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MULTICHAINS, NON-CROSSING PARTITIONS AND TREES

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Bijections are presented between certain classes of trees and multichains in non-crossing partition lattices.

1. Introduction

In a previous paper [1], we proved results about the enumeration of certain types of chains in the non-crossing partition lattice T_m and its generalizations. In this paper we present bijections to certain classes of trees which reprove one theorem [1, Corollary 3.4] and provide a combinatorial proof for the other [1, Theorem 5.3].

We begin with a review of the definitions. A set partition $X=\{B_1, B_2, \ldots, B_k\}$ of the set $\{1, 2, \ldots, m\} \equiv [m]$ is called *non-crossing* (n.c.) if there do not exist four numbers $a<b<c<d$ such that $a, c \in B_i$ and $b, d \in B_j$ and $i \neq j$. Let T_m be the set of all n.c. partitions of $[m]$ ordered by refinement. That is, $X \leq Y$ if each block of X is contained in a block of Y. T_m is a lattice and was first studied by Kreweras [5] and Poupard [8].

Define a *n.c. 2-partition* of the set $[m]$ to be a set

$$\pi=\{(A_1, B_1), (A_2, B_2), \ldots, (A_k, B_k)\}$$

such that

(i) the sets $A_1, A_2, \ldots, A_k$ form a n.c. partition of $[m]$;
(ii) the sets $B_1, B_2, \ldots, B_k$ form an ordinary partition of $[m]$;
(iii) $|A_j|=|B_j|$ for all $1 \leq j \leq k$.

Let T_m^2 be the poset of n.c. 2-partitions ordered by refinement.

For any poset P, define $Z(P; n)$, *the zeta polynomial of* P, to be the number of multichains in P of cardinality $n-1$, i.e., the number of multichains $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$ in P. That $Z(P; n)$ is a polynomial in n follows from elementary considerations. For more information about $Z(P; n)$ see [3].

2. A bijection to k-ary trees

We follow the definitions of Knuth [4, p. 305] for trees. Define a k-ary tree to be an ordered rooted tree where each node has at most k subtrees and if there are

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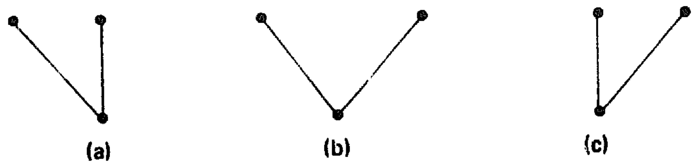


Fig. 1.

fewer than k subtrees then we distinguish between the first, second, ..., and kth subtree. For example, Fig. 1 shows three of the 3-ary trees with 3 vertices. The tree of Fig. 1(a) has an empty third subtree, on the root, (b) has an empty second subtree and (c) has an empty first subtree. The ith son of a vertex is the root of the ith subtree rooted at v.

Theorem 1.1. *The number of k-ary trees with m vertices equals $Z(T_m; k)$.*

Proof. We construct a bijection between the trees and $k-1$ element multichains in T_m as follows:

Given a k-ary tree on m vertices, pre-order the vertices with numbers from $[m]$. That is, label the vertices by the following inductive procedure:

(0) Label the root with the smallest label remaining.

(1) Label the first subtree.

(2) Label the second subtree.

⋮

(k) Label the kth subtree.

For example, the 3-ary tree in Fig. 2 has been pre-ordered. Note that this procedure is the same as labeling the vertices by depth-first search.

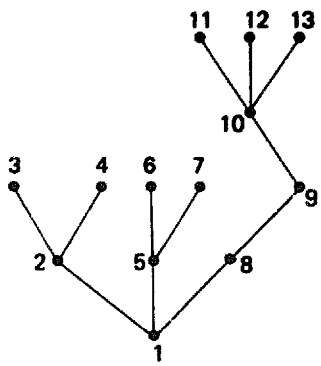


Fig. 2.

Define the ith partition in the multichain by defining a father and his $k-i+1$, $k-i+2, \ldots,$ and k sons to be in the same block, and close the blocks transitively.

For example, the chain associated with Fig. 2 is $X_1 \leq X_2$, where

$$X_1 = (1, 8, 9)(2, 4)(3)(5, 7)(6)(10, 13)(11)(12),$$
$$X_2 = (1, 5, 6, 7, 8, 9)(2, 4)(3)(10, 12, 13)(11).$$

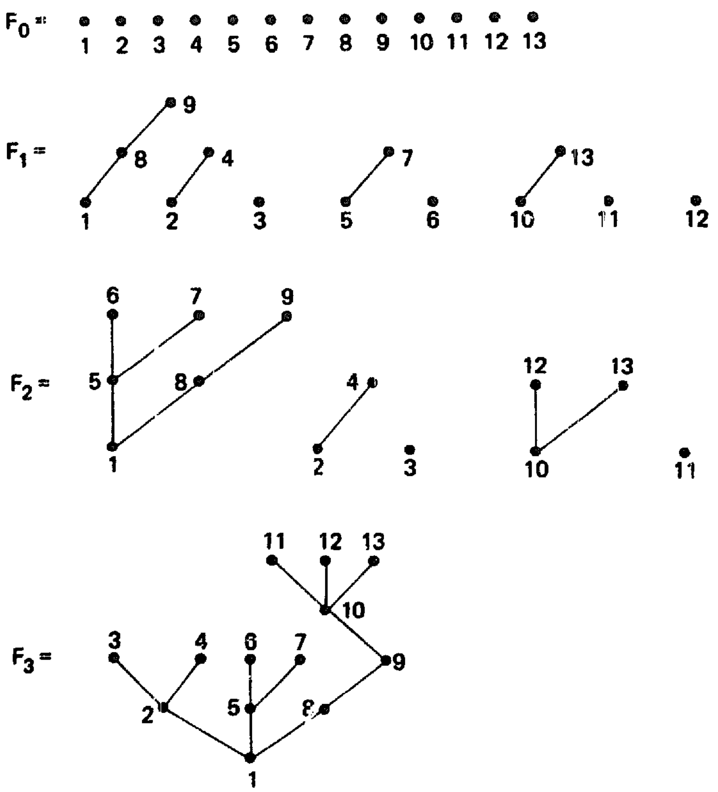
Lemma 1.2. *The partitions constructed above are all non-crossing.*

Proof. Suppose there were a crossing in the ith partition. That means that there are numbers $a < b < c < d$, where a, c are in one block and b, d are in a different block. Since a and c are in the same block, there exists an ancestor x, such that the path from x to a and the path from x to c use only sons that are identified as $k-i+1$ to k, and x is the vertex closest to the root with this property. Similarly we can find a y with the same property related to b and d. Since $a < b < c$ and b is not in the same block as x, y is in the subtree rooted at x. A path from x to y must use a 1 to $k-i$ son, or a, b, c and d would all be in the same block. Also, since b and d lie in the subtree rooted at y, c must also lie in that subtree, since the entire tree was pre-ordered. But this implies that the path from x to c contains a 1 to $k-i$ son, which is a contradiction. Hence the partition is non-crossing. □

This completes our map from trees to chains. We must invert this process to prove the bijection. Suppose we have a chain $X_0 \leq X_1 \leq \cdots \leq X_{k-1} \leq X_k$, where $X_0 = \hat{0}$ and $X_k = \hat{1}$. We will construct a sequence of labeled rooted forests of k-ary trees, $F_0, F_1, \ldots, F_k$, so that if $X_i = (B_1, B_2, \ldots, B_s)$, then $F_i = (T_1, T_2, \ldots, T_s)$ with $|B_j| = |T_j|$ for all j. The elements of B_j label the tree T_j and the root of T_j is labeled with the minimum element of B_j. F_k will be the inverse k-ary tree of the chain $X_1 \leq \cdots \leq X_{k-1}$ under our bijection, its labeling being the same as its pre-order.

Our construction will be inductive. Let $F_0 = (T_1^0, T_2^0, \ldots, T_m^0)$, where T_i^0 is the one vertex k-ary tree labeled i. Suppose we have our forest $F_i = (T_1^i, T_2^i, \ldots, T_s^i)$ corresponding to the partition X_i. Look at the partition X_{i+1}. It has a block $\bar{B}_s$ which is the merge of blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_r}$, $i_1 < i_2 < \cdots < i_r$, of x_i. (Linearly order the blocks by the smallest element they contain.) Take the root of T_{i_2} and make it the $(k-i)$th son of the largest vertex in T_{i_1} which is less than the root label of T_{i_2}. Now merge T_{i_3} into this new tree in the same manner. Repeat with T_{i_k}, $3 < k \leq r$, to produce a new tree $\bar{T}_s$. Do this procedure for each block in X_{i+1} to obtain F_{i+1}, which has all the appropriate properties. It is easy to see that F_k will be a k-ary tree labeled in pre-order.

Example 1.3. Take the chain associated with Fig. 2. The sequence of forests we get is



Finally we must show that our two bijections are inverses of each other. Suppose we start with a chain $C = (X_1 \leq X_2 \leq \cdots \leq X_{k-1})$ and from it produce a sequence of forests $F_0, F_1, \ldots, F_k$. We want to show that if we started from F_k, a k-ary tree, and used our correspondence we would produce the chain C.

The first partition in the chain we derive from F_k is defined by putting a father and its kth son in the same block and closing the blocks transitively. But this is exactly the partition we get by taking the forest F_1 and defining a block to be those labels on the same tree. This is X_1. Similarly the ith partition derived from F_k is the partition related to the forest F_i which is, by construction, X_i. So the image of F_k is C. The reverse direction is similarly easy and left to the reader.

This completes our bijection and proves Theorem 1.1.

It is well known that the number of k-ary trees with m vertices is $\binom{km}{m-1}/m$, see for example [4]. Thus Theorem 1.1 provides us with another combinatorial proof of [1, Corollary 3.4].

The bijection given as the proof in Theorem 1.1 has more power than may be immediately evident. One can see that the number of vertices which are $(k-l)$th sons of some vertex is the same as the difference in the number of blocks in X_l and X_{l+1}. Using this observation and theorems such as [1, Theorem 3.2] we can enumerate k-ary trees by the type of sons the vertices possess. A similar idea is exploited in another paper [2] where a different bijection between n.c. partitions and ordered trees is used.

3. A bijection to k-trees

A k-tree is a graph defined recursively: A complete graph on k vertices is a k-tree, and a k-tree with $n+1$ vertices is any graph obtained by joining a new vertex to k vertices mutually adjacent in a k-tree with n vertices. By a rooted, labeled, k-tree we mean a pair (T, R) consisting of a labeled k-tree T and a k element subset of the labels R, such that the vertices labeled by the set R form a clique. Thus, given T, the labeled k-tree in Fig. 3, $(T, \{-1, -2\})$, is not a rooted, labeled, k-tree where as $(T, \{1, 2\})$ is.

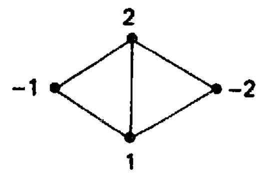


Fig. 3.

In what follows, our k-trees will be labeled with the set $\{-k, -k+1, \ldots, -1, 1, 2, \ldots, m\}$ and $R = \{-k, -k+1, \ldots, -1\}$. For convenience we will denote the set $\{-k, -k+1, \ldots, -1\}$ by $[-k]$. In Fig. 4 we see all the rooted labeled 2-trees (T, R) on 4 vertices.

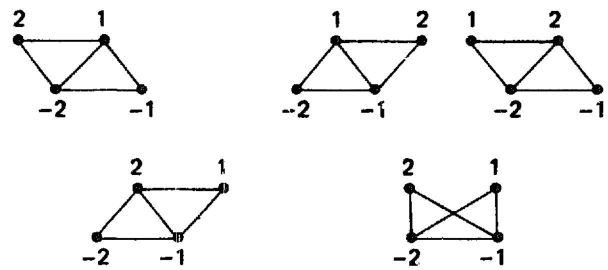


Fig. 4.

Theorem 2.1. *The number of rooted labeled k-trees (T, R) on $m+k$ vertices equals* $Z(T^2_m; k+1)$.

Proof. We will proceed in a manner analogous to the proof of Theorem 1.1. The labels from the set $[m]$ will be called p-labels and correspond to the regular partition of the required 2-partition. We must label the vertices again, in a manner similar to the pre-order labeling, to get the labels for the n.c. partition. These will be called n-labels. By $p(v)$ ($n(v)$) we mean the p-label (n-label) of the vertex v. The labels of the root are n-labels; they do not have p-labels, and will start our inductive labeling. The rest of the n-labels come from the set $[m]$.

Consider the vertices $v_1, v_2, \ldots, v_l$ which are adjacent to all k of the root vertices, and ordered such that $p(v_1) < p(v_2) < \cdots < p(v_l)$. Associated with each v_i are k sub k-trees, rooted on the n-label sets $[-k]-\{j\}+\{n(v_i)\}$ for $j \in [-k]$. We order these trees by the lexicographic order of the sets $[-k]-\{i\}$, i.e.,

$$\{-k, -k+1, \ldots, -2\} < \{-k, -k+1, -3, -1\} < \cdots$$
$$< \{-k+1, -k+2, \ldots, -2, -1\}.$$

Now we can define our n-labeling inductively.

(1) Label v_1 with smallest label remaining.

(2) Label the k subtrees of v_1 in order.

(3) Label v_2 and its k-subtrees.

⋮

$(l+1)$ Label v_l and its k-subtrees.

In Fig. 5 we see a rooted labeled 2-tree with its p-labels in (a) and its n-labels in (b).

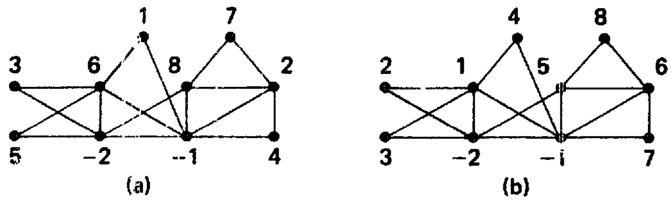


Fig. 5.

There are a couple of things to notice about this labeling. First, each vertex with an n-label is adjacent to exactly k vertices with smaller n-labels than its own, and that those vertices form a clique. We will say that the vertex is *rooted* at the k-clique. Vertices that are rooted at the same clique are called *brothers*. The vertices that are adjacent to the root of the ith subtree of a vertex v are called the ith *sons* of v. For example, in Fig. 5, the vertices p-labeled 6 and 8 are brothers as are 3 and 5. 3 and 5 are the first sons of 6, 1 is a second son of 6. 2 is a second son of 8, 4 is a first son of 2 and 7 is a second son of 2.

The other thing to notice about this labeling is that a vertex v is the son of the vertex w which is adjacent to v and has the largest n-label less than $n(v)$. Thus, if w is in a subtree of v, then there exists a unique chain $w = x_1, x_2, x_3, \ldots, x_l = v$ such that x_i is the son of x_{i+1}, and as a corollary

$$n(w) = n(x_1) > n(x_2) > \cdots > n(x_l) = n(v).$$

To define the chain of n.c. 2-partitions we will produce a chain of partitions of the non-root vertices. From each such partition we get a n.c. 2-partition by taking the n-labels for the n.c. partition and the p-labels for the regular partition.

The blocks of the first partition will be those vertices which are brothers. The blocks of the second partition are defined so that a vertex is in the same block as its brothers and also as its kth sons where the blocks are closed transitively.

In general, for the ith partition, a vertex is in the same block as its brothers and its kth, $(k-1)$st, ..., and $(k-i+2)$nd sons, with the blocks closed transitively. For example, the chain generated by the 2-tree in Fig. 5 is $X_1 \leq X_2$, where

$$X_1 = \binom{(1,5)}{(6,8)}\binom{(2,3)}{(3,5)}\binom{4}{1}\binom{6}{2}\binom{7}{4}\binom{8}{7},$$

and

$$X_2 = \binom{(1,4,5,6,8)}{(6,1,8,2,7)}\binom{(2,3)}{(3,5)}\binom{7}{4}.$$

Lemma 2.2. *The n-labels of the partitions form a non-crossing partition.*

Proof. We proceed as in the proof for Lemma 1.2. Suppose the ith partition of the n-labels crosses. Then there are numbers $a<b<c<d$, all n-labels of vertices, such that a and c are in one block and b and d are in a different block. Let v_i be the vertex such that $n(v_i)=i$. Since a and c are in the same block, we can find a vertex x such that there are two sequences $x=x_0^i, x_1^i, x_2^i, \ldots, x_l^i = s^i$ for $i=1$ and 2, where x_j and x_{j+1} are either brothers of x_{j+1} is a kth, $(k-1)$st, ..., or $(k-i+2)$nd son of x_j and $s^1=v_a$ and $s^2=v_c$. Let x be the vertex with the above property and such that $n(x)$ is minimum. Similarly we can find the vertex y with the same property for the n-labels b and d.

Since $a<b<c$, the vertex v_b is one of the sub k-trees of x or its brothers. Since b is in a different block from a and c, this implies that y is in one of the subtrees of x or its brothers. We also know that $b<c<d$, so v_c must be in a subtree rooted at y or one of its brothers. But since the chain described above exists from x to v_c, this implies that y is in the same block as x and hence that a, b, c and d are all in the same block, which is a contradiction. Hence the n-labels are non-crossing.

Now we must produce the map from chains of non-crossing 2-partitions to k-trees. Again we proceed as in the k-ary tree case. We will construct a sequence of forests of k-trees, $F_1, F_2, \ldots, F_{k+1}$ so that if the ith 2-partition $X_i = \{(A_1, B_1) \cdots (A_2, B_s)\}$, then $F_i = \{T_i, \ldots, T_s\}$ with $|A_j|+k=|B_j|+k=|T_j|$ for all j. The non-root vertices of the k-trees will be labeled by pairs $(a_1, b_i) \in [m]\times[n]$. The blocks of the two partitions will be ordered by ordering the n.c. blocks by their smallest element, as done previously.

The construction is inductive. Let $X_1 = \{(A_1, B_1) \cdots (A_s, B_s)\}$ be the finest n.c. 2-partition in the chain. Let F_1 be the forest consisting of s k-trees $(T_1, T_2, \ldots, T_s)$ so that T_i is the rooted k-tree consisting of the root and $|A_i|$ brothers. If $A_i = \{a_1, a_2, \ldots, a_l\}$ with $a_1 < a_2 < \cdots < a_l$ and $B_i = \{b_1, b_2, \ldots, b_l\}$ and $b_1 < \cdots < b_l$, label the l brothers with the pairs (a_i, b_i). For example, if we are dealing with 2-trees and the block of the 2-partition is ({1, 4, 5}, {2, 3, 4}), then the related 2-tree is shown in Fig. 6.

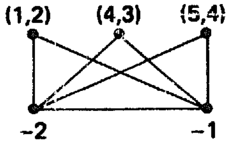


Fig. 6.

Suppose we have the forest of rooted k-trees $F_i = (T_1^i, T_2^i, \ldots, T_s^i)$ corresponding to partition X_i. Look at the partition X_{i+1}. It has a block $\bar{B}_k$ which is formed by merging blocks $B_{j_1}, B_{j_2}, \ldots, B_{j_r}$, $i_1 < \cdots < i_r$ of x_i. Remove the root of tree $T_{j_2}^i$ and root it as the $(k-i+1)$st subtree of the vertex with the largest n-label in $T_{j_1}^i$ which is less than the smallest n-label of $T_{j_2}^i$. Merge in $T_{j_3}^i$ into this new tree in the same manner and continue untill all the r trees are merged in. Do this procedure for each block of X_{i+1} to obtain F_{i+1}, with the appropriate properties. We see that the n-label of the vertex in F_{k+1} corresponds to the label produced by the algorithm defined previously.

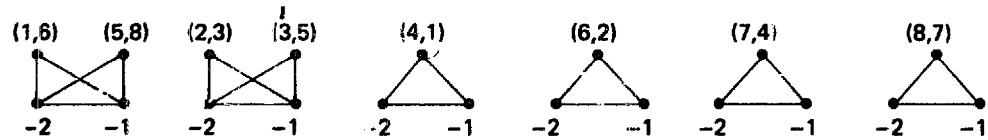
We present an example.

Example 2.3. Consider the two element chain

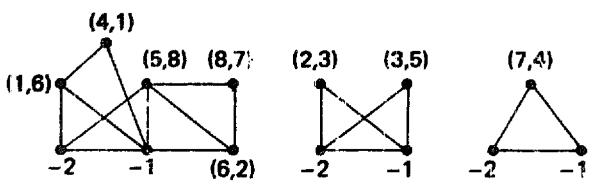
$$\binom{(1,5)}{(6,8)}\binom{(2,3)}{(3,5)}\binom{4}{1}\binom{6}{2}\binom{7}{4}\binom{8}{7} \le \binom{(1,4,5,6,8)}{(6,1,8,2,7)}\binom{(2,3)}{(3,5)}\binom{7}{4}.$$

Our scheme produces the following forests

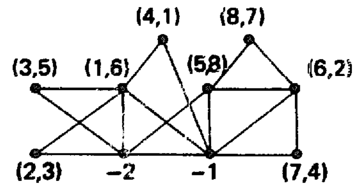
$F_1 =$



$F_2 =$



$F_3 =$



We still have to show that the bijections presented are inverses. The proof is exactly the same as in the k-ary tree correspondence and it is left to the interested reader.

The number of rooted, labeled k-trees (T, R) on $m+k$ vertices is known to be $(mk+1)^{m-1}$ (see [6]). Thus Theorem 2.1 gives a combinatorial proof of [1, Theorem 5.3]. The referee has suggested that it may be possible to obtain an alternative proof to Theorem 2.1 using the techniques of Poupard [7, Chapter IV].

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