JOURNAL OF FUNCTIONAL ANALYSIS 130, 310–333 (1995)

On Regularization of γ-Generating Pairs

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Communicated by D. Sarason

Received March 2, 1994; revised June 19, 1994

This paper gives an affirmative answer to one of D. Sarason's questions concerning the exposed points in H^1. Namely, one can infer nothing special about a from the knowledge that (a, b) is a Nehari pair. The proof essentially uses D. Z. Arov's concept of A-regular and A-singular j-inner matrix-functions and is based on an analysis of V. Katsnelson's results on their regularization. © 1995 Academic Press, Inc.

1. γ-GENERATING AND NEHARI PAIRS. STATEMENT OF THE RESULT

The following notations will be used in this paper:

$\mathbb{D}$—unit disk of the complex plane $\mathbb{C}$;

$\mathbb{T}$—unit circle, $\mathbb{T} = \partial\, \mathbb{D}$;

L^p—space of measurable functions on $\mathbb{T}$, which are pth power summable;

L^∞—space of bounded measurable functions on $\mathbb{T}$;

$H^p_\pm$—standard Hardy spaces of analytic (antianalytic) functions on $\mathbb{D}$;

$P_\pm$—orthogonal projectors from L^2 onto $H^2_\pm$;

Ball(L)—the ball of radii 1 with center at the origin in the linear metric space L;

t—independent variable on $\mathbb{T}$.

DEFINITION [5]. A *pair* (a, b) is called *γ-generating* iff

(i) $a \in H^\infty_+$, $b \in H^\infty_+$

(ii) $a \not\equiv 0$, a is an outer function

(iii) $b(0) = 0$

(iv) $|a|^2 + |b|^2 = 1$, a.e. on $\mathbb{T}$.

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0022-1236/95 $6.00

The Nehari Problem [1–3, 17, 6]. Let $\gamma_{-1}, \gamma_{-2}, \ldots$ be a given sequence of complex numbers. One wants to find all functions w, $w \in L^\infty$, and w is bounded in modulus by 1, such that

$$P_- w = \gamma_{-1} \bar{t} + \gamma_{-2} \bar{t}^2 + \cdots.$$

The necessary and sufficient conditions for solvability of the Nehari problem are well-known. One often sets the following version of the Nehari problem: Let $w_0 \in L^\infty$ of $\mathbb{T}$, and be bounded in modulus by 1. One wants to find all functions w of the same class such that $w - w_0 \in H^\infty_+$ (i.e., with the same P_- part as w_0). Existence of the solution is obvious for this version.

THEOREM. ([2b], See Also [1–3, 17, 6]). *If the Nehari problem is indeterminate (i.e., has more than one solution), then the whole solution set may be described as follows*:

$$w = \frac{a}{\bar{a}} \frac{\omega - \bar{b}}{1 - \omega b}, \qquad \omega \in \text{Ball}(H^\infty_+), \tag{1.1}$$

where (a, b) is a γ-generating pair which is defined by the data of the Nehari problem uniquely up to the following transformation:

$$(a, b) \to (a \cdot c, b \cdot c^2), \tag{1.2}$$

where c is a constant of modulus 1.

DEFINITION. A γ-generating pair (a, b) is called a *Nehari pair* iff it appears in the context of some indeterminate Nehari problem as mentioned above.

Remark. Every γ-generating pair (a, b) generates according to formula (1.1) a mapping from $\text{Ball}(H^\infty_+)$ into $\text{Ball}(L^\infty)$ that produces functions w with the same P_- part (this is the reason for the name "γ-generating"), but not every one produces the whole set of functions with this P_- part. So the class of Nehari pairs is a proper subclass of the class of γ-generating pairs.

The following theorem is proved in this paper:

MAIN THEOREM. *Let (a, b) be any γ-generating pair. Then there exists an inner function θ, such that $(a, b\theta)$ is a Nehari pair.*

Remark. It follows from this theorem that the a-element of a Nehari pair has no additional properties in comparison with the a-element of a γ-generating pair. Any outer function a, such that $|a| < 1$ on $\mathbb{D}$ and $\ln(1 - |a|^2) \in L^1$ on $\mathbb{T}$, is the a-element of some Nehari pair.

There exists a one-to-one correspondence (up to a sign of a) between Nehari pairs and exposed points of the Ball(H^1) [6, 21]. The following formula describes this correspondence

$$f = \left(\frac{a}{1-b}\right)^2,$$

where f is the exposed point of Ball(H^1), (a, b) is the Nehari pair.

D. Sarason set the following question [21]:

QUESTION. Let (a, b) be any γ-generating pair. Does there exist an inner function θ, such that $(a/(1 - b\theta))^2$ is an exposed point of Ball(H^1)?

An affirmative answer to this question follows from the theorem formulated above.

In this paper we will use a maximum principle. We formulate it here so as not to interrupt the exposition in further sections.

MAXIMUM PRINCIPLE (see, e.g., [17, Lecture 1; 10, Sect. 3]). *Let f be the ratio of two bounded analytic functions on $\mathbb{D}$,*

$$f = f_1/f_2, \qquad f_1 \in H^\infty_+, \qquad f_2 \in H^\infty_+,$$

and let the denominator f_2 be an outer function. Then $f \in L^p \Rightarrow f \in H^p_+$.

2. γ-GENERATING AND SCATTERING MATRICES

DEFINITION [23, 4, 5]. A matrix-function $\left[\begin{smallmatrix} p & q \\ q & p \end{smallmatrix}\right]$, defined a.e. on $\mathbb{T}$, is called a *γ-generating matrix* if $p = 1/a$, $q = -b/a$, where (a, b) is a γ-generating pair. This class was introduced and studied by D. Z. Arov [23, 4, 5]. Let $j = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. It is easy to check that a γ-generating matrix is j-unitary, i.e.,

$$\begin{bmatrix} \bar p & \bar q \\ q & p \end{bmatrix} j \begin{bmatrix} \bar p & \bar q \\ q & p \end{bmatrix}^* = j$$

and

$$\begin{bmatrix} \bar p & \bar q \\ q & p \end{bmatrix}^* j \begin{bmatrix} \bar p & \bar q \\ q & p \end{bmatrix} = j,$$

where * means adjoint matrix.

Any γ-generating pair (or corresponding γ-generating matrix) generates a linear-fractional mapping from the Ball(H_+^∞) into the Ball(L^∞) by the formula

$$w = \frac{a}{\bar{a}} \frac{\omega - \bar{b}}{1 - b\omega} = \frac{\bar{p}\omega + \bar{q}}{q\omega + p}, \qquad \omega \in \text{Ball}(H_+^\infty). \tag{2.1}$$

It is convenient to introduce a special notation for the image of the function $\omega \equiv 0$ under this transformation:

$$s_0 \overset{\text{def}}{=} -\frac{a}{\bar{a}} \bar{b} = \frac{\bar{q}}{p}.$$

By means of it, (2.1) may be rewritten as follows:

$$w = s_0 + a\omega(1 - b\omega)^{-1} a \tag{2.2}$$

Using the Maximum Principle (see Section 1), one can easily deduce from (2.2) that this mapping produces functions w with the same P_- part.

DEFINITION. The matrix $\left[\begin{smallmatrix} b & a \\ a & s_0 \end{smallmatrix}\right]$ is called the *scattering matrix* associated to the γ-generating pair (a, b). This matrix is defined almost everywhere on $\mathbb{T}$ and is unitary.

The concepts of A-singular and A-regular pairs were introduced by D. Z. Arov [23, 4, 5].

DEFINITION. A γ-generating pair (a, b), and the corresponding γ-generating and scattering matrices, are called *A-singular* if $s_0 \in H_+^\infty$.

It means that the mapping defined by formula (2.1) or (2.2) acts, in fact, from Ball(H_+^∞) into Ball(H_+^∞), i.e., this mapping produces analytic functions (i.e., functions with zero P_- part) but surely not necessarily all of them.

LEMMA 2.1. *If (a, b) is A-singular, then $u \overset{\text{def}}{=} \frac{a}{\bar{a}} \in H_+^\infty$, and hence it is an inner function.*

Proof. According to the definition,

$$s_0 = -\frac{a}{\bar{a}} \bar{b} \in H_+^\infty.$$

Multiplying this equality by b and using $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, one can obtain

$$s_0 b = -\frac{a}{\bar{a}}(1 - a\bar{a}).$$

Hence

$$u \equiv \frac{a}{\bar{a}} = a^2 - s_0 b \in H^\infty_+.$$

DEFINITION. A γ-generating pair (a, b), and the corresponding γ-generating and scattering matrices, are called *A-regular* iff the following decomposition is impossible:

$$\begin{bmatrix} \bar{p} & \bar{q} \\ q & p \end{bmatrix} = \begin{bmatrix} \bar{p}_1 & \bar{q}_1 \\ q_1 & p_1 \end{bmatrix} \begin{bmatrix} \bar{p}_2 & \bar{q}_2 \\ q_2 & p_2 \end{bmatrix}, \tag{2.3}$$

where the matrix to the left of the equality is the γ-generating one corresponding to the pair (a, b), the ones on the right are γ-generating matrices, and the second one is non-constant A-singular.

This decomposition can be rewritten by means of γ-generating pairs,

$$a = \frac{a_1 a_2}{1 - b_1 s_0^{(2)}}, \qquad b = b_2 + \frac{b_1 a_2^2}{1 - b_1 s_0^{(2)}}, \tag{2.4}$$

where $s_0^{(2)} = \bar{q}_2 / p_2 = -(a_2/\overline{a_2})\, \bar{b}_2 \in H^\infty_+$, because the pair (a_2, b_2) is A-singular. This decomposition of the γ-generating pair (matrix) generates a decomposition of the corresponding linear-fractional mapping.

THEOREM [4, 5]. *A γ-generating pair is A-regular iff it is a Nehari pair.*

THEOREM [4, 5]. *Any γ-generating pair (matrix) permits an A-regular—A-singular factorization of the type* (2.4) (*equivalently* (2.3)). *The A-regular part is defined by the given pair uniquely up to the normalization* (1.2).

3. j-INNER MATRICES, SARASON PROBLEM, SARASON MATRICES[1]

DEFINITION. A meromorphic 2×2 matrix-function on the unit disk $\mathbb{D}$ is called *j-inner* if it takes j-contractive values inside $\mathbb{D}$ and j-unitary boundary values on $\mathbb{T}$.

THEOREM [4, 5]. Boundary values of j-inner matrix-function admit[2] essentially unique representation of the type

$$\begin{bmatrix} \theta_2 & 0 \\ 0 & \bar{\theta}_1 \end{bmatrix} \begin{bmatrix} \bar{p} & \bar{q} \\ q & p \end{bmatrix}, \tag{$*$}$$

[1] γ-generating, Nehari and Sarason matrices and pairs were introduced in [1–3] and intensively studied in [4, 5, 23].

[2] Matrix-function ($*$) may differ from the original j-inner matrix-function by right constant j-unitary factor.

where

(i) θ_2 and θ_1 are inner functions;

(ii) $\left[\begin{smallmatrix} \bar{p} & \bar{q} \\ q & p \end{smallmatrix}\right]$ is γ-generating ($p = 1/a$, $q = -b/a$);

(iii) $s_0 \overset{\text{def}}{\equiv} \theta_2\theta_1(\bar{q}/p) \equiv -\theta_2\theta_1\frac{a}{\bar{a}}\bar{b} \in H^\infty_+$.

And vise versa, any matrix-function of that kind on $\mathbb{T}$ admits a meromorphic continuation on $\mathbb{D}$ with j-contractive values by the formula

$$\frac{1}{\theta_1 a}\cdot\begin{bmatrix} u & s_0 \\ -b & 1 \end{bmatrix},$$

where $u \overset{\text{def}}{=} \theta_2\theta_1\frac{a}{\bar{a}} = \theta_2\theta_1 a^2 - s_0 b$ (u is an inner function). In what follows we will deal with matrix-functions of the type ($*$) with properties (i)–(iii) and *will call them j-inner* (although, *normalized j-inner* would be a more precise name for them). Any j-inner matrix generates a linear-fractional mapping

$$w = \theta_2\theta_1\frac{\bar{p}\omega + \bar{q}}{q\omega + p}, \qquad \omega \in \text{Ball}(H^\infty_+). \tag{3.1}$$

It can be rewritten by means of the γ-generating pair (a, b) corresponding to the "γ-generating part" $\left[\begin{smallmatrix} \bar{p} & \bar{q} \\ q & p \end{smallmatrix}\right]$,

$$w = \theta_2\theta_1\frac{a}{\bar{a}}\frac{\omega - \bar{b}}{1 - b\omega}. \tag{3.2}$$

And, finally, this mapping can be rewritten as

$$w = s_0 + \theta_2 a\omega(1 - b\omega)^{-1} a\theta_1. \tag{3.3}$$

According to the (iii), $s_0 \in H^\infty_+$. Hence this mapping acts from $\text{Ball}(H^\infty_+)$ into $\text{Ball}(H^\infty_+)$.

DEFINITION. The matrix

$$\begin{bmatrix} b & a\theta_1 \\ \theta_2 a & s_0 \end{bmatrix}$$

is called the *scattering matrix corresponding to the j-inner matrix*. The scattering matrix is inner.

DEFINITION [4, 5, 23]. A j-inner matrix is called *A-singular* iff $\theta_2 = \theta_1 = 1$, i.e., iff it is γ-generating at the same time. So A-singular matrices are the ones which are both γ-generating and j-inner.

DEFINITION [4, 5, 23]. A j-inner matrix is called *A-regular* iff it does not permit splitting of a non-constant A-singular factor on the right.

In what follows we will deal with j-inner matrices of the type

$$\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{p} & \bar{q} \\ q & p \end{bmatrix}$$

(i.e., $\theta_2 = \theta$, $\theta_1 = 1$) and the name "j-inner matrix" will mean a j-inner matrix of this type.

We are presenting here the "weak" formulation of the Sarason Problem, but it will be enough for our purposes.

THE SARASON PROBLEM. *Let w_0 be a given analytic function on $\mathbb{D}$ bounded in modulus by 1, and let θ be an inner function. One wants to find all analytic functions w bounded in modulus by 1 such that*

$$\frac{w - w_0}{\theta} \in H^\infty_+$$

(i.e., these functions have to "coincide" at the "spectrum" of θ).

THEOREM. (3, See Also [6, 17]). *If the Sarason Problem has more than one solution, the whole solution set may be described as*

$$w = \theta \frac{\bar{p}\omega + \bar{q}}{q\omega + p} = \theta \frac{a}{\bar{a}} \frac{\omega - \bar{b}}{1 - b\omega} = s_0 + \theta a \omega (1 - b\omega)^{-1} a,$$

$$\omega \in \operatorname{Ball}(H^\infty_+), \tag{3.4}$$

where

$$\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{p} & \bar{q} \\ q & p \end{bmatrix}$$

is a j-inner matrix.

Remark. One can see from (3.3) that any j-inner matrix produces a set of functions w, such that

$$\frac{w - s_0}{\theta} \in H^\infty_+.$$

But not every one of them produces all those functions.

DEFINITION. A j-inner matrix is called a *Sarason matrix* if it appears in the context of some indeterminate Sarason Problem, as described above.

THEOREM [4, 5]. *A j-inner matrix is a Sarason matrix iff it is A-regular.*

4. Two Composition Lemmas

LEMMA 4.1. *If* $\begin{bmatrix} \bar p_1 & \bar q_1 \\ q_1 & p_1 \end{bmatrix}$ *is a* γ*-generating matrix, and* $\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar p_2 & \bar q_2 \\ q_2 & p_2 \end{bmatrix}$ *is* j*-inner, then*

$$\begin{bmatrix} \bar\theta & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \bar p_1 & \bar q_1 \\ q_1 & p_1 \end{bmatrix} \cdot \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar p_2 & \bar q_2 \\ q_2 & p_2 \end{bmatrix}$$

is a γ*-generating matrix.*

Proof. It can be checked by direct calculation that the matrix above is of the form

$$\begin{bmatrix} \bar p & \bar q \\ q & p \end{bmatrix}, \qquad p = \frac{1}{a}, \qquad q = -\frac{b}{a},$$

where

$$a = \frac{a_1 a_2}{1 - b_1 s_0^{(2)}}, \qquad b = b_2 + \theta \frac{b_1 a_2^2}{1 - b_1 s_0^{(2)}},$$

$$s_0^{(2)} = -\theta \frac{a_2}{\bar a_2} \bar b_2 \in H_+^\infty \qquad \text{(because of } j\text{-innerness)}.$$

The product of j-unitary matrices is j-unitary, hence $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Because b_1 and $s_0^{(2)}$ are in H_+^∞ and bounded in modulus by 1, $1 - b_1 s_0^{(2)}$ is outer. Using the Maximum Principle (see Section 1) one obtains $a \in H_+^\infty$, $b \in H_+^\infty$. Because a_1 and a_2 are outer, a is outer. Because $b_2(0) = 0$ and $b_1(0) = 0$, $b(0) = 0$. Hence (a, b) is γ-generating.

LEMMA 4.2. *If* $\begin{bmatrix} \bar p_1 & \bar q_1 \\ q_1 & p_1 \end{bmatrix}$ *is a Nehari matrix and* $\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar p_2 & \bar q_2 \\ q_2 & p_2 \end{bmatrix}$ *is a Sarason matrix, then*

$$\begin{bmatrix} \bar p & \bar q \\ q & p \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \bar\theta & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \bar p_1 & \bar q_1 \\ q_1 & p_1 \end{bmatrix} \cdot \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar p_2 & \bar q_2 \\ q_2 & p_2 \end{bmatrix} \tag{4.1}$$

is a Nehari matrix.

Proof. We consider the linear-fractional transformation

$$w = \frac{\bar p \omega + \bar q}{q\omega + p}, \qquad \omega \in \mathrm{Ball}(H_+^\infty). \tag{4.2}$$

We denote by s_0 the image of $\omega = 0$, $s_0 = \bar q / p = -(a/\bar a)\,\bar b$. Let w_0 be any solution of the Nehari problem with data s_0, i.e., $w_0 \in \mathrm{Ball}(L^\infty)$,

$w_0 - s_0 \in H^\infty_+$. To prove the lemma we need to find $\omega_0 \in \mathrm{Ball}(H^\infty_+)$, such that w_0 is the image of ω_0 under the transformation (4.2). Due to definition (4.1),

$$s_0 = \bar{\theta} \frac{\bar{p}_1 s_0^{(2)} + \bar{q}_1}{q_1 s_0^{(2)} + p_1},$$

where

$$s_0^{(2)} = \theta \frac{\bar{q}_2}{p_2} \in H^\infty_+.$$

Hence

$$\theta s_0 = \frac{\bar{p}_1 s_0^{(2)} + \bar{q}_1}{q_1 s_0^{(2)} + p_1}.$$

Because $\theta w_0 - \theta s_0 = \theta(w_0 - s_0) \in H^\infty_+$ and $\left[\begin{smallmatrix} \bar{p}_1 & \bar{q}_1 \\ q_1 & p_1 \end{smallmatrix}\right]$ is a Nehari matrix, there exists $\omega_1 \in \mathrm{Ball}(H^\infty_+)$ such that

$$\theta w_0 = \frac{\bar{p}_1 \omega_1 + \bar{q}_1}{q_1 \omega_1 + p_1}.$$

Hence

$$w_0 = \bar{\theta} \frac{\bar{p}_1 \omega_1 + \bar{q}_1}{q_1 \omega_1 + p_1}. \tag{4.3}$$

Due to the j-unitarity of $\left[\begin{smallmatrix} \bar{p}_1 & \bar{q}_1 \\ q_1 & p_1 \end{smallmatrix}\right]$,

$$\begin{aligned} w_0 - s_0 &= \bar{\theta} \frac{1}{q_1 \omega_1 + p_1} (\omega_1 - s_0^{(2)}) \frac{1}{q_1 s_0^{(2)} + p_1} \\ &= \bar{\theta} \frac{a_1}{1 - b_1 \omega_1} (\omega_1 - s_0^{(2)}) \frac{a_1}{1 - b_1 s_0^{(2)}}. \end{aligned}$$

Hence

$$\bar{\theta}(\omega_1 - s_0^{(2)}) = \frac{1 - b_1 \omega_1}{a_1} (w_0 - s_0) \frac{1 - b_1 s_0^{(2)}}{a_1}.$$

Due to the Maximum Principle, the right hand side is a H^∞_+ function. Hence ω_1 is the solution of the Sarason Problem with "data" $s_0^{(2)}$ and "spectrum" θ. Hence there exists ω_0 such that

$$\omega_1 = \theta \frac{\bar{p}_2 \omega_0 + \bar{q}_2}{q_2 \omega_0 + p_2}. \tag{4.4}$$

Combining (4.3) and (4.4), one obtains

$$w_0 = \frac{\bar{p}\omega_0 + \bar{q}}{q\omega_0 + p}.$$

5. More about the Sarason Problem

Let θ be an inner function, $K_\theta = H^2_+ \ominus \theta H^2_+$. Let w_0 be a given function in H^∞_+ bounded in modulus by 1. We will be concerned with the Sarason Problem with "data" w_0 and "spectrum" θ, i.e., one wants to describe all H^∞_+ functions w, bounded in modulus by 1, such that

$$\frac{w - w_0}{\theta} \in H^\infty_+.$$

It is well known (see, e.g., [17]) that this condition is equivalent to the following one:

$$P_+ \bar{w} x = P_+ \bar{w}_0 x, \qquad \forall x \in K_\theta.$$

So, we denote by W^* the linear operator from K_θ to H^2_+ acting by the formula

$$W^* x \stackrel{\text{def}}{=} P_+ \bar{w}_0 x, \qquad x \in K_\theta.$$

And we want to find all functions $w \in H^\infty_+$, w bounded in modulus by 1, such that

$$W^* x = P_+ \bar{w} x, \qquad x \in K_\theta.$$

This problem was solved by V. M. Adamyan, D. Z. Arov, and M. G. Krein [1–3] (see also [17, 6]).

Let $D(x, x)$ be the non-negative quadratic form on K_θ defined by

$$D(x, x) \stackrel{\text{def}}{=} \langle (I - WW^*) x, x \rangle \geqslant 0, \qquad x \in K_\theta,$$

where $W: H^2_+ \to K_\theta$ is the adjoint operator to W^*,

$$Wy = P_\theta w y, \qquad y \in H^2_+,$$

where P_θ is the orthogonal projection from H^2_+ onto K_θ, $\langle\ ,\ \rangle$ is the inner product on K_θ (induced from L^2).

Assumption. For our purposes we will need the following additional asumption about the given solution w_0: $\ln(1 - |w_0|^2) \in L^1$. This condition is equivalent to the existence of an outer function a_0 such that $1-|w_0|^2 = |a_0|^2$ a.e. on $\mathbb{T}$.

Under this assumption, $D(x, x)$ may be rewritten,

$$\begin{aligned} D(x,x) &= \langle x - w_0 P_+ \bar{w}_0 x, x\rangle \\ &= \langle (1 - w_0 \bar{w}_0)\, x, x\rangle + \langle w_0 P_- \bar{w}_0 x, x\rangle \\ &= \langle a_0 \bar{a}_0 x, x\rangle + \|P_- \bar{w}_0 x\|^2_{L^2} \\ &= \|\bar{a}_0 x\|^2_{L^2} + \|P_- \bar{w}_0 x\|^2_{L^2}. \end{aligned} \tag{5.1}$$

This representation permits one to introduce the following space (which is equivalent, in fact, to the completion of K_θ under the quadratic form $D(x, x)$),

$$H \stackrel{\text{def}}{=} \operatorname{clos}\left\{\begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}, x \in K_\theta\right\}.$$

We will denote vectors of H by $[{}^{h}_{h_-}]$. "clos" means the closure in the vector L^2 space of the unit circle $\mathbb{T}$, and the metric in H is induced from the L^2. We define the operator $T\colon K_\theta \to K_\theta$,

$$Tx \stackrel{\text{def}}{=} P_+ \bar{t} x, \qquad x \in K_\theta,$$

where t is the independent variable. Then one can check the identity

$$D(x, x) - D(Tx, Tx) = |x(0)|^2 - |(W^* x)(0)|^2.$$

This becomes

$$\left\|\begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}\right\|^2 - \left\|\begin{bmatrix} \bar{a}_0 Tx \\ P_- \bar{w}_0 Tx \end{bmatrix}\right\|^2 = |x(0)|^2 - |(W^* x)(0)|^2,$$

by means of representation (5.1). This identity permits one to introduce the isometric operator $V\colon H \oplus E_1 \to H \oplus E_2$ (where $E_1 = E_2 = \mathbb{C}^1$) with domain

$$d_V \stackrel{\text{def}}{=} \operatorname{clos}\left\{\begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \\ (W^* x)(0) \end{bmatrix}, x \in K_\theta\right\},$$

and the range

$$\Delta_V \stackrel{\text{def}}{=} \operatorname{clos}\left\{\begin{bmatrix} \bar{a}_0 Tx \\ P_- \bar{w}_0 Tx \\ x(0) \end{bmatrix}, x \in K_\theta\right\},$$

acting by the formula

$$V:\begin{bmatrix}\bar{a}_0 x\\ P_-\bar{w}_0 x\\ (W^*x)(0)\end{bmatrix}\mapsto\begin{bmatrix}\bar{a}_0 Tx\\ P_-\bar{w}_0 Tx\\ x(0)\end{bmatrix}.$$

The orthogonal complements of the domain and range (the so-called defect subspaces) play an important role in the investigation of the problem. We will denote them by N_{d_V} and N_{Δ_V},

$$N_{d_V}\overset{\text{def}}{\equiv}(H\oplus E_1)\ominus d_V,\qquad N_{\Delta_V}\overset{\text{def}}{\equiv}(H\oplus E_2)\ominus\Delta_V.$$

The following theorem is a special case of the D. Z. Arov's theorem [5, Theorem 1(b)].

THEOREM 5.1. $\dim N_{d_V}=1$.

Proof. It is obvious that the vector

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\notin\left\{\begin{bmatrix}\bar{a}_0 x\\ P_-\bar{w}_0 x\\ (W^*x)(0)\end{bmatrix}, x\in K_\theta\right\}.$$

We are going to show it does not belong to the closure of this set too. In fact,

$$\begin{aligned}a_0(0)\cdot(W^*x)(0)&=\langle (W^*x),\bar{a}_0\rangle_{L^2}\\&=\langle P_+\bar{w}_0 x,\bar{a}_0\rangle\\&=\langle \bar{w}_0 x-P_-\bar{w}_0 x,\bar{a}_0\rangle\\&=\langle a_0 x, w_0\rangle-\langle P_-\bar{w}_0 x,\bar{a}_0\rangle.\end{aligned}$$

If $\bar{a}_0 x_n \xrightarrow[n\to\infty]{} 0$ and $P_-\bar{w}_0 x_n\xrightarrow[n\to\infty]{} 0$ in L^2, then $a_0(0)\cdot(W^*x_n)(0)\xrightarrow[n\to\infty]{} 0$, and hence $(W^*x_n)(0)\to 0$. Thus

$$\operatorname{clos}\left\{\begin{bmatrix}\bar{a}_0 x\\ P_-\bar{w}_0 x\\ (W^*x)(0)\end{bmatrix}, x\in K_\theta\right\}\neq H\oplus E_1,$$

i.e., $\dim N_{d_V}\geqslant 1$. But if

$$\begin{bmatrix}h\\ h_-\\ c\end{bmatrix}\in N_{d_V}\qquad\text{and}\qquad c=0,$$

then $\left[\begin{smallmatrix}h\\h_-\end{smallmatrix}\right]\perp H$, and hence $\left[\begin{smallmatrix}h\\h_-\end{smallmatrix}\right]=0$. Thus $\dim N_{d_V}\leqslant 1$.

Remark. The following property was proved in [16, 10, 1]:

$$(\dim N_{d_V} = 1) \Rightarrow (\dim N_{\Delta_V} = 1).$$

LEMMA 5.2. *The vector*

$$\begin{bmatrix} h \\ h_- \\ c \end{bmatrix} \in N_{d_V}$$

iff

$$\begin{gathered} P_\theta(a_0 h + w_0 h_- + w_0 c) = 0 \\ \begin{bmatrix} h \\ h_- \end{bmatrix} \in \operatorname{clos}\left\{\begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}, x \in K_\theta\right\} = H. \end{gathered} \tag{5.2}$$

Proof. The second condition means nothing but $[{}^{h}_{h}] \in H$, and we rewrite it this way to stress the approximative sense of the definition of the space H. The first condition is a straightforward consequence of the orthogonality to d_V. Conditions (5.2) will play the key role in further constructions.

LEMMA 5.3. *Equation* (5.2) *has a unique* (*up to the constant factor*) *non-zero solution*

$$\begin{bmatrix} h^0 \\ h^0_- \\ c^0 \end{bmatrix}.$$

Proof. This is true because $\dim N_{d_V} = 1$. ∎

A Fourier representation is associated to any solution of the Sarason Problem (see [14]),

$$F^w x \stackrel{\text{def}}{=} \begin{bmatrix} 1 & w \\ \bar{w} & 1 \end{bmatrix} \begin{bmatrix} x \\ -W^* x \end{bmatrix}, x \in K_\theta.$$

It maps K_θ into de Branges–Rovnyak space H^w and (see [14]),

$$\|F^w x\|^2_{H^w} = D(x, x) = \left\| \begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix} \right\|^2. \tag{5.3}$$

Space H^w is defined as $f = [\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}] \in H^w$ if

$$f_+ \in H^2_+, \qquad f_- \in H^2_-, \qquad f(t) \in \operatorname{rank} \begin{bmatrix} 1 & w(t) \\ \bar{w}(t) & 1 \end{bmatrix}$$

for almost all $t \in \mathbb{T}$, and

$$\int_{\mathbb{T}} [\overline{f_+}(t), \overline{f_-}(t)] \begin{bmatrix} 1 & w(t) \\ \bar{w}(t) & 1 \end{bmatrix}^{[-1]} \begin{bmatrix} f_+(t) \\ f_-(t) \end{bmatrix} dm(t) < \infty,$$

where $dm(t)$ is normalized Lebesgue measure on the unit circule $\mathbb{T}$. The last integral defines the inner product on H^w and turns it into a complete Hilbert space. We refer to [11, 15, 18, 19] for details concerning the space H^w.

So, in particulary, $F^w x$ has two components

$$F^w x = \begin{bmatrix} F^w_+ x \\ F^w_- x \end{bmatrix},$$

which lie in H^2_+ and H^2_- correspondingly. Due to (5.3), one can reinterpret F^w as an isometric mapping defined on H. For the given solution w_0, this version of F^{w_0} admits a simple explicit representation. In fact,

$$\begin{aligned} F^{w_0} x &= \begin{bmatrix} x - w_0 P_+ \bar{w}_0 x \\ P_- \bar{w}_0 x \end{bmatrix} \\ &= \begin{bmatrix} (1 - w_0 \bar{w}_0)\, x + w_0 P_- \bar{w}_0 x \\ P_- \bar{w}_0 x \end{bmatrix} \\ &= \begin{bmatrix} a_0 \cdot \bar{a}_0 x + w_0 \cdot P_- \bar{w}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}. \end{aligned}$$

So, $F^{w_0} x = [\begin{smallmatrix} a_0 h + w_0 h_- \\ h_- \end{smallmatrix}]$, where $[\begin{smallmatrix} h \\ h_- \end{smallmatrix}] = [\begin{smallmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{smallmatrix}]$, $x \in K_\theta$. Hence, F^{w_0}, viewed as a mapping defined on H, is given by the formula

$$F^{w_0} \begin{bmatrix} h \\ h_- \end{bmatrix} = \begin{bmatrix} a_0 h + w_0 h_- \\ h_- \end{bmatrix}, \qquad \begin{bmatrix} h \\ h_- \end{bmatrix} \in H, \tag{5.4}$$

and

$$\left\| F^{w_0} \begin{bmatrix} h \\ h_- \end{bmatrix} \right\|^2_{H^{w_0}} = \left\| \begin{bmatrix} h \\ h_- \end{bmatrix} \right\|^2_H .$$

One can extend the isometry V to a unitary colligation $A: H \oplus E_1 \oplus N_2 \to H \oplus E_2 \oplus N_1$, where $N_1 = N_2 = \mathbb{C}$, in the following way:

$A \mid d_V = V$, $A \mid N_{d_V}$ is a unitary mapping onto N_1

$A \mid N_2$ is a unitary mapping onto N_{Δ_V}.

The scattering matrix of this colligation is defined as

$$S(\zeta) = P_{N_1 \oplus E_2}(I - \zeta A P_H)^{-1} A \mid_{N_2 \oplus E_1}.$$

$S(\zeta)$ is a 2×2 contractive inner matrix-function on $\mathbb{D}$ [1–3, 14, 22, 15]. S has the following structure (see [13]),

$$S = \begin{bmatrix} b & a \\ \theta a & s_0 \end{bmatrix},$$

where (a, b) is a Nehari pair. The solutions of the Sarason Problem are described as

$$w = s_0 + \theta a \omega (1 - b\omega)^{-1} a = \theta \frac{a}{\bar{a}} \frac{\omega - \bar{b}}{1 - \omega b} = \theta \frac{\bar{p}\omega + \bar{q}}{q\omega + p},$$
$$\omega \in \operatorname{Ball}(H_+^\infty), \tag{5.5}$$

where $p = 1/a$, $q = -b/a$.

The following formula was proved in [12, 11]:

$$F^w P_H \mid N_{d_V} + \begin{bmatrix} w \\ 1 \end{bmatrix} P_{E_1} \mid N_{d_V} = \begin{bmatrix} \theta a (1 - \omega b)^{-1} \omega \\ \bar{a}(1 - \bar{\omega}\bar{b})^{-1} \end{bmatrix} \cdot P_{N_1} A \mid N_{d_V}, \tag{5.6}$$

where w is a solution of the Sarason Problem, ω is the parameter corresponding to the w under formula (5.5). Putting the (unique) non-zero vector

$$\begin{bmatrix} h^0 \\ h_-^0 \\ c^0 \end{bmatrix} \in N_{d_V}$$

into (5.6), one obtains

$$F^w \begin{bmatrix} h^0 \\ h_-^0 \end{bmatrix} + \begin{bmatrix} w \\ 1 \end{bmatrix} c^0 = \begin{bmatrix} \theta a (1 - \omega b)^{-1} \omega \\ \bar{a}(1 - \bar{\omega}\bar{b})^{-1} \end{bmatrix} \cdot \tilde{c}^0, \tag{5.7}$$

where

$$\tilde{c}^0 = A \begin{bmatrix} h^0 \\ h_-^0 \\ c^0 \end{bmatrix}$$

is a constant, $|\tilde{c}^0|^2 = |c^0|^2 + \| [\begin{smallmatrix} h^0 \\ h_-^0 \end{smallmatrix}] \|^2 \neq 0$.

Let ω_0 be the parameter corresponding to the given solution w_0 under (5.5). By means of (5.4), equality (5.7) turns into

$$\begin{aligned} a_0 h^0 + w_0 h^0_- + w_0 c^0 &= \theta a(1-\omega_0 b)^{-1}\,\omega_0 \cdot \tilde{c}^0 \\ h^0_- + c^0 &= \bar{a}(1-\bar{\omega}_0 \bar{b})^{-1} \cdot \tilde{c}^0. \end{aligned} \tag{5.8}$$

So, we obtained some additional information about the vector

$$\begin{bmatrix} h^0 \\ h^0_- \\ c^0 \end{bmatrix} \in N_{d_V}.$$

LEMMA 5.4. *If*

$$\begin{bmatrix} h^0 \\ h^0_- \\ c^0 \end{bmatrix} \in N_{d_V}$$

then

$$a_0 h^0 + w_0 h^0_- + w_0 c^0 = \theta a(1-\omega_0 b)^{-1}\,\omega_0 \cdot \tilde{c}^0$$

$$h^0_- + c^0 = \bar{a}(1-\bar{\omega}_0 \bar{b})^{-1} \cdot \tilde{c}^0,$$

$$|\tilde{c}^0|^2 = |c^0|^2 + \left\| \begin{bmatrix} h^0 \\ h^0_- \end{bmatrix} \right\|^2, \tag{5.9}$$

$$\begin{bmatrix} h^0 \\ h^0_- \end{bmatrix} \in \operatorname{clos} \left\{ \begin{bmatrix} \bar{a}_0 x \\ P - w_0 x \end{bmatrix}, x \in K_\theta \right\} = H.$$

Now we can answer the following question: How can one recognize, given the solution w_0 and the spectrum θ, whether the corresponding parameter ω_0 is equal to zero or not?

THEOREM 5.5. $\omega_0 = 0$ *iff*

$$\begin{bmatrix} -\dfrac{\bar{a}_0}{a_0} w_0 \\ P_- \bar{a}_0 \end{bmatrix} \in \operatorname{clos} \left\{ \begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}, x \in K_\theta \right\} = H. \tag{5.10}$$

Proof. (1) Let (5.10) be true. Then the vector

$$\begin{bmatrix} h^0 \\ h^0_- \\ c^0 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} -(\bar{a}_0/a_0)\, w_0 \\ P_- \bar{a}_0 \\ \bar{a}_0(0) \end{bmatrix}$$

satisfies the equality

$$\begin{cases} a_0 h^0 + w_0 h^0_- + w_0 c^0 = 0 \\ \begin{bmatrix} h^0 \\ h^0_- \end{bmatrix} \in H. \end{cases} \tag{5.11}$$

Hence it also satsifies (5.2). Then, by Lemma 5.2 it lies in N_{d_V}. Hence, by Lemma 5.4 it has to satsify (5.9) and

$$|\tilde{c}^0|^2 = \left\| \begin{bmatrix} h^0 \\ h^0_- \end{bmatrix} \right\|^2 + |c^0|^2 = 1.$$

Comparing (5.11) and (5.9), one obtains

$$\theta a(1 - \omega_0 b)^{-1} \omega_0 = 0.$$

Hence $\omega_0 = 0$.

(2) Let $\omega_0 = 0$. By Lemma 5.4 the (unique up to the constant factor) non-trivial vector

$$\begin{bmatrix} h^0 \\ h^0_- \\ c^0 \end{bmatrix} \in N_{d_V}$$

must satisfy (5.9) with $\omega_0 = 0$, i.e.,

$$\begin{aligned} a_0 h^0 + w_0 h^0_- + w_0 c^0 &= 0 \\ h^0_- + c^0 &= \bar{a} \cdot \tilde{c}^0, \end{aligned} \tag{5.12}$$

and constant $\tilde{c}^0 \neq 0$. It follows from (5.5) that $\omega_0 = 0$ yields $w_0 = s_0$. Because S is inner, $|a|^2 = 1 - |s_0|^2$. By definition (see the Assumption at the beginning of this section) $|a_0|^2 = 1 - |w_0|^2$. Hence $|a| = |a_0|$. Because a and a_0 are outer, coinsidence of their moduli implies $a = a_0 \cdot k$, where k is a constant of modulus 1. One can choose $\tilde{c}^0 = \bar{k}$, then he will obtain from (5.12)

$$h^0_- + c^0 = \bar{a}_0$$

$$h^0 = -\frac{\bar{a}_0}{a_0} w_0,$$

or

$$\begin{bmatrix} h^0 \\ h^0_- \\ c^0 \end{bmatrix} = \begin{bmatrix} -(\bar{a}_0/a_0)\, w_0 \\ P_- \bar{a}_0 \\ \bar{a}_0(0) \end{bmatrix}$$

Hence the vector

$$\begin{bmatrix} -\dfrac{\bar{a}_0}{a_0} w_0 \\ P_- \bar{a}_0 \end{bmatrix}$$

has to lie in H.

6. Regularization of γ-Generating Pairs. Katsnelson's Approximation Approach

Let (a_0, b_0) be an A-singular γ-generaring pair, i.e.

$$w_0 \overset{\text{def}}{=} -\frac{a_0}{\bar{a}_0} \bar{b}_0 \in H^\infty_+. \tag{6.1}$$

Obviously, this function w_0 satisfies the Assumption of the previous section. In fact, $1 - |w_0|^2 = 1 - |b_0|^2 = |a_0|^2$, but $\ln |a_0|^2 \in L^1$, because $a_0 \in H^\infty_+$. In comparison with the situation we met in the previous section, we now have additional condition (6.1), which means, in other words, the "pseudo-continuability" property

$$\frac{w_0}{a_0} = -\frac{\bar{b}_0}{\bar{a}_0}, \qquad \text{a.e. on } \mathbb{T}. \tag{6.2}$$

The left side is analytic on $\mathbb{D}$, the right side is antianalytic on $\mathbb{D}$, and the boundary values coincide almost everywhere. Let $\left[\begin{smallmatrix} \bar{p}_0 & \bar{q}_0 \\ q_0 & p_0 \end{smallmatrix}\right]$ be the A-singular γ-generating matrix corresponding to the A-singular γ-generating pair (a_0, b_0). Let θ be an inner function. Then

$$\begin{bmatrix} \bar{p}_0 & \bar{q}_0 \\ q_0 & p_0 \end{bmatrix} \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{p}_0 & \overline{q_0 \theta} \\ q_0 \theta & p_0 \end{bmatrix} \tag{6.3}$$

is a j-inner matrix. The corresponding scattering matrix is $\left[\begin{smallmatrix} b_0 \theta & a_0 \\ \theta a_0 & w_0 \end{smallmatrix}\right]$.

The goal of this section is to prove the following theorem

Theorem 6.1. *For any A-singular γ-generating pair (a_0, b_0), there exists an inner function θ such that the matrix (6.3) is a Sarason matrix.*

The main theorem of this paper follows from the previous one by using composition lemmas of Section 4.

Main Theorem. *For any γ-generating pair (a, b) there exists an inner function θ such that $(a, b\theta)$ is a Nehari pair.*

Proof. Let $\left[\begin{smallmatrix} \bar{p} & \bar{q} \\ q & p \end{smallmatrix}\right]$ be the γ-generating matrix corresponding to the pair (a, b), and let $\left[\begin{smallmatrix} \bar{p} & \bar{q} \\ q & p \end{smallmatrix}\right] = \left[\begin{smallmatrix} \bar{p}_1 & \bar{q}_1 \\ q_1 & p_1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \bar{p}_2 & \bar{q}_2 \\ q_2 & p_2 \end{smallmatrix}\right]$ be its A-regular–A-singular decomposition (see Section 2). The first multiple is a Nehari matrix, the second one is an A-singular γ-generating matrix. According to Theorem 6.1, one can choose the inner function θ in such a way that the matrix.

$$\begin{bmatrix} \bar{p}_2 & \bar{q}_2 \\ q_2 & p_2 \end{bmatrix}\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{p}_2 & \overline{q_2 \theta} \\ q_2 \theta & p_2 \end{bmatrix}$$

becomes a Sarason matrix. Using composition Lemma 4.2, one obtain that the matrix

$$\begin{aligned} \begin{bmatrix} \bar{p} & \overline{q\theta} \\ q\theta & p \end{bmatrix} &= \begin{bmatrix} \bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{p} & \bar{q} \\ q & p \end{bmatrix} \cdot \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{p}_1 & \bar{q}_1 \\ q_1 & p_1 \end{bmatrix} \cdot \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{p}_2 & \overline{q_2\theta} \\ q_2\theta & p_2 \end{bmatrix}, \end{aligned}$$

is a Nehari matrix. Hence the corresponding γ-generating pair $(a, b\theta)$ is a Nehari pair.

Now we are on the way to the proof of Theorem 6.1. The main tool is the Katsnelson's approximation [10, 24]. Let (a_0, b_0) be an A-singular γ-generating pair, θ be an inner function, $w_0 = -(a_0/\bar{a}_0)\,\bar{b}_0 (\in H^{\infty}_{+})$. We consider the j-inner matrix $\left[\begin{smallmatrix} \bar{p}_0 & \bar{q}_0 \\ q_0 & p_0 \end{smallmatrix}\right]\left[\begin{smallmatrix} \theta & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and the corresponding scattering matrix

$$\begin{bmatrix} b_0\theta & a_0 \\ \theta a_0 & w_0 \end{bmatrix}. \tag{6.4}$$

One can consider the Sarason Problem with "data" w_0 and "spectrum" θ. Let

$$\begin{bmatrix} b^\theta & a^\theta \\ \theta a^\theta & s_0^\theta \end{bmatrix} \tag{6.5}$$

be the scattering matrix of this problem, i.e., the formula

$$w = s_0^\theta + \theta a^\theta \omega (1 - b^\theta \omega)^{-1} a^\theta, \qquad \omega \in \mathrm{Ball}(H^{\infty}_{+}) \tag{6.6}$$

gives the parametrization of solutions of this problem. The superscript θ shows that the pairs (a^θ, b^θ) are different for different θ.

We are interested in the case when the scattering matrix (6.4) is a Sarason scattering matrix, i.e., when

$$\begin{bmatrix} b_0\theta & a_0 \\ \theta a_0 & w_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b^\theta & a^\theta \\ \theta a^\theta & s_0^\theta \end{bmatrix} \tag{6.7}$$

are equivalent (i.e., $a^\theta = a_0 \cdot c$, $b^\theta = b_0\theta \cdot c^2$, where c is a constant of modulus 1, $w_0 = s_0^\theta$).

THEOREM 6.2. *Let ω_0^θ be the parameter corresponding to the solution w_0 under the parametrization* (6.6). *Then* (6.7) *is true iff* $\omega_0^\theta = 0$

Proof. Obviously, $\omega_0^\theta = 0 \Leftrightarrow w_0 = s_0^\theta$. If matrices (6.7) are equivalent then, in particular, $w_0 = s_0^\theta$, and hence $\omega_0^\theta = 0$. Vise versa: let $\omega_0^\theta = 0$, then $w_0 = s_0^\theta$. Because $|a_0|^2 = 1 - |w_0|^2$ and $|a^\theta|^2 = 1 - |s_0^\theta|^2$,

$$w_0 = s_0^\theta \Rightarrow |a_0|^2 = |a^\theta|^2.$$

But a_0 and $a^\theta$are outer functions. Hence

$$a^\theta = c \cdot a_0,$$

where c is a constant of modulus 1. The matrix (6.5) is inner. Hence

$$b^\theta = -\theta \frac{a^\theta}{\overline{a^\theta}} \overline{s_0^\theta}.$$

According to definition (6.1)

$$b_0 = -\frac{a_0}{\bar{a}_0} \bar{w}_0, \qquad \text{or} \qquad b_0\theta = -\theta \frac{a_0}{\bar{a}_0} \bar{w}_0.$$

Hence $b^\theta = c^2 \cdot b_0\theta$. So, the two scattering matrices are equivalent. ∎

According to Theorem 5.5, $\omega_0^\theta = 0$ iff

$$\begin{bmatrix} \bar{b}_0 \\ P_- \bar{a}_0 \end{bmatrix} \in \operatorname{clos} \left\{ \begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}, x \in K_\theta \right\}. \tag{6.8}$$

We write $\bar{b}_0$ instead of $-(\bar{a}_0/a_0)\, w_0$, according to definition (6.1). So, to prove Theorem 6.1, we have to choose θ such that (6.8) is true. Here we are following Ref. [10, Sect. 1, Sect. 7)] Let

$$\tilde{x}_\varepsilon \overset{\text{def}}{=} -\frac{w_0/a_0}{1 + \varepsilon\, |w_0/a_0|^2}, \qquad \text{a.e. on } \mathbb{T}. \tag{6.9}$$

$\tilde{x}_\varepsilon \in L^\infty$ and $\tilde{x}_\varepsilon$ permits both meromorphic (ratio of two bounded analytic functions) and antimeromorphic (complex conjugate to the ratio of two bounded analytic functions) continuations on $\mathbb{D}$ (by means of (6.2)),

$$\tilde{x}_\varepsilon = -\frac{w_0/a_0}{1 + \varepsilon(w_0/a_0) \cdot (\bar{w}_0/\bar{a}_0)} = -\frac{w_0/a_0}{1 - \varepsilon(w_0/a_0)(b_0/a_0)} = -\frac{w_0 a_0}{a_0^2 - \varepsilon w_0 b_0}$$

and

$$\tilde{x}_\varepsilon = \frac{(\bar{b}_0/\bar{a}_0)}{1-\varepsilon(\bar{b}_0/\bar{a}_0)(\bar{w}_0/\bar{a}_0)} = \frac{\overline{a_0 b_0}}{a_0^2 - \varepsilon b_0 w_0}.$$

Let $a_0^2 - \varepsilon w_0 b_0 = \theta_\varepsilon \phi_\varepsilon$ be the inner-outer factorization. Then, due to the Maximum Principle,

$$x_\varepsilon \overset{\text{def}}{=} \theta_\varepsilon \tilde{x}_\varepsilon = -\frac{w_0 a_0}{\varphi_\varepsilon} \in H_+^\infty,$$

$$y_\varepsilon \overset{\text{def}}{=} \bar{\theta}_\varepsilon \tilde{x}_\varepsilon = \frac{\overline{b_0 a_0}}{\bar{\varphi}_\varepsilon} \in H_-^\infty$$

($y_\varepsilon(0) = 0$, because $b_0(0) = 0$). But this means that $x_\varepsilon \in H_+^2$ and $\overline{\theta_\varepsilon^2} x_\varepsilon \in H_-^2$, i.e., $x_\varepsilon \in K_{\theta_\varepsilon^2}$.

Now to prove Theorem 6.1 we need two lemmas:

LEMMA 6.1. *One can choose the sequence $\varepsilon_k \downarrow 0$ such that the product $\theta = \prod_{k=1}^\infty \theta_{\varepsilon_k}^2$ is convergent in the L^2 sense, and hence defines an inner function θ.*

Remark. $x_{\varepsilon_k} \in K_{\theta_{\varepsilon_k}^2} \Rightarrow x_{\varepsilon_k} \in K_\theta$.

LEMMA 6.2.

$$\begin{bmatrix} \bar{a}_0 x_\varepsilon \\ P_- \bar{w}_0 x_\varepsilon \end{bmatrix} \xrightarrow[\varepsilon \downarrow 0]{} \begin{bmatrix} \bar{b}_0 \\ P_- \bar{a}_0 \end{bmatrix} \quad \textit{in } L^2.$$

Proof of Theorem 6.1. Combining Lemmas 6.1 and 6.2, one obtains

$$\begin{bmatrix} \bar{b}_0 \\ P_- \bar{a}_0 \end{bmatrix} \in \operatorname{clos}\left\{ \begin{bmatrix} \bar{a}_0 x \\ P_- \bar{w}_0 x \end{bmatrix}, x \in K_\theta \right\}.$$

Hence θ is the function we are searching for. This finishes the proof of Theorem 6.1.

Proof of Lemma 6.1. The function $\varphi_\varepsilon(\zeta)$, $|\zeta| < 1$, is defined by

$$\varphi_\varepsilon(\zeta) = \exp\left\{ \int_{\mathbb{T}} \frac{t+\zeta}{t-\zeta} \ln |a_0^2 - \varepsilon \cdot w_0 b_0| \, dm(t) \right\} \cdot \frac{a_0(0)}{\overline{a_0(0)}}.$$

Note that

$$|a_0^2 - \varepsilon \cdot w_0 b_0| = |a_0^2| \cdot \left| 1 + \varepsilon \left| \frac{w_0}{a_0} \right|^2 \right|.$$

Hence $|a_0|^2 \leqslant |a_0^2 - \varepsilon \cdot w_0 b_0| \leqslant 1 + \varepsilon$. The last estimates mean that the family

$$\ln |a_0^2 - \varepsilon \cdot w_0 b_0|$$

has a summable majorant. But $a_0^2 - \varepsilon w_0 b_0 \xrightarrow[\varepsilon \downarrow 0]{} a_0^2$, a.e. on $\mathbb{T}$. Hence, by the Dominated Convergence Theorem, $\varphi_\varepsilon(\zeta) \xrightarrow[\varepsilon \downarrow 0]{} a_0^2(\zeta)$, $\forall \zeta$, $|\zeta| < 1$. Hence $\theta_\varepsilon(\zeta) \to 1$, $|\zeta| < 1$. It is enough now to consider $\zeta = 0$. Due to the equality $\|1 - \theta_\varepsilon\|_{L^2}^2 = 2 - \theta_\varepsilon(0) - \overline{\theta_\varepsilon(0)}$, $\theta_\varepsilon \to 1$ in L^2 (because $\theta_\varepsilon(0) \to 1$). One can choose a sequence $\varepsilon_k \downarrow 0$ such that the product $\prod_{k=1}^{\infty} \theta_{\varepsilon_k}^2(0)$ is convergent. Hence the product $\prod_{k=1}^{\infty} \theta_{\varepsilon_k}^2$ converges in L^2.

Proof of Lemma 6.2. $\bar{a}_0 x_\varepsilon = \theta_\varepsilon \bar{b}_0 / (1 + \varepsilon |w_0/a_0|^2)$,

$$\bar{a}_0 x_\varepsilon - \bar{b}_0 = \theta_\varepsilon \left(\frac{\bar{b}_0}{1 + \varepsilon |w_0/a_0|^2} - \bar{b}_0 \right) + (\theta_\varepsilon - 1)\, \bar{b}_0,$$

$$\|\bar{a}_0 x_\varepsilon - \bar{b}_0\|_{L^2} \leqslant \left\| \frac{\bar{b}_0}{1 + \varepsilon |w_0/a_0|^2} - \bar{b}_0 \right\|_{L^2} + \|\theta_\varepsilon - 1\|_{L^2} \xrightarrow[\varepsilon \downarrow 0]{} 0.$$

(The first term in the right side tends to zero due to the Dominated Convergence Theorem, the second due to Lemma 6.1.) We should check now that $P_- \bar{w}_0 x_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} P_- \bar{a}_0$.

$$\begin{aligned} \bar{w}_0 x_\varepsilon &= -\theta_\varepsilon \frac{\bar{w}_0 w_0 / a_0}{1 + \varepsilon\, |w_0/a_0|^2} \\ &= -\theta_\varepsilon \frac{(1 - \bar{a}_0 a_0)/a_0}{1 + \varepsilon\, |w_0/a_0|^2} \\ &= -\theta_\varepsilon \frac{1/a_0}{1 + \varepsilon\, |w_0/a_0|^2} + \theta_\varepsilon \frac{\bar{a}_0}{1 + \varepsilon |w_0/a_0|^2}. \end{aligned}$$

The second term tends to $\bar{a}_0$ in L^2, due to the Dominated Convergence Theorem (to see this, one can subtract and add $\theta_\varepsilon \bar{a}_0$ and use Lemma 6.1). The first term can be tranformed as

$$\begin{aligned} -\theta_\varepsilon \frac{1/a_0}{1 + \varepsilon\, |w_0\, /a_0|^2} &= -\theta_\varepsilon \frac{1/a_0}{1 + \varepsilon (w_0/a_0) \cdot \bar{w}_0/\bar{a}_0} = -\theta_\varepsilon \frac{1/a_0}{1 - \varepsilon (w_0/a_0)(b_0/a_0)} \\ &= -\theta_\varepsilon \frac{a_0}{a_0^2 - \varepsilon w_0 b_0} = -\frac{a_0}{\varphi_\varepsilon}. \end{aligned}$$

It lies in L^∞, because of the estimate (using $|w_0|^2 = 1 - |a_0|^2$)

$$\frac{|1/a_0|}{1 + \varepsilon\, |w_0/a_0|^2} = \frac{|a_0|}{|a_0|^2 + \varepsilon |w_0|^2} = \frac{|a_0|}{\varepsilon + (1 - \varepsilon)\, |a_0|^2} \leqslant \frac{|a_0|}{\varepsilon}.$$

But φ_ε is outer, hence $a_0/\varphi_\varepsilon \in H^\infty_+$. Thus, it is "killed" by P_-. This proves $P_- \bar{w}_0 x_\varepsilon \to P_- \bar{a}_0$, and finishes the proof of the lemma.

Remark. Using the Frostman–Rudin theorem (see [9, 10]), one can choose $\varepsilon_k \downarrow 0$ such that θ_{ε_k} is a Blaschke product. And, hence, θ will be a Blaschke product.

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