Norms of commutators of self-adjoint operators

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Abstract

In this note, the estimate of norms of commutators of self-adjoint operators is established. © 2007 Elsevier Inc. All rights reserved.

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Throughout this paper, \( \mathcal{H} \) denotes a Hilbert space. The set of all bounded linear operators acting on \( \mathcal{H} \) is denoted by \( \mathcal{B}(\mathcal{H}) \). An operator \( A \) is said to be self-adjoint if \( A = A^* \). An operator \( A \) is said to be positive if \( (Ax, x) \geq 0 \) for each vector \( x \in \mathcal{H} \). An operator \( P \in \mathcal{B}(\mathcal{H}) \) is said to be an orthogonal projection if \( P = P^* = P^2 \), where \( T^* \) denotes the adjoint of \( T \). For two operators \( A, B \in \mathcal{B}(\mathcal{H}) \), the commutator of \( A \) and \( B \) is the operator \( AB - BA \). As is well known, the research about norms of operator commutators has attracted much attention of many authors (see [1–3, 5–9]). In general, by the triangle inequality and the submultiplicativity of the usual operator norm that

\[
\|AB - BA\| \leq 2\|A\|\|B\|. \tag{1}
\]

For the case when \( A \) or \( B \) is positive, the inequality (1) has been recently improved by Kittaneh in [5] as follows:

\[
\|AB - BA\| \leq \|A\|\|B\|. \tag{2}
\]

The purpose of this paper is to establish inequalities for norms of commutators of self-adjoint operators by using operator spectral theory. We shall see that our inequalities improve the inequality in [5], and they seem natural and applicable to be widely useful.

Firstly, we shall give some lemmas.

Lemma 1. (See [9].) Let \( A \) and \( B \in \mathcal{B}(\mathcal{H}) \). Then

\[
\|AX - XB\| \leq \min \left\{ \|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C} \right\} \|X\| \quad \text{for } X \in \mathcal{B}(\mathcal{H}).
\]

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Moreover, there exists a sequence \( \{X_n\} \subseteq \mathcal{B}(\mathcal{H}) \) with \( \|X_n\| = 1 \) such that
\[
\lim_{n \to \infty} \|AX_n - X_nB\| = \min \{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}.
\]

**Lemma 2.** (See [4].) If \( W \) and \( L \) are two closed subspaces of \( \mathcal{H} \) and \( P \) and \( Q \) denote the orthogonal projections on \( W \) and \( L \), respectively, then \( P \) and \( Q \) have the operator matrices
\[
P = I_1 \oplus I_2 \oplus I_5 \oplus 0 I_4 \oplus I_5 \oplus 0 I_6
\]
and
\[
Q = I_1 \oplus 0 I_2 \oplus I_5 \oplus 0 I_4 \oplus \begin{pmatrix} Q_0 & Q_0^\frac{1}{2} (I_5 - Q_0)^{\frac{1}{2}} D \\ D^* Q_0^\frac{1}{2} (I_5 - Q_0)^{\frac{1}{2}} & D^* (I_5 - Q_0) D \end{pmatrix}
\]
with respect to the space decomposition \( \mathcal{H} = \bigoplus_{i=1}^{6} \mathcal{H}_i \), respectively, where \( \mathcal{H}_1 = W \cap L, \mathcal{H}_2 = W \cap L^\perp \), \( \mathcal{H}_3 = W^\perp \cap L, \mathcal{H}_4 = W^\perp \cap L^\perp \), \( \mathcal{H}_5 = W \ominus (\mathcal{H}_1 \oplus \mathcal{H}_2) \) and \( \mathcal{H}_6 = \mathcal{H} \ominus (\bigoplus_{i=1}^{5} \mathcal{H}_i) \). \( Q_0 \) is a positive contraction on \( \mathcal{H}_5 \), 0 and 1 are not eigenvalues of \( Q_0 \), and \( D \) is a unitary from \( \mathcal{H}_6 \) onto \( \mathcal{H}_5 \). \( I_i \) is the identity on \( \mathcal{H}_i, i = 1, \ldots, 5 \).

**Theorem 3.** Let \( A \) and \( B \in \mathcal{B}(\mathcal{H}) \) be self-adjoint and denote
\[
\alpha_1 = \min \{\lambda : \lambda \in \sigma(A)\}, \quad \alpha_2 = \max \{\lambda : \lambda \in \sigma(A)\}
\]
and
\[
\beta_1 = \min \{\mu : \mu \in \sigma(B)\}, \quad \beta_2 = \max \{\mu : \mu \in \sigma(B)\}.
\]
If
\[
\frac{1}{2} (\alpha_1 + \alpha_2) \leq \frac{1}{2} (\beta_1 + \beta_2),
\]
then
\[
\|AX - XB\| \leq (\beta_2 - \alpha_1)\|X\|.
\]

**Proof.** By Lemma 1, we only need to prove that
\[
\min \{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\} = \beta_2 - \alpha_1.
\]

For convenience, the proof should be divided into four steps.

**Step 1.** Suppose \( \frac{1}{2} (\alpha_1 + \alpha_2) \leq \lambda \leq \frac{1}{2} (\beta_1 + \beta_2) \). Observe that \( \lambda - \alpha_1 \geq \alpha_2 - \lambda \) and \( \lambda - \beta_1 \leq \beta_2 - \lambda \), we have
\[
\|A - \lambda I\| = \lambda - \alpha_1
\]
and
\[
\|B - \lambda I\| = \beta_2 - \lambda.
\]

Hence
\[
\|A - \alpha_1 I\| + \|B - \lambda I\| = (\lambda - \alpha_1) + (\beta_2 - \lambda) = \beta_2 - \alpha_1.
\]

**Step 2.** Suppose \( \lambda < \frac{1}{2} (\alpha_1 + \alpha_2) \). Similarly, we get
\[
\|A - \lambda I\| = \alpha_2 - \lambda
\]
and
\[
\|B - \lambda I\| = \beta_2 - \lambda.
\]

In this case,
\[
\|A - \lambda I\| + \|B - \lambda I\| = (\alpha_2 - \lambda) + (\beta_2 - \lambda) = \alpha_2 + \beta_2 - 2\lambda.
\]
Noting that \( \alpha_2 + \beta_2 - 2\lambda \geq \alpha_2 + \beta_2 - (\alpha_1 + \alpha_2) = \beta_2 - \alpha_1 \), then
\[
\|A - \lambda I\| + \|B - \lambda I\| \geq \beta_2 - \alpha_1.
\]

**Step 3.** Suppose \( \lambda > \frac{1}{2}(\beta_1 + \beta_2) \). Similarly, we get
\[
\|A - \lambda I\| = \lambda - \alpha_1
\]
and
\[
\|B - \lambda I\| = \lambda - \beta_1.
\]
In this case,
\[
\|A - \lambda I\| + \|B - \lambda I\| = (\lambda - \alpha_1) + (\lambda - \beta_1) = 2\lambda - (\alpha_1 + \beta_1).
\]
Noting that \( 2\lambda - (\alpha_1 + \beta_1) \geq (\beta_1 + \beta_2) - (\alpha_1 + \beta_1) = \beta_2 - \alpha_1 \), then
\[
\|A - \lambda I\| + \|B - \lambda I\| \geq \beta_2 - \alpha_1.
\]

**Step 4.** Suppose \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). If \( \lambda = a + ib \), then
\[
\|A - \lambda I\| + \|B - \lambda I\| = \|A - aI - ibI\| + \|B - aI - ibI\|
\geq \|A - aI\| + \|B - aI\|
\geq \beta_2 - \alpha_1.
\]

Combining the four steps above, we have established
\[
\min\{\|A - \lambda I\| + \|B - \lambda I\|; \lambda \in \mathbb{C}\} = \beta_2 - \alpha_1.
\]

**Corollary 4.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be self-adjoint. If \( a = \min\{\lambda; \lambda \in \sigma(A)\} \) and \( b = \max\{\lambda; \lambda \in \sigma(A)\} \), then
\[
\|AX - XA\| \leq (b - a)\|X\| \quad \text{for} \quad X \in \mathcal{B}(\mathcal{H}).
\]

**Proof.** It is clear since \( \|A\| = \max\{\lambda; \lambda \in \sigma(A)\} \) and \( 0 \leq \min\{\lambda; \lambda \in \sigma(A)\} \) if \( A \) is positive. \( \square \)

**Corollary 5.** (See [5].) Let \( A \in \mathcal{B}(\mathcal{H}) \) be positive. Then
\[
\|AX - XA\| \leq \|A\|\|X\| \quad \text{for} \quad X \in \mathcal{B}(\mathcal{H}).
\]

**Proof.** If \( A \) is positive and invertible, then \( \|A\| = \max\{\lambda; \lambda \in \sigma(A)\} \) and \( \|A^{-1}\|^{-1} = \min\{\lambda; \lambda \in \sigma(A)\} \). By Corollary 4, we get
\[
\|AX - XA\| \leq \left(\|A\| - \|A^{-1}\|^{-1}\right)\|X\|. \quad \square
\]

**Corollary 6.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be positive and invertible. Then
\[
\|AX - XA\| \leq \left(\|A\| - \|A^{-1}\|^{-1}\right)\|X\| \quad \text{for} \quad X \in \mathcal{B}(\mathcal{H}).
\]

**Proof.** If \( A \) is positive and invertible, then \( \|A\| = \max\{\lambda; \lambda \in \sigma(A)\} \) and \( \|A^{-1}\|^{-1} = \min\{\lambda; \lambda \in \sigma(A)\} \). By Corollary 4, we get
\[
\|AX - XA\| \leq \left(\|A\| - \|A^{-1}\|^{-1}\right)\|X\|. \quad \square
\]

**Corollary 7.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be self-adjoint. Then
\[
\|AB - BA\| \leq \frac{1}{2}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1),
\]
where \( \alpha_1 = \min\{\lambda; \lambda \in \sigma(A)\} \), \( \alpha_2 = \max\{\lambda; \lambda \in \sigma(A)\} \) and \( \beta_1 = \min\{\mu; \mu \in \sigma(B)\} \), \( \beta_2 = \max\{\mu; \mu \in \sigma(B)\} \).

**Proof.** For any pair \( \lambda \) and \( \mu \) of complex numbers, we have
\[
AB - BA = (A - \lambda I)(B - \mu I) - (B - \mu I)(A - \lambda I).
\]
So
\[
\|AB - BA\| = \|(A - \lambda I)(B - \mu I) - (B - \mu I)(A - \lambda I)\| \leq 2\|(A - \lambda I)\|\|(B - \mu I)\|.\]
Therefore,
\[ \|AB - BA\| \leq \min\{\|A - \lambda I\|: \lambda \in \mathbb{C}\} \min\{\|B - \mu I\|: \mu \in \mathbb{C}\}. \]

Moreover, suppose \( A \) and \( B \) are self-adjoint, by Theorem 3, we have
\[ \min\{\|A - \lambda I\|: \lambda \in \mathbb{C}\} = \frac{1}{2}(\alpha_2 - \alpha_1) \]
and
\[ \min\{\|B - \mu I\|: \mu \in \mathbb{C}\} = \frac{1}{2}(\beta_2 - \beta_1). \]
Hence,
\[ \|AB - BA\| \leq \frac{1}{2}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1). \]

**Corollary 8.** (See [6].) Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be positive. Then
\[ \|AB - BA\| \leq \frac{1}{2}\|A\|\|B\|. \]

**Proof.** By Corollary 7, it is clear. \( \square \)

**Remarks.**
1. The inequality of Corollary 8 is sharp. For example, if \( 2 \times 2 \) matrices \( A \) and \( B \) are as follows:
\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}; \]
it is clear that \( A \) and \( B \) are positive, and \( \|A\| = \|B\| = 1 \). In this case,
\[ AB - BA = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \]
So
\[ \|AB - BA\| = \frac{1}{2}. \]

2. By Corollary 8, if \( A, B \) are two positive contractions and \( \|AB - BA\| = \frac{1}{2} \), then \( \|A\| = \|B\| = 1 \).

3. Let \( P \) and \( Q \) be orthogonal projections. Then \( \|PQ - PQ\| = \frac{1}{2} \) if and only if \( \frac{1}{2} \in \sigma(PQ) \). In this case, by Lemma 2,
\[ PQ - PQ = 0I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} 0 & Q_{0}^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}D \\ -D^{*}Q_{0}^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & 0 \end{pmatrix}, \]
so
\[ \|PQ - PQ\| = \|Q_{0}^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}\|. \]
Since \( 0 \leq Q_0 \leq 1 \), \( \|Q_{0}^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}\| \leq \frac{1}{2} \). Moreover, \( \|Q_{0}^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}\| = \frac{1}{2} \) if and only if \( \frac{1}{2} \in \sigma(Q_0) \). We know that \( \frac{1}{2} \in \sigma(Q_0) \) if and only if \( \frac{1}{2} \in \sigma(PQ) \). Hence, \( \|PQ - PQ\| = \frac{1}{2} \) if and only if \( \frac{1}{2} \in \sigma(PQ) \).

4. In (2), note that \( 0 \leq Q_0 \leq 1 \). Then we get that
\[ \{\|PQ - PQ\|: P \text{ and } Q \text{ are orthogonal projections in } \mathcal{B}(\mathcal{H})\} = \left[ 0, \frac{1}{2} \right]. \]

**Corollary 9.** (See [5].) Let \( A \) and \( B \) be positive contractions. Then
\[ \|AX - XB\| \leq \|X\|. \]

**Proof.** Because \( A \) and \( B \) are positive contractions, \( 0 \leq \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 1 \), so \( |\beta_2 - \alpha_1| \leq 1 \) and \( |\alpha_2 - \beta_1| \leq 1 \). By Theorem 3,
\[ \|AX - XB\| \leq \|X\|. \]
\( \square \)
Corollary 10. Let $A$ and $B \in B(H)$ be positive and invertible. Then
\[ \|A - B\| \leq \max\{\|A\| - \|B^{-1}\|^{-1}, \|B\| - \|A^{-1}\|^{-1}\}. \]

Proof. If $\frac{1}{2}(\|A\| + \|A^{-1}\|^{-1}) \geq \frac{1}{2}(\|B\| + \|B^{-1}\|^{-1})$, then $\|A - B\| \leq \|A\| - \|B^{-1}\|^{-1}$.
Conversely, if $\frac{1}{2}(\|B\| + \|B^{-1}\|^{-1}) \geq \frac{1}{2}(\|A\| + \|A^{-1}\|^{-1})$, then $\|A - B\| \leq \|B\| - \|A^{-1}\|^{-1}$. \(\square\)

Corollary 11. If $T = A + iB$ is the Cartesian decomposition of an operator $T$, then
\[ \|T^*T - TT^*\| \leq 4 \min\{\|A - \lambda I\|: \lambda \in \mathbb{C}\} \min\{\|B - \lambda I\|: \lambda \in \mathbb{C}\}. \]

Proof. Let $\alpha_0 = \min\{\alpha: \alpha \in \sigma(A)\}$ and $\beta_0 = \min\{\beta: \beta \in \sigma(B)\}$ and denote $\lambda_0 = \alpha_0 + i\beta_0$. Since $T^*T - TT^* = (T - \lambda_0)^*(T - \lambda_0) - (T - \lambda_0)(T - \lambda_0)^*$ and $A - \alpha_0 \geq 0$ and $B - \beta_0 \geq 0$, it follows that
\[ \|T^*T - TT^*\| = \|T - \lambda_0\|^* (T - \lambda_0) - (T - \lambda_0)(T - \lambda_0)^* \]
\[ = 2\| (A - \alpha_0 I)(B - \beta_0 I) - (B - \beta_0 I)(A - \alpha_0 I) \| \]
\[ \leq \|A - \alpha_0\| \|B - \beta_0\| \]
\[ = 4 \min\{\|A - \lambda I\|: \lambda \in \mathbb{C}\} \min\{\|B - \lambda I\|: \lambda \in \mathbb{C}\}. \]
\(\square\)

Corollary 12. (See [6].) Let $T = A + iB$ be the Cartesian decomposition of an operator $T$. If $A$ and $B$ are positive, then
\[ \|T^*T - TT^*\| \leq \|A\| \|B\|. \]

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