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STRUCTURE AND RECOGNITION OF DOMISHOLD GRAPHS

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1. Introduction

Aim of this paper is to investigate the structure of domishold graphs and find a recognition algorithm that does not need to examine the entire adjacency structure of the graph, but considerably less parameters associated to it.

The relevant background material is collected is Section 2.

In Section 3, a canonical cell structure for domishold graphs and an easy characterization of cell are given. Every cell will be characterized by two parameters that can be derived by the adjacency structure of the graph. It is shown that, by identifying the vertices of each cell to one, we get a threshold graph of specified type.

In Section 4, we will prove that, if G is a domishold graph, the graph G', obtained from G by a finite number of interchanges (i.e. with the same degree sequence), is not necessarily domishold. Nevertheless it is possible to characterize the interchanges that keep a domishold graph in the class and to show that all domishold graphs with the same degree sequence are isomorphic.

In Section 5, finally, we will give a recognition algorithm, that, starting from the parameters characterizing each cell, works in time $O(h)$, where h is the number of cells of the graph.

2. Preliminaries and notations

In this paper we consider only finite, simple, loopless, undirected graphs $G = (V, E)$, where V is the vertex set of G and E is the edge set of G. For any $x \in V$, we denote by $N(x)$ the set of vertices adjacent to x and by $M(x)$ the set of vertices of G not belonging to $x \cup N(x)$. Furthermore, for any $x \in V$, we denote by d_x its degree and by S_x the sum of the degrees of all vertices adjacent to x.

We denote by I_k, K_k, J_{2k} respectively the edgeless graph on k vertices, the complete graph with k vertices and the complement of the perfect matching on $2k$

vertices. For two graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ with $V_1 \cap V_2 = \Phi$, we define their direct sum G_1+G_2 as being $(V_1 \cup V_2,\ E_1 \cup E_2)$ and their direct product $G_1 \times G_2$ as being $(V_1 \cup V_2,\ E_1 \cup E_2 \cup E_{12})$, where E_{12} is the set of all edges linking points in V_1 to the points in V_2.

A subset U of the vertex set V of a graph G is called a dominating set of G ($U \operatorname{dom} G$) if any vertex $x \notin U$ is adjacent to at least one vertex $y \in U$.

In the following we are interested to the class of domishold graphs.

A graph is domishold if it is possible to find positive real numbers associated to their vertices so that U is dominating if and only if the sum of the corresponding weights of vertices of U exceeds a certain threshold θ.

Benzaken and Hammer [1] have proved the following result:

Proposition 1. *The following properties are equivalent:*

(i) *G is domishold.*

(ii) *G has no induced subgraph isomorphic to H_1, H_2, H_3, H_4, H_5 (see Fig. 1).*

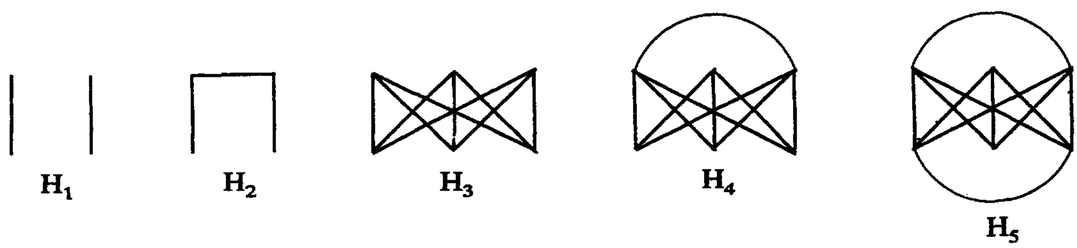

Fig. 1.

(iii) *G is built from the empty graph by the repeated application of $G' \to G''$, where*

$$G'' = (G' + I_p) \times J_{2r} \times K_q \quad (p+q+r \neq 0).$$

The class of domishold graphs properly contains the class of threshold graphs.

In fact for threshold graphs, the following result holds [3]:

Proposition 2. *Any threshold graph can be generated from the empty graph for repeated application of $G' \to G''$, where*

$$G'' = (G' + I_p) \times K_q \quad (p+q \neq 0).$$

3. Cell structure of domishold graphs

The possibility of building any domishold graph from the empty graph by repeated application of $G' \to G''$ where $G'' = (G' + I_p) \times J_{2r} \times K_q$ $(p+q+r \neq 0)$ and the associativity and commutativity of $+$ and $\times$, suggest the idea of introducing the following concept of cell.

Definition 3. We call cells of a domishold graph the *maximal subsets* of vertices, constituting one I_p or one J_{2r} or one K_q, that correspond to a construction of G by (iii) within a minimum number of steps.

Remark 4. Using associativity and commutativity of $+$ and $\times$, a minimum (steps) construction is entirely defined by a sequence of triples:

$$(p_0, 2r_0, q_0)\ (p_1, 2r_1, q_1) \cdots (p_t, 2r_t, q_t),$$

where

(a) $\quad p_i \neq 0, \quad i = 1, \ldots, t, \qquad 2r_i + q_i \neq 0, \quad i = 0, \ldots, t-1,$

(b) $\quad p_0 = 0 \quad \text{or} \quad p_0 \geq 3.$

Condition (a) guarantees minimality of the construction. Unfortunately this minimum sequence is not unique unless we satisfy condition (b), because:

$$(G_0 + I_1) \times J_{2r_0} \times K_{q_0} = G_0 \times J_{2r_0} \times K_{q_0+1},$$
$$(G_0 + I_2) \times J_{2r_0} \times K_{q_0} = G_0 \times J_{2(r_0+1)} \times K_{q_0}.$$

To distinguish the different types of cells we assign them different colours: green, red and black according as the induced subgraph is an edgeless graph, the complement of a perfect matching or a complete graph respectively. We denote by C_i^G, C_i^R, C_i^B the green, red and black cell generated at the ith step of the construction of G.

Remark 5. The cells interconnections structure may be derived immediately from (iii). The cells are in fact joined as follows:

$$C_i^G \text{ to } C_k^R, C_k^B, \quad k \geq i,$$
$$C_i^R \text{ to all } C_j^R, C_j^B \quad (\text{and } C_l^G, l \leq i),$$
$$C_i^B \text{ to all } C_j^R, C_j^B \quad (\text{and } C_l^G, l \leq i).$$

Remark 6. All the vertices belonging to a cell have the same degree in G. This follows from the regularity of the subgraph induced by the vertices of a cell and the interconnection structure of the cells. We will denote then by $d_{C_i^G}, d_{C_i^R}, d_{C_i^B}$ the degree of a green, red and black cell (i.e. the degree of its vertices) constructed at the ith step of the generation of G.

Lemma 7. *Let G be a domishold graph and x, y two vertices of G. If $d_x = d_y$, then x and y belong to the same cell or $x \in C_i^B$, $y \in C_{i+1}^R$ and $|C_{i+1}^G| = 1$.*

Proof. The degree of the cells can be computed from Remark 5 as follows:

$$d_{C_i^G} = \sum_{k=i}^{t} 2r_k + q_k, \tag{3.1}$$

$$d_{C_i^R} = \sum_{k=0}^{t} (2r_k + q_k) - 2 + \sum_{l=0}^{i} p_l, \tag{3.2}$$

$$d_{C_i^B} = \sum_{k=0}^{t} (2r_k + q_k) - 1 + \sum_{l=0}^{i} p_l. \tag{3.3}$$

From condition (b) we have:

$$d_{C_0^G} < d_{C_0^R} < d_{C_0^B}.$$

From condition (a) we have:

$$d_{C_{i+1}^G} < d_{C_i^G}, \qquad d_{C_i^R} < d_{C_{i+1}^R}, \qquad d_{C_i^B} < d_{C_{i+1}^B},$$

and furthermore:

$$d_{C_i^R} < d_{C_i^B} \leqslant d_{C_{i+1}^R} < d_{C_{i+1}^B},$$

where the equality holds only in the case $p_{i+1} = 1$. Therefore two vertices x, y have the same degree only if they belong to the same cell or $x \in C_i^B$, $y \in C_{i+1}^R$ and $|C_{i+1}^G| = 1$. $\square$

Theorem 8. *Let G be a domishold graph and x, y two vertices of G. Then x and y belong to the same cell if and only if $d_x = d_y$ and $S_x = S_y$.*

Proof. If two vertices x and y belong to the same cell, they have the same degree from Remark 6. Furthermore it is immediate to see that either

$$N(x) - \{y\} = N(y) - \{x\} \quad \text{or} \quad |(N(x) - \{y\}) \oplus (N(y) - \{x\})| = |\{u, v\}| = 2,$$

where u and v belong to the same cell as x and y. In any case we have

$$S_x = \sum_{z \in N(x)} d_z = \sum_{z \in N(y)} d_z = S_y.$$

Now suppose that x and y, such that $d_x = d_y$, $S_x = S_y$, do not belong to the same cell. From Lemma 7 we have that one of them, say x, must belong to cell C_i^B for some i, y must belong to C_{i+1}^R, while there must exist a green cell C_{i+1}^G of cardinality 1 whose vertex z is joined to y and not to x. However in this case we have $S_x - S_y = d_y - d_z > 0$, contradicting the hypothesis. Therefore x and y must belong to the same cell and the two parameters d, S identify the cell in an unique way. $\square$

Proposition 9. *Let G be a domishold graph with n vertices and $C_0, C_1, \ldots, C_h$, the cells of G ordered for non increasing values of d and, if two cells C_i and C_j have the same degree, for increasing values of S, then one of the following cases must occur:*

(1) *C_h is a green cell with $d_h = 0$ and $S_h = 0$,*

(2) *C_0 is a black cell with $d_0 = n - 1$ and $S_0 = \sum_{x \in V} d_x - (n - 1)$,*

(3) *C_0 is a red cell with $d_0 = n - 2$ and $S_0 = \sum_{x \in V} d_x - 2(n - 2)$.*

Proof. It follows from the fact that at the last step t of the construction of G we have either $2r_t+q_t=0$, $p_t\neq 0$ and therefore case (1) occurs or $q_t\neq 0$ and case (2) is verified or $q_t=0$ and $2r_t\neq 0$ and case (3) is verified. □

Finally we will show that, by identifying the vertices of each cell of a domishold graph to one, we get a threshold graph of specified type.

We will call frame of a domishold graph $G(V, E)$, the graph $F(V_F, E_F)$ whose vertices are the maximal cells of G and there is an edge connecting two vertices u_i and $u_j \in V_F$ if the bipartite graph with bipartition (C_i, C_j) is a complete bipartite graph in G.

At each vertex of F we will assign the same colour of the corresponding cell in G.

Theorem 10. *Let $F(V_F, E_F)$ be a graph whose vertices are coloured green, red and black and let the degree partition of V_F given by $V_F=D_0+D_1+\cdots+D_m$, where D_i is the set of all vertices of degree δ_i with $0=\delta_0<\delta_1<\cdots<\delta_m<|V_F|$. Only D_0 is possibly empty. F with its colours is the frame of some domishold graph G if and only if:*

(a) *F is a threshold graph.*

(b) *The green vertices form a stable set of F.*

(c) *The red and black vertices form a complete set of F.*

(d) *The vertices belonging to $D_0, D_1, \ldots, D_{\lfloor m/2\rfloor}$ are green and $|D_1|=|D_2|=\cdots=|D_{\lfloor m/2\rfloor}|=1$.*

The vertices belonging to $D_{\lceil m/2\rceil+1}, \ldots, D_m$ are red and (or) black and $|D_{\lceil m/2\rceil+1}|, \ldots, |D_m|\leqslant 2$. In the case $|D_i|=2$, the colours of the two vertices are unlike. The subset $D_{\lceil m/2\rceil}$ has cardinality $\leqslant 3$.

In the case $|D_{\lceil m/2\rceil}|=1$ the vertex is either red or black. In all other cases the vertices have different colours.

Proof. Let G be a domishold graph. From Proposition 1 and from the associativity and commutativity of $+$ and $\times$, the graph G can always be built as follows:

$$G_0=\Phi,$$
$$G_{i+1}=(G_i+C_i^G)\times C_i^R\times C_i^B, \quad i=0,\ldots,t-1,$$
$$G=G_t,$$

where C_i^G, C_i^R, C_i^B are cells whose colours are green, red and black respectively.

The maximality of the cells implies that, at each step $i=1,\ldots,t-2$, it must be $p_i\neq 0$ and $r_i+q_i\neq 0$. Only for the first step may be $p_i=0$ and for the last step $r_i+q_i=0$. The corresponding frame F, then, can be generated from the empty graph by adding at each step a green vertex as an isolated vertex and a red or (and) a black vertex as an universal vertex. Hence, the graph F is a threshold graph. The green vertices form a stable set, the black and the red vertices form a

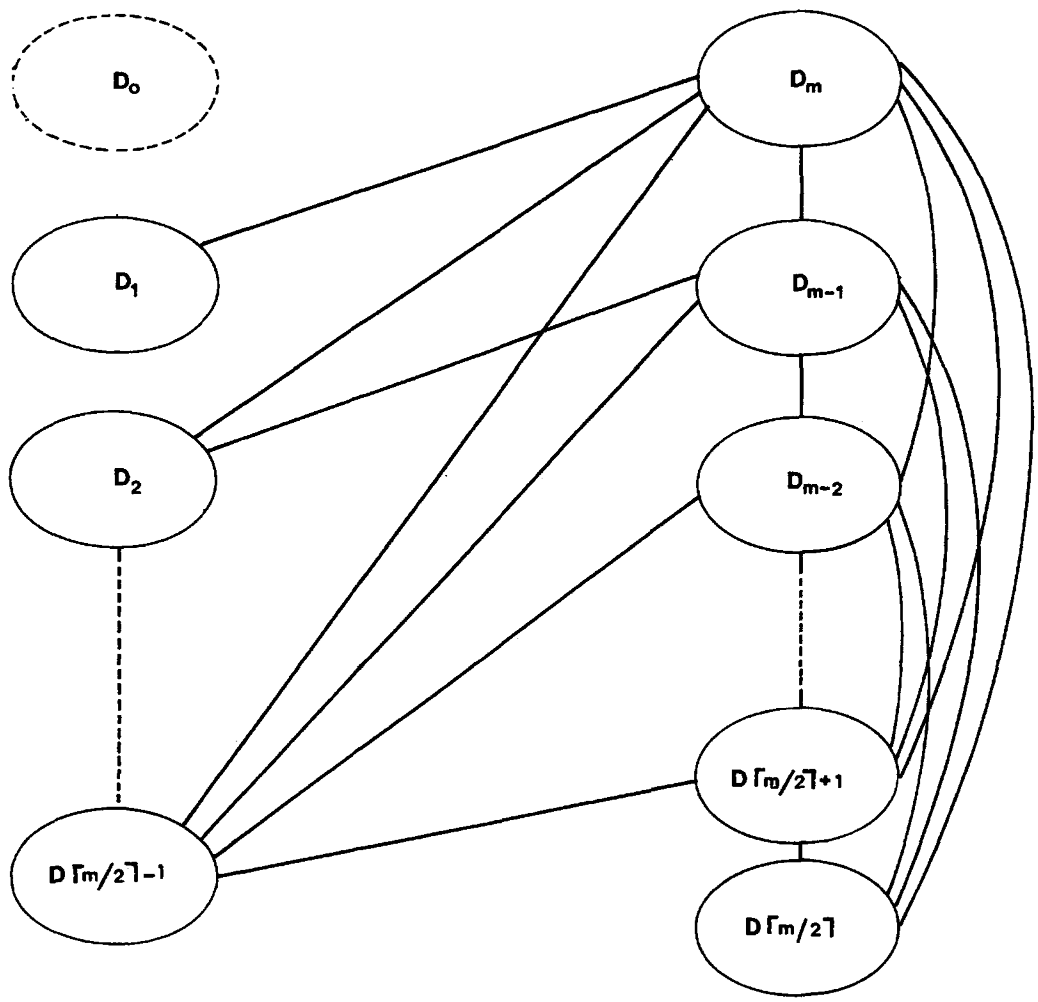

Fig. 2.

complete set. Since F is threshold, its structure is entirely determined by the indices of the degree partition [5]. Furthermore the adjacencies possess a natural containement as illustrated in Fig. 2.

The vertices added at Step 0 will have the same degree in F and will belong to $D_{\lceil m/2\rceil}$ and therefore $|D_{\lceil m/2\rceil}|\leqslant 3$. For any other Step $i>0$, the green vertex added will belong to $D_{\lceil m/2\rceil-i}$ while the red and (or) black vertex will belong to $D_{\lceil m/2\rceil+i}$. Since must be $p_i\neq 0$ and $r_i+q_i\neq 0$, we have $|D_{\lceil m/2\rceil-i}|=1$ and $|D_{\lceil m/2\rceil+i}|\leqslant 2$. If, in the last iteration, it occurs that $r_i+q_i=0$, then D_0 exists and $|D_0|=1$.

Now suppose that F is a threshold graph with properties (a), (b), (c) and (d).

Since F is threshold, it is always possible to find an ordering of the vertices in such a way that the graph F can be built from the empty graph by adding each vertex in the order as an isolated vertex if it belongs to the stable set and as an universal vertex if it belongs to the complete set [3]. Thus the corresponding graph G built from the empty graph according to the same order, by means of the operators $+$ and $\times$, replacing each green vertex with an edgeless graph and each red (black) vertex with the complement of a perfect matching (complete graph) is domishold. Furthermore, from (d), the domishold graph is obtained in a minimum number of steps. $\square$

4. Degree sequences of domishold graphs

In this section we will show that domisholdness is not a property of the degree sequence.

This means that we can find both domishold graphs and non domishold graphs with the same degree sequence.

It is however possible to characterize the interchanges that, starting from a domishold graph, generate a domishold graph. Finally we will prove that if two domishold graphs have the same degree sequence, they are isomorphic.

By a theorem due to Fulkerson, Hoffmann and McAndrew [4] we know that if two graphs G and G' have the same degree sequence, then G' can be obtained from G by a finite number of interchanges, where an interchange is a replacement of edges xu and yz and non-edges xz and yu by the edges xz and yu and the non-edges xu and yz (see Fig. 3).

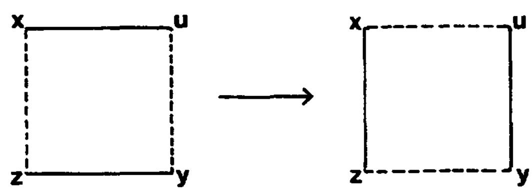

Fig. 3.

Remark 11. If an interchange xu, yz may work on a domishold graph then $xyzu$ forms a chordless cycle on 4 vertices, otherwise G would contain the forbidden configurations H_1 or H_2.

In the following lemma we want to investigate at which cells the vertices x, u, z, y of a domishold graph must belong in order to have the possibility to operate an interchange.

Lemma 12. *If G is a domishold graph any chordless cycle on 4 vertices has at least two non adjacent vertices belonging to the same red cell. If three of them belong to the same red cell, then the fourth must also belong to it. If a third vertex belongs to a black cell, then the fourth must necessarily belong to a green cell.*

Proof. It is immediately seen that at least one vertex belongs to a red cell. In fact suppose that none of them belongs to it. Clearly one vertex at least must belong to a black cell, x for example, then z belongs to a green cell, y to a black and this brings a contradiction for the vertex u.

Suppose that there is not a couple of vertices that belong to the same red cell. Since there is at least one, suppose it is x, then z must belong to a green cell, y to a black cell or to a different red cell since it is connected to z while x is not. In both cases we have a contradiction for the vertex u.

Hence either x and z or y and u belong to the same red cell.

It is immediately seen that if there are 3 vertices belonging to the same red cell, then the fourth must also belong to it.

If x, z belong to the same red cell and u (or y) belong to a black cell, then y (or u) must necessarily belong to a green cell. □

In the following theorem we want to characterize those interchanges that, starting from a domishold graph, generate a new graph that is still domishold.

Theorem 13. *If G is a domishold graph, the only interchanges that generate a graph G' still domishold are those where at least three vertices have the same degree. This implies that either the four vertices of the interchange belong to the same red cell or two vertices belong to a red cell, a third to a black cell with the same degree and the fourth vertex belongs to a green cell of cardinality* 1.

In both cases G' is isomorphic with G. Any other interchange generates a graph G' that has an induced subgraph isomorphic with H_2.

Proof. Let n be the number of vertices of G. If $n \leq 4$, then either it is not possible to operate any interchange or G is the complement of a perfect matching and G' is isomorphic with G.

Suppose then $n > 4$. From Lemma 12, in order to operate an interchange there must be at least two non-adjacent vertices belonging to the same red cell. Suppose they are x and z.

Then either y (or u) must be connected to all vertices p to which x and z are connected. In fact if this does not happen, after the interchange the induced subgraph $G'_{x,y,z,p}$ is isomorphic with the forbidden configuration H_2 (see Fig. 4).

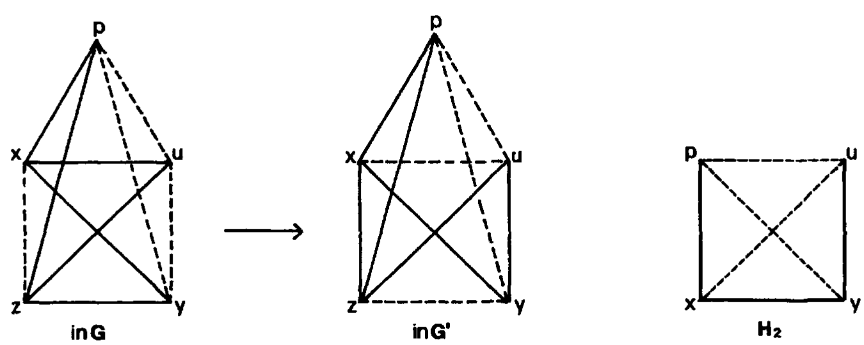

Fig. 4.

This implies that either $d_u \geq d_x = d_z$ or $d_y \geq d_x = d_z$. On the other side, for each vertex q to which x and z are not connected neither u nor y may be connected. In fact if this is not so, after an interchange we get in G' the subgraph $G'_{u,x,q,z}$ isomorphic with H_2 (see Fig. 5).

Then $d_u \leq d_x = d_z$ and $d_y \leq d_x = d_z$. Then at least three vertices must have the same degree. It follows that we have only two possible interchanges that generate a domishold graph. From Theorem 8, we have either all the vertices belong to

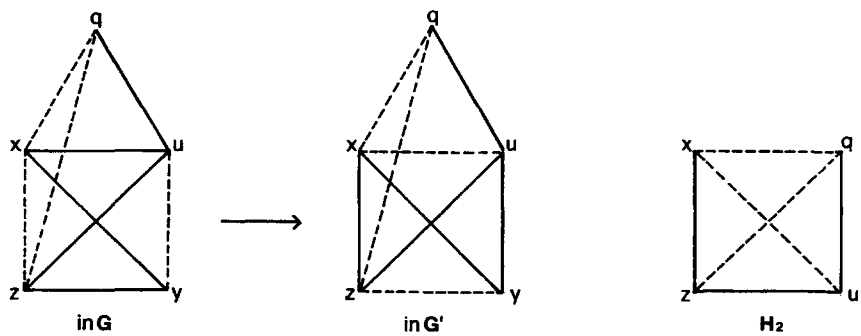

Fig. 5.

the same red cell or two vertices belong to a red cell, the third belongs to a black cell with the same degree and the fourth belongs to a green cell of cardinality 1.

In both cases G' is isomorphic with G, by $\sigma: V \to V$, defined as $c = \sigma(b)$, $b = \sigma(c)$, $x = \sigma(x)$ (where c and b are two connected vertices of the same degree in G). □

Theorem 14. *If G is a domishold graph with sequence (d, S), any other graph G' with the same sequence is domishold.*

Proof. Since G' has the same degree sequence as G it must be obtained from G by a finite number of interchanges.

Consider the interchange illustrated in Fig. 6.

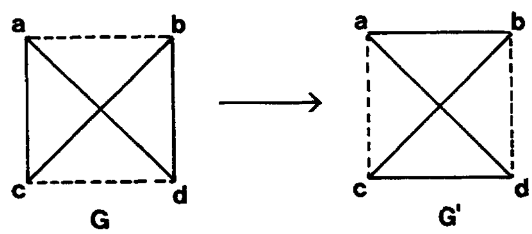

Fig. 6.

For any $x \in V \neq a, b, c, d$, the vertex x has the same adjacency list in G as in G'. Since an interchange does not modify the degrees of the vertices, x has the same value of S in G as in G'.

Let

$$\bar{S}_x = \sum_{\substack{y \in N(x) \\ y \neq a,b,c,d}} d_y.$$

In G we have

$$\begin{bmatrix} S_a = \bar{S}_a + d_c + d_d, & S_b = \bar{S}_b + d_c + d_d, \\ S_c = \bar{S}_c + d_b + d_a, & S_d = \bar{S}_d + d_b + d_a, \end{bmatrix} \tag{4.1}$$

and in G'

$$\begin{bmatrix} S'_a=\bar{S}_a+d_b+d_d, & S'_b=\bar{S}_b+d_a+d_c, \\ S'_c=\bar{S}_c+d_b+d_d, & S'_d=\bar{S}_d+d_a+d_c. \end{bmatrix} \tag{4.2}$$

We will show now that those interchanges that keep a domishold graph in its class are the only one's that do not modify the (d, S) sequence.

Since G is domishold, from Lemma 12 there must be at least two nonadjacent vertices, let say a and b, such that $d_a=d_b$, $S_a=S_b$. From Theorem 13, in order to get a graph G' still domishold either c and d belong also to the same red cell or one, say c, belongs to a black cell and d to a green cell with cardinality 1. In the first case, the values d_x, S_x, for $x=a, b, c, d$ are not changed by the interchange. In the second case we have in G;

$$d_a=d_b=d_c, \qquad \bar{S}_a=\bar{S}_b=\bar{S}_c \tag{4.3}$$

and taking in consideration (4.3), from (4.1) and (4.2) we get:

$$(d_a, S_a)=(d_a, S'_a), \qquad (d_b, S_b)=(d_c, S'_c),$$
$$(d_c, S_c)=(d_b, S'_b), \qquad (d_d, S_d)=(d_d, S'_d),$$

and therefore the overall sequence (d, S) is not changed in G'. For any other interchange we have

$$d_c \neq d_a=d_b \quad \text{and} \quad d_d \neq d_a=d_b$$

and this implies that the sequence (d, S) is modified in G'.

Then the only interchanges that preserve the sequence (d, S) are those that also preserve domisholdness. □

5. An algorithm for recognizing domishold graph

In this section we develop an algorithm to decide if a given graph G is domishold. The algorithm will actually give a way for generating iteratively a domishold graph. Before giving the algorithm we extend the definition of cell for a general graph.

Definition 15. We call *cells* of a graph G the equivalence classes of vertices defined by $d_x=d_y$, $S_x=S_y$.

Theorem 16. *The following properties are equivalent:*

(a) *G is domishold.*

(b) *There is an ordering of the cells and a partition of them into three disjoint subsets R, B and Q such that: every $v_h \in C_j \in B$ or R is adjacent to all vertices $v_l \in C_i$ with $i>j$ and every $v_h \in C_j \in Q$ is adjacent to none of the vertices $v_l \in C_i$ with $i>j$.*

Any cell $C_i \in B$ induces a complete graph (black cell), any cell $C_i \in R$ induces the complement of a perfect matching (red cell), any $C_i \in Q$ induces an edgeless graph (green cell).

Proof. The implication (b) $\Rightarrow$ (a) follows from (iii) of Proposition 1. Indeed let G_i denote the subgraph of G induced by $\{C_1, C_2, \ldots, C_i\}$; if $C_{i+1} \in B$ or R, then $G_{i+1} = G_i \times C_{i+1}$, if $C_{i+1} \in Q$, then $G_{i+1} = G_i + C_{i+1}$. Hence by induction on i, every G is domishold.

It remains to be proved that (a) $\Rightarrow$ (b).

We shall accomplish this by means of an algorithm which finds for every domishold graph G the cells, their ordering and the partition described in (b); otherwise it stops pointing out that the graph is not domishold. □

Given a graph G, associate to each vertex its degree d and the sum S of the degrees of the vertices adjacent to it. Consider the partition of the vertex set $V = C_1 + C_2 + \cdots C_h$, where C_i are the cells of G characterized by (d_i, S_i) and let $m_i = |C_i|$.

The input of the algorithm will be the sequence

$$(d_1, S_1)^{m_1} (d_2, S_2)^{m_2} \cdots (d_h, S_h)^{m_h},$$

ordered for non increasing values of d and, for the same value of d, for increasing values of S.

By Proposition 9, the algorithm will find at each step either a cell of maximal degree or a cell or minimal degree. The vertices of the cell will be added to the appropriate subset B, R or Q depending on its colour. The process will be then iteratively repeated on the induced subgraph obtained by deleting from G the cell found.

With a proper book-keeping we do not actually need to compute the new subsequence (d, S) for the induced subgraph. All we need are two pointers p and q to the first and last cell not yet examined, and a counter i for the number of cells C_i identified so far.

Algorithm

Step 0. $p \leftarrow 1$, $q \leftarrow h$, $\nu = m_1 + m_2 + \cdots + m_h$, $i \leftarrow 0$, $B = R = Q = \Phi$,

$S = d_1 m_1 + d_2 m_2 + \cdots + d_h m_h$, $S_0 = 0$.

Step 1. If $\nu = 0$, stop: G is domishold; otherwise

if $d_p = |R| + |B| + \nu - 1$ and $S_p = S - d_p$, go to Step 2,

if $d_q = |R| + |B|$ and $S_q = S_0$, go to Step 3,

if $d_p = |R| + |B| + \nu - 2$ and $S_p = S - 2d_p$, go to Step 4,

else stop: G is not domishold.

Step 2. $i \leftarrow i+1$, $C_i \leftarrow V_p$, $B \leftarrow B \cup V_p$, $\nu \leftarrow \nu - m_p$,

$S_0 \leftarrow S_0 + m_p d_p$, $p \leftarrow p+1$, go to Step 1.

Step 3. $i \leftarrow i+1$, $C_i \leftarrow V_q$, $Q \leftarrow Q \cup V_q$, $\nu \leftarrow \nu - m_q$,

$S \leftarrow S - m_q d_q$, $q \leftarrow q-1$, go to Step 1.

Step 4. if m_p odd stop: G is not domishold; otherwise

$i \leftarrow i+1$, $C_i \leftarrow V_p$, $R \leftarrow R \cup V_p$, $\nu \leftarrow \nu - m_p$,

$S_0 \leftarrow S_o + m_p d_p$, $p \leftarrow p+1$, go to Step 1.

Let us prove that the Algorithm is correct. Suppose first that the Algorithm stops declaring that the sequence (d, S) is not domishold and in spite of this, it is domishold. Since, at any step, the algorithm works on the sequence (d, S) of an induced subgraph, this must be domishold as well. Hence one of the conditions of Proposition 9 must be verified, contradicting the fact that the algorithm stops.

Conversely suppose that the algorithm works through the end. It builds up a graph G', with the same sequence (d, S) as G, the appropriate cell structure described in Theorem 16 and therefore domishold. Since from Theorem 14 any other graph with the same (d, S) sequence can be obtained only with interchanges that preserve domisholdness, G is also domishold and isomorphic with G', from Theorem 13.

It is immediate to see that the algorithm works in only $0(h)$ steps where h is the number of cells in input. In order to check whether a given graph G with n vertices and m edges is domishold, we can first generate in time $0(\max(m, n \log n))$ the ordered sequence (d, S) and then apply the above algorithm. The overall complexity is again $0(\max(m, n \log n))$.

Example. Consider the sequence

$$(d, S) = [(12, 124)^4, (12, 132)^3, (9, 112)^4, (7, 84)^2, (4, 48)^1]$$

and let V be the corresponding partition of the vertex set

$$V = \{\{1, 2, 3, 4\}, \{5, 6, 7\}, \{8, 9, 10, 11\}, \{12, 13\}, \{14\}\}.$$

From the algorithm we have

$$\begin{array}{ll} C_1 = \{1, 2, 3, 4\}, & \\ C_2 = \{14\}, & B = \{C_3\}, \\ C_3 = \{5, 6, 7\}, & R = \{C_1 \cup C_5\}, \\ C_4 = \{12, 13\}, & Q = \{C_2 \cup C_4\}, \\ C_5 = \{8, 9, 10, 11\}. & \end{array}$$

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