3-2) + f(0) - 0 = 3$ and corners occur at $(6, 3)$ and $(3, 6)$. Similarly for $B = \{1, 1, 2, 1, 2\}$, $h(6) = 2$ in Fig. 1(b).

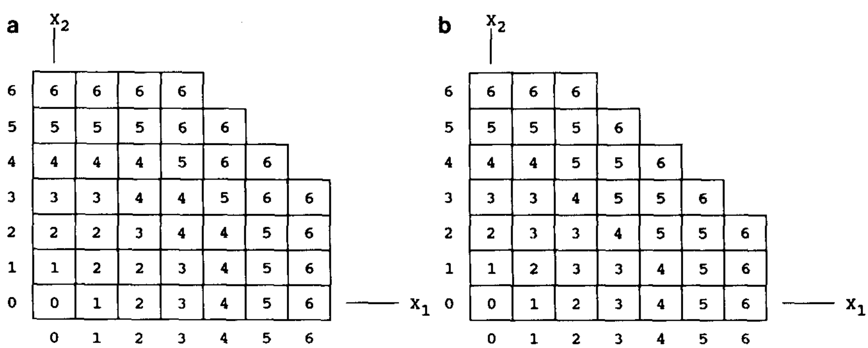

FIG. 1. First quadrants of digital octagons: (a) Octagon of $B = \{1, 2\}$ for radius $r = 6$. Note that $h(6) = 3$ and corners occur at $(6, 3)$ and $(3, 6)$. (b) Octagon of $B = \{1, 1, 2, 1, 2\}$ for radius $r = 6$. Note that $h(6) = 2$ and corners occur at $(6, 2)$ and $(2, 6)$.

Interestingly the asymptotic relative size of $h(r)$ with respect to r tends to a constant defined by the B:

$$\lim_{r \to \infty} h(r)/r = f(p)/p - 1 = m_B - 1 = h(p).$$

This constant $m_B = f(p)/p$ is termed the *neighbourhood parameter*. It is invariant under the reordering of the elements of B and plays an important role in the approximate analysis. Conceptually m_B is the average neighbourhood value in the expansion of every minimal path determined by this B. In fact the asymptotic values of the area and perimeter errors of the disks are functions solely of m_B [2]. Note that m_B is related to the *characteristic value* Δ_B of a B as defined in [2] via the relation $m_B = \Delta_B + 1$.

3. Simple Octagonal Distances

We find from Theorem 1 that the summation part of the distance function is a fairly complex integer function. For example if $B = \{1, 1, 2, 2\}$ then the sum is $\lfloor (a+1)/6 \rfloor + \lfloor (a+3)/6 \rfloor + \lfloor (a+4)/6 \rfloor + \lfloor (a+5)/6 \rfloor$, for $a = x_1 + x_2$; whereas for $B = \{1, 2\}$, it is $\lfloor (a+1)/3 \rfloor + \lfloor (a+2)/3 \rfloor = \lceil 2a/3 \rceil$. Hence there is enough reason to expect that for some B's the sum turns out to be a single ceiling function [3, p. 37]. Such distances are obviously easy to handle and efficient to perform computations with. So we call them *simple* distances. In the following theorem we show that for every $p = |B|$ and $f(p)$, $p \leqslant f(p) \leqslant 2p$ there exists a unique B which defines a simple $d(B)$.

THEOREM 3. *$d((x_1, x_2); B)$ is simple, i.e., of the form* $\max(x_1, x_2, \lceil (x_1 + x_2)/m \rceil)$, *iff $b(i) = \lfloor if(p)/p \rfloor - \lfloor (i-1) f(p)/p \rfloor$, $1 \leqslant i \leqslant p$, where $1 < m < 2$, $m = f(p)/p$, $f(p)$ and p are relatively prime, $x_1, x_2 \in Z$, and $x_1, x_2 \geqslant 0$. In addition, for $m = 1$, $B = \{1\}$ and $d((x_1, x_2); B) = x_1 + x_2$ and for $m = 2$, $B = \{2\}$ and $d((x_1, x_2); B) = \max(x_1, x_2)$ are also simple.*

Proof. First express B in terms of $f(i)$'s and $g(i)$'s, $1 \leqslant i \leqslant p$, as

$$f(i) = \sum_{j=1}^{i} b(j) = \lfloor if(p)/p \rfloor, \qquad 1 \leqslant i \leqslant p$$

$$\begin{aligned} g(i) &= f(p) - f(i-1) - 1 = f(p) - \lceil (i-1) f(p)/p \rceil, \qquad 2 \leqslant i \leqslant p-1 \\ &= f(p) - 1, \qquad i = 1. \end{aligned}$$

Now, $d((x_1, x_2); B) = \max(x_1, x_2, \sum_{i=1}^{p} \lfloor ((x_1 + x_2) + g(i))/f(p) \rfloor)$. Hence we need to show that the above $g(i)$ satisfies the integer equation

$$\sum_{i=1}^{p} \lfloor (a + g(i))/f(p) \rfloor = \lceil pa/f(p) \rceil, \qquad a \in Z, a \geqslant 0.$$

Let $a = rf(p) + s$, $0 \leqslant s \leqslant f(p) - 1$, $r \geqslant 0$. So we need to prove that

$$\sum_{i=1}^{p} \lfloor (s + g(i))/f(p) \rfloor = \lceil ps/f(p) \rceil, \qquad 0 \leqslant s \leqslant f(p) - 1.$$

Now clearly $f(i) > f(j)$, $i > j$, and $g(i) < g(j)$, $i > j$. Moreover

$$1 \leqslant f(i) - f(i-1) \leqslant 2, \qquad 2 \leqslant i \leqslant p,$$

$$1 \leqslant g(i) - g(i+1) \leqslant 2, \qquad 1 \leqslant i \leqslant p-1 \text{ and } g(1) = f(p) - 1.$$

Consider two cases now.

Case 1. $\exists j$, $1 \leqslant j \leqslant p$ such that $s = f(p) - g(j)$. So LHS $= \sum_{i=1}^{p} \lfloor (f(p) - g(j) + g(i))/f(p) \rfloor = j =$ RHS provided $\lceil (f(p) - g(j))\, p/f(p) \rceil = j$ or $\lfloor g(j)\, p/f(p) \rfloor = p - j$.

Case 2. $\exists j$, $1 \leqslant j \leqslant p$, such that $s = f(p) - g(j) + 1 = f(p) - g(j+1) - 1$. So in this case we require to prove that

$$\lfloor (g(j) - 1)\, p/f(p) \rfloor = p - j, \qquad \text{where} \quad g(j) = g(j+1) + 2.$$

It may be noted that either of the above two cases must occur. Now we prove that $\lfloor g(j)\, p/f(p) \rfloor = p - j$ given that

$$g(j) = f(p) - \lceil (j-1)\, f(p)/p \rceil.$$

Substituting the expression for $g(j)$,

$$\lfloor g(j)\, p/f(p) \rfloor = p - \lceil \lceil (j-1)\, f(p)/p \rceil (p/f(p)) \rceil.$$

So we have to establish that

$$\lceil \lceil (j-1)x \rceil / x \rceil = j \qquad \text{for} \quad 1 < x < 2$$

$$\begin{aligned} \lceil (j-1)x \rceil &= \lceil (j-1)x + 1 \rceil - 1 = \lceil jx - (x-1) \rceil - 1 \\ &\leqslant \lceil jx \rceil - 1 \\ &< jx \end{aligned}$$

and $\lceil (j-1)x \rceil \geqslant (j-1)x$. So $\lceil (j-1)x \rceil / x < j \leqslant \lceil (j-1)x \rceil / x + 1$. Hence $\lceil \lceil (j-1)x \rceil / x \rceil = j$ and $\lfloor g(j)\, p/f(p) \rfloor = p - j = \text{RHS}$. Next we prove that $\lfloor (g(j)-1)\, p/f(p) \rfloor = p - j$ if $g(j) = g(j+1) + 2$. We have

$$\begin{aligned} \text{LHS} &= \lfloor (g(j+1)+1)\, p/f(p) \rfloor = \lfloor (f(p) - \lceil jf(p)/p \rceil + 1)\, p/f(p) \rfloor \\ &= p - \lceil (\lceil jf(p)/p \rceil - 1)\, p/f(p) \rceil = p - j \end{aligned}$$

provided $\lceil (\lceil jf(p)/p \rceil - 1)\, p/f(p) \rceil = j$, i.e., $\lceil (\lceil jx \rceil - 1)/x \rceil = j$, $1 < x < 2$. Also

$$\begin{aligned} \lceil jx \rceil < jx + 1 \text{ and } \lceil jx \rceil - 1 &= \lfloor jx \rfloor = \lfloor (j-1)x + (x-1) \rfloor + 1 \\ &\geqslant \lfloor (j-1)x \rfloor + 1 \\ &> (j-1)x. \end{aligned}$$

So $(\lceil jx \rceil - 1)/x < j \leqslant (\lceil jx \rceil - 1)/x + 1$. Hence $\lceil (\lceil jx \rceil - 1)/x \rceil = j$ and $\lfloor (g(j)-1)\, p/f(p) \rfloor = p - j = \text{RHS}$. Finally note that $m = 1$ and $m = 2$ are also special cases of the general form. Q.E.D.

For example, let $p = 5$ and $f(p) = 7$. So $b(1) = \lfloor 7/5 \rfloor - 0 = 1$, $b(2) = \lfloor 14/5 \rfloor - \lfloor 7/5 \rfloor = 2 - 1 = 1$, $b(3) = \lfloor 21/5 \rfloor - \lfloor 14/5 \rfloor = 4 - 2 = 2$, $b(4) = b(5) = 1$, and $B = \{1, 1, 2, 1, 2\}$ is simple with $d((x_1, x_2); B) = \max(x_1, x_2, \lceil 5(x_1 + x_2)/7 \rceil)$.

The $d(B)$'s in the above form are referred to as "simple" $d(B)$'s corresponding to "simple" N-sequences. It is interesting to note that given p and $f(p)$ there are $p!/(f(p)-p)!\,(2p - f(p))! = \binom{p}{f(p)-p}$ $d(B)$'s out of which only one is simple.

Simple $d(B)$'s not only give simple, easy to handle analytical distance functions, but at the same time they help to avoid the metricity test.

LEMMA 2. *If B is simple then $d(B)$ is a metric.*

Proof. From Theorem 2 we know that $d(B)$ is a metric if and only if B is well-behaved. So here we prove that every simple B is well-behaved. From Theorem 3, $f(i) = \lfloor if(p)/p \rfloor$, $1 \leqslant i \leqslant p$. So

$$\begin{aligned} f(i) + f(j) &= \lfloor if(p)/p \rfloor + \lfloor jf(p)/p \rfloor \\ &\leqslant \lfloor (i+j)\, f(p)/p \rfloor, \qquad \text{since } \lfloor x \rfloor + \lfloor y \rfloor \leqslant \lfloor x + y \rfloor \\ &\leqslant f(i+j). \end{aligned}$$

Again

$$f(i)+f(j)\leqslant\lfloor(i+j)f(p)/p\rfloor$$
$$\leqslant\lfloor(i+j-p)f(p)/p\rfloor+f(p), \qquad p\leqslant i+j<2p$$
$$\leqslant f(p)+f(i+j-p).$$

Hence if B is simple then B is well-behaved and thus $d(B)$ is a metric. The metricity of a simple $d(B)$ can also be proved from the functional form of $d(B)$ using the fact that $1\leqslant m_B\leqslant 2$ and $\lceil x\rceil+\lceil y\rceil\geqslant\lceil x+y\rceil$. Q.E.D.

The disks of a simple $d(B)$ are also easy to compute.

LEMMA 3. *For a simple B, $h(r)=\lfloor rm_B\rfloor-r$, $r\geqslant 0$.*

Proof. From Lemma 1, $h(r)=\lfloor r/p\rfloor(f(p)-p)+f(r \bmod p)-(r \bmod p)$. Let $r=sp+t$, $0\leqslant t\leqslant p-1$. So $h(r)=s(f(p)-p)+f(t)-t=sf(p)+f(t)-r=sf(p)+\lfloor tf(p)/p\rfloor-r=\lfloor rf(p)/p\rfloor-r=\lfloor rm_B\rfloor-r$. Note that $h(r)=\lfloor rm_B\rfloor-r$ also holds for $t=0$ of $f(p)=p$ or $2p$. Hence the result. Q.E.D.

For example, if $m_B=7/5$ then $h(6)=\lfloor 42/5\rfloor-6=2$ as shown in Fig. 1(b).

4. DIRECT AND AVERAGE ERROR ESTIMATIONS

The estimation of the direct/average absolute or relative difference between a simple $d(B)$ and the true euclidean distance e is rather difficult to carry out in general. However, frequently, we are interested in the asymptotic values of these quantities. Actually for wide applicability in domains (subsets of Z^2) of any size, it is often preferable that we choose a $d(B)$ which minimizes the asymptotic errors. So for this purpose of error analysis we approximate every simple $d(B)$ by distance d_m in the real domain where $m=f(p)/p$. Clearly $d_m\colon R^2\times R^2\to R^+$ and $d_m((x_1,x_2))=\max(|x_1|,|x_2|,(|x_1|+|x_2|)/m)$ approaches $d(B)$ for sufficiently large values of x_1 and x_2, where R and R^+ are sets of real and positive real numbers, respectively.

In this section we analyze four kinds of errors for a $d(B)$: two absolute and two relative. Since absolute error turns out to be a function of the domain size (say $M\times M$) over which the computation is carried out we normalize it to get proper bounded error functions. So for $m_B=m=f(p)/p$ the following four error estimates are used for approximation.

Direct Absolute (Normalized) Error:

$$\alpha(m)=\max_{0\leqslant x_1,\,x_2\leqslant M}\{|e((x_1,x_2))-d_m((x_1,x_2))|\}/M,\ M>0.$$

Direct Relative Error:

$$\sigma(m) = \max_{0 \leqslant x_1, x_2 \leqslant M} \{|e((x_1, x_2)) - d_m((x_1, x_2))|/e((x_1, x_2))\}$$
$$= \max_{0 \leqslant x_1, x_2 \leqslant M} \{|1 - d_m((x_1, x_2))/e((x_1, x_2))|\}.$$

Average Absolute (Normalized) Error:

$$A(m) = \frac{\int_0^M \int_0^{x_1} \{|e((x_1, x_2)) - d_m((x_1, x_2))|/M\} \, dx_2 \, dx_1}{\int_0^M \int_0^{x_1} dx_2 \, dx_1}.$$

Average Relative Error:

$$R(m) = \frac{\int_0^M \int_0^{x_1} |1 - d_m((x_1, x_2))/e((x_1, x_2))| \, dx_2 \, dx_1}{\int_0^M \int_0^{x_1} dx_2 \, dx_1}.$$

The expressions for these errors in terms of the neighbourhood parameter m have been derived in the next four theorems using the following lemma:

LEMMA 4. *The following definite integrals are true*:

(i) $$I_1(r, s) = \int_0^M \int_{rx_1}^{sx_1} \sqrt{(x_1^2 + x_2^2)} \, dx_2 \, dx_1$$
$$= ((s\sqrt{(1+s^2)} - r\sqrt{(1+r^2)}) + \ln((s + \sqrt{(1+s^2)})/(r + \sqrt{(1+r^2)}))) \, M^3/6.$$

(ii) $$I_2(r, s) = \int_0^M \int_{rx_1}^{sx_1} x_1 \, dx_2 \, dx_1 = (s - r) \, M^3/3.$$

(iii) $$I_3(r, s) = \int_0^M \int_{rx_1}^{sx_1} x_2 \, dx_2 \, dx_1 = (s^2 - r^2) \, M^3/6.$$

(iv) $$I_4(r, s) = \int_0^M \int_{rx_1}^{sx_1} dx_2 \, dx_1 = (s - r) \, M^2/2.$$

(v) $$I_5(r, s) = \int_0^M \int_{rx_1}^{sx_1} x_1/\sqrt{(x_1^2 + x_2^2)} \, dx_2 \, dx_1$$
$$= \ln((s + \sqrt{(1+s^2)})/(r + \sqrt{(1+r^2)})) \, M^2/2.$$

(vi) $$I_6(r, s) = \int_0^M \int_{rx_1}^{sx_1} x_2/\sqrt{(x_1^2 + x_2^2)} \, dx_2 \, dx_1$$
$$= (\sqrt{(1+s^2)} - \sqrt{(1+r^2)}) \, M^2/2.$$

Proof. Follows from the following indefinite integrals:

$$\int \sqrt{(x^2+a^2)}\,dx = (x/2)\sqrt{(x^2+a^2)} + (a^2/2)\ln(x+\sqrt{(x^2+a^2)}),$$

$$\int x/\sqrt{(x^2+a^2)}\,dx = \sqrt{(x^2+a^2)},$$

and

$$\int 1/\sqrt{(x^2+a^2)}\,dx = \ln(x+\sqrt{(x^2+a^2)}). \qquad \text{Q.E.D.}$$

THEOREM 4.

$$\begin{aligned}\alpha(m) &= \max_{0\leqslant x_1,\, x_2\leqslant M} \{|e((x_1, x_2)) - d_m((x_1, x_2))|\}/M \\ &= \max(\sqrt{(1+(m-1)^2)}-1,\ |2/m-\sqrt{2}|),\\ &\qquad \textit{where}\quad M>0 \quad \textit{and} \quad 1\leqslant m\leqslant 2.\end{aligned}$$

Proof. Clearly,

$$\begin{aligned}&\max_{0\leqslant x_1,\, x_2\leqslant M} |e((x_1, x_2)) - d_m((x_1, x_2))|/M\\ &= \max_{0\leqslant x_2\leqslant x_1\leqslant M} |e((x_1, x_2)) - d_m((x_1, x_2))|/M\\ &= \max_{0\leqslant x_1\leqslant M} \{\max_{0\leqslant x_2\leqslant x_1} |\sqrt{(x_1^2+x_2^2)} - \max(x_1, (x_1+x_2)/m)|/M\}\\ &= \max_{0\leqslant x_1\leqslant M} x_1 \{\max_{\substack{0\leqslant x\leqslant 1\\ x=x_2/x_1}} |\sqrt{(1+x^2)} - \max(1, (1+x)/m)|/M\}\\ &= \max_{0\leqslant x\leqslant 1} f_A(x) \text{ where } f_A(x) = |\sqrt{(1+x^2)} - \max(1, (1+x)/m)|.\end{aligned}$$

Now

$$\begin{aligned}f_A(x) &= \sqrt{(1+x^2)} - 1,\ 0\leqslant x\leqslant m-1\\ &= |\sqrt{(1+x^2)} - (1+x)/m|,\ m-1\leqslant x\leqslant 1.\end{aligned}$$

Now let $g(x) = \sqrt{(1+x^2)} - (1+x)/m$. Therefore $dg/dx = x/\sqrt{(1+x^2)} - 1/m = 0$, i.e., $x = 1/\sqrt{(m^2-1)}$ and $d^2g/dx^2 = 1/(1+x)^{3/2} > 0$. Hence $g(x)$ has a minimum at $x = 1/\sqrt{(m^2-1)}$. Since $1/\sqrt{(m^2-1)} \leqslant 1$ for $\sqrt{2}\leqslant m\leqslant 2$, $g(x)$ is decreasing in the interval $[0, 1/\sqrt{(m^2-1)}]$ and increasing in the interval $[1/\sqrt{(m^2-1)}, 1]$ for $\sqrt{2}\leqslant m\leqslant 2$. Also $g(x = 1/\sqrt{(m^2-1)}) = (\sqrt{(m^2-1)} - 1)/m \geqslant 0$ for $\sqrt{2}\leqslant m\leqslant 2$. So the maximum of $|g(x)|$ in the interval $[m-1, 1]$ occurs at either extreme point $x = m-1$ or $x = 1$. Now if $1 < m < \sqrt{2}$, $1/\sqrt{(m^2-1)} > 1$. Consequently, $g(x)$ is monotonically decreasing in the interval $[0, 1]$. So the maximum of $|g(x)|$ in $[m-1, 1]$ again occurs at either $x = m-1$ or at $x = 1$. Finally for $m = 1$, $g(0) = 0$ and $g(x) < 0$ for $x > 0$ with $dg/dx < 0$. Again $|g(x)|$ maximizes at $x = 1$. Combining all cases we get

$$\max_{m-1 \leqslant x \leqslant 1} |g(x)| = \max(\sqrt{(1+(m-1)^2)} - 1, |2/m - \sqrt{2}|).$$

Since $\sqrt{(1+x^2)} - 1$ is an increasing function, we get

$$\max_{0 \leqslant x \leqslant 1} f_A(x) = \max(\sqrt{(1+(m-1)^2)} - 1, \max_{m-1 \leqslant x \leqslant 1} |g(x)|).$$

That is, $\alpha(m) = \max(\sqrt{(1+(m-1)^2)} - 1, |2/m - \sqrt{2}|)$. Q.E.D.

So the maximum of the normalized absolute error $\alpha(m)$ minimizes at m_{opt}, the solution of the equation

$$\sqrt{(1+(m-1)^2)} - 1 = |2/m - \sqrt{2}|.$$

We have solved the equations numerically (graphically in Fig. 2) to get two

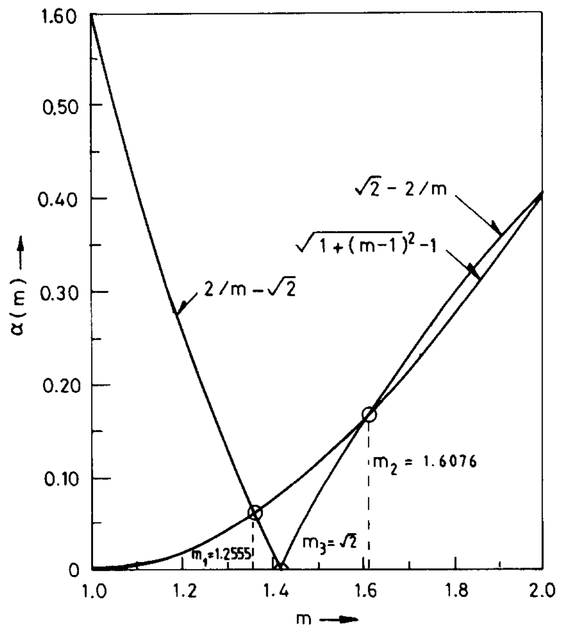

FIG. 2. Variation of direct absolute (normalized) error $\alpha(m)$ with m. Note the solutions of $\sqrt{1+(m-1)^2} - 1 = |2/m - \sqrt{2}|$ for minimum $\alpha(m)$.

solutions, $m = 1.3555$ or 1.6076. Hence $m_{opt} = 1.3555$ and minimum absolute error $\alpha(m_{opt}) = \min\{\alpha(m) \mid 1 \leqslant m \leqslant 2\} = 0.0613 = 6.13\%$.

THEOREM 5.

$$\sigma(m) = \max_{0 \leqslant x_1, x_2 \leqslant M} \{|1 - d_m((x_1, x_2))/e((x_1, x_2))|\}$$
$$= \max(1 - 1/\sqrt{(1+(m-1)^2)}, |1 - \sqrt{2}/m|),$$
$$\textit{where} \quad M > 0 \textit{ and } 1 \leqslant m \leqslant 2.$$

Proof. First show that $\sigma(m) = \max_{0 \leqslant x \leqslant 1} f_R(x)$, where

$$f_R(x) = 1 - 1/\sqrt{(1+x^2)}, \qquad 0 \leqslant x \leqslant m-1$$
$$= |1 - (1+x)/m\sqrt{(1+x^2)}|, \qquad m-1 \leqslant x \leqslant 1.$$

Proceeding as in the previous theorem the result immediately follows.

Q.E.D.

In the case of relative error m_{opt} is the solution of

$$1 - 1/\sqrt{(1+(m-1)^2)} = |1 - \sqrt{2}/m|.$$

That is, $m = 1.3420$ or 2.0. Hence $m_{opt} = 1.3420$ and minimum relative error $= \sigma(m_{opt}) = \min\{\sigma(m) \mid 1 \leqslant m \leqslant 2\} = 0.0538 = 5.38\%$ (see Fig. 3).

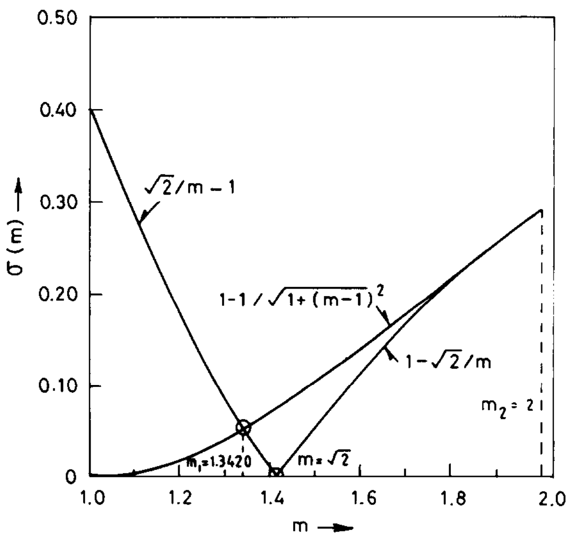

FIG. 3. Variation of direct relative error $\sigma(m)$ with m. Note the solutions of $1 - 1/\sqrt{1+(m-1)^2} = |1 - \sqrt{2}/m|$.

THEOREM 6.

$$A(m) = (1/3)((2-\sqrt{2}) - \ln(\sqrt{2}+1) - (m^2 - 2 + 4t(m) + 4t^2(m))/m + 2t(m)\sqrt{(1+t^2(m))} + 2\ln(t(m) + \sqrt{(1+t^2(m))})), \quad 1 \leqslant m \leqslant \sqrt{2}$$
$$= (1/3)((2+\sqrt{2}) + \ln(\sqrt{2}+1) - (m^2+4)/m), \quad \sqrt{2} \leqslant m \leqslant 2,$$

where $t(m) = (1 - m\sqrt{(2-m^2)})/(m^2-1)$.

Proof. To evaluate $A(m)$ we first derive the expression of the integral $\int_0^M \int_0^{x_1} |e((x_1, x_2)) - d_m((x_1, x_2))| \, dx_2 \, dx_1$. Consider two cases.

Case 1. $0 \leqslant x_2 \leqslant (m-1)\, x_1$. Therefore

$$d_m((x_1, x_2)) = x_1 \leqslant \sqrt{(x_1^2 + x_2^2)} \leqslant e((x_1, x_2)).$$

So we get, using Lemma 4,

$$\int_0^M \int_0^{(m-1)x_1} (\sqrt{(x_1^2 + x_2^2)} - x_1) \, dx_2 \, dx_1 = I_1(0, m-1) - I_2(0, m-1).$$

Case 2. $(m-1)\, x_1 \leqslant x_2 \leqslant x_1$. Clearly $d_m((x_1, x_2)) = (x_1 + x_2)/m$. Now two cases occur depending on whether $d_m((x_1, x_2)) \geqslant e((x_1, x_2))$ or $\leqslant e((x_1, x_2))$. Hence

Subcase 1. $1 \leqslant m \leqslant \sqrt{2}$. Now

$$\begin{aligned} m^2(e^2((x_1, x_2)) - d_m^2((x_1, x_2))) &= (m^2-1)\, x_1^2 - 2x_1 x_2 + (m^2-1)\, x_2^2 \\ &= (x_2 - t(m)\, x_1)(x_2 - (1/t(m))\, x_1)(m^2-1) \geqslant 0 \end{aligned}$$

implies either $x_2 \geqslant (1/t(m))\, x_1$ or $x_2 \leqslant t(m)\, x_1$, where $t(m) = (1 - m\sqrt{(2-m^2)})/(m^2-1)$. Again as $x_2 \leqslant x_1$ and $t(m) \leqslant 1$, $x_2 \geqslant (1/t(m))\, x_1$ is not feasible. It is also easy to show that $m-1 \leqslant t(m) \leqslant 1$. Thus,

$$\begin{aligned} e((x_1, x_2)) &\geqslant d_m((x_1, x_2)) \quad &\text{if } (m-1)\, x_1 \leqslant x_2 \leqslant t(m)\, x_1 \\ &\leqslant d_m(x_1, x_2) \quad &\text{if } t(m)\, x_1 \leqslant x_2 \leqslant x_1. \end{aligned}$$

Hence we get, using Lemma 4,

$$\int_0^M \int_{(m-1)x_1}^{t(m)x_1} (\sqrt{(x_1^2+x_2^2)} - (x_1+x_2)/m)\, dx_2\, dx_1$$

$$+ \int_0^M \int_{t(m)x_1}^{x_1} ((x_1+x_2)/m - \sqrt{(x_1^2+x_2^2)})\, dx_2\, dx_1$$

$$= I_1(m-1, t(m)) - I_1(t(m), 1)$$

$$+ (I_2(t(m), 1) - I_2(m-1, t(m))$$

$$+ I_3(t(m), 1) - I_3(m-1, t(m)))/m.$$

Subcase 2. $\sqrt{2} \leqslant m \leqslant 2$. Now

$$m^2(e^2((x_1, x_2)) - d_m^2((x_1, x_2)))$$

$$= (m^2-1)\, x_1^2 - 2x_1x_2 + (m^2-1)\, x_2^2 \geqslant (x_1 - x_2)^2 \geqslant 0.$$

Hence,

$$\int_0^M \int_{(m-1)x_1}^{x_1} (\sqrt{(x_1^2+x_2^2)} - (x_1+x_2)/m)\, dx_2\, dx_1$$

$$= I_1(m-1, 1) - (I_2(m-1, 1) + I_3(m-1, 1))/m.$$

Combining both cases and substituting I_1, I_2, I_3 from Lemma 4 the result follows. Q.E.D.

To estimate the minima of the normalized average absolute error we have plotted $A(m)$ in Fig. 4. Solving numerically we get that $A(m)$ minimizes for $m = 1.400001$ with the minimum error $0.015950 \approx 1.6\%$. Also note that $A(1) = 0.234804$, $A(\sqrt{2}) = 0.017649$, and $A(2) = 0.098529$.

In the next theorem we estimate the average relative error $R(m)$.

THEOREM 7.

$$\begin{aligned} R(m) &= (2t(m) - 1) - (1 - 1/m) \ln(m - 1 + \sqrt{(1 + (m-1)^2)}) \\ &\quad + \sqrt{(1 + (m-1)^2)}/m + (1/m)(\sqrt{2} + \ln(\sqrt{2} + 1)) \\ &\quad - (2/m)(\ln(t(m) + \sqrt{(1 + t^2(m))}) + \sqrt{(1 + t^2(m))}), \qquad 1 \leqslant m \leqslant \sqrt{2} \\ &= 1 - (1 - 1/m) \ln(m - 1 + \sqrt{(1 + (m-1)^2)}) \\ &\quad - (1/m)(\sqrt{2} + \ln(\sqrt{2} + 1)) + \sqrt{(1 + (m-1)^2)}/m, \qquad \sqrt{2} \leqslant m \leqslant 2, \end{aligned}$$

where $t(m) = (1 - m\sqrt{(2 - m^2)})/(m^2 - 1)$.

Proof. Similar to the previous theorem. In this case I_4, I_5, and I_6 are used and are substituted from Lemma 4.

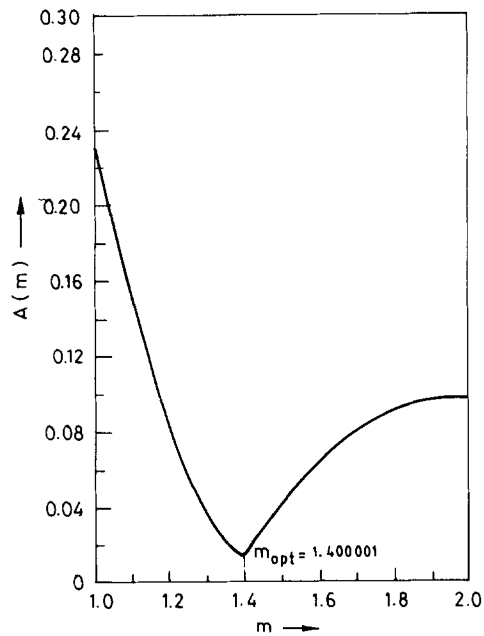

FIG. 4. Variation of average absolute (normalized) error $A(m)$ with m. Note that $A(m)$ minimizes for $m = 1.400001$.

$R(m)$, surprisingly, minimizes for $m = 1.400001$, i.e., at the minimum point of $A(m)$. Minimum relative error is found to be $0.021651 \approx 2.2\%$. The nature of $R(m)$ has been illustrated in Fig. 5. Also $R(1) = 0.295587$, $R(\sqrt{2}) = 0.024047$, and $R(2) = 0.118627$.

Though asymptotic analysis provides the necessary trend of the error, it is also interesting to observe the actual errors for some finite values of M using the actual octagonal distance $d(B)$ in place of d_m. In this case the average errors are computed in the digital domain as follows. For simple B with $m = f(p)/p$,

$$A_D(m) = \left(\sum_{x_1=0}^{M} \sum_{x_2=0}^{x_1} |e((x_1, x_2)) - d((x_1, x_2); B)| \Big/ \sum_{x_1=0}^{M} \sum_{x_2=0}^{x_1} 1 \right) \Big/ M$$

$$= (2/(M(M+1)(M+2)))$$

$$\times \sum_{x_1=0}^{M} \sum_{x_2=0}^{x_1} |e((x_1, x_2)) - d((x_1, x_2); B)| \qquad \text{and}$$

$$R_D(m) = (2/((M+1)(M+2)))$$

$$\times \sum_{x_1=0}^{M} \sum_{x_2=0}^{x_1} |1 - d((x_1, x_2); B)/e((x_1, x_2))|.$$

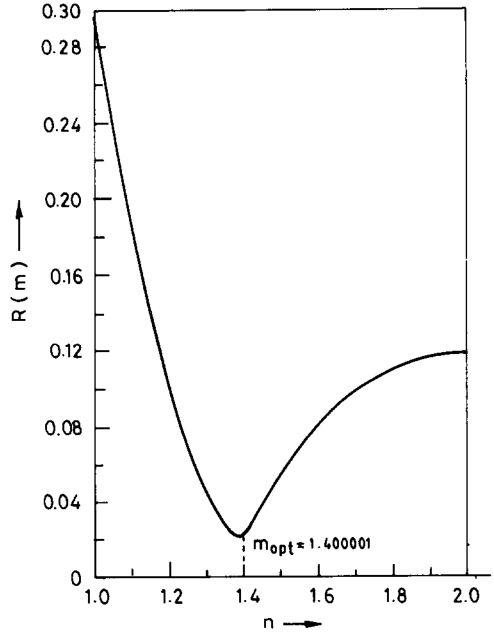

FIG. 5. Variation of average relative error $R(m)$ with m. Note that $R(m)$ minimizes for $m = 1.400001$.

We have tabulated $A_D(m)$ and $R_D(m)$ for $M = 4$, 16, 64, 256, and ∞ in Tables I and II, respectively, for all simple B's having length up to 11 and having distinct $m_B = f(p)/p$. So $B = \{1, 2, 1, 2\}$ has been omitted in preference to $B = \{1, 2\}$ and so on. Note that such B's can be easily generated as $(f(p)/p - 1) = (m_B - 1)$ forms a Farey series [3, p. 157] of order 11. Now the validity of the above asymptotic analysis can be derived from these tables where errors approach the limiting value with the increase of M.

5. Best Simple Distances

Equipped with the results of various error analyses we are now ready to select the best simple distances to be used in practical applications. From Tables I and II, we formulate Table III, where for every $p = 1, 2, \ldots, 11$, the simple $d(B)$ having minimum average error has been shown. Since $A(m)$ and $R(m)$ have very similar natures, a $d(B)$ which lowers $A(m)$ also lowers $R(m)$ and vice versa. In addition the direct errors have been calculated in

TABLE I

Average Absolute (Normalized) Errors $A_D(m)$ for Finite Pictures

$f(p)/p$	B	$M=4$	$M=16$	$M=64$	$M=256$	$M=\infty$
1/ 1 = 1.000	{1}	0.2210	0.2311	0.2339	0.2346	0.2348
12/11 = 1.091	{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2}	0.2210	0.1778	0.1603	0.1557	0.1542
11/10 = 1.100	{1, 1, 1, 1, 1, 1, 1, 1, 1, 2}	0.2210	0.1713	0.1532	0.1487	0.1472
10/ 9 = 1.111	{1, 1, 1, 1, 1, 1, 1, 1, 2}	0.2210	0.1624	0.1448	0.1403	0.1388
9/ 8 = 1.125	{1, 1, 1, 1, 1, 1, 1, 2}	0.2210	0.1515	0.1346	0.1303	0.1288
8/ 7 = 1.143	{1, 1, 1, 1, 1, 1, 2}	0.2043	0.1387	0.1221	0.1179	0.1165
7/ 6 = 1.167	{1, 1, 1, 1, 1, 2}	0.1876	0.1227	0.1065	0.1024	0.1011
13/11 = 1.182	{1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2}	0.1877	0.1151	0.0975	0.0933	0.0919
6/ 5 = 1.200	{1, 1, 1, 1, 2}	0.1543	0.1015	0.0864	0.0827	0.0814
11/ 9 = 1.222	{1, 1, 1, 1, 2, 1, 1, 1, 2}	0.1543	0.0913	0.0747	0.0709	0.0695
5/ 4 = 1.250	{1, 1, 1, 2}	0.1251	0.0742	0.0603	0.0571	0.0561
14/11 = 1.273	{1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2}	0.1251	0.0664	0.0508	0.0473	0.0462
9/ 7 = 1.286	{1, 1, 1, 2, 1, 1, 2}	0.1251	0.0597	0.0452	0.0421	0.0411
13/10 = 1.300	{1, 1, 1, 2, 1, 1, 2, 1, 1, 2}	0.1251	0.0540	0.0399	0.0368	0.0359
4/ 3 = 1.333	{1, 1, 2}	0.0805	0.0379	0.0282	0.0262	0.0256
15/11 = 1.364	{1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2}	0.0805	0.0318	0.0211	0.0194	0.0190
11/ 8 = 1.375	{1, 1, 2, 1, 1, 2, 1, 2}	0.0805	0.0284	0.0189	0.0176	0.0173
7/ 5 = 1.400	{1, 1, 2, 1, 2}	0.0638	0.0219	0.0159	0.0158	<u>0.0160</u>
10/ 7 = 1.429	{1, 1, 2, 1, 2, 1, 2}	0.0638	0.0191	0.0185	0.0212	0.0223
13/ 9 = 1.444	{1, 1, 2, 1, 2, 1, 2, 1, 2}	0.0638	0.0200	0.0231	0.0262	0.0273
16/11 = 1.455	{1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2}	0.0638	0.0205	0.0261	0.0292	0.0303
3/ 2 = 1.500	{1, 2}	0.0455	0.0327	0.0400	0.0422	0.0430
17/11 = 1.545	{1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 2}	0.0455	0.0401	0.0503	0.0530	0.0540
14/ 9 = 1.556	{1, 2, 1, 2, 1, 2, 1, 2, 2}	0.0455	0.0429	0.0527	0.0553	0.0562
11/ 7 = 1.571	{1, 2, 1, 2, 1, 2, 2}	0.0455	0.0468	0.0562	0.0587	0.0596
8/ 5 = 1.600	{1, 2, 1, 2, 2}	0.0508	0.0537	0.0622	0.0645	0.0652
13/ 8 = 1.625	{1, 2, 1, 2, 2, 1, 2, 2}	0.0507	0.0582	0.0667	0.0690	0.0697
18/11 = 1.636	{1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2}	0.0507	0.0599	0.0687	0.0709	0.0716
5/ 3 = 1.667	{1, 2, 2}	0.0543	0.0673	0.0741	0.0758	0.0763
17/10 = 1.700	{1, 2, 2, 1, 2, 2, 1, 2, 2, 2}	0.0543	0.0710	0.0786	0.0803	0.0809
12/ 7 = 1.714	{1, 2, 2, 1, 2, 2, 2}	0.0543	0.0738	0.0805	0.0821	0.0827
19/11 = 1.727	{1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2}	0.0543	0.0751	0.0821	0.0837	0.0842
7/ 4 = 1.750	{1, 2, 2, 2}	0.0709	0.0795	0.0850	0.0862	0.0866
16/ 9 = 1.778	{1, 2, 2, 2, 1, 2, 2, 2, 2}	0.0709	0.0824	0.0877	0.0889	0.0893
9/ 5 = 1.800	{1, 2, 2, 2, 2}	0.0709	0.0853	0.0899	0.0908	0.0911
20/11 = 1.818	{1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2}	0.0709	0.0869	0.0913	0.0922	0.0925
11/ 6 = 1.833	{1, 2, 2, 2, 2, 2}	0.0709	0.0885	0.0925	0.0933	0.0935
13/ 7 = 1.857	{1, 2, 2, 2, 2, 2, 2}	0.0709	0.0906	0.0941	0.0947	0.0949
15/ 8 = 1.875	{1, 2, 2, 2, 2, 2, 2, 2, }	0.0709	0.0926	0.0952	0.0956	0.0958
17/ 9 = 1.889	{1, 2, 2, 2, 2, 2, 2, 2, 2, }	0.0709	0.0934	0.0959	0.0962	0.0964
19/10 = 1.900	{1, 2, 2, 2, 2, 2, 2, 2, 2, 2}	0.0709	0.0938	0.0964	0.0967	0.0968
21/11 = 1.909	{1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}	0.0709	0.0943	0.0969	0.0971	0.0971
2/ 1 = 2.000	{2}	0.1124	0.1022	0.0995	0.0988	0.0985

Note. $M \to \infty$ shows the asymptotic value derived in Theorem 6.

TABLE II

Average Relative Errors $R_D(m)$ for Finite Pictures

$f(p)/p$	B	$M = 4$	$M = 16$	$M = 64$	$M = 256$	$M = \infty$
1/ 1 = 1.000	{1, 2}	0.2403	0.2825	0.2926	0.2949	0.2956
12/11 = 1.091	{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2}	0.2403	0.2246	0.2013	0.1941	0.1917
11/10 = 1.100	{1, 1, 1, 1, 1, 1, 1, 1, 1, 2}	0.2403	0.2165	0.1923	0.1851	0.1827
10/ 9 = 1.111	{1, 1, 1, 1, 1, 1, 1, 1, 2}	0.2403	0.2056	0.1815	0.1745	0.1721
9/ 8 = 1.125	{1, 1, 1, 1, 1, 1, 1, 2}	0.2403	0.1926	0.1686	0.1616	0.1593
8/ 7 = 1.143	{1, 1, 1, 1, 1, 1, 2}	0.2285	0.1765	0.1527	0.1459	0.1437
7/ 6 = 1.167	{1, 1, 1, 1, 1, 2}	0.2152	0.1569	0.1329	0.1263	0.1242
13/11 = 1.182	{1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2}	0.2152	0.1484	0.1219	0.1149	0.1126
6/ 5 = 1.200	{1, 1, 1, 1, 2}	0.1846	0.1304	0.1076	0.1015	0.0995
11/ 9 = 1.222	{1, 1, 1, 1, 2, 1, 1, 1, 2}	0.1846	0.1188	0.0933	0.0868	0.0847
5/ 4 = 1.250	{1, 1, 1, 2}	0.1539	0.0971	0.0751	0.0698	0.0681
14/11 = 1.273	{1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2}	0.1539	0.0886	0.0637	0.0579	0.0561
9/ 7 = 1.286	{1, 1, 1, 2, 1, 1, 2}	0.1539	0.0805	0.0569	0.0515	0.0500
13/10 = 1.300	{1, 1, 1, 2, 1, 1, 2, 1, 1, 2}	0.1539	0.0743	0.0504	0.0452	0.0437
4/ 3 = 1.333	{1, 1, 2,}	0.1043	0.0522	0.0359	0.0326	0.0317
15/11 = 1.364	{1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2}	0.1043	0.0460	0.0280	0.0250	0.0243
11/ 8 = 1.375	{1, 1, 2, 1, 1, 2, 1, 2}	0.1043	0.0420	0.0255	0.0230	0.0226
7/ 5 = 1.400	{1, 1, 2, 1, 2}	0.0910	0.0343	0.0221	0.0214	<u>0.0217</u>
10/ 7 = 1.429	{1, 1, 2, 1, 2, 1, 2}	0.0910	0.0311	0.0251	0.0281	0.0297
13/ 9 = 1.444	{1, 1, 2, 1, 2, 1, 2, 1, 2}	0.0910	0.0319	0.0302	0.0340	0.0357
16/11 = 1.455	{1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2}	0.0910	0.0325	0.0336	0.0377	0.0394
3/ 2 = 1.500	{1, 2}	0.0663	0.0430	0.0501	0.0533	0.0546
17/11 = 1.545	{1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 2}	0.0663	0.0503	0.0619	0.0661	0.0676
14/ 9 = 1.556	{1, 2, 1, 2, 1, 2, 1, 2, 2}	0.0663	0.0534	0.0647	0.0688	0.0703
11/ 7 = 1.571	{1, 2, 1, 2, 1, 2, 2}	0.0663	0.0578	0.0689	0.0728	0.0742
8/ 5 = 1.600	{1, 2, 1, 2, 2}	0.0700	0.0655	0.0760	0.0765	0.0808
13/ 8 = 1.625	{1, 2, 1, 2, 2, 1, 2, 2}	0.0700	0.0703	0.0813	0.0848	0.0860
18/11 = 1.636	{1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2}	0.0700	0.0721	0.0835	0.0871	0.0883
5/ 3 = 1.667	{1, 2, 2}	0.0739	0.0808	0.0900	0.0928	0.0937
17/10 = 1.700	{1, 2, 2, 1, 2, 2, 1, 2, 2, 2}	0.0739	0.0844	0.0950	0.0980	0.0989
12/ 7 = 1.714	{1, 2, 2, 1, 2, 2, 2}	0.0739	0.0878	0.0973	0.1001	0.1010
19/11 = 1.727	{1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2}	0.0739	0.0890	0.0991	0.1018	0.1027
7/ 4 = 1.750	{1, 2, 2, 2}	0.0872	0.0946	0.1025	0.1047	0.1055
16/ 9 = 1.778	{1, 2, 2, 2, 1, 2, 2, 2, 2}	0.0872	0.0974	0.1056	0.1078	0.1084
9/ 5 = 1.800	{1, 2, 2, 2, 2}	0.0872	0.1009	0.1081	0.1099	0.1105
20/11 = 1.818	{1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2}	0.0872	0.1024	0.1097	0.1115	0.1120
11/ 6 = 1.833	{1, 2, 2, 2, 2, 2}	0.0872	0.1044	0.1111	0.1127	0.1131
13/ 7 = 1.857	{1, 2, 2, 2, 2, 2, 2}	0.0872	0.1066	0.1129	0.1143	0.1147
15/ 8 = 1.875	{1, 2, 2, 2, 2, 2, 2, 2}	0.0872	0.1085	0.1141	0.1153	0.1156
17/ 9 = 1.889	{1, 2, 2, 2, 2, 2, 2, 2, 2}	0.0872	0.1093	0.1150	0.1160	0.1163
19/10 = 1.900	{1, 2, 2, 2, 2, 2, 2, 2, 2, 2}	0.0872	0.1098	0.1156	0.1165	0.1167
21/11 = 1.909	{1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}	0.0872	0.1103	0.1160	0.1169	0.1171
2/ 1 = 2.000	{2}	0.1221	0.1214	0.1195	0.1188	0.1186

Note. $M \to \infty$ is from Theorem 7.

TABLE III

Selection of Best Simple Octagonal Distance

p	$f(p)$	$f(p)/p$	B	$A(m)$	$R(m)$	$\alpha(m)$	$\sigma(m)$
1	2	2.0000	{2}	0.09853	0.11863	0.41421	0.29289
2	3	1.5000	{1, 2}	0.04297	0.05456	0.11803	0.10557
3	4	1.3333	{1, 1, 2}	0.02562	0.03167	0.08579	0.06066
4	6	1.5000	{1, 2, 1, 2}	0.04297	0.05456	0.11803	0.10557
5	7	1.4000	{1, 1, 2, 1, 2}	<u>0.01595</u>	<u>0.02165</u>	0.07703	0.07152
6	8	1.3333	{1, 1, 2, 1, 1, 2}	0.02562	0.03167	0.08579	0.06066
7	10	1.4286	{1, 1, 2, 1, 2, 1, 2}	0.02234	0.02974	0.08797	0.08086
8	11	1.3750	{1, 1, 2, 1, 1, 2, 1, 2}	0.01734	0.02260	0.06800	0.06367
9	12	1.3333	{1, 1, 2, 1, 1, 2, 1, 1, 2}	0.02562	0.03167	0.08579	0.06066
10	14	1.4000	{1, 1, 2, 1, 2, 1, 1, 2, 1, 2}	0.01595	0.02165	0.07703	0.07152
11	15	1.3636	{1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2}	0.01899	0.02430	0.06406	0.06021

Table III to give some idea about the other kinds of errors. From Table III we make the following recommendations:

$p=1$: $B=\{2\}$ with $A(m)=9.9\%$ and $R(m)=11.9\%$.

$p=2$: $B=\{1, 2\}$ with $A(m)=4.3\%$ and $R(m)=5.5\%$.

$p=3$: $B=\{1, 1, 2\}$ with $A(m)=2.6\%$ and $R(m)=3.2\%$.

$p=4$: $B=\{1, 2, 1, 2\}$ is no different from $B=\{1, 2\}$ and has a worse performance than $B=\{1, 1, 2\}$. So no B with $p=4$ is advised.

$p=5$: $B=\{1, 1, 2, 1, 2\}$ with $A(m)=1.6\%$ and $R(m)=2.2\%$. In this case $m_B=f(p)/p=7/5=1.4$ which is extremely close to the minima point of $A(m)$ and $R(m)$ (at $m=1.400001$). Thus this $d(B)$ has an exellent performance. And from $m_{\text{opt}}=1.400001$, we can easily foresee that increasing p to a reasonable extent would not see any considerable improvement in the performance. Moreover, larger p's offer additional processing time for the computation of the distance transformation and hence we always try to restrict p to small values. Moreover, this B also keeps the direct errors fairly small. In particular, $\alpha(m=7/5)=7.70\%$ (minimum possible is 6.13%) and $\sigma(m=7/5)=7.15\%$ (minimum possible is 5.38%).

So we recommend the use of $\{2\}$, $\{1, 2\}$, $\{1, 1, 2\}$, and $\{1, 1, 2, 1, 2\}$ for more and more accurate results and we *do not* recommend any longer B at all. It may be noted here from Table III that in the selection of the best metric we have given more importance to average errors than to direct errors. This is truly justified since in general large aberrations at a limited number of isolated points may be acceptable if the majority of the points in a domain get closely approximated distance values.

6. Conclusions

Analyzing the class of octagonal distances in 2-D digital pictures we have identified best approximate distances which are simple in functional form, metric in nature, and easy to compute. A generalization of these results in n dimensions using hyperoctagonal distances [1] remains an interesting open problem.

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