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A Generalization of the Rectangular Bounding Method for Continuous Location Models

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Abstract—We develop a generalized bounding method for the Weiszfeld iterative procedure used to solve the hyperbolically approximated ℓ_p-norm single- and multifacility minimum location problems. We also show that, at optimality, the solution to the bound problem coincides with the solution to the original location problem. We use this result to show that the rectangular bound value converges to the single-facility location problem optimal objective function value.

Keywords—Facility location, Weiszfeld, Bounding.

1. INTRODUCTION

A single facility minimum location problem (SFMLP) in the Euclidean plane ($\Re^2$) is stated as follows:

$$\min S(\mathbf{x}) = \sum_{j=1}^{n} w_j d(\mathbf{x}, \mathbf{a}_j), \tag{1}$$

where n is the number of fixed facilities; $\mathbf{a}_j = (a_{j1}, a_{j2})$, $j = 1, \dots, n$ are the fixed facility locations; $\mathbf{x} = (x_1, x_2)$ is the sought-after location of the new facility; $w_j > 0$, $j = 1, \dots, n$ is the weight (demand) associated with fixed facility j; and $d(\mathbf{u}, \mathbf{v})$ is some distance function used to calculate the distance between any two points $\mathbf{u}, \mathbf{v} \in \Re^2$. Using similar notation, the multifacility

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minimum location problem (MFMLP) is given by

$$\min M(\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij} d(\mathbf{x}_i, \mathbf{a}_j) + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir} d(\mathbf{x}_i, \mathbf{x}_r), \tag{2}$$

where m is the number of new facilities; w_{1ij} converts the distance between new and existing facilities i and j into a cost, where $i = 1, \dots, m$, $j = 1, \dots, n$; and w_{2ir} converts the distance between two new facilities i and r into a cost, where $i = 1, \dots, m-1$, $r = i+1, \dots, m$. A comprehensive review of continuous location theory is found in [1; 2, Chapter 11]. As is readily seen in formulations (1) and (2), distance predicting functions are an important part of the objective function of a continuous location model. Since the model should represent the real situation as closely as possible, the accuracy of the distance predicting function employed plays a crucial role in terms of the validity and the applicability of the locational decisions. The importance of an accurate representation of demand, the other ingredient of the model, is discussed in [2, Chapter 2].

Norms are usually employed as the basis for distance predicting functions in continuous location models. The main reason for this lies in the basic properties of a norm [3] which makes it well suited for distance predictions. Moreover, since norms are convex functions, incorporating a norm in the objective function of a continuous location problem provides the useful property of convexity in the optimization model. Love and Morris [4,5] present several distance predicting functions which are mostly *round* norms weighted by an inflation factor (stretch factor) to account for the amount of nonlinearity in the transportation network. A significant conclusion of their study is that an empirical distance function should be tailored to a given region whenever a premium is placed on accuracy. This result is based on statistical analyses showing that the weighted ℓ_p-norm outperforms both the weighted Euclidean and the weighted rectangular norms. Recall that the ℓ_p distance between any two points $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is given by

$$\ell_p(\mathbf{u}, \mathbf{v}) = [|u_1 - v_1|^p + |u_2 - v_2|^p]^{1/p}, \qquad p \geq 1. \tag{3}$$

Using the notation given in (1) and (2), the objective functions of the single- and multifacility location problems with the ℓ_p-norm are represented by $S(\mathbf{x})$ and $M(\mathbf{X})$, respectively.

2. WEISZFELD PROCEDURES

The Weiszfeld procedure depends upon the convexity of the Euclidean metric, and thus, utilizes the first-order necessary and sufficient conditions. Since it is impossible to express the unknown variables x_1 and x_2 in closed form equations, the first-order derivatives cannot be solved directly. Instead, an iteration function is obtained by using these derivatives. Morris and Verdini [6] generalized the Weiszfeld procedure to solve $S(\mathbf{x})$ and $M(\mathbf{X})$, and provided several properties for that instance of the problem.

In order to eliminate the obvious difficulty caused by the discontinuities in the derivatives, we use an approximation of the ℓ_p-norm in the objective function $S(\mathbf{x})$. Using an approximation is discussed for rectangular distances by Wesolowsky and Love [7] and for Euclidean and rectangular distances by Eyster *et al.* [8]. Similar approximations are given for the ℓ_p-norm by Love and Morris [9], and Morris and Verdini [6]. Verdini [10] shows that the approximation given by Eyster *et al.* (for the Euclidean distance case) and Love and Morris is not appropriate when the Weiszfeld procedure is used for the ℓ_p distances problem with $p \geq 1$. Therefore, the approximation given here follows the one given by Morris and Verdini [6]. We employ the following hyperbolic approximation of the ℓ_p-norm. Using the notation given in (3),

$$\tilde{\ell}_p(\mathbf{u}, \mathbf{v}) = \left[\left((u_1 - v_1)^2 + \epsilon\right)^{p/2} + \left((u_2 - v_2)^2 + \epsilon\right)^{p/2}\right]^{1/p}, \qquad \text{where } p \geq 1,\ \epsilon > 0. \tag{4}$$

Note that as shown by Brimberg and Love [11,12], the convergence of the Weiszfeld procedure for the ℓ_p distance problem is only guaranteed for $1 \leq p \leq 2$. For location problems where $p > 2$, the convergence properties of the Weiszfeld procedure is provided by Üster and Love [13].

We use the notation $\tilde{S}(\mathbf{x})$and $\tilde{M}(\mathbf{X})$ to denote the objective functions of the approximated ℓ_p-norm SFMLP and MFMLP, respectively. Weiszfeld's one-point iteration scheme for minimizing $\tilde{S}(\mathbf{x})$ is given by

$$x_t^{k+1} = \frac{\sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt} \right)^2 + \epsilon \right)^{(p/2)-1} \left(\tilde{\ell}_p \left(\mathbf{x}^k, \mathbf{a}_j \right) \right)^{1-p} a_{jt}}{\sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt} \right)^2 + \epsilon \right)^{(p/2)-1} \left(\tilde{\ell}_p \left(\mathbf{x}^k, \mathbf{a}_j \right) \right)^{1-p}}, \qquad t = 1, 2, \tag{5}$$

and for minimizing $\tilde{M}(\mathbf{X})$, it is given by

$$x_{rt}^{k+1} = \frac{NF_{rt}^k + NS_{rt}^k}{DF_{rt}^k + DS_{rt}^k}, \qquad r = 1, \ldots, m, \quad t = 1, 2, \tag{6}$$

where

$$NF_{rt}^k = \sum_{j=1}^{n} w_{1rj} \left(\left(x_{rt}^k - a_{jt} \right)^2 + \epsilon \right)^{(p-2)/2} \left(\tilde{\ell}_p \left(\mathbf{x}_r^k, \mathbf{a}_j \right) \right)^{1-p} a_{jt},$$

$$DF_{rt}^k = \sum_{j=1}^{n} w_{1rj} \left(\left(x_{rt}^k - a_{jt} \right)^2 + \epsilon \right)^{(p-2)/2} \left(\tilde{\ell}_p \left(\mathbf{x}_r^k, \mathbf{a}_j \right) \right)^{1-p},$$

$$NS_{rt}^k = \sum_{i=1}^{m} w_2 \left(\left(x_{rt}^k - x_{it}^k \right)^2 + \epsilon \right)^{(p-2)/2} \left(\tilde{\ell}_p \left(\mathbf{x}_r^k, \mathbf{x}_i^k \right) \right)^{1-p} x_{it}^k,$$

$$DS_{rt}^k = \sum_{i=1}^{m} w_2 \left(\left(x_{rt}^k - x_{it}^k \right)^2 + \epsilon \right)^{(p-2)/2} \left(\tilde{\ell}_p \left(\mathbf{x}_r^k, \mathbf{x}_i^k \right) \right)^{1-p},$$

and

$$w_2 = \begin{cases} w_{2ri}, & \text{if } r < i, \\ w_{2ir}, & \text{if } r > i. \end{cases}$$

It should be noted that in order to deal with a well-formulated problem, we assume that all new facilities are chained [14]. New facility i is chained if there exists a positive w_{1ij} where j is any existing facility or if there exists a positive w_{2ir} (or w_{2ri}) where r is any *chained* new facility.

3. BOUNDING METHOD

The Weiszfeld procedure is basically an iterative steepest-descent algorithm with a predetermined step size. Therefore, to terminate the iterative procedure, a stopping rule or a bound for the best objective function value is required. The rectangular bound at an iteration is obtained by solving a rectangular distance location problem. The bound problem involves locating the same number of facilities in the original problem with respect to the existing facility locations with newly created weights. At each iteration, the percent difference between the optimum objective function value of the rectangular bound problem and the current objective function value of the original problem is calculated. If this difference is smaller than a termination value prespecified by the user, the procedure is terminated.

Several bounding methods have been proposed for single- and multifacility continuous location models. Love and Yeong [15] and Juel [16] developed stopping rules that can be used with ℓ_p distance continuous location models. Drezner [17] introduced a stopping rule, which involves the solution of a rectangular distance location problem, for the single-facility Euclidean distance location problem. Dowling and Love [18] provided the extension of Drezner's rectangular bound to multifacility Euclidean distance models. Wendell and Peterson [19] utilized the dual of the ℓ_p distance minimum location problem to calculate a lower bound for the primal objective function. Finally, Love and Dowling extended the rectangular bound to the single-facility ℓ_p distance model [20] and the multifacility ℓ_p distance model [21]. Both extensions involve the unapproximated ℓ_p distances. Among those methods, it has been shown by Love and Dowling [18,20,21] that the rectangular bound is more efficient than the others. Based on this result in this section, we first present some new results regarding the rectangular bounding methods for the ℓ_p-norm SFMLP and MFMLP. In particular, we develop rectangular bounding methods for the iterative solution methods of these problems with the *approximated* distance function $\tilde{\ell}_p$-norm. We also show that, at optimality, the solution to the bound problem for $\tilde{S}(\mathbf{x})$ coincides with the solution to the original location problem. We use this result to show that, at optimality, objective function values of $\tilde{S}(\mathbf{x})$ and the rectangular bound problem are equivalent.

3.1. Bound for SFMLP

A rectangular bound for the iterative procedure (5) can be obtained by using the Hölder inequality given by

$$\sum_{i=1}^{N} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{N} |\alpha_i|^p\right)^{1/p} \left(\sum_{i=1}^{N} |\beta_i|^q\right)^{1/q},$$

where α and β are N-dimensional vectors, $p > 1$ and $1/p + 1/q = 1$. Taking $N = 2$ for the planar location model and letting

$$\alpha_1 = \left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2},$$
$$\beta_1 = \left((x_1^k - a_{j1})^2 + \epsilon\right)^{(p-1)/2},$$
$$\alpha_2 = \left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2},$$

and

$$\beta_2 = \left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2},$$

we obtain

$$\begin{aligned}\left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2} \left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2} &+ \left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2} \left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2} \\ &\le \left(\left(\left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2}\right)^p + \left(\left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2}\right)^p\right)^{1/p} \\ &\times \left(\left(\left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2}\right)^q + \left(\left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2}\right)^q\right)^{1/q}.\end{aligned}$$

Rearranging terms, we have

$$\begin{aligned}\tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j)&\left(\left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{p/2} + \left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{p/2}\right)^{1/q} \\ &\ge \left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2} \left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2} \\ &+ \left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2} \left(\left(x_2^k - a_{j2}\right) + \epsilon\right)^{(p-1)/2}.\end{aligned}$$

Rewriting the second term on the left-hand side, we obtain

$$\begin{aligned}\tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j) \left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1} &\ge \left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2} \left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2} \\ &+ \left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2} \left(\left(x_2^k - a_{j2}\right) + \epsilon\right)^{(p-1)/2}.\end{aligned}$$

In order to obtain the cost function of the minimum model, we multiply both sides by w_j and sum for $j = 1, \ldots, n$. Thus, we have

$$\begin{aligned}\tilde{S}(\mathbf{x}) &= \sum_{j=1}^{n} w_j \tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j) \\ &\geq \sum_{j=1}^{n} w_j \frac{\left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2} \left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}} \\ &+ \sum_{j=1}^{n} w_j \frac{\left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2} \left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}}.\end{aligned}$$

Minimizing both sides of the inequality over $\mathbf{x}$ gives

$$\tilde{S}\left(\mathbf{x}^*\right) \geq \min_{\mathbf{x}} \left\{ \sum_{j=1}^{n} w_j \frac{\left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2} \left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}} + \sum_{j=1}^{n} w_j \frac{\left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2} \left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}} \right\}.$$

Without changing the direction of the inequality, the terms $((x_1 - a_{j1})^2 + \epsilon)^{1/2}$ and $((x_2 - a_{j2})^2 + \epsilon)^{1/2}$ can be simplified as $|x_1 - a_{j1}|$ and $|x_2 - a_{j2}|$, respectively. Thus, the bound as a rectangular distance problem, $\tilde{S}B^k(\mathbf{x})$, is found as

$$\tilde{S}B^k\left(\mathbf{x}^R\right) = \min_{x_1^R} \sum_{j=1}^{n} u_j \left|x_1^R - a_{j1}\right| + \min_{x_2^R} \sum_{j=1}^{n} v_j \left|x_2^R - a_{j2}\right|, \tag{7}$$

where

$$u_j = w_j \frac{\left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}}, \qquad \text{and}$$

$$v_j = w_j \frac{\left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}}, \qquad j = 1, \ldots, n.$$

For notational convenience, we denote the solution of a rectangular distance location problem by $\mathbf{x}^R$, and thus, the bound at an iteration k is given by $\tilde{S}B^k(\mathbf{x}^{R^*})$.

Let $\tilde{S}_j(\mathbf{x})$, $j = 1, \ldots, n$, denote the terms in $\tilde{S}(\mathbf{x})$. Then the first derivatives of $\tilde{S}_j(\mathbf{x})$ with respect to x_1 and x_2 are

$$\frac{\partial \tilde{S}_j(\mathbf{x})}{\partial x_1} = w_j \frac{\left((x_1 - a_{j1})^2 + \epsilon\right)^{(p-1)/2}}{(\tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j))^{p-1}} \left(\frac{(x_1 - a_{j1})}{\left((x_1 - a_{j1})^2 + \epsilon\right)^{1/2}} \right), \qquad \text{and}$$

$$\frac{\partial \tilde{S}_j(\mathbf{x})}{\partial x_2} = w_j \frac{\left((x_2 - a_{j2})^2 + \epsilon\right)^{(p-1)/2}}{(\tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j))^{p-1}} \left(\frac{(x_2 - a_{j2})}{\left((x_2 - a_{j2})^2 + \epsilon\right)^{1/2}} \right), \qquad j = 1, \ldots, n.$$

By letting $\epsilon \to 0$ and using the equality $(x_t - a_{jt}) = \text{sign}\,(x_t - a_{jt})|x_t - a_{jt}|$, for $t = 1, 2$, we can simplify the last terms and obtain

$$\frac{\partial \tilde{S}_j(\mathbf{x})}{\partial x_1} = w_j \frac{\left((x_1 - a_{j1})^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j)\right)^{p-1}} \text{sign}\,(x_1 - a_{j1}), \qquad \text{and}$$

$$\frac{\partial \tilde{S}_j(\mathbf{x})}{\partial x_2} = w_j \frac{\left((x_2 - a_{j2})^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j)\right)^{p-1}} \text{sign}\,(x_2 - a_{j2}), \qquad j = 1, \dots, n.$$

Thus, u_j and v_j can be rewritten as

$$u_j = \left| \frac{\partial \tilde{S}_j\left(\mathbf{x}^k\right)}{\partial x_1} \right| \quad \text{and} \quad v_j = \left| \frac{\partial \tilde{S}_j\left(\mathbf{x}^k\right)}{\partial x_2} \right|, \qquad j = 1, \dots, n. \tag{8}$$

PROPERTY 3.1. Let $\mathcal{S}^{\mathcal{R}}$ be the set of all the points which are optimal for the rectangular bound problem. Then, $\mathbf{x}^* \in \mathcal{S}^{\mathcal{R}}$.

PROOF. We order the existing facility locations $\mathbf{a}_j$, $j = 1, \dots, n$ in their coordinates in the x_1- and x_2-directions as follows:

$$a_{j1}^{[1]} < \cdots < a_{j1}^{[\mu]} < a_{j1}^{[\mu+1]} < \cdots < a_{j1}^{[n]}$$

and

$$a_{j2}^{[1]} < \cdots < a_{j2}^{[\nu]} < a_{j2}^{[\nu+1]} < \cdots < a_{j2}^{[n]},$$

where the bracketed superscripts denote the ordering and ties are broken arbitrarily. Let $\mathcal{H}^{\mathcal{C}}$ denote the convex hull of the existing facility locations. It is well known that $\mathbf{x}^* \in \mathcal{H}^{\mathcal{C}}$ [22]. There are four cases to consider.

CASE 1. Suppose that $\mathbf{x}^* \in \mathcal{H}^{\mathcal{I}}$ where $\mathcal{H}^{\mathcal{I}} = \mathcal{H}^{\mathcal{C}} \cap \mathcal{H}^{\mathcal{R}}$ and $\mathcal{H}^{\mathcal{R}}$ is a rectangular hull defined by the edges $x_1 = a_{j1}^{[\mu]}$, $x_1 = a_{j1}^{[\mu+1]}$, $x_2 = a_{j2}^{[\nu]}$, $x_2 = a_{j2}^{[\nu+1]}$. The first-order necessary and sufficient conditions for the SFMLP state that

$$\sum_{j=1}^{n} \frac{\partial \tilde{S}_j(\mathbf{x}^*)}{\partial x_1} = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \frac{\partial \tilde{S}_j(\mathbf{x}^*)}{\partial x_2} = 0.$$

Since

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} > 0, \quad \text{for } \sigma \le \mu, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} < 0, \quad \text{for } \sigma \ge \mu + 1,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} > 0, \quad \text{for } \sigma \le \nu, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} < 0, \quad \text{for } \sigma \ge \nu + 1,$$

we must have

$$\sum_{\sigma \le \mu} \left| \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} \right| = \sum_{\sigma \ge \mu+1} \left| \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} \right|, \tag{9}$$

$$\sum_{\sigma \le \nu} \left| \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} \right| = \sum_{\sigma \ge \nu+1} \left| \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} \right|. \tag{10}$$

Using (8), (9) and (10) can be rewritten as

$$\sum_{\sigma \le \mu} u_j^{[\sigma]} = \sum_{\sigma \ge \mu+1} u_j^{[\sigma]}, \tag{11}$$

$$\sum_{\sigma \le \nu} v_j^{[\sigma]} = \sum_{\sigma \ge \nu+1} v_j^{[\sigma]}, \tag{12}$$

respectively. Observe that (11) and (12) imply the optimality conditions for the rectangular bound problem. *Any* point in $\mathcal{H}^{\mathcal{I}}$ is an optimal solution to the rectangular bound problem, i.e., $\mathcal{S}^{\mathcal{R}} = \mathcal{H}^{\mathcal{I}}$, and thus, $\mathbf{x}^* \in \mathcal{S}^{\mathcal{R}}$.

CASE 2. Suppose that $\mathbf{x}^* \in \mathcal{H}^{\mathcal{I}}$, where $\mathcal{H}^{\mathcal{I}} = \mathcal{H}^{\mathcal{C}} \cap \mathcal{H}^{\mathcal{L}}$ and $\mathcal{H}^{\mathcal{L}}$ is a line segment on the hyperplane $x_1 = a_{j1}^{[\mu]}$ between $x_2 = a_{j2}^{[\nu]}$ and $x_2 = a_{j2}^{[\nu+1]}$. Using the first-order necessary and sufficient conditions for the SFMLP, (8) and the relations

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} > 0, \quad \text{for } \sigma \le \mu - 1, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} = 0, \quad \text{for } \sigma = \mu,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} < 0, \quad \text{for } \sigma \ge \mu + 1, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} > 0, \quad \text{for } \sigma \le \nu,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} < 0, \quad \text{for } \sigma \ge \nu + 1,$$

we must have

$$\sum_{\sigma \le \mu-1} u_j^{[\sigma]} = \sum_{\sigma \ge \mu+1} u_j^{[\sigma]}, \tag{13}$$

$$\sum_{\sigma \le \nu} v_j^{[\sigma]} = \sum_{\sigma \ge \nu+1} v_j^{[\sigma]}. \tag{14}$$

Observe that (13) and (14) imply the optimality conditions for the rectangular bound problem. We have $\mathcal{S}^{\mathcal{R}} = \mathcal{H}^{\mathcal{EI}}$ where $\mathcal{H}^{\mathcal{EI}} = \mathcal{H}^{\mathcal{C}} \cap \mathcal{H}^{\mathcal{ER}}$ and $\mathcal{H}^{\mathcal{ER}}$ is a rectangular hull defined by the edges $x_1 = a_{j1}^{[\mu-1]}$, $x_1 = a_{j1}^{[\mu+1]}$, $x_2 = a_{j2}^{[\nu]}$, and $x_2 = a_{j2}^{[\nu+1]}$. Obviously, $\mathcal{H}^{\mathcal{I}} \subset \mathcal{H}^{\mathcal{EI}}$, and thus, $\mathbf{x}^* \in \mathcal{S}^{\mathcal{R}}$.

CASE 3. Similar to Case 2, suppose that $\mathbf{x}^* \in \mathcal{H}^{\mathcal{I}}$ where $\mathcal{H}^{\mathcal{I}} = \mathcal{H}^{\mathcal{C}} \cap \mathcal{H}^{\mathcal{L}}$ and $\mathcal{H}^{\mathcal{L}}$ is a line segment on the hyperplane $x_2 = a_{j2}^{[\nu]}$ between $x_1 = a_{j1}^{[\mu]}$ and $x_1 = a_{j1}^{[\mu+1]}$ Using the first-order necessary and sufficient conditions for the SFMLP, (8) and the relations

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} > 0, \quad \text{for } \sigma \le \mu, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} < 0, \quad \text{for } \sigma \ge \mu + 1,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} > 0, \quad \text{for } \sigma \le \nu - 1, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} = 0, \quad \text{for } \sigma = \nu,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} < 0, \quad \text{for } \sigma \ge \nu + 1,$$

we must have

$$\sum_{\sigma \le \mu} u_j^{[\sigma]} = \sum_{\sigma \ge \mu+1} u_j^{[\sigma]}, \tag{15}$$

$$\sum_{\sigma \le \nu-1} v_j^{[\sigma]} = \sum_{\sigma \ge \nu+1} v_j^{[\sigma]}. \tag{16}$$

Observe that (15) and (16) imply the optimality conditions for the rectangular bound problem. We have $\mathcal{S}^{\mathcal{R}} = \mathcal{H}^{\mathcal{EI}}$, where $\mathcal{H}^{\mathcal{EI}} = \mathcal{H}^{\mathcal{C}} \cap \mathcal{H}^{\mathcal{ER}}$ and $\mathcal{H}^{\mathcal{ER}}$ is a rectangular hull defined by the edges $x_1 = a_{j1}^{[\mu]}$, $x_1 = a_{j1}^{[\mu+1]}$, $x_2 = a_{j2}^{[\nu-1]}$, and $x_2 = a_{j2}^{[\nu+1]}$. Obviously, $\mathcal{H}^{\mathcal{I}} \subset \mathcal{H}^{\mathcal{EI}}$, and thus, $\mathbf{x}^* \in \mathcal{S}^{\mathcal{R}}$.

CASE 4. Suppose that $\mathbf{x}^*$ is the intersection point of hyperplanes $x_1 = a_{j1}^{[\mu]}$ and $x_2 = a_{j2}^{[\nu]}$. Using the first-order necessary and sufficient conditions for the SFMLP, (8) and the relations

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} > 0, \quad \text{for } \sigma \le \mu - 1, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} = 0, \quad \text{for } \sigma = \mu,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_1} < 0, \quad \text{for } \sigma \ge \mu + 1, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} > 0, \quad \text{for } \sigma \le \nu - 1,$$

$$\frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} = 0, \quad \text{for } \sigma = \nu, \qquad \frac{\partial \tilde{S}_j^{[\sigma]}(\mathbf{x}^*)}{\partial x_2} < 0, \quad \text{for } \sigma \ge \nu + 1,$$

we must have

$$\sum_{\sigma \le \mu - 1} u_j^{[\sigma]} = \sum_{\sigma \ge \mu + 1} u_j^{[\sigma]}, \tag{17}$$

$$\sum_{\sigma \le \nu - 1} v_j^{[\sigma]} = \sum_{\sigma \ge \nu + 1} v_j^{[\sigma]}. \tag{18}$$

Observe that (17) and (18) imply the optimality conditions for the rectangular bound problem. We have $\mathcal{S}^{\mathcal{R}} = \mathcal{H}^{\mathcal{EI}}$, where $\mathcal{H}^{\mathcal{EI}} = \mathcal{H}^{\mathcal{C}} \cap \mathcal{H}^{\mathcal{ER}}$ and $\mathcal{H}^{\mathcal{ER}}$ is a rectangular hull defined by the edges $x_1 = a_{j1}^{[\mu-1]}$, $x_1 = a_{j1}^{[\mu+1]}$, $x_2 = a_{j2}^{[\nu-1]}$, and $x_2 = a_{j2}^{[\nu+1]}$. Obviously, $\mathcal{H}^{\mathcal{I}} \subset \mathcal{H}^{\mathcal{EI}}$, and thus, $\mathbf{x}^* \in \mathcal{S}^{\mathcal{R}}$. ∎

PROPERTY 3.2. If $\mathbf{x}^*$ is optimal for $\tilde{S}(\mathbf{x})$, then $\lim_{k\to\infty} \tilde{SB}(\mathbf{x}^*) = \tilde{S}(\mathbf{x}^*)$, for all $p > 1$.

PROOF. We first rewrite (7) at optimality in its original form by introducing $\epsilon > 0$,

$$\tilde{SB}^k\left(\mathbf{x}^{R^*}\right) = \sum_{j=1}^n w_j \frac{\left(\left(x_1^{R^*} - a_{j1}\right)^2 + \epsilon\right)^{1/2} \left(\left(x_1^k - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}}$$
$$+ \sum_{j=1}^n w_j \frac{\left(\left(x_2^{R^*} - a_{j2}\right)^2 + \epsilon\right)^{1/2} \left(\left(x_2^k - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{p-1}}.$$

By Property 3.1, we have

$$\lim_{k\to\infty} \tilde{SB}^k\left(\mathbf{x}^*\right) = \sum_{j=1}^n w_j \frac{\left(\left(x_1^* - a_{j1}\right)^2 + \epsilon\right)^{1/2} \left(\left(x_1^* - a_{j1}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^*, \mathbf{a}_j\right)\right)^{p-1}}$$
$$+ \sum_{j=1}^n w_j \frac{\left(\left(x_2^* - a_{j2}\right)^2 + \epsilon\right)^{1/2} \left(\left(x_2^* - a_{j2}\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^*, \mathbf{a}_j\right)\right)^{p-1}},$$

and by simplifying, we obtain

$$\lim_{k\to\infty} \tilde{SB}^k\left(\mathbf{x}^*\right) = \sum_{j=1}^n w_j \frac{\left(\left(x_1^* - a_{j1}\right)^2 + \epsilon\right)^{p/2} + \left(\left(x_2^* - a_{j2}\right)^2 + \epsilon\right)^{p/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}^*, \mathbf{a}_j\right)\right)^{p-1}}$$
$$= \sum_{j=1}^n w_j \frac{\left(\tilde{\ell}_p\left(\mathbf{x}^*, \mathbf{a}_j\right)\right)^p}{\left(\tilde{\ell}_p\left(\mathbf{x}^*, \mathbf{a}_j\right)\right)^{p-1}}$$
$$= \tilde{S}\left(\mathbf{x}^*\right).$$

∎

3.2. Bound for MFMLP

We develop a rectangular bound for the iterative procedure (6) used to solve $\tilde{M}(\mathbf{X})$. The Hölder inequality is used in the same way as in the single-facility case. Considering the first part of the objective function, let

$$\alpha_1 = \left((x_{i1} - a_{j1})^2 + \epsilon\right)^{1/2},$$
$$\beta_1 = \left((x_{i1}^k - a_{j1})^2 + \epsilon\right)^{(p-1)/2},$$
$$\alpha_2 = \left((x_{i2} - a_{j2})^2 + \epsilon\right)^{1/2},$$

and

$$\beta_2 = \left((x_{i2}^k - a_{j2})^2 + \epsilon\right)^{(p-1)/2}.$$

Then it follows that:

$$\begin{aligned}\tilde{\ell}_p(\mathbf{x}_i, \mathbf{a}_j)&\left(\left((x_{i1}^k - a_{j1})^2 + \epsilon\right)^{p/2} + \left((x_{i2}^k - a_{j2})^2 + \epsilon\right)^{p/2}\right)^{1/q} \\ &\geq \left((x_{i1} - a_{j1})^2 + \epsilon\right)^{1/2}\left((x_{i1}^k - a_{j1})^2 + \epsilon\right)^{(p-1)/2} \\ &+ \left((x_{i2} - a_{j2})^2 + \epsilon\right)^{1/2}\left((x_{i2}^k - a_{j2}) + \epsilon\right)^{(p-1)/2},\end{aligned}$$

and using similar steps as those for the single-facility case, we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}\tilde{\ell}_p(\mathbf{x}_i, \mathbf{a}_j) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} u_{1ij}|x_{i1} - a_{j1}| + \sum_{i=1}^{m}\sum_{j=1}^{n} v_{1ij}|x_{i2} - a_{j2}|, \tag{19}$$

where

$$u_{1ij} = w_{1ij}\frac{\left((x_{i1}^k - a_{j1})^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p(\mathbf{x}_i^k, \mathbf{a}_j)\right)^{p-1}}, \qquad \text{and}$$

$$v_{1ij} = w_{1ij}\frac{\left((x_{i2}^k - a_{j2})^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p(\mathbf{x}_i^k, \mathbf{a}_j)\right)^{p-1}}, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

For the second part of the objective function, we define

$$\alpha_1 = \left((x_{i1} - x_{r1})^2 + \epsilon\right)^{1/2},$$
$$\beta_1 = \left((x_{i1}^k - x_{r1}^k)^2 + \epsilon\right)^{(p-1)/2},$$
$$\alpha_2 = \left((x_{i2} - x_{r2})^2 + \epsilon\right)^{1/2},$$

and

$$\beta_2 = \left((x_{i2}^k - x_{r2}^k)^2 + \epsilon\right)^{(p-1)/2}.$$

Using the Hölder inequality, it follows that:

$$\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\tilde{\ell}_p(\mathbf{x}_i, \mathbf{x}_r) \geq \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} u_{2ir}|x_{i1} - x_{r1}| + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} v_{2ir}|x_{i2} - x_{r2}|, \tag{20}$$

where

$$u_{2ir} = w_{2ir} \frac{\left(\left(x_{i1}^k - x_{r1}^k\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}_i^k, \mathbf{x}_r^k\right)\right)^{p-1}}, \quad \text{and}$$

$$v_{2ir} = w_{2ir} \frac{\left(\left(x_{i2}^k - x_{r2}^k\right)^2 + \epsilon\right)^{(p-1)/2}}{\left(\tilde{\ell}_p\left(\mathbf{x}_i^k, \mathbf{x}_r^k\right)\right)^{p-1}}, \quad i = 1, \dots, m-1, \quad r = i+1, \dots, m.$$

Combining the two results (19) and (20), we obtain

$$\begin{aligned} \tilde{M}(\mathbf{X}) \geq & \sum_{i=1}^{m} \sum_{j=1}^{n} u_{1ij} |x_{i1} - a_{j1}| + v_{1ij} |x_{i2} - a_{j2}| \\ & + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} u_{2ir} |x_{i1} - x_{r1}| + v_{2ir} |x_{i2} - x_{r2}|. \end{aligned} \tag{21}$$

Minimizing both sides of (21) and defining $\mathbf{X}^R$ as in the SFMLP case, we have

$$\tilde{M}\left(\mathbf{X}^*\right) \geq \min_{\mathbf{X}^R} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} u_{1ij} \left|x_{i1}^R - a_{j1}\right| + v_{1ij} \left|x_{i2}^R - a_{j2}\right| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} u_{2ir} \left|x_{i1}^R - x_{r1}\right| + v_{2ir} \left|x_{i2}^R - x_{r2}\right| \right\}.$$

Let $\tilde{M}_{1ij}(\mathbf{X})$ and $\tilde{M}_{2ir}(\mathbf{X})$ denote the terms of the first and second sum in $\tilde{M}(\mathbf{X})$, respectively. Then, similar to the SFMLP case, the created weights are given by

$$u_{1ij} = \left| \frac{\partial \tilde{M}_{1ij}\left(\mathbf{X}^k\right)}{\partial x_{i1}} \right|, \quad v_{1ij} = \left| \frac{\partial \tilde{M}_{1ij}\left(\mathbf{X}^k\right)}{\partial x_{i2}} \right|, \quad i = 1, \dots, m, \quad j = 1, \dots, n,$$

$$u_{2ir} = \left| \frac{\partial \tilde{M}_{2ir}\left(\mathbf{X}^k\right)}{\partial x_{i1}} \right|, \quad v_{2ir} = \left| \frac{\partial \tilde{M}_{2ir}\left(\mathbf{X}^k\right)}{\partial x_{i2}} \right|, \quad i = 1, \dots, m-1, \quad r = i+1, \dots, m.$$

If we denote the right-hand side of inequality (21) by $\tilde{MB}^k(\mathbf{X}^R)$, then a bound for the multifacility location problem is obtained at any iteration k by solving the multifacility rectangular distance location problem $\min_{\mathbf{X}^R} \tilde{MB}^k(\mathbf{X}^R)$. The equality of the objective function values for the MFMLP and the rectangular bound problem at optimality, i.e., $\tilde{MB}(\mathbf{X}^*) = \tilde{M}(\mathbf{X}^*)$, can be obtained as in the single facility case (Property 3.2).

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