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Closed conformal Killing–Yano tensor and Kerr–NUT–de Sitter space–time uniqueness

Tsuyoshi Houri [a], Takeshi Oota [b,*], Yukinori Yasui [a]

[a] Department of Mathematics and Physics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan
[b] Osaka City University Advanced Mathematical Institute (OCAMI), 3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan

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Abstract

We study space–times with a closed conformal Killing–Yano tensor. It is shown that the D-dimensional Kerr–NUT–de Sitter space–time constructed by Chen–Lü–Pope is the only space–time admitting a rank-2 closed conformal Killing–Yano tensor with a certain symmetry.

Higher-dimensional black hole solutions have attracted renewed interests in the recent developments of supergravity and superstring theories [1–6]. Recently, the D-dimensional Kerr–NUT–de Sitter metrics were constructed by [6]. All the known vacuum type D black hole solutions are included in these metrics [7]. Kerr–NUT–de Sitter metrics are also interesting from the point of view of AdS/CFT correspondence. Indeed, odd-dimensional metrics lead to Sasaki–Einstein metrics by taking BPS limit [6,8–10] and even-dimensional metrics lead to Calabi–Yau metrics in the limit [6,11,12]. Especially, the five-dimensional Sasaki–Einstein metrics have emerged quite naturally in the AdS/CFT correspondence.

On the other hand, it has been shown that geodesic motion in the Kerr–NUT–de Sitter space–time is integrable for all dimensions [13–18]. Indeed, the constants of motion that are in involution can be explicitly constructed from a rank-2 closed conformal Killing–Yano (CKY) tensor. In this Letter, using a geometric characterisation of the separation of variables in the Hamilton–Jacobi equation [19], we study space–times with a rank-2 closed CKY tensor.

The rank-2 CKY tensor is defined as a two-form

$$h = \frac{1}{2} h_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b, \quad h_{ab} = -h_{ba} \tag{1}$$

satisfying the equation [20]

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2\xi_c g_{ab} - \xi_a g_{bc} - \xi_b g_{ac}. \tag{2}$$

The vector field ξ_a is called the associated vector of h_{ab}, which is given by

$$\xi_a = \frac{1}{D-1} \nabla^b h_{ba}. \tag{3}$$

The following is our main result.

Theorem 1. *Let us assume the existence of a single rank-2 CKY tensor h for D-dimensional space–time (M, g) satisfying the conditions,*

$$\text{(a) } dh = 0, \qquad \text{(b) } \mathcal{L}_\xi g = 0, \qquad \text{(c) } \mathcal{L}_\xi h = 0. \tag{4}$$

Then, M is only the Kerr–NUT–de Sitter space–time.[1]

The Kerr–NUT–de Sitter metric takes the form [6]:

(a) $D = 2n$

$$g = \sum_{\mu=1}^{n} \frac{\mathrm{d}x_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{j=0}^{n-1} A_\mu^{(j)}\, \mathrm{d}\psi^j \right)^2. \tag{5}$$

* Corresponding author.
E-mail addresses: houri@sci.osaka-cu.ac.jp (T. Houri), toota@sci.osaka-cu.ac.jp (T. Oota), yasui@sci.osaka-cu.ac.jp (Y. Yasui).

[1] We require a further technical condition which will be detailed in the proof. See the assumption below Eq. (14).

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(b) $D=2n+1$

$$g=\sum_{\mu=1}^{n}\frac{dx_\mu^2}{Q_\mu}+\sum_{\mu=1}^{n}Q_\mu\left(\sum_{j=0}^{n-1}A_\mu^{(j)}\,d\psi^j\right)^2+S\left(\sum_{j=0}^{n}A^{(j)}\,d\psi^j\right)^2. \tag{6}$$

The functions Q_μ are given by

$$Q_\mu=\frac{X_\mu}{U_\mu},\qquad U_\mu=\prod_{\substack{\nu=1\\(\nu\neq\mu)}}^{n}\left(x_\mu^2-x_\nu^2\right), \tag{7}$$

where X_μ is an arbitrary function depending only on x_μ[2] and

$$A_\mu^{(k)}=\sum_{\substack{1\leqslant\nu_1<\cdots<\nu_k\leqslant n\\(\nu_i\neq\mu)}}x_{\nu_1}^2x_{\nu_2}^2\cdots x_{\nu_k}^2,\qquad A^{(k)}=\sum_{1\leqslant\nu_1<\cdots<\nu_k\leqslant n}x_{\nu_1}^2x_{\nu_2}^2\cdots x_{\nu_k}^2 \tag{8}$$

($A_\mu^{(0)}=A^{(0)}=1$) and $S=c/A^{(n)}$ with a constant c.

In the following we briefly describe the proof of the theorem (see [21] for detailed analysis). The wedge product of two CKY tensors is again a CKY tensor and so the wedge powers $h^{(j)}=h\wedge\cdots\wedge h$ are CKY tensors. The condition (a) means that the Hodge dual $(D-2j)$-forms $f^{(j)}=*h^{(j)}$ are Killing–Yano tensors:

$$\nabla_{(a_1}f^{(j)}_{a_2)a_3\cdots a_{D-2j+1}}=0. \tag{9}$$

These Killing–Yano tensors generate the rank-2 Killing tensors $K^{(j)}$ obeying the equation $\nabla_{(a}K^{(j)}_{bc)}=0$. Under the condition (a) the Killing tensors $K^{(j)}$ are mutually commuting [17,18],

$$\left[K^{(i)},K^{(j)}\right]_S=0. \tag{10}$$

The bracket $[\,,\,]_S$ represents a symmetric Schouten product. The equation can be written as

$$K^{(i)}_{d(a}\nabla^dK^{(j)}_{bc)}-K^{(j)}_{d(a}\nabla^dK^{(i)}_{bc)}=0. \tag{11}$$

Let us define the vector fields $\eta^{(j)}$ by

$$\eta_a^{(j)}=K_a^{(j)b}\xi_b. \tag{12}$$

Then we have

$$\nabla_{(a}\eta^{(j)}_{b)}=\frac{1}{2}\mathcal{L}_\xi K^{(j)}_{ab}-\nabla_\xi K^{(j)}_{ab}, \tag{13}$$

which vanishes by the conditions (b) and (c), i.e., $\eta^{(j)}$ are Killing vectors. We can show that Killing vectors $\eta^{(i)}$ and Killing tensors $K^{(j)}$ are mutually commuting [19],

$$\left[\eta^{(i)},K^{(j)}\right]_S=0,\qquad \left[\eta^{(i)},\eta^{(j)}\right]=0. \tag{14}$$

Here, we assume that the Killing tensors $K^{(j)}$ and $K^{(ij)}=\eta^{(i)}\otimes\eta^{(j)}+\eta^{(j)}\otimes\eta^{(i)}$ are independent. Therefore all the separability conditions of the geodesic Hamilton–Jacobi equation are satisfied [19].

Let y^a be geodesic separable coordinates of $D=(n+k)$-dimensional space–time M:

$$y^a=\left(x^\mu,\psi^i\right),\quad \mu=1,2,\ldots,n,\ i=0,1,\ldots,k-1, \tag{15}$$

where $k=n$ $(k=n+1)$ for D even (odd). In these coordinates the commuting Killing vectors $\eta^{(j)}$ $(j=0,1,\ldots,k-1)$ are written as $\eta^{(j)}=\partial/\partial\psi^j$. From [22–24] the inverse metric components are of the form,

$$g^{\mu\mu}=\bar\phi^{\mu}_{(0)}(x),\qquad g^{ij}=\sum_{\mu=1}^{n}\zeta_\mu^{ij}\left(x^\mu\right)\bar\phi^{\mu}_{(0)}(x), \tag{16}$$

and the components of the Killing tensors $K^{(j)}$ $(j=0,1,\ldots,n-1)$ are given by

$$K^{(j)\mu\nu}=\delta^{\mu\nu}\bar\phi^{\mu}_{(j)}(x),\qquad K^{(j)\mu i}=0,\qquad K^{(j)i\ell}=\sum_{\mu=1}^{n}\zeta_\mu^{i\ell}\left(x^\mu\right)\bar\phi^{\mu}_{(j)}(x). \tag{17}$$

Here, $\bar\phi^{\mu}_{(j)}$ is the (μ,j)-component of the inverse of an $n\times n$ Stäckel matrix $(\phi_\mu^{(j)})$, i.e., each element depends on the variable corresponding to the lower index only: $\phi_\mu^{(j)}(x^\mu)$. It should be noticed that the Killing tensors are constructed from CKY tensors, so that they obey the following recursion relations as linear operators [19]:

$$K^{(j)}=A^{(j)}I-QK^{(j-1)}, \tag{18}$$

where I is an identity operator and Q is defined by

$$Q^a_b=-h^a_ch^c_b. \tag{19}$$

Here $A^{(j)}$ is given by

$$\det{}^{1/2}(I+\beta Q)=\sum_{j=0}^{n}A^{(j)}\beta^j. \tag{20}$$

Note that Eq. (2) with the condition (a) is equivalent to

$$\nabla_ah_{bc}=\xi_cg_{ab}-\xi_bg_{ac}. \tag{21}$$

We can further restrict the unknown functions $\bar\phi^{\mu}_{(0)}$ and ζ_μ^{ij} in the metric (16). This is analyzed by considering Eq. (21) with $\xi=\eta^{(0)}$, and finally we find the Kerr–NUT–de Sitter metric (5) or (6).

As a crosscheck of our theorem, we confirmed by the direct calculation that a CKY tensor satisfying the assumptions (a), (b) and (c) does not exist in the five-dimensional black ring background [25].

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[2] We call the metric Kerr–NUT–de Sitter for an arbitrary X_μ. The existence of h does not restrict the metric to be Einstein.

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