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The strong Lefschetz property for Artinian algebras with non-standard grading

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Abstract

Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra, where $A_c \neq (0)$ and $\operatorname{char} K = 0$. (*The grading may not necessarily be standard.*) Then A has the strong Lefschetz property if there exists an element $g \in A_1$ such that the multiplication $\times g^{c-2i} : A_i \to A_{c-i}$ is bijective for every $i = 0, 1, \ldots, [c/2]$. The main results obtained in this paper are as follows:

1. A has the strong Lefschetz property if and only if there is a linear form $z \in A_1$ such that $\operatorname{Gr}_{(z)}(A)$ has the strong Lefschetz property.
2. If A is Gorenstein, then A has the strong Lefschetz property if and only if there is a linear form $z \in A$ such that all central simple modules of (A, z) have the strong Lefschetz property.
3. A finite free extension of an Artinian K-algebra with the strong Lefschetz property has the strong Lefschetz property if the fiber does.
4. The complete intersection defined by power sums of consecutive degrees has the strong Lefschetz property.

Keywords: Hilbert function; Artinian algebra; Gorenstein algebra; Lefschetz property; Nilpotent matrix

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1. Introduction

By now a considerable amount of results has been obtained for the strong and weak Lefschetz properties for Artinian graded K-algebras over a field K of characteristic zero (e.g. [1,10]). So far all algebras considered for the strong Lefschetz property have the standard grading. In this paper we consider the Lefschetz properties for algebras with grading which may *not necessarily be standard*.

The definition of the strong or weak Lefschetz property can be made verbatim, provided that the algebra has a linear form, as with the case of the standard grading. With this definition, one notices that some basic facts of the Lefschetz property in the case of standard grading easily fail to generalize. This is because a non-standard grading can force tough restrictions on the behavior of a linear element as a multiplication operator on the algebra. By the same reason, however, there are situations where the strong Lefschetz property is easier to prove for algebras with a non-standard grading.

To explain in more detail, let $R = K[x_1, \dots, x_n]$ be the polynomial ring, and let

$$I = (p_d, p_{d+1}, \dots, p_{d+n-1})$$

be the ideal of R, where p_d is the power sum symmetric function of degree d, i.e.,

$$p_d = x_1^d + \dots + x_n^d.$$

Since the generators of I are symmetric forms, they are contained in the ring of invariants $S \subset R$ under the action of the symmetric group. Notice that S, being generated by the elementary symmetric polynomials, has no longer the standard grading as a graded subring of R. Put $A = R/I$ and $B = S/I \cap S$. Since $e_1 = x_1 + \dots + x_n$ is a unique linear form of S, it is the only candidate for a strong Lefschetz element for $S/I \cap S$. In this particular case it happens that it is in fact a strong Lefschetz element. It is easier to prove the strong Lefschetz property for $S/I \cap S$ rather than for R/I. Once it is known that $S/I \cap S$ has the strong Lefschetz property, it determines the central simple modules of $(R/I, e_1)$, and it reduces the problem to a lower dimensional case. Our results of this paper enables us to conclude that R/I indeed has the strong Lefschetz property.

Generally speaking, it may well occur that the algebra $S/I \cap S$ does not have the strong Lefschetz property even if I can be generated by a regular sequence of invariant forms. For example if the ideal contains a power of e_1 as a member of a minimal generating set, then obviously it cannot have the strong Lefschetz property. Nonetheless it reduces the problem to prove the strong Lefschetz property to a lower dimensional case at least.

Much the same results of [2] and [4] are valid for algebras with any grading as long as the algebra possesses a linear form. This is not surprising as the strong or weak Lefschetz property is a property of a single linear form as an operator for a graded vector space.

It seems natural to define the strong Lefschetz property for a finite graded vector space (rather than for an Artinian K-algebra)

$$V = \bigoplus_{i=a}^{b} V_i.$$

For V we call $g \in \mathrm{End}(V)$ a strong Lefschetz element if it is a grade-preserving map of degree one and if the restricted map

$$g^{b-a-2i}\big|_{V_{a+i}} : V_{a+i} \longrightarrow V_{b-i}$$

is bijective for all i such that $0 \leqslant 2i \leqslant b-a$.

If A is a graded algebra and if V is a graded module over A, then we have the regular representation

$$\times : A \longrightarrow \mathrm{End}(V).$$

Namely, $\times z$ is the map $V \to V$ defined by $a \mapsto za$ for $a \in A$. Then we will say that V has the strong Lefschetz property as an A-module if there exists a strong Lefschetz element for V in the set $\times A \subset \mathrm{End}(V)$. In this case we say that a linear form $z \in A$ is a strong Lefschetz element for V if $\times z$ is. This way it is possible to extend the definition of the strong Lefschetz property to graded modules over graded algebras whose grading may not be standard. In our previous paper [4] we defined the central simple modules,

$$U_1, \ldots, U_s,$$

of a pair (A, z) of an Artinian algebra A and a linear form $z \in A$, in which case A was assumed to have the standard grading. We show that much the same results can be extended to Artinian algebras with non-standard grading. Thus it is possible to establish the chain of implications:

$$A \text{ has SLP} \quad \Leftarrow \quad \mathrm{Gr}_{(z)}(A) \text{ has SLP} \quad \Leftarrow \quad \bigoplus \widetilde{U}_i \text{ has SLP} \quad \Leftarrow \quad U_i \text{ has SLP for } \forall i,$$

where A is an Artinian Gorenstein algebra and z is any linear form in A. We have the same implications for an arbitrary Artinian algebra A with certain additional conditions for $\widetilde{U}_i$. For definition of U_i and $\widetilde{U}_i$ see Section 5.

More important than the individual results is the fact that they can be used to prove the strong Lefschetz property for Artinian algebras with the *standard* grading. An example is the complete intersection defined by power sums of consecutive degrees as mentioned earlier. This we discuss in Section 7. In Section 8 we show more examples where the consideration of a non-standard grading works effectively.

In Section 3, we prove that $V_1 \otimes V_2$ has the strong Lefschetz property if and only if V_1 and V_2 have the strong Lefschetz property, where V_i are finite graded vector spaces (Theorem 3.10). The "if" part is fairly well known for algebras and the "only if" part does not seem to have been written anywhere. One finds that the "if" part, in the end, reduces to the simplest case which asserts that the algebra

$$K[X, Y]/\big(X^r, Y^s\big)$$

has the strong Lefschetz property with the linear form $X + Y$ as a strong Lefschetz element. This itself is by no means trivial as the assertion is essentially equivalent to what is known as the Gordan–Clebsch decomposition of the tensor product

$$V(r) \otimes V(s)$$

of irreducible modules $V(r)$ and $V(s)$ over the Lie algebra sl_2. One other method to prove it is to use the fact which says that the generic initial ideal is Borel fixed [1]. For completeness we show a new method to prove it using a theorem due to Ikeda in Appendix A. The proof in itself is of considerable interest. For other results of this paper the discussion of Section 3 is indispensable.

In Section 4 we show that the strong Lefschetz property of $\mathrm{Gr}_{(z)}(A)$ implies that of A, where z is any linear form (Theorem 4.6).

In Section 5 we generalize the notion of the central simple modules for non-standard grading and characterize the strong Lefschetz property in terms of it in two theorems.

In Section 6 we show that a finite free extension of an Artinian algebra with the strong Lefschetz property has the strong Lefschetz property if the fiber does. It is stated in Theorem 6.1. For the standard grading this was proved in [2] with a correction made in [3]. In this paper it is proved for any grading and the proof is substantially simplified. This is another example which shows that the "central simple modules" are useful.

Section 2 is preliminaries, where we briefly review basic definitions and re-produce proofs for some lemmas to be used in the sequel.

Some related results can be found in Maeno [6] and Morita–Wachi–Watanabe [7], where one finds a vast amount of examples of graded vector spaces with a strong Lefschetz element. This was really the starting point of the present paper.

2. Hilbert functions and the Lefschetz properties

The Hilbert function of a graded vector space $V = \bigoplus_{i=a}^{b} V_i$ is the map $i \mapsto \dim V_i$. If V has finite dimension, then its Hilbert series is the polynomial

$$h_V(q) = \sum_{i=a}^{b} (\dim V_i) q^i.$$

We define the *Sperner number* of V by

$$\mathrm{Sperner}(V) = \mathrm{Max}\{\dim V_a, \dim V_{a+1}, \ldots, \dim V_b\}$$

and *CoSperner number* by

$$\mathrm{CoSperner}(V) = \sum_{i=a}^{b-1} \mathrm{Min}\{\dim V_i, \dim V_{i+1}\}.$$

The Hilbert function $h_V(q)$ of $V = \bigoplus_{i=a}^{b} V_i$ (where $V_a \neq (0)$ and $V_b \neq (0)$) is symmetric if $\dim V_{a+i} = \dim V_{b-i}$ for all $i = 0, 1, \ldots, [(b-a)/2]$. Then we call the half integer $(a+b)/2$ the reflecting degree of $h_V(q)$. The Hilbert function of V is unimodal if there exists an integer m ($a \leqslant m \leqslant b$) such that

$$\dim V_a \leqslant \dim V_{a+1} \leqslant \cdots \leqslant \dim V_m \geqslant \dim V_{m+1} \geqslant \cdots \geqslant \dim V_b.$$

Definition 2.1. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra. Suppose that $V = \bigoplus_{i=a}^{b} V_i$ is a finite graded A-module with $V_a \neq (0)$ and $V_b \neq (0)$.

(i) V has the *weak Lefschetz property* (WLP) as an A-module if there is a linear form $g \in A_1$ such that the multiplication $\times g : V_i \to V_{i+1}$ is either injective or surjective for all $i = a, a + 1, \dots, b - 1$.
(ii) V has the *strong Lefschetz property* (SLP) as an A-module if there is a linear form $g \in A_1$ such that the multiplication $\times g^{b-a-2i} : V_{a+i} \to V_{b-i}$ is bijective for all $i = 0, 1, \dots, [(b - a)/2]$.

Remark 2.2.

(1) It is easy to see that if A has the WLP and if A has the standard grading, then the Hilbert function of A is unimodal. If the grading of A is not standard, the WLP of A does not imply the unimodality of the Hilbert function of A. Whatever the grading of A is, the WLP of an A-module V does not imply the unimodality of the Hilbert function.
(2) If V has a unimodal Hilbert function, then $\mathrm{CoSperner}(V) = \dim V - \mathrm{Sperner}(V)$.
(3) If V has the SLP, then the Hilbert function of V is symmetric and unimodal.

The following is a characterization of a weak Lefschetz element.

Lemma 2.3. *Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra and let $V = \bigoplus_{i=a}^{b} V_i$ be a finite graded A-module with a unimodal Hilbert function, where $V_a \neq (0)$ and $V_b \neq (0)$. Then the following are equivalent.*

(1) *A linear form $\ell \in A_1$ is a weak Lefschetz element of V.*
(2) $\dim V/\ell V = \mathrm{Sperner}(V)$.
(3) $\mathrm{rank}(\times\ell) = \mathrm{CoSperner}(V)$.

Proof. First note that for any linear form $\ell \in A_1$ we have $\dim V/\ell V \geqslant \mathrm{Sperner}(V)$ or equivalently, $\mathrm{rank}(\times\ell) \leqslant \mathrm{CoSperner}(V)$. Now one sees that the equality holds if and only if $\ell : V_i \to V_{i+1}$ is either surjective or injective for every i (cf. Proposition 3.2 in [9]). □

With the same notation as Definition 2.1, put

$$\mathrm{SP}_k(V) = \sum_{i=a}^{b} \mathrm{Max}\{\dim V_i - \dim V_{i-k}, 0\}$$

for $1 \leqslant k \leqslant b - a$, where $\dim_K V_j = 0$ for all $j < a$. We call

$$\mathbf{SP}(V) = \big(\mathrm{SP}_1(V), \mathrm{SP}_2(V), \dots, \mathrm{SP}_{b-a}(V)\big)$$

the *Sperner vector* of V. Note that $\mathrm{SP}_1(V)$ is equal to the Sperner number of V when V has a unimodal Hilbert function.

Using the Sperner vector we can characterize a strong Lefschetz element as follows.

Lemma 2.4. *Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra and let $V = \bigoplus_{i=a}^{b} V_i$ be a finite graded A-module with a symmetric unimodal Hilbert function, where $V_a \neq (0)$ and $V_b \neq (0)$. Then the following conditions are equivalent.*

(1) *A linear form* $g \in A_1$ *is a strong Lefschetz element of* V.
(2) $\dim V/g^k V = \mathrm{SP}_k(V)$ *for all* $k = 1, 2, \ldots, b-a$.
(3) $\mathrm{rank}(\times g^k) = \dim V - \mathrm{SP}_k(V)$ *for all* $k = 1, 2, \ldots, b-a$.

Proof. Note that we have $\dim_K V/fV \geqslant \mathrm{SP}_k(V)$ or equivalently, $\mathrm{rank}(\times f) \leqslant \dim V - \mathrm{SP}_k(V)$ for any homogeneous form f of degree k, with $1 \leqslant k \leqslant b-a$. Now suppose that $f = g^k$ for a linear form g. Then it is easy to see that the equality holds if and only if $g^k : V_i \to V_{i+k}$ has the full rank for all i, and that it is equivalent to claiming that g is a strong Lefschetz element (cf. Lemma 2.2 in [4]). □

For later use we would like to summarize the basic facts used in the proofs of the above lemmas.

Lemma 2.5. *Let* A *be a graded Artinian* K*-algebra and let* $V = \bigoplus_{i=a}^{b} V_i$ *be a finite graded* A*-module with a symmetric unimodal Hilbert function, where* $V_a \neq (0)$ *and* $V_b \neq (0)$. *Then*

(1) $\dim V/\ell V \geqslant \mathrm{Sperner}(V)$ *for all* $\ell \in A_1$.
(2) $\mathrm{rank}(\times \ell) \leqslant \mathrm{CoSperner}(V)$ *for all* $\ell \in A_1$.
(3) $\dim_K V/fV \geqslant \mathrm{SP}_k(V)$ *for all* $f \in A_k$, *where* $1 \leqslant k \leqslant b-a$.
(4) $\mathrm{rank}(\times f) \leqslant \dim V - \mathrm{SP}_k(V)$ *for any* $f \in A_k$, *where* $1 \leqslant k \leqslant b-a$.

3. The tensor product of modules with the strong Lefschetz property

In this section, we discuss the tensor product of modules with the strong Lefschetz property. An important result is Proposition 3.9 which often enables us to reduce an issue of the strong Lefschetz property to that of the weak Lefschetz property.

Notation and Remark 3.1. Let A be a graded Artinian K-algebra, let V be a finite graded A-module and let z be a linear form of A. Since V is a finite graded A-module, the linear map $\times z \in \mathrm{End}(V)$ is nilpotent. Hence the Jordan canonical form J of $\times z$ is the matrix of the following form:

$$J = \begin{bmatrix} J(0,n_1) & & & \text{\Large 0} \\ & J(0,n_2) & & \\ & & \ddots & \\ \text{\Large 0} & & & J(0,n_r) \end{bmatrix},$$

where $J(0,m)$ is the Jordan block of size $m \times m$,

$$J(0,m) = \begin{bmatrix} 0 & 1 & & \text{\Large 0} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \text{\Large 0} & & & 0 \end{bmatrix}.$$

Then we denote the *Jordan decomposition* of $\times z$ by writing

$$P(\times z) = n_1 \oplus n_2 \oplus \cdots \oplus n_r.$$

We note that $r = \dim_K V/zV$ and $\dim V = \sum n_i$. Two decompositions $n_1 \oplus n_2 \oplus \cdots \oplus n_r$ and $n'_1 \oplus n'_2 \oplus \cdots \oplus n'_{r'}$ are regarded as the same if they are the same as multisets.

Lemma 3.2. *Let A and B be graded Artinian K-algebras. Let V be a finite graded A-module and $g \in A$ a linear form, and similarly W and h for B. Assume that*

(i) *the Jordan decomposition of $\times g \in \mathrm{End}(V)$ is the same as that of $\times h \in \mathrm{End}(W)$,*
(ii) *there is an integer m such that $h_V(q) = q^m h_W(q)$.*

Then g is a weak (respectively strong) Lefschetz element of V if and only if h is a weak (respectively strong) Lefschetz element of W.

Proof. Since $h_V(q) = q^m h_W(q)$, the Sperner vectors of V and W are the same. Hence, using the assumption (i), the assertion follows from Lemmas 2.3 and 2.4. □

Following is another characterization of a strong Lefschetz element.

Lemma 3.3. *Let A be a graded Artinian K-algebra and let $V = \bigoplus_{i=a}^{b} V_i$ be a finite graded A-module, where $V_a \neq (0)$ and $V_b \neq (0)$. Put $r = \mathrm{Sperner}(V)$. Let g be a linear form of A. Then the following conditions are equivalent.*

(1) *g is a strong Lefschetz element for V.*
(2) *There are graded vector subspaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r$ of V with $V = \bigoplus_{i=1}^{r} \mathcal{V}_i$ which satisfy the following conditions for each $i = 1, 2, \ldots, r$.*
   (i) *$g\mathcal{V}_i \subset \mathcal{V}_i$.*
   (ii) *The Jordan canonical form of $\times g \in \mathrm{End}(\mathcal{V}_i)$ is a single Jordan block.*
   (iii) *The reflecting degree of $h_{\mathcal{V}_i}(q)$ is equal to that of $h_V(q)$.*

In this case $P(\times g) = \dim \mathcal{V}_1 \oplus \dim \mathcal{V}_2 \oplus \cdots \oplus \dim \mathcal{V}_r$.

Proof. (1) ⇒ (2) Assume g is a strong Lefschetz element for V. Let $h_V(q) = \sum_{i=a}^{b} h_i q^i$ be the Hilbert function of V. A basis for the Jordan decomposition of $\times g \in \mathrm{End}(V)$ is obtained as follows. Let $v_1, v_2, \ldots, v_{h_a}$ be a basis of V_a. By the SLP of V, the elements

$$\{g^k v_j \mid 1 \leqslant j \leqslant h_a,\ 0 \leqslant k \leqslant b-a\},$$

being linearly independent, will be a part of the basis. Next let $\{v'_{h_a+1}, v'_{h_a+2}, \ldots, v'_{h_{a+1}}\}$ be a basis of $\mathrm{Ker}[V_{b-1} \xrightarrow{\times g} V_b]$ (if it exists). By the SLP of V, there exist elements $\{v_{h_a+1}, v_{h_a+2}, \ldots, v_{h_{a+1}}\}$ of V_{a+1} such that $g^{b-a-2} v_j = v'_j$ for all $j = h_a+1, h_a+2, \ldots, h_{a+1}$. Then the elements

$$\{g^k v_j \mid h_a + 1 \leqslant j \leqslant h_{a+1},\ 0 \leqslant k \leqslant b-a-2\},$$

none of these being dependent of the previously chosen basis elements, will be another part of the basis. We repeat the same to expand basis elements. We may carry over this process to decompose $\times g$ into Jordan blocks. Since

$$h_a + (h_{a+1} - h_a) + (h_{a+2} - h_{a+1}) + \cdots + (h_{[(b-a)/2]} - h_{[(b-a)/2]-1}) = r,$$

one sees that there are r Jordan blocks.

Now let $\mathcal{V}_i$ be the subspace of V spanned by $\{v_i, gv_i, \ldots, g^{b-a-2(d_i-a)}v_i\}$ for $i = 1, 2, \ldots, r$, where $d_i = \deg(v_i)$. Then it is easy to verify the conditions stated in (2).

(2) ⇒ (1) By (iii) it suffices to prove this for each $\mathcal{V}_i$, which is obvious. □

Proposition 3.4. *Let K be a field of characteristic zero. Let A and B be graded Artinian K-algebras, let V be a finite graded A-module and W a finite graded B-module. If V and W have the SLP, then $V \otimes_K W$ also has the SLP as an $A \otimes_K B$-module.*

First we prove a lemma.

Lemma 3.5. *With the same notation as in Proposition* 3.4*, let $g \in A$ be any linear form and let $\mathcal{V}$ be a graded subspace of V such that $g\mathcal{V} \subset \mathcal{V}$. Similarly let $h \in B$ be a linear form and $\mathcal{W} \subset W$ a graded subspace such that $h\mathcal{W} \subset \mathcal{W}$. Assume that $\times g \in \mathrm{End}(\mathcal{V})$ is a single Jordan block and the same for $\times h \in \mathrm{End}(\mathcal{W})$. Moreover let $m = \dim \mathcal{V}$, $n = \dim \mathcal{W}$ and $s = \mathrm{Sperner}(\mathcal{V} \otimes \mathcal{W})$.*

Then there exist graded vector subspaces $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_s$ of $\mathcal{V} \otimes \mathcal{W}$ such that $\mathcal{V} \otimes \mathcal{W} = \bigoplus_{i=1}^{s} \mathcal{U}_i$, which satisfy the following conditions for $i = 1, 2, \ldots, s$.

(i) $(g \otimes 1 + 1 \otimes h)\mathcal{U}_i \subset \mathcal{U}_i$.
(ii) *The Jordan canonical matrix of* $\times(g \otimes 1 + 1 \otimes h) \in \mathrm{End}(\mathcal{U}_i)$ *is a single Jordan block.*
(iii) *The reflecting degree of* $h_{\mathcal{U}_i}(q)$ *is equal to that of* $h_{\mathcal{V}\otimes\mathcal{W}}(q)$.

Proof. By Lemma 3.3, it is enough to show that $g \otimes 1 + 1 \otimes h$ is a strong Lefschetz element for $\mathcal{V} \otimes \mathcal{W}$. Let d and e be the initial degrees of $\mathcal{V}$ and $\mathcal{W}$, respectively. Then the Hilbert functions of $\mathcal{V}$ and $\mathcal{W}$ are

$$\begin{cases} h_{\mathcal{V}}(q) = q^d + q^{d+1} + \cdots + q^{d+m-1}, \\ h_{\mathcal{W}}(q) = q^e + q^{e+1} + \cdots + q^{e+n-1}. \end{cases}$$

Let $K[x]$ be the polynomial ring in one variable. As a graded vector space we may choose an isomorphism $\mathcal{V} \to K[x]/(x^m)(-d)$ so that we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\times g} & \mathcal{V} \\ \downarrow & & \downarrow \\ (K[x]/(x^m))(-d) & \xrightarrow{\times x} & (K[x]/(x^m))(-d). \end{array}$$

Likewise we may choose an isomorphism $\mathcal{W} \to K[y]/(y^n)(-e)$ to get the similar diagram for $\mathcal{W}$.

These diagrams give rise the following commutative diagram:

$$\begin{array}{ccc} \mathcal{V}\otimes\mathcal{W} & \xrightarrow{\times(g\otimes 1+1\otimes h)} & \mathcal{V}\otimes\mathcal{W} \\ \downarrow & & \downarrow \\ (K[x,y]/(x^m,y^n))(-(d+e)) & \xrightarrow{\times(x+y)} & (K[x,y]/(x^m,y^n))(-(d+e)), \end{array}$$

where the vertical maps are isomorphisms as graded vector spaces. Since the characteristic of K is zero, it follows by Proposition A.1 in Appendix A that $x+y$ is a strong Lefschetz element for $K[x,y]/(x^m,y^n)$. Hence $g\otimes 1+1\otimes h$ is a strong Lefschetz element for $\mathcal{V}\otimes\mathcal{W}$ as well as $x+y$ is a strong Lefschetz element for $(K[x,y]/(x^m,y^n))$. $\square$

Proof of Proposition 3.4. Let $V=\bigoplus_{i=1}^{r}\mathcal{V}_i$ and $W=\bigoplus_{i=j}^{u}\mathcal{W}_j$ be the direct sum decomposition of V and W constructed in Lemma 3.3 with respect to the strong Lefschetz elements $g\in A_1$ and $h\in B_1$ for V and W, where $r=\mathrm{Sperner}(V)$ and $u=\mathrm{Sperner}(W)$. Then we see that $V\otimes W=\bigoplus_{i,j}(\mathcal{V}_i\otimes\mathcal{W}_j)$ and $\mathcal{V}_i\otimes\mathcal{W}_j$ is closed by the multiplication $\times(g\otimes 1+1\otimes h): V\otimes W\to V\otimes W$ for all i and j. Noting

$$\begin{cases} h_{V\otimes W}(q)=h_V(q)h_W(q)=\left(\sum_{i=1}^{r}h_{\mathcal{V}_i}(q)\right)\left(\sum_{j=1}^{u}h_{\mathcal{W}_j}(q)\right), \\ h_{\mathcal{V}_i\otimes\mathcal{W}_j}(q)=h_{\mathcal{V}_i}(q)h_{\mathcal{W}_j}(q), \end{cases}$$

we see that the reflecting degree of $h_{\mathcal{V}_i\otimes\mathcal{W}_j}(q)$ is equal to that of $h_{V\otimes W}(q)$. By Lemma 3.5, for each pair (i,j), there are subspaces $\mathcal{U}_k^{(ij)}$ satisfying the conditions (i)–(iii) of Lemma 3.5. We note that the Sperner number of $V\otimes W$ is equal to the sum of those of $\mathcal{V}_i\otimes\mathcal{W}_j$. Hence, by Lemma 3.3, the linear form $g\otimes 1+1\otimes h$ is a strong Lefschetz element for $V\otimes W$. $\square$

Lemma 3.6. *Let*

$$u_1\geqslant u_2\geqslant\cdots\geqslant u_r$$

be a descending sequence of positive integers, all even or all odd. Let a,b be any integers such that $b-a+1=u_1$. Let

$$h(q)=\frac{1}{1-q}\sum_{i=1}^{r}\left(q^{(a+b+1-u_i)/2}-q^{(a+b+1+u_i)/2}\right). \tag{3.1}$$

Then $h(q)$ is a symmetric unimodal polynomial with reflecting degree $(a+b)/2$ and the coefficient h_{a+k} of q^{a+k} in $h(q)$ for $0\leqslant k\leqslant(b-a)/2$ is given by

$$h_{a+k}=\#\{u_i\mid u_i\geqslant b-a+1-2k\}.$$

Conversely suppose that $h(q)=\sum_{i=a}^{b}h_iq^i$ is a symmetric unimodal polynomial with positive integers as coefficients and $h_ah_b\neq 0$. Put $r=\mathrm{Max}\{h_a,h_{a+1},\dots,h_b\}$. Then $h(q)$ can be written *as above with positive integers $u_1, \dots, u_r$. These integers are uniquely determined as a multiset by $h(q)$. In particular* $\mathrm{Max}\{u_i\} = b - a + 1$ *and* $(a+b)/2$ *is the reflecting degree of* $h(q)$.

Proof. Notice that the ith summand (after divided by $1-q$) on the right-hand side of (3.1) is

$$p_i(q) := q^{d_i} + q^{d_i+1} + \cdots + q^{d_i+(u_i-1)},$$

where $d_i = (a+b+1-u_i)/2$. In particular, $p_1(q) = q^a + q^{a+1} + \cdots + q^b$. Details are left to the reader. □

Lemma 3.7. *Suppose that V is a finite graded module with the Hilbert function $h_V(q)$ over a graded Artinian algebra A. Let $u_1, \dots, u_r$ be the positive integers satisfying* (3.1) *in the above lemma. Let $g \in A_1$ and let $P(\times g) = n_1 \oplus \cdots \oplus n_{r'}$. Then g is a strong Lefschetz element for V if and only if $\{u_i\}$ and $\{n_i\}$ are the same as multisets.*

Proof. Assume that g is a strong Lefschetz element for V. Let

$$\mathcal{V}_1, \dots, \mathcal{V}_r$$

be the subspaces given in Lemma 3.3. Then $\mathcal{V}_1, \dots, \mathcal{V}_r$ correspond to Jordan blocks for $\times g \in \mathrm{End}(V)$. Thus $n_i = \dim \mathcal{V}_i$, if they are arranged in decreasing order. By construction of the bases for $\mathcal{V}_i$ one sees that $\{n_i\}$ and $\{u_i\}$ coincide as multisets. The converse follows from Lemma 2.4. □

Lemma 3.8. *Let A and B be graded Artinian K-algebras, where* $\mathrm{char}(K) = 0$. *Let V be a finite graded A-module and W a finite graded B-module.*

(1) *Let $z \in A_1$ and $z' \in B_1$. If $P(\times z) = d_1 \oplus d_2 \oplus \cdots \oplus d_r$ and $P(\times z') = f_1 \oplus f_2 \oplus \cdots \oplus f_{r'}$, then*

$$P\bigl(\times(z \otimes 1 + 1 \otimes z')\bigr) = \bigoplus_{i,j} \bigoplus_{k=1}^{\mathrm{Min}\{d_i, f_j\}} (d_i + f_j + 1 - 2k).$$

In particular

$$\mathrm{corank}\bigl(\times(z \otimes 1 + 1 \otimes z')\bigr) = \sum_{i,j} \mathrm{Min}\{d_i, f_j\}.$$

(2) *Let $\{u_i\}$ be the integers for $h_V(q)$ given in Lemma* 3.6 *and similarly $\{v_j\}$ for W. Then*

$$\mathrm{Sperner}(V \otimes W) = \sum_{i,j} \mathrm{Min}\{u_i, v_j\}.$$

Proof. (1) This can be proved in the same way as Proposition 10 in [2]. (2) $\mathrm{Sperner}(V \otimes W)$ is equal to the corank of the linear map $\times(z \otimes 1 + 1 \otimes z')$, provided that it is a strong Lefschetz element. This is the case if $P(\times z) = u_1 \oplus \cdots \oplus u_r$ and $P(\times z') = v_1 \oplus \cdots \oplus v_{r'}$, where r and r' are the Sperner numbers of V and W, respectively. □

Proposition 3.9. *Let K be a field of characteristic zero, let A be a graded Artinian K-algebra and let $V = \bigoplus_{i=a}^{b} V_i$ be a finite graded A-module with a symmetric unimodal Hilbert function, where $V_a \neq (0)$ and $V_b \neq (0)$ and $b - a \geqslant 1$. Let t be a new variable. Then the following conditions are equivalent.*

(1) *V has the SLP.*
(2) *$V \otimes_K K[t]/(t^\alpha)$ has the WLP as an $A \otimes_K K[t]$-module for all positive integers α.*
(3) *$V \otimes_K K[t]/(t^\alpha)$ has the WLP as an $A \otimes_K K[t]$-module for all $\alpha = 1, 2, \ldots, b - a$.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 3.4. (2) $\Rightarrow$ (3) is trivial. We show (3) $\Rightarrow$ (1). Put $r = \mathrm{Sperner}(V)$. Let $u_1, u_2, \ldots, u_r$ be the decreasing sequence of positive integers for $h_V(q)$ given in Lemma 3.6. By way of contradiction assume that no linear form of A is a strong Lefschetz element for V. Let $z \in A_1$ be any element. Suppose that $\times z \in \mathrm{End}(V)$ decomposes as $P(\times z) = n_1 \oplus n_2 \oplus \cdots \oplus n_{r'}$, where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{r'}$. By Lemma 3.7, the two multisets $\{u_i\}$ and $\{n_j\}$ are different since $\times z$ is not a strong Lefschetz element.

Let j be the least integer for which $u_j \neq n_j$. We claim that $u_j > n_j$. If $j = 1$ or $r = 1$, then the claim is obvious. Assume that $n_1 = u_1$. Let Z be a variable. We may regard V as a graded module over $K[Z]$ via the algebra map $K[Z] \to A$ defined by $Z \mapsto z$. Since $n_1 = u_1 = b - a + 1$, there is a Jordan block $\mathcal{V} \subset V$ for $\times z \in \mathrm{End}(V)$ such that the Hilbert function is

$$h_{\mathcal{V}}(q) = \left(q^a + q^{a+1} + \cdots + q^b\right) = \frac{1}{1-q}\left(q^{(a+b+1-u_1)/2} - q^{(a+b+1+u_1)/2}\right).$$

Let $V' \subset V$ be a $K[Z]$-module such that $V = V' \oplus \mathcal{V}$ and such that the Jordan decomposition of $\times z \in \mathrm{End}(V')$ is given by $n_2 \oplus \cdots \oplus n_{r'}$. Since the Hilbert function of V' is $h_V(q) - h_{\mathcal{V}}(q)$, which is symmetric unimodal with the maximal part one less than that of $h_V(q)$, proof of the claim is now complete by induction of r.

Now choose the least j such that $u_j > n_j$. Put $\alpha = n_j$. Since $u_1 = b - a + 1$ and $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_j > n_j = \alpha$, we see $\alpha \leqslant b - a$. Consider the $A \otimes K[t]$-module

$$W = V \otimes K[t]/\left(t^\alpha\right).$$

By Lemma 3.8(1), the Jordan decomposition of $\times(z \otimes 1 + 1 \otimes t) \in \mathrm{End}(W)$ is given by

$$P\left(\times(z \otimes 1 + 1 \otimes t)\right) = \bigoplus_{i=1}^{r'} \left[\bigoplus_{k=1}^{\mathrm{Min}\{n_i, \alpha\}} (n_i + \alpha + 1 - 2k) \right].$$

This implies

$$\dim W/(z \otimes 1 + 1 \otimes t)W = \sum_{i=1}^{r'} \mathrm{Min}\{n_i, \alpha\}.$$

On the other hand, since $\sum_{i=1}^{r'} n_i = \sum_{i=1}^{r} u_i$, we have that $\sum_{i=j+1}^{r'} n_i > \sum_{i=j+1}^{r} u_i$. Hence

$$\dim W/(z \otimes 1 + 1 \otimes u)W = \sum_{i=1}^{r'} \mathrm{Min}\{n_i, \alpha\}$$

$$\begin{aligned} &= \underbrace{\alpha + \cdots + \alpha}_{j} + n_{j+1} + \cdots + n_{r'} \\ &> \underbrace{\alpha + \cdots + \alpha}_{j} + u_{j+1} + \cdots + u_r \\ &\geqslant \sum_{i=1}^{r} \operatorname{Min}\{u_i, \alpha\}. \end{aligned}$$

By Lemma 3.8(2), we have

$$\sum_{i=1}^{r} \operatorname{Min}\{u_i, \alpha\} = \operatorname{Sperner}(W).$$

Thus we have shown that no linear form of $A \otimes K[t]/(t^{\alpha})$ is a weak Lefschetz element for W, which contradicts the assumption (3). □

Theorem 3.10. *Let K be a field of characteristic zero. Let A and B be graded Artinian K-algebras, and let V be a finite graded A-module and W a finite graded B-module. Suppose that the Hilbert functions of V and W are symmetric and unimodal. Then $V \otimes_K W$ has the SLP as an $A \otimes_K B$-module, if and only if V and W have the SLP.*

Proof. The "if" part was proved in Proposition 3.4. We prove the "only if" part. By way of contradiction assume that V does not have the SLP. By Proposition 3.9, the module $V' = V \otimes K[t]/(t^n)$ does not have the WLP for some integer $n > 0$.

Let $v_1, v_2, \ldots, v_s$ be the integers for $h_W(q)$ given in Lemma 3.6. We may assume that $v_s = 1$. In fact, if $v_s > 1$, we may replace W by $W \otimes K[u]/(u^m)$ where m is the number of maximal part in the Hilbert function of W.

Let $u_1, u_2, \ldots, u_r$ be the integers for $h_{V'}(q)$ given in Lemma 3.6. Let z be any linear form of $A \otimes K[t]$ and $P(\times z) = d_1 \oplus d_2 \oplus \cdots \oplus d_\alpha$ the Jordan decomposition of $\times z \in \operatorname{End}(V')$. Here note that $r \leqslant \alpha$. Also note that if $V' = \bigoplus_{1=1}^{\alpha} \mathcal{V}_i$ is a decomposition into Jordan blocks, then the Hilbert function of $\mathcal{V}_i$ is of the form

$$q^a + q^{a+1} + \cdots + q^b$$

for some integers a, b (depending on i). Bearing this in mind, we see that

$$\begin{aligned} u_1 + u_2 + \cdots + u_k &= \sum_{i=-\infty}^{\infty} \operatorname{Min}\bigl\{k, \text{ the coefficient of } q^i \text{ in } h_{V'}(q)\bigr\} \\ &\geqslant d_1 + d_2 + \cdots + d_k \end{aligned}$$

for all $k = 1, 2, \ldots, r$. Hence, noticing that $\sum_{i=1}^{r} u_i = \sum_{i=1}^{\alpha} d_i$, we see

$$u_k + u_{k+1} + \cdots + u_\alpha \leqslant d_k + d_{k+1} + \cdots + d_\alpha$$

for all $k = 1, 2, \ldots, \alpha$, where we have put $u_i = 0$ for $i > r$. Using this, we can easily verify

$$\sum_{i=1}^{r} \operatorname{Min}\{u_i, f\} \leqslant \sum_{i=1}^{\alpha} \operatorname{Min}\{d_i, f\} \tag{3.2}$$

for any integer $f > 0$. Let y be any linear form of B and $P(\times y) = f_1 \oplus f_2 \oplus \cdots \oplus f_\beta$ the Jordan decomposition of $\times y \in \operatorname{End}(W)$. Similarly we get

$$\sum_{j=1}^{s} \operatorname{Min}\{d, v_j\} \leqslant \sum_{j=1}^{\beta} \operatorname{Min}\{d, f_j\} \tag{3.3}$$

for any integer $d > 0$.

Recall that r is the Sperner number of V' and α is equal to the corank of the linear map $\times z : V' \to V'$. So we have $r < \alpha$ by Lemma 2.3 as z is not a weak Lefschetz element of V'. Using the inequalities (3.2) and (3.3) above, we obtain

$$\begin{aligned} \sum_{i,j} \operatorname{Min}\{u_i, v_j\} &= \left\{ \sum_{j=1}^{s-1} \left(\sum_{i=1}^{r} \operatorname{Min}\{u_i, v_j\} \right) \right\} + \sum_{i=1}^{r} \operatorname{Min}\{u_i, 1\} \\ &< \left\{ \sum_{j=1}^{s-1} \left(\sum_{i=1}^{\alpha} \operatorname{Min}\{d_i, v_j\} \right) \right\} + \sum_{i=1}^{\alpha} \operatorname{Min}\{d_i, 1\} \\ &= \sum_{i,j} \operatorname{Min}\{d_i, v_j\} \\ &\leqslant \sum_{i,j} \operatorname{Min}\{d_i, f_j\}. \end{aligned}$$

This shows that $z \otimes 1 + 1 \otimes y$ is not a weak Lefschetz element of $V' \otimes W$, since

$$\text{Sperner } (V' \otimes W) = \sum_{i,j} \operatorname{Min}\{u_i, v_j\}$$

and since

$$\dim(V' \otimes W)/(z \otimes 1 + 1 \otimes y)(V' \otimes W) = \sum_{i,j} \operatorname{Min}\{d_i, f_j\}.$$

This means that $V' \otimes W$ does not have the WLP. However, since $V \otimes W$ has the SLP, Proposition 3.9 implies that $V' \otimes W$ has the WLP. This is a contradiction. □

4. The strong Lefschetz property for $\mathrm{Gr}_{(z)}(A)$

We need some preparations for the proof of Theorem 4.6.

Notation and Remark 4.1. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra. For any linear form $z \in A_1$, consider the associated graded ring

$$\mathrm{Gr}_{(z)}(A) = A/(z) \oplus (z)/\left(z^2\right) \oplus \left(z^2\right)/\left(z^3\right) \oplus \cdots \oplus \left(z^{p-1}\right)/\left(z^p\right),$$

where p is the least integer such that $z^p = 0$. For a non-zero element $a \in A$ there is i such that $a \in (z^i) \setminus (z^{i+1})$. In this case we write $a^* \in \mathrm{Gr}_{(z)}(A)$ for the natural image of a in $(z^i)/(z^{i+1})$.

As is well known $\mathrm{Gr}_{(z)}(A)$ is endowed with a commutative ring structure. The multiplication in $\mathrm{Gr}_{(z)}(A)$ is given by

$$\left(a + \left(z^{i+1}\right)\right)\left(b + \left(z^{j+1}\right)\right) = ab + \left(z^{i+j+1}\right),$$

where $a \in (z^i) \setminus (z^{i+1})$ and $b \in (z^j) \setminus (z^{j+1})$. Note that

$$\left(z^i\right)/\left(z^{i+1}\right) \cong z^i A_0 \oplus \left(z^i A_1/z^{i+1} A_0\right) \oplus \left(z^i A_2/z^{i+1} A_1\right) \oplus \cdots \oplus \left(z^i A_{c-i}/z^{i+1} A_{c-i-1}\right)$$

as graded vector spaces for all $i = 0, 1, \ldots, p-1$. Furthermore note that $\mathrm{Gr}_{(z)}(A)$ inherits a grading from A. More precisely, $\mathrm{Gr}_{(z)}(A) = \bigoplus_{i=0}^{c} [\mathrm{Gr}_{(z)}(A)]_i$, where

$$\left[\mathrm{Gr}_{(z)}(A)\right]_i \cong (A_i/zA_{i-1}) \oplus \left(zA_{i-1}/z^2 A_{i-2}\right) \oplus \cdots \oplus \left(z^{i-1} A_1/z^i A_0\right) \oplus z^i A_0$$

as graded vector spaces for all $i = 0, 1, \ldots, c$. Hence, $\mathrm{Gr}_{(z)}(A)$ and A have the same Hilbert function.

Notation and Remark 4.2. Let $S = K[Y, Z]$ be the polynomial ring in two variables over an infinite field K. Here we regard S as a standard graded K-algebra with $\deg(Y) = \deg(Z) = 1$. Let V be a finite graded S-module. Write $V \cong F/N$, where

$$F = \bigoplus_{j=1}^{s} \left[K[Y, Z](-d_j)\right]$$

is a free graded S-module of rank s and N a graded submodule of F. An element $\mathbf{f} \in F$ can be written uniquely as $\mathbf{f} = \mathbf{a}_0 + \mathbf{a}_1 Z + \cdots + \mathbf{a}_d Z^d$ with $\mathbf{a}_i \in \oplus^s K[Y]$. Denote by $\mathrm{In}'(\mathbf{f})$ the term $\mathbf{a}_j Z^j$ for the minimal j such that $\mathbf{a}_j \neq \mathbf{0}$. Furthermore we define $\mathrm{In}'(N)$ to be the graded submodule of F generated by the set $\{\mathrm{In}'(\mathbf{f})\}$, where $\mathbf{f}$ runs over homogeneous forms of N. Put

$$\mathrm{Gr}_{(Z)}(V) = V/ZV \oplus ZV/Z^2 V \oplus Z^2 V/Z^3 V \oplus \cdots.$$

Then, we have that $\mathrm{Gr}_{(Z)}(V) \cong F/\mathrm{In}'(N)$ as finite graded S-modules. Suppose that

$$F_1 \xrightarrow{\phi} F_0 \longrightarrow V \longrightarrow 0$$

is a finite presentation of V. Let $\Delta_{\mathrm{MAX}}(V)$ be the ideal of S generated by the maximal minors of ϕ. As a Fitting ideal it does not depend on the choice of finite presentation.

The following is a key to the proof of Theorem 4.6.

Proposition 4.3. *(See [4, Proposition 3.3].) With the same notation as above, let g be a general linear form of S. Assume that K is an infinite field. Then*

$$\dim_K V/gV \leqslant \dim_K \mathrm{Gr}_{(Z)}(V)/g\mathrm{Gr}_{(Z)}(V).$$

Lemma 4.4. *Let $A=\bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra with the WLP (respectively SLP) and let $y, z \in A_1$ be two linear forms of A. If y is a weak Lefschetz element (respectively strong Lefschetz element) for A, then so is $y+\lambda z$ for a general element $\lambda \in K$.*

Proof. Let M and N be the square matrices of $\times y \in \mathrm{End}(A)$ and $\times z \in \mathrm{End}(A)$ for a same basis of A, respectively. Suppose that y is a strong Lefschetz element for A. By assumption and Lemma 2.4, we have that $\mathrm{rank}(M^k)=\dim A-\mathrm{SP}_k(A)$ for all $k=1,2,\dots,c$. We would like to show that $\mathrm{rank}(M+\lambda N)^k=\dim A-\mathrm{SP}_k(A)$ for a general element $\lambda\in K$ and all $k=1,2,\dots,c$. Let Q_0 be an $n\times n$ square submatrix of M^k such that $\mathrm{rank}(Q_0)=\mathrm{rank}\, M^k$ and Q_0 is regular. Put

$$P=(M+\lambda N)^k=M^k+M^{k-1}N\lambda+\cdots+MN^{k-1}\lambda^{k-1}+N^k\lambda^k,$$

and let P' and Q_i be the submatrices of P and $M^{k-i}N^i$ $(1\leqslant i\leqslant k)$, respectively, obtained by deleting the same rows and columns as Q_0. Obviously

$$P'=Q_0+Q_1\lambda+\cdots+Q_k\lambda^k.$$

Hence, we have that

$$\det(P')=\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2+\cdots+\alpha_{nk}\lambda^{nk}$$

for some $\alpha_i\in K$ $(0\leqslant i\leqslant nk)$, where $\alpha_0=\det(Q_0)$. Thus, noting that $\det(Q_0)\neq 0$, our assertion follows from Lemmas 2.4 and 2.5.

Next suppose that y is a weak Lefschetz element for A. By assumption and Lemma 2.3, we have that $\mathrm{rank}(M)=\mathrm{CoSperner}(A)$. We would like to show that $\mathrm{rank}(M+\lambda N)=\mathrm{CoSperner}(A)$ for a general element $\lambda\in K$. This also follows by the same idea as above. □

Notation and Remark 4.5. Let $A=\bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra, let $z\in A_1$ be a linear form of A and let $P(\times z)=n_1\oplus n_2\oplus\cdots\oplus n_r$ be the Jordan decomposition of $\times z\in\mathrm{End}(A)$. Now we can take homogeneous elements $a_i\in A\setminus(z)$ $(i=1,2,\dots,r)$ such that the set

$$\bigcup_{i=1}^{r}\{a_i, a_iz, a_iz^2,\dots,a_iz^{n_i-1}\} \tag{4.1}$$

is a basis for A as a vector space, that is, the matrix of $\times z\in\mathrm{End}(A)$ for the basis above coincides with the Jordan canonical from of $\times z$. Then it is easy to check that the set

$$\bigcup_{i=1}^{r}\{a_i^*, a_i^*z^*, a_i^*(z^*)^2,\dots,a_i^*(z^*)^{n_i-1}\}$$

is a basis for $\mathrm{Gr}_{(z)}(A)$. Hence the Jordan canonical form of $\times z^*\in\mathrm{End}(\mathrm{Gr}_{(z)}(A))$ is the same as that of $\times z\in\mathrm{End}(A)$.

Theorem 4.6. *Let K be a field of characteristic zero and let A be a graded Artinian K-algebra. Then A has the WLP (respectively SLP) if and only if* $\mathrm{Gr}_{(z)}(A)$ *has the WLP (respectively SLP) for some linear form z of A.*

Proof. ($\Rightarrow$) Let z be a Lefschetz element of A. Recall that A and $\mathrm{Gr}_{(z)}(A)$ have the same Hilbert function. Then Remark 4.5 and Lemma 3.2 show that z^* is a Lefschetz element of $\mathrm{Gr}_{(z)}(A)$.

($\Leftarrow$) *Step* 1: First we prove that if $\mathrm{Gr}_{(z)}(A)$ has the WLP then so does A. Let g be a weak Lefschetz element of $\mathrm{Gr}_{(z)}(A)$. Since $[\mathrm{Gr}_{(z)}(A)]_1 = A_1/zA_0 \oplus zA_0$, g can be written as $g = y^* + \lambda_0 z^*$ for some $y \in A_1$ and $\lambda_0 \in K$. Let $S = K[Y, Z]$ be the polynomial ring in two variables. Define the algebra homomorphism $S \to \mathrm{End}(A)$ by $Y \mapsto \times y$ and $Z \mapsto \times z$. Then we may regard A as a finite graded S-module. From Proposition 4.3, it follows that

$$\dim A/(y + \lambda z)A \leqslant \dim \mathrm{Gr}_{(z)}(A)/(y^* + \lambda z^*)\mathrm{Gr}_{(z)}(A) \tag{4.2}$$

for a general element $\lambda \in K$. Furthermore, it follows from Lemmas 2.3 and 4.4 that

$$\dim \mathrm{Gr}_{(z)}(A)/(y^* + \lambda z^*)\mathrm{Gr}_{(z)}(A) = \mathrm{Sperner}\big(\mathrm{Gr}_{(z)}(A)\big)$$

for a general element $\lambda \in K$.

On the other hand, since $\mathrm{Sperner}(\mathrm{Gr}_{(z)}(A)) = \mathrm{Sperner}(A)$ and since $\mathrm{Sperner}(A) \leqslant \dim A/(y + \lambda z)A$ by Lemma 2.5, we have

$$\mathrm{Sperner}(A) = \dim A/(y + \lambda z)A,$$

proving that A has the WLP.

Step 2: It still remains to show that A has the SLP assuming that $\mathrm{Gr}_{(z)}(A)$ has the SLP. Let t be a new variable and let $\widetilde{A} = A[t]/(t^\alpha)$, where α is any positive integer. Then, since we have

$$\mathrm{Gr}_{(z)}(\widetilde{A}) \cong \mathrm{Gr}_{(z)}(A) \otimes_K K[t]/\big(t^\alpha\big)$$

and since the SLP is preserved by tensor product, it follows that $\mathrm{Gr}_{(z)}(\widetilde{A})$ has the SLP. By Step 1, this implies that $\widetilde{A}$ has the WLP for all $\alpha > 0$. Hence the SLP of A follows by Proposition 3.9. □

5. Central simple modules and the strong Lefschetz property

In this section we prove Theorems 5.2 and 5.4.

Definition and Remark 5.1. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra, let $z \in A_1$ be a linear form of A and let $P(\times z) = n_1 \oplus n_2 \oplus \cdots \oplus n_r$ be the Jordan decomposition of $\times z \in \mathrm{End}(A)$. Furthermore, let $(f_1, f_2, \ldots, f_s)$ be the finest subsequence of $(n_1, n_2, \ldots, n_r)$ such that $f_1 > f_2 > \cdots > f_s$. Then we rewrite the same Jordan decomposition $P(\times z)$ as

$$P(\times z) = n_1 \oplus n_2 \oplus \cdots \oplus n_r = \underbrace{f_1 \oplus \cdots \oplus f_1}_{m_1} \oplus \underbrace{f_2 \oplus \cdots \oplus f_2}_{m_2} \oplus \cdots \oplus \underbrace{f_s \oplus \cdots \oplus f_s}_{m_s}.$$

We call the graded A-module

$$U_i = \frac{(0 : z^{f_i}) + (z)}{(0 : z^{f_{i+1}}) + (z)}$$

the *ith central simple module* of (A, z), with $1 \leqslant i \leqslant s$ and $f_{s+1} = 0$. Note that these are defined for a pair of the algebra A and a linear form $z \in A_1$. By the definition, it is easy to see that the modules $U_1, U_2, \ldots, U_s$ are the non-zero terms of the successive quotients of the descending chain of ideals

$$A = \left(0 : z^{f_1}\right) + (z) \supset \left(0 : z^{f_1 - 1}\right) + (z) \supset \cdots \supset (0 : z) + (z) \supset (z).$$

For $1 \leqslant i \leqslant s$, define $\widetilde{U_i}$ by $\widetilde{U_i} = U_i \otimes_K K[t]/(t^{f_i})$.

Theorem 5.2. *Let K be a field of characteristic zero and let A be a graded Artinian K-algebra. Then the following conditions are equivalent.*

(i) *A has the SLP.*
(ii) *There exists a linear form z of A such that all the central simple modules U_i of (A, z) have the SLP and the reflecting degree of the Hilbert function of $\widetilde{U_i}$ coincides with that of A for $i = 1, 2, \ldots, s$.*

Proof. (i) $\Rightarrow$ (ii): Assume that A has the SLP and z is a strong Lefschetz element. Put $\overline{A}_i = A_i / zA_{i-1}$. Then we may write

$$A/(z) = \bigoplus_{i=0}^{c'} \overline{A_i}$$

where c' is the largest integer such that $(A/(z))_{c'} \neq 0$. Then, noting that A has the SLP, one sees easily that (A, z) has $c' + 1$ central simple modules $U_1, \ldots, U_{c'+1}$ and that $U_i \cong \overline{A_{i-1}}$. This shows that U_i has only one non-trivial graded piece concentrated at the degree $i - 1$. Hence U_i has the SLP for trivial reasons. Also it is easy to show that the reflecting degree of the Hilbert function of $\widetilde{U_i}$ coincides with that of A for all $i = 1, 2, \ldots, c' + 1$.

(ii) $\Rightarrow$ (i): By Theorem 4.6 it suffices to prove that $\mathrm{Gr}_{(z)}(A)$ has the SLP. We divide the proof into three steps.

Step 1: Choose $g \in A_1$ general enough so that g is a strong Lefschetz element of U_i for any $i = 1, 2, \ldots, s$. Let $m_i = \dim U_i$ $(1 \leqslant i \leqslant s)$. Now we take a basis of U_i as a vector space for all $1 \leqslant i \leqslant s$,

$$\{\overline{e_{i1}}, \overline{e_{i2}}, \ldots, \overline{e_{im_i}}\},$$

where $e_{ij} \in (0 : z^{f_i}) + (z)$, $j = 1, \ldots, m_i$. Then the set

$$\left\{\overline{e_{ij}} \otimes \bar{t}^k \,\middle|\, 1 \leqslant j \leqslant m_i,\ 0 \leqslant k \leqslant f_i - 1\right\}$$

is a basis of $\widetilde{U}_i = U_i \otimes_K K[t]/(t^{f_i})$ for all i such that $1 \leqslant i \leqslant s$. Put $\widetilde{U} = \bigoplus_{i=1}^{s} \widetilde{U}_i$. The set

$$\bigcup_{i=1}^{s} \left\{ \overline{e_{ij}} \otimes \bar{t}^k \;\middle|\; 1 \leqslant j \leqslant m_i,\ 0 \leqslant k \leqslant f_i - 1 \right\} \tag{5.1}$$

is a basis of $\widetilde{U}$. We may consider $\widetilde{U}$ as a graded module over $A \otimes K[t]$. Here we calculate a matrix of the multiplication $\times(g \otimes 1 + 1 \otimes t) : \widetilde{U} \to \widetilde{U}$ as an endomorphism. Let P_i be the square matrix of $\times(g \otimes 1 + 1 \otimes t) : \widetilde{U}_i \to \widetilde{U}_i$ using the basis above. Since $(g \otimes 1 + 1 \otimes t)\widetilde{U}_i \subset \widetilde{U}_i$, a matrix for $\times(g \otimes 1 + 1 \otimes t) \in \operatorname{End}(\widetilde{U})$ is of the following form:

$$P = \begin{bmatrix} P_1 & & & 0 \\ & P_2 & & \\ & & \ddots & \\ 0 & & & P_s \end{bmatrix}.$$

Hence it follows that

$$P^h = \begin{bmatrix} P_1^h & & & 0 \\ & P_2^h & & \\ & & \ddots & \\ 0 & & & P_s^h \end{bmatrix}$$

for all h.

Step 2: Let g^* be the initial form of g in $\operatorname{Gr}_{(z)}(A)$. We calculate a matrix of the multiplication $\times g^* : \operatorname{Gr}_{(z)}(A) \to \operatorname{Gr}_{(z)}(A)$. First we note that the set

$$\bigcup_{i=1}^{s} \left\{ \left(e_{ij}^*\right)(z^*)^k \;\middle|\; 1 \leqslant j \leqslant m_i,\ 0 \leqslant k \leqslant f_i - 1 \right\} \tag{5.2}$$

is a basis of $\operatorname{Gr}_{(z)}(A)$. Let V_i be the subspace of $A/(z)$ in $\operatorname{Gr}_{(z)}(A)$ which is generated by $\{e_{i1}^*, e_{i2}^*, \ldots, e_{im_i}^*\}$ for all $i = 1, 2, \ldots, s$. Furthermore, let V_i^* be the subspace of $\operatorname{Gr}_{(z)}(A)$ which is generated by

$$\left\{ \left(e_{ij}^*\right)(z^*)^k \;\middle|\; 1 \leqslant j \leqslant m_i,\ 0 \leqslant k \leqslant f_i - 1 \right\}$$

for all $i = 1, 2, \ldots, s$. Then, since $g^* V_i \subset \bigoplus_{j=i}^{s} V_j$ for all $i = 1, 2, \ldots, s$, we have that $g^* V_i^* \subset \bigoplus_{j=i}^{s} V_j^*$. Hence, a matrix of $\times(g^* + z^*) : \operatorname{Gr}_{(z)}(A) \to \operatorname{Gr}_{(z)}(A)$ is of the following form:

$$Q = \begin{bmatrix} P_1 & & & * \\ & P_2 & & \\ & & \ddots & \\ 0 & & & P_s \end{bmatrix}.$$

Thus, we obtain that

$$Q^h = \begin{bmatrix} P_1^h & & & * \\ & P_2^h & & \\ & & \ddots & \\ 0 & & & P_s^h \end{bmatrix}$$

for all h.

Step 3: Note that $\widetilde{U}$ and $\mathrm{Gr}_{(z)}(A)$ have the same Hilbert function, since $\deg \overline{e_{ij}} = \deg e_{ij}^*$ for all i and j and since the sets (5.1) and (5.2) are bases for $\widetilde{U}$ and $\mathrm{Gr}_{(z)}(A)$, respectively.

Next we claim that $\widetilde{U} = \oplus \widetilde{U}_i$ has the SLP as an $A \otimes_K K[t]$-module. Indeed, by Proposition 3.4, every $\widetilde{U}_i$ has the SLP as an $A \otimes_K K[t]$-module with $g \otimes 1 + 1 \otimes t$ a strong Lefschetz element. By assumption the reflecting degree of $h_{\widetilde{U}_i}(q)$ coincides with that of $h_{\widetilde{U}}(q)$ for any i. This proves that the element $g \otimes 1 + 1 \otimes t$ is a strong Lefschetz element for $\widetilde{U} = \bigoplus \widetilde{U}_i$.

Now if $\operatorname{rank} P^h \leqslant \operatorname{rank} Q^h$ for all h, it implies that $\mathrm{Gr}_{(z)}(A)$ has the SLP by Lemma 2.5(4). So we prove that $\operatorname{rank} P^h \leqslant \operatorname{rank} Q^h$ for all h.

Let P'_{ih} be a square submatrix of P_i^h such that P'_{ih} is of full rank and $\operatorname{rank} P_i^h = \operatorname{rank} P'_{ih}$ for all i and h. Then we have

$$\begin{aligned} \operatorname{rank} P^h &= \sum_{i=1}^{s} \operatorname{rank} P_i^h \\ &= \sum_{i=1}^{s} \operatorname{rank} P'_{ih} \\ &= \operatorname{rank} P'_h, \end{aligned}$$

where

$$P'_h = \begin{bmatrix} P'_{1h} & & & 0 \\ & P'_{2h} & & \\ & & \ddots & \\ 0 & & & P'_{sh} \end{bmatrix}.$$

Let Q'_h be the square submatrix of Q^h consisting of the same rows and columns as P'_h, so that Q'_h is of the following form:

$$Q'_h = \begin{bmatrix} P'_{1h} & & & * \\ & P'_{2h} & & \\ & & \ddots & \\ 0 & & & P'_{sh} \end{bmatrix}.$$

Then, since $\det Q'_h = \prod_{i=1}^{s} \det P'_{ih} \neq 0$, it follows that $\operatorname{rank} Q'_h = \operatorname{rank} P'_h = \operatorname{rank} P^h$. This means that $\operatorname{rank} P^h \leqslant \operatorname{rank} Q^h$ for all h. $\square$

Proposition 5.3. *Let K be any field and suppose that A is a graded Artinian Gorenstein K-algebra. Suppose that z is a linear form of A and let*

$$P(\times z) = \underbrace{f_1 \oplus \cdots \oplus f_1}_{m_1} \oplus \underbrace{f_2 \oplus \cdots \oplus f_2}_{m_2} \oplus \cdots \oplus \underbrace{f_s \oplus \cdots \oplus f_s}_{m_s}$$

be as in Notation 5.1. *Furthermore let U_i be the ith central simple module of (A, z). Then, for any i with $1 \leqslant i \leqslant s$, the graded module U_i has a symmetric Hilbert function and the module $\widetilde{U}_i = U_i \otimes K[t]/(t^{f_i})$ has a symmetric Hilbert function with the same reflecting degree as that of A.*

Proof. Let c be the socle degree of A. We induct on c. First note that $A/0:z$ is Gorenstein and the socle degree of $A/0:z$ is $c-1$, as long as $A/0:z \neq 0$. To see this consider the exact sequence

$$0 \longrightarrow (A/0:z)(-1) \longrightarrow A \longrightarrow A/zA \longrightarrow 0,$$

where the first map sends 1 to z. Since the zero ideal of A is irreducible, the sequence shows that the zero ideal of $A/0:z$ is irreducible also. This implies that $A/0:z$ is a Gorenstein algebra (cf. [8]). Recall that A_c, the socle, is the unique minimal ideal of A. Hence the ideal $zA \cong (A/0:z)(-1)$ contains A_c, from which it follows that $(A/zA)_c = 0$. This shows that $zA_{c-1} \neq 0$. On the other hand, obviously, it holds that $zA_c = 0$. Thus the socle degree of $A/0:z$ is precisely $c-1$.

If $A/0:z = 0$, then $s = 1$, $f_1 = 1$ and $U_1 = \widetilde{U}_1 = A$, and the assertions are clear from the fact that A has a symmetric Hilbert function. Now we assume that $A/0:z \neq 0$. Note that the Jordan-block-size for $\times\bar{z} \in \operatorname{End}(A/0:z)$ is given by

$$P(\times\bar{z}) = \underbrace{f_1 - 1 \oplus \cdots \oplus f_1 - 1}_{m_1} \oplus \underbrace{f_2 - 1 \oplus \cdots \oplus f_2 - 1}_{m_2} \oplus \cdots \oplus \underbrace{f_s - 1 \oplus \cdots \oplus f_s - 1}_{m_s}.$$

First assume that $f_s > 1$. Then $(A/0:z, \bar{z})$ and (A, z) have the same central simple modules. Thus U_i has the symmetric Hilbert function for all i such that $1 \leqslant i \leqslant s$. Moreover the reflecting degrees of $A/0:z$ and A differ by $1/2$. Hence the second assertion is also clear. Now assume that $f_s = 1$. It is possible to regard $U_1, \ldots, U_{s-1}$ as the central simple modules for $(A/0:z, \bar{z})$. Thus, by induction hypothesis, $U_i \otimes K[t]/(t^{f_i - 1})$, for all $i = 1, 2, \ldots, s-1$, have the same reflecting degree as that of $A/0:z$. Hence the same is true for A and $\widetilde{U}_i$ with $i = 1, 2, \ldots, s-1$. It remains to prove that, with the assumption $f_s = 1$, the modules U_s and $\widetilde{U}_s$ have symmetric Hilbert functions, and $\widetilde{U}_s$ with the same reflecting degree as that of A. Since $f_s = 1$, we have $\widetilde{U}_s = U_s$. Notice that

$$h_A(q) = \sum_{j=1}^{s} h_{\widetilde{U}_j}(q).$$

We may apply the induction hypothesis to the first $s-1$ summands. Hence the last summand $h_{\widetilde{U}_s}(q)$ also has the reflecting degree equal to that of A. □

Theorem 5.4. *Let K be a field of characteristic zero and let A be a graded Artinian Gorenstein K-algebra. Then the following conditions are equivalent.*

(i) *A has the SLP.*
(ii) *There exists a linear form z of A such that all the central simple modules U_i of (A,z) have the SLP.*

Proof. This follows from Theorem 5.2 and Proposition 5.3. □

6. Finite free extensions of a graded Artinian K-algebra

Theorem 6.1 below is an extension of Theorem 28 in [3]. We can now give another proof simplifying very much the proof given in [2] and [3].

Theorem 6.1. *Let K be a field of characteristic zero, let B be a graded Artinian K-algebra and let A be a finite flat algebra over B such that the algebra map $\varphi : B \to A$ preserves grading. Assume that both B and A/mA have the SLP, where m is the maximal ideal of B. Then A has the SLP.*

First we prove a lemma.

Lemma 6.2. *We use the same notation as in Theorem* 6.1. *Let z' be any linear form of B and put $z = \varphi(z')$. Let U_i' and U_i be the ith central simple modules of (B,z') and (A,z), respectively. Then $U_i' \otimes_B A \cong U_i$.*

Proof. By assumption, we can write $A \cong Be_1 \oplus Be_2 \oplus \cdots \oplus Be_k$ for some homogeneous elements $e_i \in A$. Let

$$P(\times z') = \underbrace{f_1 \oplus \cdots \oplus f_1}_{m_1} \oplus \underbrace{f_2 \oplus \cdots \oplus f_2}_{m_2} \oplus \cdots \oplus \underbrace{f_s \oplus \cdots \oplus f_s}_{m_s}$$

be the Jordan decomposition of $\times z' \in \operatorname{End}(B)$. Then it follows immediately that

$$P(\times z) = \underbrace{f_1 \oplus \cdots \oplus f_1}_{m_1 \times k} \oplus \underbrace{f_2 \oplus \cdots \oplus f_2}_{m_2 \times k} \oplus \cdots \oplus \underbrace{f_s \oplus \cdots \oplus f_s}_{m_s \times k}.$$

Hence, noting that

$$\left(0 :_A z^m\right) + zA = \left\{\left(0 :_B (z')^m\right) + z'B\right\}e_1 \oplus \cdots \oplus \left\{\left(0 :_B (z')^m\right) + z'B\right\}e_k$$

for all integers $m > 0$, we have that

$$\begin{aligned} U_i' \otimes_B A &\cong \left[\left\{\left(0 :_B (z')^{f_i}\right) + z'B\right\}/\left\{\left(0 :_B (z')^{f_{i+1}}\right) + z'B\right\}\right] \otimes_B \left(\bigoplus_{i=1}^{k} Be_i\right) \\ &\cong \bigoplus_{i=1}^{k}\left[\left\{\left(0 :_B (z')^{f_i}\right) + z'B\right\}/\left\{\left(0 :_B (z')^{f_{i+1}}\right) + z'B\right\}\right]e_i \end{aligned}$$

$$\cong \{(0:_A z^{f_i}) + zA\}/\{(0:_A z^{f_{i+1}}) + zA\}$$
$$\cong U_i$$

for all $i = 1, 2, \ldots, s$. □

Proof of Theorem 6.1. Let z' be a strong Lefschetz element of B and put $z = \varphi(z')$. From the proof (i) ⇒ (ii) of Theorem 5.2, every central simple module U_i' of (B, z') has only one non-trivial graded piece. Hence U_i' has the SLP. Also, since U_i' is annihilated by m, we have by Lemma 6.2 that $U_i' \otimes_K A/mA \cong U_i$, where U_i is the ith central simple module of (A, z). Thus, using our assumption that A/mA has the SLP, it follows by Proposition 3.4 that every central simple module of (A, z) has the SLP.

To proceed with the proof, we use the same notation as in the proof of Lemma 6.2. Fix i such that $1 \leqslant i \leqslant s$ and put

$$\widetilde{U}_i = U_i \otimes_K K[t]/(t^{f_i}) \quad \text{and} \quad \widetilde{U}_i' = U_i' \otimes_K K[t]/(t^{f_i}).$$

Let a, b be the initial and end degrees of the Hilbert function of $\widetilde{U}_i'$ so that

$$h_{\widetilde{U}_i'}(q) = h_a q^a + (\text{mid terms}) + h_b q^b.$$

Similarly let

$$h_B(q) = 1 + (\text{mid terms}) + q^c, \qquad h_{A/mA}(q) = 1 + (\text{mid terms}) + q^d$$

be the Hilbert functions of B and A/mA. Since $h_{\widetilde{U}_i}(q) = h_{\widetilde{U}_i'}(q) h_{A/mA}(q)$ and $h_A(q) = h_B(q) h_{A/mA}(q)$, the Hilbert functions of $\widetilde{U}_i$ and A are of the following form:

$$h_{\widetilde{U}_i}(q) = h_a q^a + (\text{mid terms}) + h_b q^{b+d},$$
$$h_A(q) = 1 + (\text{mid terms}) + q^{c+d}.$$

Since the reflecting degree of $h_{\widetilde{U}_i'}(q)$ coincides with that of $h_B(q)$, we have $(a + b)/2 = c/2$. Hence we obtain $(a + b + d)/2 = (c + d)/2$. This means that the reflecting degree of $h_{\widetilde{U}_i}(q)$ coincides with that of $h_A(q)$ independent of i. Thus by Theorem 5.2 the proof is finished. □

Example 6.3. In general the converse of Theorem 6.1 is not true. We give such an example. Let $A = K[x_1, x_2]/(e_1^2, e_2^2)$ and $B = K[e_1, e_2]/(e_1^2, e_2^2)$, where $e_1 = x_1 + x_2$ and $e_2 = x_1 x_2$. Note that A has the standard grading, but B does not. By Lemma 7.5 below, A is a finite free module over B. Furthermore, by Proposition 4.4 of [1], both A and A/mA have the SLP. However, B does not have the SLP. In fact e_1 is the only candidate for a strong Lefschetz element for B while we have $h_B(q) = 1 + q + q^2 + q^3$ and $\overline{e_1}^2 = 0$ in B.

Remark 6.4. The converse of Theorem 6.1 is true under some assumptions. We use the same notation as in Theorem 6.1. Assume that there is a linear form z' of B such that

(i) all central simple modules of $(A, \varphi(z'))$ have the SLP,

(ii) every central simple module of (B, z') has a symmetric unimodal Hilbert function and
(iii) A/mA has a symmetric unimodal Hilbert function.

Then, using Theorems 5.2 and 3.10 and the same idea as in the proof of Theorem 6.1, we can easily prove that B and A/mA have the SLP.

7. Complete intersections defined by power sums of consecutive degrees

Using Theorem 6.1 we prove the following:

Proposition 7.1. *Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field K of characteristic zero with $\deg(x_i) = 1$ for all i. Let a be a positive integer. Let*

$$I = \big(p_a(x_1, \dots, x_n), p_{a+1}(x_1, \dots, x_n), \dots, p_{a+n-1}(x_1, \dots, x_n)\big),$$

where $p_d = x_1^d + x_2^d + \dots + x_n^d$ is the power sum symmetric function of degree d. Then $A = R/I$ is a complete intersection and has the SLP.

The proof is given at the end of this section after a series of lemmas.

Lemma 7.2. *With the same notation as in Proposition 7.1, the ideal I is a complete intersection.*

Proof. The well-known identity

$$p_m = -\sum_{j=1}^{n}(-1)^j e_j p_{m-j}$$

where e_i is the elementary symmetric function of degree i, implies that $p_m \in I$ for all $m > a + n - 1$. Hence we have $(p_m, p_{2m}, \dots, p_{nm}) \subset I$. It is easy to see that $(p_m, p_{2m}, \dots, p_{nm})$ is a complete intersection and hence so is I. □

Lemma 7.3. *Let $C = \bigoplus_{i=0}^{c} C_i$ be a graded Artinian K-algebra with the SLP, where $C_c \neq (0)$. Let $B = \bigoplus_{i=0}^{c} B_i$ be a graded subalgebra of C, where $B_c \neq (0)$. If B contains a strong Lefschetz element for C and if the Hilbert function of B is symmetric, then B has the SLP.*

Proof. Suppose $z \in B$ is a strong Lefschetz element for C. Then the multiplication $\times z^{c-2i} : B_i \to B_{c-i}$ is injective for all $i \leqslant [c/2]$. Hence, noting that $\dim B_i = \dim B_{c-i}$, we have that $\times z^{c-2i}$ is bijective for all i. □

Notation and Remark 7.4. Let K be a field of characteristic zero and $R = K[x_1, x_2, \dots, x_n]$ the polynomial ring over K with $\deg(x_i) = 1$ for all i. Let $e_i = e_i(x_1, \dots, x_n)$ be the elementary symmetric function of degree i in R, i.e.,

$$e_i(x_1, \dots, x_n) = \sum_{j_1 < j_2 < \dots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}$$

for all $i = 1, 2, \dots, n$.

We denote by S the subring $S = K[e_1, e_2, \dots, e_n]$ of R. Put $S_j = S \cap R_j$. Then we have $S = \bigoplus_{j \geqslant 0} S_j$, which we regard as defining the grading of S, so that the natural injection $S \subset R$ is a grade-preserving algebra map. Let $\mathcal{H}$ be the set of harmonic functions in R. Namely, $\mathcal{H}$ is the vector space spanned by the partial derivatives of the alternating polynomial $\prod_{i<j}(x_i - x_j)$. The following are well known.

(1) The ring S contains all symmetric functions of R.
(2) $\mathcal{H}$ is isomorphic to $R/(e_1, \dots, e_n)$ as a graded vector space.
(3) The map

$$\mathcal{H} \otimes_K S \ni h \otimes e \mapsto he \in R$$

is an isomorphism as graded vector spaces. Hence it follows that R is a finite free module over S.

Lemma 7.5. *With the same notation as above, let $f_1, f_2, \dots, f_n \in S$ be homogeneous polynomials. Put $I = (f_1, f_2, \dots, f_n)R$ and $J = (f_1, f_2, \dots, f_n)S$. Then R/I is a finite free module over S/J.*

Proof. This follows immediately from the last statement of Notation and Remark 7.4. □

Lemma 7.6. *The Artinian complete intersection*

$$B = K[e_1, e_2, \dots, e_n]/(p_a, p_{a+1}, \dots, p_{a+n-1})$$

has the SLP, where $p_d = x_1^d + x_2^d + \cdots + x_n^d$ is the power sum of degree d.

Proof. Since

$$h_B(q) = \frac{(1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} \quad \text{and} \quad q^{an-n} h_B\left(q^{-1}\right) = h_B(q),$$

B has a symmetric Hilbert function and the socle degree of B is equal to $an - n$. Put

$$C = K[x_1, x_2, \dots, x_n]/\left(x_1^a, x_2^a, \dots, x_n^a\right).$$

One notices that B and C have the same socle degree.

Next we show that B is naturally a graded subring of C. The symmetric group $G = S_n$ acts on C by permutation of the variables. Consider the exact sequence

$$0 \longrightarrow \left(x_1^a, \dots, x_n^a\right) \longrightarrow R \longrightarrow C \longrightarrow 0.$$

Since $\mathrm{char}(K) = 0$, we have the exact sequence

$$0 \longrightarrow \left(x_1^a, \dots, x_n^a\right)^G \longrightarrow R^G \longrightarrow C^G \longrightarrow 0,$$

where M^G denotes the invariant subspace for any G-module M. Note that $R^G = S$. Hence it follows that $C^G \cong S/((x_1^a, \dots, x_n^a) \cap S)$. We would like to prove that

$$\left(x_1^a, \dots, x_n^a\right) \cap S = (p_a, p_{a+1}, \dots, p_{a+n-1}),$$

or equivalently, the natural surjection

$$\psi : B \longrightarrow S/\left(\left(x_1^a, \dots, x_n^a\right) \cap S\right) \cong C^G$$

is an isomorphism. By way of contradiction assume that $\mathrm{Ker}(\psi) \neq (0)$. Since B is Gorenstein, B_{an-n} is the unique minimal ideal of B and it should be contained in $\mathrm{Ker}(\psi)$. Hence the socle degree of $B/\mathrm{Ker}(\psi)$ is less than $an - n$. However, since the element $(x_1 x_2 \cdots x_n)^{a-1} \in C$ lies in C^G, the socle degree of C^G is equal to $an - n$. This is a contradiction. We have proved $\mathrm{Ker}(\psi) = (0)$.

It is known that the image of e_1 in C is a strong Lefschetz element for C (cf. Corollary 3.5 in [9]). Thus it follows that e_1 is a strong Lefschetz element for B by Lemma 7.3. □

Proof of Proposition 7.1. The first assertion is proved in Lemma 7.2. By Example 6.4 of [4], the algebra

$$A/mA = K[x_1, x_2, \dots, x_n]/(e_1, e_2, \dots, e_n)$$

has the SLP. Hence the second assertion follows from Lemmas 7.5, 7.6 and Theorem 6.1. □

8. Some more applications

Throughout this section we fix R, S to be the same as in Notation and Remark 7.4. Suppose that $f_1, \dots, f_n$ is a regular sequence of S. Put $B = S/(f_1, \dots, f_n)S$ and $A = R/(f_1, \dots, f_n)R$. Let m be the maximal ideal of B. Then we have (1) A is a finite flat over B and (2) A/mA has the SLP. Thus, by Theorem 6.1, if B has the SLP, then A has the SLP. This is the idea of the proof of Proposition 7.1. We give more applications of Theorem 6.1.

Proposition 8.1. *Let $f \in S$ be a homogeneous element of degree d. Suppose that $(e_2, e_3, \dots, e_n, f)$ is a complete intersection in S. Then $R/(e_2, e_3, \dots, e_n, f)R$ has the SLP.*

Proof. Put $A = R/(e_2, \dots, e_n, f)R$ and $B = S/(e_2, \dots, e_n, f)$. It is easy to see that A is a finite flat over B. Note that B has embedding dimension one. Hence it has the SLP, as one easily notices. Since the fiber of $B \to A$ is $R/(e_1, \dots, e_n)R$, it has the SLP. By Theorem 6.1 the assertion is proved. □

Lemma 8.2. *Suppose that*

$$f_2, f_3, \dots, f_n, f_d$$

is a regular sequence in S such that degree $f_i = i$ *for $i = 2, 3, \dots, n, d$ with $d > n$. Put $B = S/(f_2, \dots, f_n, f_d)$. Then the following conditions are equivalent.*

(1) *B has the strong Lefschetz property.*
(2) *The embedding dimension of B is* 1.

Proof. Since B is a complete intersection, we have

$$h_B(q) = 1 + q + \cdots + q^{d-1}.$$

Assume (2). Then as a K-algebra, B is generated by $\overline{e_1}$. Hence B is isomorphic to $K[X]/(X^d)$. Thus (1) follows. Assume (1). The maximal ideal of B is generated by $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_n}$. Since B has the strong Lefschetz property, we get that $\overline{e_k}$ is a constant multiple of $\overline{e_1}^k$. Hence the maximal ideal is generated by a single element. □

Proposition 8.3. *Using the same notation as in the previous lemma, assume that d is a prime number. Then we have*

(1) *$B = S/(f_2, f_3, \ldots, f_n, f_d)S$ has the strong Lefschetz property.*
(2) *$R/(f_2, f_3, \ldots, f_n, f_d)R$ has the strong Lefschetz property.*

Proof. (2) follows form (1) by Theorem 6.1. To prove (1) it suffices to show that the embedding dimension of B is one by Lemma 8.2. Write $f_2 = \alpha e_1^2 + \beta e_2$, with $\alpha, \beta \in K$. Assume $\beta = 0$. Then since $(e_1^2, f_3, \ldots, f_n, f_d)$ is a complete intersection, so is $(e_1, f_3, \ldots, f_n, f_d)$. Hence, if we put $B' = S/(e_1, f_3, \ldots, f_n, f_d)$, then

$$h_{B'}(q) = \frac{(1-q)(1-q^3)\cdots(1-q^n)(1-q^d)}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1 + q + \cdots + q^{d-1}}{1-q}.$$

This forces the numerator of the last function be divisible by $1 - q$, a contradiction, since d is a prime. This means that $\beta \neq 0$. Thus we may replace e_2 by f_2 as a generator of the algebra B. Hence, by modifying the elements $f_3, \ldots, f_n, f_d$ suitably, we have

$$B \cong K[e_1, e_3, \ldots, e_n]/(f_3, \ldots, f_n, f_d).$$

Now suppose that $f_3 = \alpha e_1^3 + \beta e_3$. As with the preceding case it does not occur that $\beta = 0$. Hence f_3 may replace e_3 as a generator of the algebra. Hence we have

$$B \cong K[e_1, e_4, \ldots, e_n]/(f_4, \ldots, f_n, f_d),$$

with modification of the generators $f_4, \ldots, f_n, f_d$. We may repeat the same argument to obtain

$$B \cong K[e_1]/\left(e_1^d\right). \qquad \square$$

Remark 8.4. Denote by $[a]_q$ the q-integer $[a]_q = 1 + q + \cdots + q^{a-1}$. Then in the statement of the above proposition, it is the same if we say $[d]_q$ is not divisible by any one of $[2]_q, \ldots, [n]_q$, instead of the assumption "d is a prime number."

Appendix A

We show a new proof for the following

Proposition A.1. *$K[X, Y]/(X^r, Y^s)$ has the strong Lefschetz property with $X + Y$ a strong Lefschetz element, where* $\operatorname{char} K = 0$ *or* $\geqslant \operatorname{Max}\{r, s\}$ *and* $\deg X = \deg Y = 1$.

Proof. H. Ikeda [5] proved the following: If A is a standard graded ring and if $x \in A$ is a strong Lefschetz element for A, then $A[y]/(y^2)$ has the strong Lefschetz property with $x + \bar{y}$ a strong Lefschetz element. (In fact, this is a special case of Theorem 2.8 in [5].) We would like to emphasize that her proof does not use the theory of sl_2 or the theory of Groebner bases. Using her theorem we immediately get that

$$A := K[z_1, \dots, z_n]/\left(z_1^2, \dots, z_n^2\right)$$

has the strong Lefschetz property with $\bar{z}_1 + \dots + \bar{z}_n$ a strong Lefschetz element. Now choose n so that $n = r + s - 2$ and consider the ring homomorphism

$$\phi : K[X, Y] \longrightarrow A$$

defined by $X \mapsto \bar{z}_1 + \dots + \bar{z}_{r-1}$ and $Y \mapsto \bar{z}_r + \dots + \bar{z}_n$. Put $J = \ker\phi$. We claim that $J = (X^r, Y^s)$. It is easy to see the inclusion

$$J \supset \left(X^r, Y^s\right). \tag{A.1}$$

We have to show that it is the equality. By way of contradiction assume that J is strictly larger than (X^r, Y^s). Then the homogeneous part of degree $r + s - 2$ of R/J is 0, since $R/(X^r, Y^s)$ has the unique minimal ideal, the socle, at that degree. On the other hand we have

$$\phi\left(X^{r-1}Y^{s-1}\right) = (r-1)!(s-1)!\overline{z_1 z_2 \cdots z_n} \neq 0.$$

This is a contradiction. Thus the inclusion (A.1) is in fact the equality. We have just shown that the subring $B := K[\phi(X), \phi(Y)]$ of A is isomorphic to

$$K[X, Y]/\left(X^r, Y^s\right).$$

Note that B has a symmetric Hilbert function and moreover shares the same socle with A. Then, since the Lefschetz element $\bar{z}_1 + \dots + \bar{z}_n$ for A lies in B, it is a Lefschetz element for B also. □

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