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Note

Norms of commutators of self-adjoint operators ☆

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Abstract

In this note, the estimate of norms of commutators of self-adjoint operators is established.

Keywords: Norm inequality; Commutator; Self-adjoint operator

Throughout this paper, $\mathcal{H}$ denotes a Hilbert space. The set of all bounded linear operators acting on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. An operator A is said to be self-adjoint if $A = A^*$. An operator A is said to be positive if $\langle Ax, x\rangle \geqslant 0$ for each vector $x \in \mathcal{H}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be an orthogonal projection if $P = P^* = P^2$, where T^* denotes the adjoint of T. For two operators $A, B \in \mathcal{B}(\mathcal{H})$, the commutator of A and B is the operator $AB - BA$. As is well known, the research about norms of operator commutators has attracted much attention of many authors (see [1–3, 5–9]). In general, by the triangle inequality and the submultiplicativity of the usual operator norm that

$$\|AB - BA\| \leqslant 2\|A\|\|B\|. \tag{1}$$

For the case when A or B is positive, the inequality (1) has been recently improved by Kittaneh in [5] as follows:

$$\|AB - BA\| \leqslant \|A\|\|B\|. \tag{2}$$

The purpose of this paper is to establish inequalities for norms of commutators of self-adjoint operators by using operator spectral theory. We shall see that our inequalities improve the inequality in [5], and they seem natural and applicable to be widely useful.

Firstly, we shall give some lemmas.

Lemma 1. *(See [9].) Let A and $B \in \mathcal{B}(\mathcal{H})$. Then*

$$\|AX - XB\| \leqslant \min\bigl\{\|A - \lambda I\| + \|B - \lambda I\|:\ \lambda \in \mathbb{C}\bigr\}\|X\| \quad \text{for } X \in \mathcal{B}(\mathcal{H}).$$

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Moreover, there exists a sequence $\{X_n\} \subseteq \mathcal{B}(\mathcal{H})$ *with* $\|X_n\| = 1$ *such that*

$$\lim_{n\to\infty} \|AX_n - X_nB\| = \min\{\|A - \lambda I\| + \|B - \lambda I\|\colon \lambda \in \mathbb{C}\}.$$

Lemma 2. *(See [4].) If* W *and* L *are two closed subspaces of* $\mathcal{H}$ *and* P *and* Q *denote the orthogonal projections on* W *and* L*, respectively, then* P *and* Q *have the operator matrices*

$$P = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus I_5 \oplus 0I_6 \tag{3}$$

and

$$Q = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} D \\ D^* Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix} \tag{4}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=1}^6 \mathcal{H}_i$*, respectively, where* $\mathcal{H}_1 = W \cap L$*,* $\mathcal{H}_2 = W \cap L^\perp$*,* $\mathcal{H}_3 = W^\perp \cap L$*,* $\mathcal{H}_4 = W^\perp \cap L^\perp$*,* $\mathcal{H}_5 = W \ominus (\mathcal{H}_1 \oplus \mathcal{H}_2)$ *and* $\mathcal{H}_6 = \mathcal{H} \ominus (\bigoplus_{j=1}^5 \mathcal{H}_j)$*,* Q_0 *is a positive contraction on* $\mathcal{H}_5$*,* 0 *and* 1 *are not eigenvalues of* Q_0*, and* D *is a unitary from* $\mathcal{H}_6$ *onto* $\mathcal{H}_5$*.* I_i *is the identity on* $\mathcal{H}_i$*,* $i = 1, \ldots, 5$*.*

Theorem 3. *Let* A *and* $B \in \mathcal{B}(\mathcal{H})$ *be self-adjoint and denote*

$$\alpha_1 = \min\{\lambda\colon \lambda \in \sigma(A)\}, \qquad \alpha_2 = \max\{\lambda\colon \lambda \in \sigma(A)\}$$

and

$$\beta_1 = \min\{\mu\colon \mu \in \sigma(B)\}, \qquad \beta_2 = \max\{\mu\colon \mu \in \sigma(B)\}.$$

If

$$\frac{1}{2}(\alpha_1 + \alpha_2) \leqslant \frac{1}{2}(\beta_1 + \beta_2),$$

then

$$\|AX - XB\| \leqslant (\beta_2 - \alpha_1)\|X\|.$$

Proof. By Lemma 1, we only need to prove that

$$\min\{\|A - \lambda I\| + \|B - \lambda I\|\colon \lambda \in \mathbb{C}\} = \beta_2 - \alpha_1.$$

For convenience, the proof should be divided into four steps.

Step 1. Suppose $\frac{1}{2}(\alpha_1 + \alpha_2) \leqslant \lambda \leqslant \frac{1}{2}(\beta_1 + \beta_2)$. Observe that $\lambda - \alpha_1 \geqslant \alpha_2 - \lambda$ and $\lambda - \beta_1 \leqslant \beta_2 - \lambda$, we have

$$\|A - \lambda I\| = \lambda - \alpha_1$$

and

$$\|B - \lambda I\| = \beta_2 - \lambda.$$

Hence

$$\|A - \lambda I\| + \|B - \lambda I\| = (\lambda - \alpha_1) + (\beta_2 - \lambda) = \beta_2 - \alpha_1.$$

Step 2. Suppose $\lambda < \frac{1}{2}(\alpha_1 + \alpha_2)$. Similarly, we get

$$\|A - \lambda I\| = \alpha_2 - \lambda$$

and

$$\|B - \lambda I\| = \beta_2 - \lambda.$$

In this case,

$$\|A - \lambda I\| + \|B - \lambda I\| = (\alpha_2 - \lambda) + (\beta_2 - \lambda) = \alpha_2 + \beta_2 - 2\lambda.$$

Noting that $\alpha_2+\beta_2-2\lambda \geqslant \alpha_2+\beta_2-(\alpha_1+\alpha_2)=\beta_2-\alpha_1$, then

$$\|A-\lambda I\|+\|B-\lambda I\| \geqslant \beta_2-\alpha_1.$$

Step 3. Suppose $\lambda > \frac{1}{2}(\beta_1+\beta_2)$. Similarly, we get

$$\|A-\lambda I\|=\lambda-\alpha_1$$

and

$$\|B-\lambda I\|=\lambda-\beta_1.$$

In this case,

$$\|A-\lambda I\|+\|B-\lambda I\|=(\lambda-\alpha_1)+(\lambda-\beta_1)=2\lambda-(\alpha_1+\beta_1).$$

Noting that $2\lambda-(\alpha_1+\beta_1) \geqslant (\beta_1+\beta_2)-(\alpha_1+\beta_1)=\beta_2-\alpha_1$, then

$$\|A-\lambda I\|+\|B-\lambda I\| \geqslant \beta_2-\alpha_1.$$

Step 4. Suppose $\lambda \in \mathbb{C}\setminus\mathbb{R}$. If $\lambda=a+ib$, then

$$\begin{aligned}\|A-\lambda I\|+\|B-\lambda I\| &= \|A-aI-ibI\|+\|B-aI-ibI\| \\ &\geqslant \|A-aI\|+\|B-aI\| \\ &\geqslant \beta_2-\alpha_1.\end{aligned}$$

Combining the four steps above, we have established

$$\min\bigl\{\|A-\lambda I\|+\|B-\lambda I\|\colon \lambda\in\mathbb{C}\bigr\}=\beta_2-\alpha_1. \qquad \square$$

Corollary 4. *Let $A\in\mathcal{B}(\mathcal{H})$ be self-adjoint. If $a=\min\{\lambda\colon \lambda\in\sigma(A)$ and $b=\max\{\lambda\colon \lambda\in\sigma(A)\}$, then*

$$\|AX-XA\| \leqslant (b-a)\|X\| \quad \text{for } X\in\mathcal{B}(\mathcal{H}).$$

Corollary 5. *(See [5].) Let $A\in\mathcal{B}(\mathcal{H})$ be positive. Then*

$$\|AX-XA\| \leqslant \|A\|\|X\| \quad \text{for } X\in\mathcal{B}(\mathcal{H}).$$

Proof. It is clear since $\|A\|=\max\{\lambda\colon \lambda\in\sigma(A)\}$ and $0\leqslant\min\{\lambda\colon \lambda\in\sigma(A)\}$ if A is positive. $\square$

Corollary 6. *Let $A\in\mathcal{B}(\mathcal{H})$ be positive and invertible. Then*

$$\|AX-XA\| \leqslant \bigl(\|A\|-\bigl\|A^{-1}\bigr\|^{-1}\bigr)\|X\| \quad \text{for } X\in\mathcal{B}(\mathcal{H}).$$

Proof. If A is positive and invertible, then $\|A\|=\max\{\lambda\colon \lambda\in\sigma(A)\}$ and $\|A^{-1}\|^{-1}=\min\{\lambda\colon \lambda\in\sigma(A)\}$. By Corollary 4, we get

$$\|AX-XA\| \leqslant \bigl(\|A\|-\bigl\|A^{-1}\bigr\|^{-1}\bigr)\|X\|. \qquad \square$$

Corollary 7. *Let $A, B\in\mathcal{B}(\mathcal{H})$ be self-adjoint. Then*

$$\|AB-BA\| \leqslant \frac{1}{2}(\alpha_2-\alpha_1)(\beta_2-\beta_1),$$

where $\alpha_1=\min\{\lambda\colon \lambda\in\sigma(A)\}$, $\alpha_2=\max\{\lambda\colon \lambda\in\sigma(A)\}$ and $\beta_1=\min\{\mu\colon \mu\in\sigma(B)\}$, $\beta_2=\max\{\mu\colon \mu\in\sigma(B)\}$.

Proof. For any pair λ and μ of complex numbers, we have

$$AB-BA=(A-\lambda I)(B-\mu I)-(B-\mu I)(A-\lambda I).$$

So

$$\|AB-BA\|=\bigl\|(A-\lambda I)(B-\mu I)-(B-\mu I)(A-\lambda I)\bigr\| \leqslant 2\bigl\|(A-\lambda I)\bigr\|\bigl\|(B-\mu I)\bigr\|.$$

Therefore,

$$\|AB - BA\| \leqslant 2\min\{\|(A-\lambda I)\|: \lambda \in \mathbb{C}\}\min\{\|(B-\mu I)\|: \mu \in \mathbb{C}\}.$$

Moreover, suppose A and B are self-adjoint, by Theorem 3, we have $\min\{\|(A-\lambda I)\|: \lambda \in \mathbb{C}\} = \frac{1}{2}(\alpha_2 - \alpha_1)$ and $\min\{\|(B-\mu I)\|: \lambda \in \mathbb{C}\} = \frac{1}{2}(\beta_2 - \beta_1)$. Hence,

$$\|AB - BA\| \leqslant \frac{1}{2}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1). \qquad \square$$

Corollary 8. *(See [6].) Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then*

$$\|AB - BA\| \leqslant \frac{1}{2}\|A\|\|B\|.$$

Proof. By Corollary 7, it is clear. $\square$

Remarks. (1) The inequality of Corollary 8 is sharp. For example, if 2×2 matrices A and B are as follows:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix};$$

it is clear that A and B are positive, and $\|A\| = \|B\| = 1$. In this case,

$$AB - BA = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

So

$$\|AB - BA\| = \frac{1}{2}.$$

(2) By Corollary 8, if A, B are two positive contractions and $\|AB - BA\| = \frac{1}{2}$, then $\|A\| = \|B\| = 1$.

(3) Let P and Q be orthogonal projections. Then $\|PQ - QP\| = \frac{1}{2}$ if and only if $\frac{1}{2} \in \sigma(PQ)$.

In this case, by Lemma 2,

$$PQ - QP = 0I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} 0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}D \\ -D^*Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & 0 \end{pmatrix},$$

so

$$\|PQ - QP\| = \|Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}\|.$$

Since $0 \leqslant Q_0 \leqslant 1$, $\|Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}\| \leqslant \frac{1}{2}$. Moreover, $\|Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}\| = \frac{1}{2}$ if and only if $\frac{1}{2} \in \sigma(Q_0)$. We know that $\frac{1}{2} \in \sigma(Q_0)$ if and only if $\frac{1}{2} \in \sigma(PQ)$. Hence, $\|PQ - QP\| = \frac{1}{2}$ if and only if $\frac{1}{2} \in \sigma(PQ)$.

(4) In (2), note that $0 \leqslant Q_0 \leqslant 1$. Then we get that

$$\{\|PQ - QP\|: P \text{ and } Q \text{ are orthogonal projections in } \mathcal{B}(\mathcal{H})\} = \left[0, \frac{1}{2}\right].$$

Corollary 9. *(See [5].) Let A and B be positive contractions. Then*

$$\|AX - XB\| \leqslant \|X\|.$$

Proof. Because A and B are positive contractions, $0 \leqslant \alpha_1, \alpha_2, \beta_1, \beta_2 \leqslant 1$, so $|\beta_2 - \alpha_1| \leqslant 1$ and $|\alpha_2 - \beta_1| \leqslant 1$. By Theorem 3,

$$\|AX - XB\| \leqslant \|X\|. \qquad \square$$

Corollary 10. *Let A and $B \in \mathcal{B}(\mathcal{H})$ be positive and invertible. Then*

$$\|A-B\| \leqslant \max\left\{\|A\| - \left\|B^{-1}\right\|^{-1}, \|B\| - \left\|A^{-1}\right\|^{-1}\right\}.$$

Proof. If $\frac{1}{2}(\|A\| + \|A^{-1}\|^{-1}) \geqslant \frac{1}{2}(\|B\| + \|B^{-1}\|^{-1})$, then $\|A-B\| \leqslant \|A\| - \|B^{-1}\|^{-1}$.

Conversely, if $\frac{1}{2}(\|B\| + \|B^{-1}\|^{-1}) \geqslant \frac{1}{2}(\|A\| + \|A^{-1}\|^{-1})$, then $\|A-B\| \leqslant \|B\| - \|A^{-1}\|^{-1}$. $\square$

Corollary 11. *If $T = A + iB$ is the Cartesian decomposition of an operator T, then*

$$\|T^*T - TT^*\| \leqslant 4\min\left\{\|A - \lambda I\|: \lambda \in \mathbb{C}\right\}\min\left\{\|B - \lambda I\|: \lambda \in \mathbb{C}\right\}.$$

Proof. Let $\alpha_0 = \min\{\alpha: \alpha \in \sigma(A)\}$ and $\beta_0 = \min\{\beta: \beta \in \sigma(B)\}$ and denote $\lambda_0 = \alpha_0 + i\beta_0$. Since $T^*T - TT^* = (T-\lambda_0)^*(T-\lambda_0) - (T-\lambda_0)(T-\lambda_0)^*$ and $A - \alpha_0 \geqslant 0$ and $B - \beta_0 \geqslant 0$, it follows that

$$\begin{aligned}
\|T^*T - TT^*\| &= \left\|(T-\lambda_0)^*(T-\lambda_0) - (T-\lambda_0)(T-\lambda_0)^*\right\| \\
&= 2\left\|(A-\alpha_0 I)(B-\beta_0 I) - (B-\beta_0 I)(A-\alpha_0 I)\right\| \\
&\leqslant \|A-\alpha_0\|\|B-\beta_0\| \\
&= 4\min\left\{\|A-\lambda I\|: \lambda \in \mathbb{C}\right\}\min\left\{\|B-\lambda I\|: \lambda \in \mathbb{C}\right\}. \qquad \square
\end{aligned}$$

Corollary 12. *(See [6].) Let $T = A + iB$ be the Cartesian decomposition of an operator T. If A and B are positive, then*

$$\|T^*T - TT^*\| \leqslant \|A\|\|B\|.$$

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