BOOK REVIEW


This timely monograph might well have been titled “Non-linear Age-dependent Population Dynamics and Semigroup Theory”. At a time when there are almost daily contributions to the age-dependent population literature, each with its own ad hoc theory, a unifying work, and especially one of the quality of Webb’s, is needed and welcome. Of course one must be willing to grapple with the semigroups. But Webb is a master who has pity on the uninitiated, laying the prerequisite groundwork clearly and thoroughly.

The book is devoted to the study of the following problem. Let \( p(a, t) \) denote the age density of a population at time \( t \), so that the population of individuals who are between ages \( a_1 \) and \( a_2 \) at time \( t \) is given by

\[
\int_{a_1}^{a_2} p(a, t) \, da.
\]

More generally, \( p \) could be a list of \( n \) population densities,

\[
p(a, t) = (p_1(a, t), p_2(a, t), \ldots, p_n(a, t)).
\]

In order to determine \( p \) for \( a > 0 \) and \( t > 0 \), it is assumed first that an initial distribution is given,

\[
p(a, 0) = p_0(a),
\]

where \( p_0 \in \mathcal{L} = L^1((0, \infty), \mathbb{R}^n) \). The birth rate, \( p(0, t) \), is assumed to be determined by

\[
p(0, t) = F(p(\cdot, t)),
\]

where \( F: \mathcal{L} \to \mathbb{R}^n \) is the birth modulus. Finally, the dynamics take the form

\[
Dp(a, t) = G(p(\cdot, t))(a),
\]

where \( D \) is the derivative in the \((1, 1)\) direction, i.e.

\[
Dp = \frac{\partial p}{\partial a} + \frac{\partial p}{\partial t}
\]

when the partial derivatives are continuous. Equations (1)–(3) above constitute the Age-dependent Population problem, or, as Webb calls it, problem (ADP).

Webb makes nice use of examples. Early on he introduces the reader to three which have received a lot of attention in the literature. These are revisited later at various places in order to illustrate the theory as it develops. The examples are all of the form, for \( p \in \mathcal{R} (n = 1) \),

\[
p(a, 0) = p_0(a),
\]

\[
p(0, t) = \int_0^\infty \beta(P(t))k(a)p(a, t) \, da,
\]

\[
Dp(a, t) = -\mu(P(t), a)p(a, t)
\]

and

\[
P(t) = \int_0^\infty p(a, t) \, da,
\]

where \( k(a) \) takes on various forms for reproductive capacity. In addition, the last chapter is devoted to examples and applications. Species interaction, population genetics, epidemic populations and logistic population are treated separately.

Consider the basic problems of existence, uniqueness and regularity of solutions to problem (ADP). If \( F(p(\cdot, t)) \) and \( G(p(\cdot, t))(a) \) were known exactly, \textit{a priori}, as functions \( f(t) \) and \( g(a, t) \) respectively, then along
each characteristic line where \( a - t = \text{const} \), the problem (ADP) would be reduced to the trivial ordinary differential equation

\[
\frac{d}{ds}p(a + s, t + s) = g(a + s, t + s)
\]  

with the initial condition

\[
p(0, t - a) = f(t - a)
\]

in the case \( t > a \), but with the initial condition

\[
p(a - t, 0) = p_0(a - t)
\]

in the case \( a \geq t \). Alternatively, since \( F \) and \( G \) are never known in this fashion in problems of interest, suppose that \( f \) and \( g \) give approximations to \( F \) and \( G \). For instance, suppose that \( p^n(a, t) \) is an approximate solution of problem (ADP) and define

\[
f(t) = F(p^n(\cdot, t))
\]

and

\[
g(a, t) = G(p^n(\cdot, t))(a).
\]

In this case the solutions to the initial-value problems (4)–(6) along all the characteristics give a reasonable candidate, \( p^{n+1}(a, t) \), for the next, and hopefully better, approximate solution. In fact this iteration is exactly the scheme of successive approximations which corresponds to the fixed-point argument that Webb uses to obtain existence and uniqueness of solutions to problem (ADP) (under suitable Lipschitz continuity assumptions \( F \) and \( G \)).

The solutions fit into a semigroup context as soon as the needed regularity properties (continuity of solutions with respect to initial conditions) and global existence and uniqueness are established. It is worth noting that existence, uniqueness and regularity are used to establish the semigroup properties, and not vice versa. One hears all too often the complaint that it is not worth the effort of struggling with semigroups just to get existence etc. As Webb so ably shows, the power of the semigroup theory lies in establishing further properties of solutions, such as in their long-time behavior. Letting \( S(t) \) denote the nonlinear semigroup determined by problem (ADP), we follow Webb in assuming conditions on \( F \) and \( G \) which assure that \( S(t) \) maps the positive cone of \( L^1 \) back into itself.

Webb uses a variety of hypotheses to develop a correspondingly varied theory. However, for the purposes of this review, we will limit discussion to the cases where the birth and death moduli have the forms

\[
F(p(\cdot, t)) = \int_0^\infty \beta(a, p(\cdot, t))p(a, t) \, da
\]

and

\[
G(p(\cdot, t))(a) = -\mu(a, p(\cdot, t))p(a, t),
\]

with some very mild conditions on the (matrix-valued) birth and death moduli \( \beta \) and \( \mu \). This "quasi-linearity" (linearity in \( p(a, t) \), given \( p(\cdot, t) \)) occurs in much of the literature on applications and is convenient for theoretical development. Furthermore, we insist that \( F \) and \( G \) satisfy certain conditions which guarantee that the forward solution \( S(t)q \) is defined and remains in the positive cone of \( L^1 \) whenever the initial state \( q \) is positive. Then we have the following:

Theorem. Assume hypotheses as indicated above. Further, assume that for each \( t > 0 \) and each bounded subset \( M \) of the positive cone of \( L^1 \), the set \( \{S(s)q: 0 \leq s \leq t, q \in M\} \) is bounded in \( L^1 \). Then each bounded trajectory of the semigroup in the positive cone, \( \{S(t)q: t \geq 0\} \), has compact closure in \( L^1 \).

As a result of this theorem, many of the properties of limits of solutions of ordinary differential equations in finite dimensional spaces hold in this context as well. For example, the omega-limit set of a bounded trajectory is consequently nonempty, connected and compact.

General existence theorems for steady states of problem (ADP) are scarce. Webb gives a nice exposition of a theorem due to J. Prüss, which relates the vanishing of components of the population vector (in the long-time limit) to the expected net reproduction rates. As always, all the hypotheses are clearly laid out. A
required fixed-point theorem of Amann is stated in detail and referenced down to page number. (This care in referencing is standard operating procedure for Webb.)

In considering stability of steady states, Webb reviews Lyapunov stability and the invariance principle, applying the main results to problem (ADP). Like most other sections of the book, this one goes further in reviewing the background theory than is necessary for the immediate task at hand. The result is an accessible introduction for any reader who has the mathematical sophistication of, say, a second-year graduate student.

The book ends with 315 references, an index to notation and a subject index. This well-written book is clearly deserving of a place on many private bookshelves as well as libraries. Unfortunately, it is not likely to find many such placements because of its price of over 22c/page. It deserves far better.

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