A condition for diagonalization modulo arbitrary norm ideals

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Abstract

We give a spectral condition which is sufficient for the simultaneous diagonalization of a commuting tuple of self-adjoint operators modulo a given norm ideal. For diagonalization modulo certain norm ideals this condition is also necessary, while for other norm ideals this condition seems to be close to being necessary. Moreover, this condition is easy to verify in applications.

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0. Introduction

Let $\mathcal{C}$ be a norm ideal and let $(A_1, \ldots, A_n)$ be a commuting tuple of self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Continuing our previous investigations [12–15], we consider the problem of determining whether or not $(A_1, \ldots, A_n)$ can be simultaneously diagonalized modulo $\mathcal{C}$. Voiculescu showed that $(A_1, \ldots, A_n)$ is simultaneously diagonalizable modulo $\mathcal{C}$ if and only if the quantity

$$k_{\mathcal{C}}(A_1, \ldots, A_n) = \liminf_{A \in \mathbb{R}_+^n} \sum_{j=1}^n \| [A, A_j] \|_{\mathcal{C}}$$

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is 0. See [9] and [11, Proposition 2.6]. Our task is to determine whether or not $k_C(A_1, \ldots, A_n)$ vanishes in terms of the spectral measure of the tuple.

From the extensive body of literature [2–5, 9–15] on the subject we know that this problem always reduces to the consideration of multiplication operators on some $L^2(R^n, \mu)$, where $\mu$ is a compactly supported regular Borel measure without point masses. Consider the commuting tuple $(M_1, \ldots, M_n)$, where

$$(M_j f)(x_1, \ldots, x_n) = x_j f(x_1, \ldots, x_n)$$

for $f \in L^2(R^n, \mu)$ and $1 \leq j \leq n$. In this setting the measure $\mu$ contains all the spectral information of $(M_1, \ldots, M_n)$. Intuitively, it is easy to see what kind of results one expects: if $\mu$ is concentrated on a “small” set, then $(M_1, \ldots, M_n)$ should be diagonalizable modulo “small” norm ideals. But saying that $\mu$ is concentrated on a “small” set is just another way of saying that $\mu$ is “singular” in some sense. Thus, invariably, whether or not $(M_1, \ldots, M_n)$ is diagonalizable modulo a given $C$ depends on the asymptotic behavior of $\mu(B(x, r))$ as $r \downarrow 0$. In actual estimates, because of the Euclidean structure of $R^n$, the ball $B(x, r)$ can be replaced by dyadic cubes $Q_w$, which offer obvious advantages.

The connection between singularity and diagonalizability began with the classic result of Kato, Rosenblum [7] and Carey, Pincus [4]: a single self-adjoint operator $A$ is diagonalizable modulo the trace class if and only if $A$ is purely singular with respect to the Lebesgue measure on $R$. For the Schatten class $C_p$ with $1 < p < \infty$, $(M_1, \ldots, M_n)$ is simultaneously diagonalizable modulo $C_p$ if and only if $\mu$ is $p$-singular [13]. This result was recently generalized to Orlicz ideals. In [15], we showed that for certain Orlicz ideals $C_G$, $(M_1, \ldots, M_n)$ is simultaneously diagonalizable modulo $C_G$ if and only if $\mu$ is $G$-singular.

What then about diagonalization modulo other norm ideals? In this paper we will take what we think is a significant step towards answering this general question. We will give a sufficient condition for diagonalization modulo an arbitrary norm ideal $C$. This condition is in fact a singularity condition on $\mu$. In other words, this is the kind of condition one expects for diagonalization problems. It can be shown that this condition is actually necessary for diagonalization modulo the Orlicz ideals $C_G$ considered in [15]. For a much larger class of norm ideals, this condition seems to be close to being necessary. Moreover, the condition itself is easy to verify in applications.

Let us now describe the result. First of all, it suffices to consider the case where the support of $\mu$ is contained in a unit cube $Q$. Let $\{Q_w: w \in \mathcal{W}\}$ be the system of dyadic decomposition of $Q$ that we used in [13–15] (which will be recalled in Section 2 below). Let $C$ be a norm ideal and let $\Phi_C$ denote the symmetric gauge function associated with $C$. The main result of the paper is that if there is a set of non-negative numbers $\{\lambda_w \}_{w \in \mathcal{W}}$ such that

$$\Phi_C(\{\lambda_w \}_{w \in \mathcal{W}}) < \infty$$

and

$$\sum_{w \in \mathcal{W}} 2^{|w|} \lambda_w \chi_{Q_w}(x) = \infty \quad \text{for } \mu\text{-a.e. } x \in Q,$$

then the tuple $(M_1, \ldots, M_n)$ on $L^2(R^n, \mu)$ is simultaneously diagonalizable modulo $C$.

The proof of our main result is a significant improvement of the techniques we used in [13,15]. What makes the proof possible is Lemma 3.2, which is a quantitative refinement of
[15, Lemma 6.5]. More specifically, what is new here is the quantitative condition (a) in Lemma 3.2. This condition basically ensures that the set $\mathcal{F}$ in Lemma 3.2 does not contain anything unwanted. This not only makes the subsequent estimate of certain $C$-norms possible, but also greatly simplifies the estimate itself. The reader should compare the proof of Lemma 3.3 here with the proof of [15, Lemma 7.1].

The rest of the paper is organized as follows. We recall the necessary definitions in Section 1. Our main result is stated in Section 2. Sections 3 and 4 contain the proof of the main result. In Section 5 we discuss the question of whether the condition in the main result is necessary.

1. Preliminaries

Following [6], let $\hat{c}$ denote the linear space of sequences $\{a_j\}_{j \in \mathbb{N}}$, where $a_j \in \mathbb{R}$ and for each sequence we have $a_j \neq 0$ only for a finite number of $j$’s. A map $\Phi : \hat{c} \to [0, \infty)$ is said to be a symmetric gauge function if it has the following properties:

(a) $\Phi$ is a norm on $\hat{c}$.
(b) $\Phi([1, 0, \ldots, 0, \ldots]) = 1$.
(c) $\Phi([a_j]_{j \in \mathbb{N}}) = \Phi([|a_{\pi(j)}|]_{j \in \mathbb{N}})$ for every bijection $\pi : \mathbb{N} \to \mathbb{N}$.

See [6, p. 71]. For a finite index set $F = \{s_1, \ldots, s_m\}$, we define $\Phi([c_s]_{s \in F})$ by the formula

$$\Phi([c_s]_{s \in F}) = \Phi([c_{s_1}, \ldots, c_{s_m}, 0, \ldots, 0, \ldots]).$$

For an arbitrary index set $E$, we further define

$$\Phi([c_s]_{s \in E}) = \sup\{\Phi([c_s]_{s \in F}) : F \subset E, \text{card}(F) < \infty\}.$$ 

More generally, for any map $a : E \to \mathbb{R}$, we write

$$\Phi(a) = \Phi([a(s)]_{s \in E}). \quad (1.1)$$

We will see later that (1.1) is a convenient notation for certain estimates.

All Hilbert spaces in this paper are assumed to be separable. Let $\mathcal{H}$ be a Hilbert space. A norm ideal is a two-sided ideal $C$ in $B(\mathcal{H})$ equipped with a norm $\| \cdot \|_C$ which has the following properties:

(a) For any $S, T \in B(\mathcal{H})$ and $A \in C$, we have $\|SAT\|_C \leq \|S\|_C \|A\|_C \|T\|$.
(b) If $A \in C$, then $A^* \in C$ and $\|A^*\|_C = \|A\|_C$.
(c) For any $A \in C$, $\|A\| \leq \|A\|_C$, and the equality holds when $\text{rank}(A) = 1$.
(d) $C$ is complete with respect to $\| \cdot \|_C$.
(e) $C \neq [0]$.

The term “norm ideal” is due to Schatten [8]. Elsewhere, such a $C$ is also called a symmetrically normed ideal [6].

Given a norm ideal $C$, let $C^{(0)}$ denote the $\| \cdot \|_C$-closure of the finite-rank operators in $C$. It is well known that $C^{(0)}$ can be a proper subset of $C$ [6].
Norm ideals and symmetric gauge functions are related through the s-numbers of operators. Given a symmetric gauge function Φ, define
\[ \|A\|_\Phi = \Phi(\{s_j(A)\}_{j \in \mathbb{N}}) \]
where \(s_1(A), \ldots, s_j(A), \ldots\) are the s-numbers of \(A\). Then, on any Hilbert space \(\mathcal{H}\),
\[ \mathcal{C}_\Phi = \{ A \in B(\mathcal{H}) : \|A\|_\Phi < \infty \} \]
is a norm ideal [6, Section III.4]. Conversely, if we begin with a norm ideal \(\mathcal{C}\) on a Hilbert space \(\mathcal{H}\) and if \(\{\xi_j\}_{j \in \mathbb{N}}\) is an orthonormal set in \(\mathcal{H}\), then the formula
\[ \Phi_C((a_j)_{j \in \mathbb{N}}) = \left\| \sum_{j=1}^\infty a_j \xi_j \otimes \xi_j \right\|_C \]
defines a symmetric gauge function and we have \(\|A\|_C = \|A\|_{\Phi_C}\) for every \(A \in \mathcal{C}(0)\). We will call \(\Phi_C\) the symmetric gauge function associated with \(\mathcal{C}\).

Let \(\ell^2_+\) be the Hilbert space of complex sequences \(\{a_1, \ldots, a_j, \ldots\}\) with \(\sum_{j=1}^\infty |a_j|^2 < \infty\).
Given a bounded sequence \(c_1, \ldots, c_j, \ldots\) in \(\mathbb{C}\), define the operator \(\text{diag}(c_j)_{j=1}^\infty\) on \(\ell^2_+\) by the formula
\[ \text{diag}(c_j)_{j=1}^\infty = \{c_1 a_1, \ldots, c_j a_j, \ldots\}. \]

An operator \(D\) on a Hilbert space \(\mathcal{H}\) is said to be diagonal if it is unitarily equivalent to a \(\text{diag}(c_j)_{j=1}^\infty\) on \(\ell^2_+\). Denote \(\ell^0_+ = \{a_1, \ldots, a_j, 0, \ldots, 0, \ldots\} : j \in \mathbb{N}, a_1, \ldots, a_j \in \mathbb{C}\).

For a Lipschitz function \(f\) on a subset \(E\) of \(\mathbb{R}^n\), denote
\[ L(f) = \sup\{|f(x) - f(y)|/|x - y| : x, y \in E, x \neq y\}. \]
The collection of Lipschitz functions on \(E\) will be denoted by \(\text{Lip}(E)\).

As usual, the operator of multiplication by a function \(f\) will be denoted by \(M_f\).

**Definition 1.1.** Let \(\mu\) be a regular Borel measure without point masses on \(\mathbb{R}^n\). Suppose that the support of \(\mu\) is contained in a compact set \(\mathcal{X}\). Let \(\mathcal{C}\) be a norm ideal of compact operators on \(L^2(\mathcal{X}, \mu) = L^2(\mathcal{X}, \mu)\). Then \(\mu\) is said to be \(\mathcal{C}\)-discrete if for every \(\epsilon > 0\), there exist a recurrent sequence \(y_1, \ldots, y_j, \ldots\) in the support of \(\mu\) and a unitary operator \(U : L^2(\mathcal{X}, \mu) \to \ell^2_+\) which have the following properties:

(i) \(U^* \ell^0_+ \subseteq L^\infty(\mathcal{X}, \mu)\).
(ii) For every \(f \in \text{Lip}(\mathcal{X})\), \(M_f - U^* \text{diag}(f(y_j))_{j=1}^\infty U \subseteq \mathcal{C}\) and
\[ \|M_f - U^* \text{diag}(f(y_j))_{j=1}^\infty U\|_\mathcal{C} \leq \epsilon L(f). \]

The above is the original definition of \(\mathcal{C}\)-discreteness [12, Definition 1.3]. But both the requirement that the sequence \(y_1, \ldots, y_j, \ldots\) be recurrent in the support of \(\mu\) and the requirement \(U^* \ell^0_+ \subseteq L^\infty(\mathcal{X}, \mu)\) are actually redundant [12, Theorem 3.5]. Later, Bercovici and Kostov showed that the requirement \(U^* \ell^0_+ \subseteq L^\infty(\mathcal{X}, \mu)\) is redundant in more ways than one [2].
2. The main result

As in [13–15], let $Q$ denote a unit cube in $\mathbb{R}^n$. That is,

$$Q = [0, 1)^n + v_0 = [0, 1) \times \cdots \times [0, 1) + v_0,$$

where $v_0$ is a vector in $\mathbb{R}^n$. Let us recall the labelling system for the cubes in the dyadic decomposition of $Q$ used in [13–15]. For each $\ell \in \mathbb{N}$, let $W_\ell$ denote the collection of words of length $\ell$ with $\{1, 2, 3, \ldots, 2^n\}$ being the alphabet. That is,

$$W_\ell = \{w_1 \ldots w_\ell : w_j \in \{1, 2, 3, \ldots, 2^n\}, j = 1, \ldots, \ell\}.$$

We denote the length of each word $w$ by $|w|$, i.e., $|w| = \ell$ for $w \in W_\ell$. Let

$$W = \bigcup_{\ell=1}^{\infty} W_\ell.$$  

For $w = w_1 \ldots w_\ell \in W_\ell$ and $u = u_1 \ldots u_k \in W_k$, we define the word

$$wu = w_1 \ldots w_\ell u_1 \ldots u_k \in W_{\ell+k}.$$  

Let $\gamma_1, \ldots, \gamma_{2^n}$ be an enumeration of the vectors $\{ (\epsilon_1, \ldots, \epsilon_n) : \epsilon_i \in \{0, 1\}, i = 1, \ldots, n \}$. For each $w = w_1 \ldots w_\ell \in W_\ell$, defined the cube

$$Q_w = Q_{w_1 \ldots w_\ell} = [0, 2^{-\ell})^n + 2^{-1} \gamma_{w_1} + 2^{-2} \gamma_{w_2} + \cdots + 2^{-\ell} \gamma_{w_\ell} + v_0.$$  

For arbitrary $w, w' \in W$, we have either $Q_w \cap Q_{w'} = \emptyset$, or $Q_w \supset Q_{w'}$, or $Q_{w'} \supset Q_w$. Although this labelling system for cubes is quite cumbersome, we saw in [13–15] and will see again that this system solves problems.

With the above preparation, we can now state the main result of the paper.

**Theorem 2.1.** Let $C$ be a norm ideal of compact operators and let $\Phi_C$ be the symmetric gauge function associated with $C$. Let $\mu$ be a regular Borel measure without point masses on $\mathbb{R}^n$. Furthermore, suppose that $\mu(\mathbb{R}^n \setminus Q) = 0$. If there exists a set of non-negative numbers $\{\lambda_w\}_{w \in W}$ such that

$$\Phi_C(\{\lambda_w\}_{w \in W}) < \infty \quad (2.1)$$

and

$$\sum_{w \in W} 2^{|w|} \lambda_w \chi_{Q_w}(x) = \infty \quad \text{for } \mu\text{-a.e. } x \in Q. \quad (2.2)$$

then $\mu$ is $C$-discrete.

An obvious question is whether or not the condition in Theorem 2.1 is also necessary for diagonalization modulo $C$. This will be the subject of discussion in Section 5. We have no examples where this condition fails to be necessary. In any case, we can at least find satisfaction in the fact that Theorem 2.1 makes no assumption about the norm ideal $C$. 
3. Technical steps

In this section, \( \mu \) will denote a regular Borel measure on \( \mathbb{R}^n \) whose support is contained in a compact set \( \mathcal{X} \). As in [12–15], for any Borel set \( \Delta \) in \( \mathbb{R}^n \), let \( \mu_\Delta \) denote the measure defined by the formula

\[
\mu_\Delta(A) = \mu(\Delta \cap A).
\]

We begin with a slight variation of [10, Theorem 1.2] and [12, Theorem 3.3(b)].

**Proposition 3.1.** Suppose that \( \mu \) has no point masses. Let \( C \) be a norm ideal of compact operators. Suppose that there is a sequence \( \{A_k\} \) of finite-rank operators on \( L^2(\mathbb{R}^n, \mu) \) which satisfies the following three conditions:

(i) There is a \( d > 0 \) such that \( |\langle A_k1, 1 \rangle| \geq d \) for every \( k \), where \( 1 \) denotes the function of constant value \( 1 \) in \( L^2(\mathbb{R}^n, \mu) \).

(ii) The numerical sequence \( \{\|A_k\|\} \) is bounded.

(iii) \( \lim_{k \to \infty} \sup\{\|[A_k, Mf]\|_C : f \in \text{Lip}(\mathcal{X}), \ L(f) \leq 1\} = 0 \).

Then there exists a Borel set \( \Delta \) in \( \mathbb{R}^n \) with \( \mu(\Delta) > 0 \) such that the measure \( \mu_\Delta \) is \( C \)-discrete.

**Proof.** Because of (ii), replacing \( \{A_k\} \) by a subsequence if necessary, we may assume that the sequence \( \{A_k1\} \) weakly converges to a vector \( \varphi \in L^2(\mathbb{R}^n, \mu) \). From \( \{A_k\} \) we can construct a sequence of finite-rank operators \( \{B_k\} \) such that each \( B_k \) is a finite convex combination of operators in \( \{A_k, A_{k+1}, A_{k+2}, \ldots\} \) and such that \( \lim_{k \to \infty} \|\varphi - B_k1\| = 0 \). For such a sequence \( \{B_k\} \), it follows from (iii) that

\[
\lim_{k \to \infty} \sup\{\|[B_k, Mf]\|_C : f \in \text{Lip}(\mathcal{X}), \ L(f) \leq 1\} = 0. \tag{3.1}
\]

Condition (i) guarantees that \( \|\varphi\| > 0 \). Thus there is an \( \epsilon > 0 \) such that if we let \( \Delta = \{x \in \mathbb{R}^n : |\varphi(x)| \geq \epsilon\} \), then \( \mu(\Delta) > 0 \). Define

\[
\psi(x) = \begin{cases} 
\frac{1}{\varphi(x)} & \text{if } x \in \Delta, \\
0 & \text{if } x \notin \Delta.
\end{cases}
\]

Then \( \psi \in L^\infty(\mathbb{R}^n, \mu) \). Define

\[
T_k = M_{\psi} B_k, \quad k \in \mathbb{N}.
\]

By (3.1), for the sequence \( \{T_k\} \) we have

\[
\lim_{k \to \infty} \sup\{\|[T_k, Mf]\|_C : f \in \text{Lip}(\mathcal{X}), \ L(f) \leq 1\} = 0. \tag{3.2}
\]

Thus, by [12, Theorem 3.3(b)], to show that \( \mu_\Delta \) is \( C \)-discrete, it suffices to show that the compression of the sequence \( \{T_k\} \) to the subspace \( L^2(\Delta, \mu) \) converges to 1 strongly. This will follow if we prove the strong convergence

\[
s-lim_{k \to \infty} T_k = M_{\chi_\Delta} \tag{3.3}
\]
on the Hilbert space $L^2(\mathbb{R}^n, \mu)$. Note that (3.2) implies
\begin{equation}
\lim_{k \to \infty} \| T_k f - Mf T_k 1 \| = \lim_{k \to \infty} \|[T_k, Mf] 1\| = 0, \quad f \in \text{Lip}(\mathcal{X}).
\end{equation}

On the other hand, since $\phi \psi = \chi_{\Delta}$, for each $f \in \text{Lip}(\mathcal{X})$ we also have
\begin{equation}
\lim_{k \to \infty} \| M_{\chi_{\Delta}} f - Mf T_k 1 \| = \lim_{k \to \infty} \| Mf M_{\psi}(\phi - B_k 1) \| = 0.
\end{equation}

By (ii) and the construction above, the numerical sequence \{\|T_k\|\} is bounded. Thus (3.3) follows from (3.4), (3.5) and the fact that Lip($\mathcal{X}$) is dense in $L^2(\mathbb{R}^n, \mu)$. 

As in [13–15], we will use the convention that $\sum_{u \in \{\text{empty set}\}} \ldots$ means 0.

**Lemma 3.2.** Let $a : W \to [0, \infty)$ be a map such that $2^{|w|} a(w) \leq 1/2$ for every $w \in W$. Let
\begin{equation}
R = \left\{ x \in Q : \sum_{w \in W} 2^{|w|} a(w) \chi_{Q_w}(x) = \infty \right\}.
\end{equation}

Then there exists a finite subset $\mathcal{F}$ of $\mathcal{W}$ which has the following properties:

(a) For all $w \in \mathcal{F}$ and $\ell \in \mathbb{N},$
\begin{equation}
\left(3\right) \frac{3}{4} a(w) \mu(Q_w) > \sum_{wu \in \mathcal{F}, |u| = \ell} a(wu) \mu(Q_{wu}).
\end{equation}

(b) $\sum_{w \in \mathcal{F}} 2^{|w|} a(w) \chi_{Q_w}(x) < 2$ for every $x \in Q$.

(c) $\sum_{w \in \mathcal{F}} 2^{|w|} a(w) \mu(Q_w) \geq (1/6) \mu(R)$.

**Proof.** Obviously, there is an $L \in \mathbb{N}$ such that if we set $\varphi = \sum_{1 \leq |w| \leq L} 2^{|w|} a(w) \chi_{Q_w}$ and $E = \{ x \in Q : \varphi(x) \geq 2 \}$, then $\mu(E) \geq (1/2) \mu(R)$. For each $w \in W_L$, $\varphi$ is a constant on $Q_w$. Therefore there is a subset $V$ of $W_L$ such that $E = \bigcup_{w \in V} Q_w$.

We define a subset $\mathcal{D}$ of $\{ w \in \mathcal{W} : 1 \leq |w| \leq L \}$ as follows. For any $1 \leq \ell \leq L$, a word $w_1 \ldots w_\ell$ of length $\ell$ belongs to $\mathcal{D}$ if and only if
\begin{equation}
\sum_{i=1}^\ell 2^{|w_1 \ldots w_\ell|} a(w_1 \ldots w_\ell) < 2.
\end{equation}

We claim that $\mathcal{D}$ has the following properties:

(b') $\sum_{w \in \mathcal{D}} 2^{|w|} a(w) \chi_{Q_w}(x) < 2$ for every $x \in Q$.

(c') $\sum_{w \in \mathcal{D}} 2^{|w|} a(w) \chi_{Q_w}(x) > 1$ if $x \in E$. 

To verify this claim, consider any given word \( u = u_1 \ldots u_L \in W_L \). Let \( \ell(u) \) be the largest integer in the set \( \{1, 2, \ldots, L\} \) such that

\[
\sum_{i=1}^{\ell(u)} 2^i a(u_1 \ldots u_i) < 2.
\]

(Since \( 2a(u_1) \leq 1/2 \), such an \( \ell(u) \) exists.) Obviously, we have \( u_1 \ldots u_i \in D \) if \( 1 \leq i \leq \ell(u) \) and \( u_1 \ldots u_j \notin D \) if \( \ell(u) < j \leq L \). Also, if \( w \in W_{L} \), \( 1 \leq \ell \leq L \), and if \( Q_w \cap Q_u \neq \emptyset \) (which happens if and only if \( Q_w \supset Q_u \)), then \( w = u_1 \ldots u_{\ell} \). Hence for any \( x \in Q_u \) we have

\[
\sum_{w \in D} 2^{|w|} a(w) \chi_{Q_w}(x) = \sum_{i=1}^{\ell(u)} 2^i a(u_1 \ldots u_i) < 2.
\]

This verifies \( (b') \). To verify \( (c') \), we now suppose \( u \in V \). Then

\[
\sum_{i=1}^{L} 2^i a(u_1 \ldots u_i) = \varphi(x) \geq 2 \quad \text{for } x \in Q_u.
\]

Thus, by the definition of \( \ell(u) \), we have \( \ell(u) < L \) and

\[
2^{\ell(u)+1} a_{u_1 \ldots u_{\ell(u)+1}} + \sum_{i=1}^{\ell(u)+1} 2^i a_{u_1 \ldots u_i} = \sum_{i=1}^{\ell(u)+1} 2^i a_{u_1 \ldots u_i} \geq 2.
\]

By assumption, \( 2^{\ell(u)+1} a_{u_1 \ldots u_{\ell(u)+1}} \leq 1/2 \). Therefore

\[
\sum_{w \in D} 2^{|w|} a(w) \chi_{Q_w}(x) = \sum_{i=1}^{\ell(u)} 2^i a(u_1 \ldots u_i) > 1
\]

if \( x \in Q_u \) and \( u \in V \). This verifies \( (c') \).

The desired \( F \) will be obtained as a subset of \( D \) in the following way. Define \( D_{\ell} = D \cap W_{\ell} \) for \( 1 \leq \ell \leq L \). Let \( F_L = \{ w \in D_L : a(w) \mu(Q_w) > 0 \} \). Suppose that \( 1 < k \leq L \) and that we have defined the subset \( F_{\ell} \) of \( D_{\ell} \) for every \( k \leq \ell \leq L \). Define

\[
U_{k-1} = \left\{ w \in D_{k-1} : \text{there is a } j \in \{1, \ldots, L - k + 1\} \text{ such that} \right. \\
(3/4)^j a(w) \mu(Q_w) \leq \sum_{wu \in F_{k-1+j}} a(wu) \mu(Q_{wu}) \left. \right\}.
\]

We then let \( F_{k-1} = D_{k-1} \setminus U_{k-1} \). Thus we have inductively defined \( F_L, \ldots, F_2, F_1 \). Define \( F = \bigcup_{\ell=1}^{L} F_{\ell} \). Since \( F \subset D \), \( (b) \) follows from \( (b') \).

From the definitions of \( F_L \) and \( U_{k-1} \) it is clear that \( (a) \) holds for \( F \). To verify \( (c) \), note that if \( w, w' \in U_{k-1} \) and \( w \neq w' \), then \( wu \neq w'u' \) for all \( u, u' \in \mathcal{W} \). Hence
\[
\sum_{w \in U_{k-1}} 2^{\|w\|} a(w) \mu(Q_w) \leq \sum_{j=1}^{L-k+1} \left( \frac{4}{3} \right)^{j} \sum_{v \in F_{k-1+j}} 2^{k-1} a(v) \mu(Q_v)
\]
\[
= \sum_{\ell=k}^{L} \left( \frac{2}{3} \right)^{\ell-k+1} \sum_{v \in F_{\ell}} 2^{v} a(v) \mu(Q_v).
\]
Therefore
\[
\sum_{j=1}^{L-1} \sum_{w \in U_j} 2^{\|w\|} a(w) \mu(Q_w) \leq \sum_{j=1}^{L-1} \sum_{\ell=j+1}^{L} \left( \frac{2}{3} \right)^{\ell-j} \sum_{v \in F_{\ell}} 2^{v} a(v) \mu(Q_v)
\]
\[
= \sum_{\ell=2}^{L} \sum_{j=1}^{\ell-1} \left( \frac{2}{3} \right)^{\ell-j} \sum_{v \in F_{\ell}} 2^{v} a(v) \mu(Q_v)
\]
\[
\leq 2 \sum_{\ell=2}^{L} \sum_{v \in F_{\ell}} 2^{v} a(v) \mu(Q_v) \leq 2 \sum_{v \in F} 2^{v} a(v) \mu(Q_v).
\]

Since \( F_j \cup U_j = D_j \) for \( 1 \leq j \leq L - 1 \) and \( F_L = \{ w \in D_L : a(w) \mu(Q_w) > 0 \} \), this gives us
\[
(1 + 2) \sum_{v \in F} 2^{v} a(v) \mu(Q_v) \geq \sum_{w \in D} 2^{\|w\|} a(w) \mu(Q_w) \geq \mu(E),
\]
where the second \( \geq \) follows from (c'). Since \( \mu(E) \geq (1/2) \mu(R) \), (c) is verified. \( \Box \)

Recall from [13–15] that for each \( w \in W \), \( e_w \) denotes the vector in \( L^2(\mathbb{R}^n, \mu) \) defined by the formula
\[
e_w = \begin{cases} (\mu(Q_w))^{-1/2} \chi_{Q_w} & \text{if } \mu(Q_w) > 0, \\ 0 & \text{if } \mu(Q_w) = 0. \end{cases} \tag{3.6}
\]

**Lemma 3.3.** Let \( \mathcal{F} \) be a finite subset of \( W \) and let \( a : \mathcal{F} \to [0, \infty) \) be a map such that
\[
\left( \frac{3}{4} \right)^{\ell} a(w) \mu(Q_w) > \sum_{wu \in \mathcal{F}, |u| = \ell} a(wu) \mu(Q_{wu}) \quad \text{for all } w \in \mathcal{F} \text{ and } \ell \in \mathbb{N}. \tag{3.7}
\]

Let \( \{ f_w : w \in W \} \) be functions in \( L^\infty(\mathbb{R}^n, \mu) \) such that \( \|f_w\|_\infty \leq 1 \) for every \( w \in W \). Let \( \{ \xi_w : w \in W \} \) be an orthonormal set. Then for any norm ideal \( \mathcal{C} \), the operator
\[
T = \sum_{w \in \mathcal{F}} a^{1/2}(w) \xi_w \otimes (f_w e_w)
\]
satisfies the estimate \( \|TT^*\|_\mathcal{C} \leq C_{3.3} \Phi_C(a) \), where \( C_{3.3} = 1 + 2 \sum_{\ell=1}^{\infty} (3/4)^{\ell/2} \) and \( \Phi_C \) is the symmetric gauge function associated with \( \mathcal{C} \).
Proof. Define
\[ b(w) = \begin{cases} a^{1/2}(w) & \text{if } w \in \mathcal{F}, \\ 0 & \text{if } w \in \mathcal{W} \setminus \mathcal{F}. \end{cases} \] (3.8)

We claim that
\[ \sum_{u \in \mathcal{W}_\ell} b^2(w) b^2(wu) \left| \langle f_we_w, f_wu e_{wu} \rangle \right|^2 \leq \left( \frac{3}{4} \right)^\ell b^4(w) \] (3.9)
for all \( w \in \mathcal{W} \) and \( \ell \in \mathbb{N} \). Let \( w \) and \( \ell \) be such that the left-hand side of (3.9) is greater than 0. Then it follows that \( w \in \mathcal{F} \) and \( \mu(Q_w) > 0 \). Since \( \| f_v \|_{\infty} \leq 1 \) for every \( v \in \mathcal{W} \), we have \( \left| \langle f_we_w, f_wu e_{wu} \rangle \right|^2 \leq \mu(Q_{wu})/\mu(Q_w) \) for such a \( w \). By (3.8) and (3.7),
\[ \sum_{u \in \mathcal{W}_\ell} b^2(wu) \left| \langle f_we_w, f_wu e_{wu} \rangle \right|^2 \leq \sum_{w \in \mathcal{F}, |u| = \ell} a(wu) \frac{\mu(Q_{wu})}{\mu(Q_w)} \left( \frac{3}{4} \right)^\ell a(w) = \left( \frac{3}{4} \right)^\ell b^2(w) \]
for such a pair of \( w \) and \( \ell \). This proves (3.9).

With the map \( b : \mathcal{W} \to [0, \infty) \) defined by (3.8), we can rewrite \( T \) as
\[ T = \sum_{w \in \mathcal{W}} b(w) \xi_w \otimes (f_we_w). \]

Using the operators \( T_k = \sum_{w \in \mathcal{W}_k} b(w) \xi_w \otimes (f_we_w), k \in \mathbb{N} \), we have
\[ TT^* = \sum_{k=1}^\infty T_k T_k^* + \sum_{\ell=1}^\infty \sum_{k=1}^\infty (T_k + \ell T_k^*) + (T_k + \ell T_k^*) = A_0 + \sum_{\ell=1}^\infty (A_\ell + A_\ell^*), \] (3.10)
where
\[ A_0 = \sum_{w \in \mathcal{W}} b^2(w) \langle f_we_w, f_w e_w \rangle \xi_w \otimes \xi_w, \]
\[ A_\ell = \sum_{w \in \mathcal{W}} \left( \sum_{u \in \mathcal{W}_\ell} b(w) b(wu) \langle f_we_w, f_wu e_{wu} \rangle \xi_{wu} \right) \otimes \xi_w, \quad \ell \in \mathbb{N}. \]

Since \( |\langle f_we_w, f_w e_w \rangle| \leq 1 \) for every \( w \), we obviously have
\[ \| A_0 \|_C \leq \Phi_C(b^2) = \Phi_C(a). \] (3.11)

To estimate \( \| A_\ell \|_C \), define the vectors
\[ \varphi_{\ell,w} = \sum_{u \in \mathcal{W}_\ell} b(w) b(wu) \langle f_we_w, f_wu e_{wu} \rangle \xi_{wu}. \]
$w \in \mathcal{W}$ and $\ell \in \mathbb{N}$. Then, according to (3.9),
\[
\|\varphi_{\ell, w}\|_2^2 = \sum_{u \in \mathcal{W}_\ell} b^2(w)b^2(wu)(f_{w e^w}, f_{wu e^wu})^2 \leq \left(\frac{3}{4}\right) \ell b^4(w) \tag{3.12}
\]
for all $w \in \mathcal{W}$ and $\ell \in \mathbb{N}$. We have
\[
A_\ell = \sum_{w \in \mathcal{W}} \varphi_{\ell, w} \otimes \xi_w,
\]
$\ell \in \mathbb{N}$. Note that $\varphi_{\ell, w} \perp \varphi_{\ell, w'}$ whenever $w \neq w'$. Applying (3.12), we have
\[
\|A_\ell\|_C = \Phi_C\left(\left\{\|\varphi_{\ell, w}\| \right\}_{w \in \mathcal{W}}\right) \leq \left(\frac{3}{4}\right)^{\ell/2} \Phi_C(b^2)^{\ell/2} = \left(\frac{3}{4}\right)^{\ell/2} \Phi_C(a) \tag{3.13}
\]
for every $\ell \in \mathbb{N}$. Combining (3.10), (3.11) and (3.13), the lemma follows. \hfill \Box

**Lemma 3.4.** Let $\mathcal{F}$ be a finite subset of $\mathcal{W}$ and let $a : \mathcal{F} \to [0, \infty)$ be a map such that (3.7) holds. For $i = 1, 2$, let $\{f^{(i)}_w : w \in \mathcal{W}\}$ be functions in $L^\infty(\mathbb{R}^n, \mu)$ such that $\|f^{(i)}_w\|_\infty \leq 1$ for every $w \in \mathcal{W}$. Then for any norm ideal $C$, the operator
\[
Y = \sum_{w \in \mathcal{F}} a(w)(f^{(1)}_w e^w) \otimes (f^{(2)}_w e^w)
\]
satisfies the estimate $\|Y\|_C \leq C_{3.3} \Phi_C(a)$, where $C_{3.3} = 1 + 2 \sum_{\ell=1}^\infty (3/4)^{\ell/2}$.

**Proof.** Let $\{\xi_w : w \in \mathcal{W}\}$ be an orthonormal set and define
\[
T_i = \sum_{w \in \mathcal{F}} a^{1/2}(w)\xi_w \otimes (f^{(i)}_w e^w), \quad i = 1, 2.
\]
Then $Y = T_1^*T_2$. By a standard argument, $\|Y\|_C \leq \{\|T_1^*T_1\|_C \|T_2^*T_2\|_C\}^{1/2}$ (see, e.g., [14, p. 382]). It is well known that $\|X^*X\|_C = \|XX^*\|_C$ for any finite rank operator $X$. Applying Lemma 3.3 to $T_i$, we have $\|T_i^*T_i\|_C = \|T_i T_i^*\|_C \leq C_{3.3} \Phi_C(a)$ for $i = 1, 2$. Hence $\|Y\|_C \leq C_{3.3} \Phi_C(a)$. \hfill \Box

**4. Proof of Theorem 2.1**

We may assume $\mu(Q) > 0$, for otherwise there is nothing to prove.

(1) We first prove that there exists a Borel set $\Delta$ with $\mu(\Delta) > 0$ such that the measure $\mu_\Delta$ defined by the formula $\mu_\Delta(A) = \mu(\Delta \cap A)$ is $\mathcal{C}$-discrete. For this purpose we invoke Proposition 3.1. Thus our task is to find a sequence of finite-rank operators $\{A_k\}$ which satisfy conditions (i)–(iii) in Proposition 3.1.

To construct such a sequence, let $k \in \mathbb{N}$ be given. We then separate the “good words” in $\mathcal{W}$ from the “bad words.” That is, we define
\[
\mathcal{G}^{(k)} = \{w \in \mathcal{W} : k^{-1/2} |w| \lambda_w \leq 1/2\} \quad \text{and} \quad \mathcal{B}^{(k)} = \{w \in \mathcal{W} : k^{-1/2} |w| \lambda_w > 1/2\}.
\]
The desired operator $A_k$ will have the form $G_k + B_k$, where $G_k$ and $B_k$ are constructed from $G^{(k)}$ and $B^{(k)}$, respectively.

To construct $G_k$, define the map $a_k : \mathcal{W} \to [0, \infty)$ by the formula

$$a_k(w) = \begin{cases} k^{-1} \lambda w & \text{if } w \in G^{(k)}, \\ 0 & \text{if } w \in B^{(k)}. \end{cases}$$

(4.1)

Then, of course, $2^{|w|} a_k(w) \leq 1/2$ for every $w \in \mathcal{W}$. Let

$$R_k = \left\{ x \in \mathcal{Q} : \sum_{w \in \mathcal{W}} 2^{|w|} a_k(w) \chi_{Q_w}(x) = \infty \right\}.$$ 

(4.2)

By Lemma 3.2, there exists a finite subset $\mathcal{F}^{(k)}$ of $\mathcal{W}$ such that

$$\left( \frac{3}{4} \right)^{\ell} a_k(w) \mu(Q_w) > \sum_{w \in \mathcal{F}^{(k)}} a_k(wu) \mu(Q_{wu}) \quad \text{for all } w \in \mathcal{F}^{(k)} \text{ and } \ell \in \mathbb{N};$$

(4.3)

$$\sum_{w \in \mathcal{F}^{(k)}} 2^{|w|} a_k(w) \chi_{Q_w}(x) < 2 \quad \text{for every } x \in \mathcal{Q};$$

(4.4)

$$\sum_{w \in \mathcal{F}^{(k)}} 2^{|w|} a_k(w) \mu(Q_w) \geq \frac{1}{6} \mu(R_k).$$

(4.5)

We now define

$$G_k = \sum_{w \in \mathcal{F}^{(k)}} 2^{|w|} a_k(w)e_w \otimes e_w,$$

where $e_w$ is given by (3.6). Let us first estimate $\|[G_k, M_f]\|_C$ for $f \in \text{Lip}(\mathcal{X})$, where $\mathcal{X}$ denotes the support of $\mu$. Given an $f \in \text{Lip}(\mathcal{X})$, we extend it to a function on $\mathbb{R}^n$ by setting $f = 0$ on $\mathbb{R}^n \setminus \mathcal{X}$. For each $w \in \mathcal{W}$, if $Q_w \cap \mathcal{X} \neq \emptyset$, we pick an $x_w \in Q_w \cap \mathcal{X}$ and define $f_w = 2^{|w|}(\sqrt{n}L(f))^{-1}(f - f(x_w))\chi_{Q_w}$; if $Q_w \cap \mathcal{X} = \emptyset$, we define $f_w = 0$. Then $\|f_w\|_{\infty} \leq 1$ for every $w \in \mathcal{W}$. On the Hilbert space $L^2(\mathbb{R}^n, \mu) = L^2(\mathcal{X}, \mu)$ we have

$$2^{|w|}[e_w \otimes e_w, M_f] = \sqrt{n}L(f)\left\{ e_w \otimes (\bar{f}_w e_w) - (f_w e_w) \otimes e_w \right\}$$

and, therefore,

$$[G_k, M_f] = L(f)\sqrt{n} \sum_{w \in \mathcal{F}^{(k)}} a_k(w)\left\{ e_w \otimes (\bar{f}_w e_w) - (f_w e_w) \otimes e_w \right\}.$$ 

Because of (4.3), we can now apply Lemma 3.4 to conclude that

$$\|[G_k, M_f]\|_C \leq 2L(f)\sqrt{n}C_{3,3} \Phi_C(a_k).$$
Recalling (4.1), this gives us
\[ \| [G_k, M_f] \|_C \leq 2L(f) \sqrt{nC3.3} \frac{1}{k} \Phi_C(\{\lambda_w \}_{w \in \mathcal{W}}), \quad f \in \text{Lip}(X). \] (4.6)

To estimate the operator norm of $G_k$, write it as an integral operator. That is,
\[ (G_k \xi)(x) = \int K_k(x, y) \xi(y) d\mu(y) \]

for $\xi \in L^2(\mathbb{R}^n, \mu)$, where
\[ K_k(x, y) = \sum_{w \in \mathcal{F}(k)} 2^{|w|} a_k(w) e_w(x) e_w(y). \]

Obviously, $K_k(x, y) \geq 0$ for all $x, y$. By (3.6) and (4.4), we have
\[ \int K_k(x, y) d\mu(y) \leq \sum_{w \in \mathcal{F}(k)} 2^{|w|} a_k(w) \chi_{Q_w}(x) < 2 \quad \text{for every } x \in Q, \] (4.7)

and
\[ \int K_k(x, y) d\mu(x) \leq \sum_{w \in \mathcal{F}(k)} 2^{|w|} a_k(w) \chi_{Q_w}(y) < 2 \quad \text{for every } y \in Q. \] (4.8)

It is well known that (4.7) and (4.8) together imply $\|G_k\| \leq 2$.

By (3.6), $\langle 1, e_w \rangle = \sqrt{\mu(Q_w)}$ for all $w \in \mathcal{W}$. Combining this with (4.5), we have
\[ \langle G_k 1, 1 \rangle = \sum_{w \in \mathcal{F}(k)} 2^{|w|} a_k(w) \mu(Q_w) \geq \frac{1}{6} \mu(R_k). \] (4.9)

To construct $B_k$, we begin with the set
\[ S_k = \bigcup_{w \in \mathcal{B}(k)} Q_w. \] (4.10)

There is an integer $m \geq 1$ such that if we set $\mathcal{B}_m^{(k)} = \{w \in \mathcal{B}(k) : 1 \leq |w| \leq m\}$, then
\[ \mu \left( \bigcup_{w \in \mathcal{B}_m^{(k)}} Q_w \right) \geq \frac{1}{2} \mu(S_k). \] (4.11)

Let $\mathcal{V}^{(k)}$ be a subset of $\mathcal{B}_m^{(k)}$ which is minimal with respect to the property
\[ \bigcup_{w \in \mathcal{V}^{(k)}} Q_w = \bigcup_{w \in \mathcal{B}_m^{(k)}} Q_w; \]
such a $V^{(k)}$ exists because $\text{card}(B_{m}^{(k)}) < \infty$. By (4.11), we have

$$
\mu \left( \bigcup_{w \in V^{(k)}} Q_{w} \right) \geq \frac{1}{2} \mu(S_k).
$$

(4.12)

For any $w, w' \in \mathcal{W}$, if $Q_{w} \cap Q_{w'} \neq \emptyset$, then we have either $Q_{w} \subset Q_{w'}$ or $Q_{w'} \subset Q_{w}$. Hence the minimality of $V^{(k)}$ implies that

$$
Q_{w} \cap Q_{w'} = \emptyset \quad \text{if} \quad w, w' \in V^{(k)} \text{ and } w \neq w'.
$$

(4.13)

Define

$$
B_k = \sum_{w \in V^{(k)}} e_w \otimes e_w.
$$

By (3.6) and (4.13), $B_k$ is an orthogonal projection.

For each $f \in \text{Lip}(\mathcal{X})$, we have

$$
[B_k, M_f] = L(f) \sqrt{n} \sum_{w \in V^{(k)}} 2^{-|w|} \left\{ e_w \otimes (\bar{f}_w e_w) - (f_w e_w) \otimes e_w \right\},
$$

where $f_w$ is the same as in the estimate of $\| [G_k, M_f] \|_{C}$. In particular, $\| f_w \|_{\infty} \leq 1$. Thus it follows from (3.6) and (4.13) that

$$
\| [B_k, M_f] \|_{C} \leq 2L(f) \sqrt{n} \Phi_C \left( \{ 2^{-|w|} \}_{w \in V^{(k)}} \right).
$$

Since $V^{(k)} \subset B^{(k)}$, the definition of $B^{(k)}$ ensures $2^{-|w|} < 2k^{-1} \lambda_w$ if $w \in V^{(k)}$. Hence

$$
\| [B_k, M_f] \|_{C} \leq 4L(f) \sqrt{n} \Phi_C \left( \{ \lambda_w \}_{w \in \mathcal{W}} \right), \quad f \in \text{Lip}(\mathcal{X}).
$$

(4.14)

By (3.6) and (4.12), we have

$$
\langle B_k 1, 1 \rangle = \sum_{w \in V^{(k)}} \mu(Q_w) \geq \frac{1}{2} \mu(S_k).
$$

(4.15)

As we have already mentioned, we define the desired operator $A_k$ by the formula $A_k = G_k + B_k$. Let us verify that the sequence $\{ A_k \}$ satisfies conditions (i)–(iii) in Proposition 3.1. First of all, condition (iii) follows from (4.6), (4.14) and (2.1). Since $B_k$ is an orthogonal projection, and since we showed that $\| G_k \| \leq 2$, it follows that $\| A_k \| \leq 3$, verifying condition (ii). To verify condition (i), we note that

$$
\langle A_k 1, 1 \rangle \geq \frac{1}{6} \left( \mu(R_k) + \mu(S_k) \right)
$$

by (4.9) and (4.15). Thus the verification will be complete if we can show that

$$
\mu(R_k) + \mu(S_k) \geq \mu(Q).
$$

(4.16)
This is where we use (2.2). Suppose that \( z \) is a point in \( Q \setminus S_k \) such that
\[
\sum_{w \in W} \frac{1}{k} 2^{|w|} |w| \chi_{Q_w}(z) = \infty.
\] (4.17)

By (4.10), we have
\[
\sum_{w \in B^{(k)}} \frac{1}{k} 2^{|w|} |w| \chi_{Q_w}(z) = 0.
\]

By (4.1) and (4.2), this means \( z \in R_k \). But (2.2) tells us that there is a Borel subset \( N \) of \( Q \) with \( \mu(N) = 0 \) such that if \( z \in \{ Q \setminus S_k \} \setminus N \), then (4.17) holds. Thus \( Q \setminus S_k \subset R_k \cup N \) and, consequently, \( \mu(R_k) \geq \mu(Q \setminus S_k) \). This proves (4.16) and completes the verification of conditions (i)–(iii) in Proposition 3.1. Thus we conclude that there is a Borel set \( \Delta \) with \( \mu(\Delta) > 0 \) such that the measure \( \mu_\Delta \) is \( C \)-discrete.

(2) To show that \( \mu \) itself is \( C \)-discrete, we invoke Proposition 4.3 in [12]. Let \( E \) be a Borel subset of \( Q \) with \( \mu(E) > 0 \). To complete the proof, according to [12, Proposition 4.3], it suffices to find a Borel subset \( E' \) of \( E \) with \( \mu(E') > 0 \) such that the measure \( \mu_{E'} \) defined by the formula \( \mu_{E'}(A) = \mu(E' \cap A) \) is \( C \)-discrete.

For the given \( E \), write \( \nu \) for \( \mu_{E} \). That is, \( \nu(A) = \mu(E \cap A) \). Then (2.2) implies that
\[
\sum_{w \in W} 2^{|w|} |w| \chi_{Q_w}(x) = \infty \quad \text{for } \nu \text{-a.e. } x \in Q.
\]

Also, \( \nu(Q) = \mu(E) > 0 \). Thus we can apply the conclusion we proved in part (1) to the measure \( \nu \). That is, by (1), there is a Borel set \( \Delta \) with \( \nu(\Delta) > 0 \) such that the measure \( \nu_\Delta \) defined by the formula \( \nu_\Delta(A) = \nu(\Delta \cap A) \) is \( C \)-discrete. Now set \( E' = E \cap \Delta \). Then \( \mu(E') = \nu(\Delta) > 0 \) and, since \( \mu_{E'} = \nu_\Delta \), the measure \( \mu_{E'} \) is \( C \)-discrete. Thus we have found the desired subset \( E' \). This completes the proof of Theorem 2.1. \( \square \)

5. Is the condition in Theorem 2.1 necessary?

For any non-negative numbers \( \{ \lambda_w \}_{w \in W} \), (2.2) is obviously equivalent to the condition
\[
\sum_{w \in W} 2^{|w|} |w| \lambda_w \mu(E \cap Q_w) = \infty \quad \text{for every Borel set } E \subset Q \text{ with } \mu(E) > 0.
\] (5.1)

With this in mind, the purpose of this section is to show that, for a large class of norm ideals, the condition in Theorem 2.1 seems to be close to being necessary. These are norm ideals satisfying condition (QK) which we introduced in [14].

Definition 5.1. (See [14, Definition 2.1].) A norm ideal \( C \) is said to satisfy condition (QK) if there exist constants \( 0 < t < 1 \) and \( 0 < B < \infty \) such that
\[
\| X \oplus \cdots \oplus X \|_C \leq B k^t \| X \|_C
\]
for every finite-rank operator \( X \) and every \( k \in \mathbb{N} \).
As we explained in [14], the class of norm ideals satisfying condition (QK) is quite large. This class includes the Lorentz ideals $L^{q,s}(q,s) \in ((1,\infty) \times (0,1)) \cup ((1,\infty) \times \{0\})$, and their duals [14, Section 5]. The Orlicz ideals we considered in [15] also satisfy condition (QK).

Suppose that $C$ is a norm ideal. Recall that a commuting tuple $(A_1,\ldots,A_n)$ of bounded self-adjoint operators is said to be simultaneously diagonalizable modulo $C$ if there exists a commuting tuple $(D_1,\ldots,D_n)$ of self-adjoint diagonal operators such that $A_j - D_j \in C$ for every $j \in \{1,\ldots,n\}$. From [11, Proposition 2.6] it follows that if $(A_1,\ldots,A_n)$ is simultaneously diagonalizable modulo $C$, then $(A_1,\ldots,A_n)$ is also simultaneously diagonalizable modulo $C^{(0)}$. (Recall that $C^{(0)}$ can be a proper subset of $C$ [6].)

Let $\mu$ be a compactly supported regular Borel measure on $\mathbb{R}^n$. On the Hilbert space $L^2(\mathbb{R}^n, \mu)$, we have the commuting tuple of self-adjoint operators $(M_1,\ldots,M_n)$, where

$$(M_j f)(x_1,\ldots,x_n) = x_j f(x_1,\ldots,x_n), \quad f \in L^2(\mathbb{R}^n, \mu),$$

for $1 \leq j \leq n$.

**Theorem 5.2.** Let $C$ be a norm ideal satisfying condition (QK). Let $\mu$ be a compactly supported regular Borel measure on $\mathbb{R}^n$. Suppose that the commuting tuple of self-adjoint operators $(M_1,\ldots,M_n)$ on $L^2(\mathbb{R}^n, \mu)$ is simultaneously diagonalizable modulo $C$. Then for any Borel set $E \subset Q$ with $\mu(E) > 0$, there exists a set of non-negative numbers $\{a_w\}_{w \in \mathcal{W}}$, which may depend on $E$, such that

$$\Phi_C(\{a_w\}_{w \in \mathcal{W}}) < \infty$$

and

$$\sum_{w \in \mathcal{W}} 2^{[w]} a_w \mu(E \cap Q_w) = \infty.$$  

**Proof.** Let $C'$ be the dual of $C^{(0)}$ [6, Section III.11] and let $\Phi_{C'}$ be the symmetric gauge function associated with $C'$. Let $E$ be a Borel subset of $Q$ such that $\mu(E) > 0$. Define the measure $\mu_E$ by the formula $\mu_E(A) = \mu(E \cap A)$ as before. We want to show that

$$\Phi_C(\{2^{[w]} \mu_E(Q_w)\}_{w \in \mathcal{W}}) = \infty.$$  

By the duality between $C'$ and $C^{(0)}$ and the uniform boundedness principle, (5.5) implies that there exists a set of non-negative numbers $\{a_w\}_{w \in \mathcal{W}}$ for which (5.3) and (5.4) hold.

To prove (5.5), consider the singular integral operators

$$(T_j f)(x) = \chi_E(x) \int \frac{x_j - y_j}{|x - y|^2} f(y) d\mu_E(y)$$

on $L^2(\mathbb{R}^n, \mu)$, $1 \leq j \leq n$. Since $C$ satisfies condition (QK), if (5.5) did not hold, then by [14, Theorem 2.3] we would have $T_1,\ldots,T_n \in C'$. Since $\sum_{j=1}^n [M_j, T_j] = \chi_E \otimes \chi_E$ and $\text{tr}(\chi_E \otimes \chi_E) = \mu(E) > 0$, by [11, Proposition 2.1], the condition $T_1,\ldots,T_n \in C'$ contradicts the assumption that the tuple $(M_1,\ldots,M_n)$ on $L^2(\mathbb{R}^n, \mu)$ is simultaneously diagonalizable modulo $C$. This proves (5.5) and completes the proof of the theorem. □
Obviously, the difference between (5.1) and (5.4) is that the set of numbers \( \{\lambda_w\}_{w \in W} \) in (5.1) is independent of \( E \). In light of Theorems 2.1 and 5.2, let us introduce

**Definition 5.3.** Let \( \mu \) be a regular Borel measure without point masses on \( \mathbb{R}^n \). Furthermore, suppose that \( \mu(\mathbb{R}^n \setminus Q) = 0 \). Let \( C \) be a norm ideal. We say that \( \mu \) belongs to the class \( \Omega(C) \) if it has the following two properties:

(a) If \( \{\lambda_w\}_{w \in W} \) is a set of non-negative numbers such that \( \Phi_C(\{\lambda_w\}_{w \in W}) < \infty \), then \( \mu(\{x \in Q : \sum_{w \in W} 2^{\|w\|} \lambda_w \chi_{Q_w}(x) < \infty\}) > 0 \).

(b) For any Borel set \( E \subset Q \) with \( \mu(E) > 0 \), there exists a set of non-negative numbers \( \{a_w\}_{w \in W} \) such that \( \Phi_C(\{a_w\}_{w \in W}) < \infty \) and \( \sum_{w \in W} 2^{|w|} a_w \mu(E \cap Q_w) = \infty \).

It is easy to see that property (a) in Definition 5.3 is equivalent to

(a') there is a Borel set \( B \subset Q \) with \( \mu(B) > 0 \) such that if \( \{\lambda_w\}_{w \in W} \) is a set of non-negative numbers satisfying the condition \( \Phi_C(\{\lambda_w\}_{w \in W}) < \infty \), then \( \sum_{w \in W} 2^{|w|} \lambda_w \chi_{Q_w}(x) < \infty \) for \( \mu \)-a.e. \( x \in B \).

Theorems 2.1 and 5.2 together tell us that if \( C \) is a norm ideal which satisfies condition (QK) and if \( \mu \) is supported in \( Q \) but does not belong to the measure class \( \Omega(C) \), then the question of the \( C \)-discreteness of \( \mu \) is completely settled. Thus for the class of norm ideals satisfying condition (QK), the diagonalization problem is reduced to

**Problem 5.4.** (1) Does there exist a norm ideal \( C \) for which \( \Omega(C) \) is not empty?

(2) If there exists a norm ideal \( C \) such that \( \Omega(C) \neq \emptyset \) and if \( \mu \in \Omega(C) \), then is \( \mu \) \( C \)-discrete? Or for such a \( \mu \) is there obstruction to the simultaneous diagonalization modulo \( C \) of the tuple \( (M_1, \ldots, M_n) \) on \( L^2(\mathbb{R}^n, \mu) \) defined by (5.2)?

Recall that, as non-commutative analogues of Orlicz spaces [1], one can define Orlicz ideals. For the Orlicz ideals \( C_G \) that we considered in [15], the measure class \( \Omega(C_G) \) is actually empty. Thus for such a \( C_G \), the condition in Theorem 2.1 is necessary. But we will omit the proof of this fact here for the reason that a necessary and sufficient condition for diagonalization modulo \( C_G \) was already given in [15].

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**References**


