On stochastic setting of stationary phase method

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ABSTRACT

A probability model \[
\int_{\mathbb{R}} \exp \left( i \left[nP(x) \right] \right) d\Phi_n(x)
\] with \( \Phi_n(x) \) the distribution function of random variable \( \zeta_n = \frac{1}{n} \sum_{k=1}^{n} \xi_k \) (\( \xi_k \) is i.i.d. sequence of r.v.’s with zero expectation and unit variance), being in a framework of stationary phase method is analyzed. The asymptotic expansion in CLT and Hörmander’s theorem play crucial role in asymptotic analysis of the model.

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1. Introduction

In this paper, we consider stochastic version of the stationary phase method

\[
\int_{\mathbb{R}} u(x) e^{i[nP(x)]} d\Phi_n(x) = \mathbb{E} u(\zeta_n) e^{inP(\zeta_n)},
\]

where \( P(x) \) is a real valued sufficiently smooth function, \( \Phi_n \) is a distribution function of random variable \( \zeta_n = \frac{1}{n} \sum_{k=1}^{n} \xi_k \), and \( (\xi_k)_{k \geq 1} \) is i.i.d. sequence of random variables. It is possible to give a physical interpretation of the formulation, however, this issue is outside of our presentation. We just mention that W. Kelvin introduced stationary phase method in 1887 to calculate the trace of a ship in a calm water, so one can take the stochastic setting as a step towards a solution of this problem for the case of a turbulent water.

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Our setting is reminiscent of large deviations, therefore to give reasons for the approach we start with the Laplace integral

\[ f(\omega) = \int_a^b u(x) e^{\omega f(x)} \, dx, \]

where \( f(x), u(x) \) are real valued functions, \( \omega > 0 \) is a large real parameter. Assume that \( f(x) \) is sufficiently smooth function and \( \max_{x \in [a, b]} f(x) = f(x_0) \) for a unique \( x_0 \in [a, b], f''(x_0) < 0 \) and \( u(x_0) \neq 0 \). It is well known (see e.g. [8]) that

\[ J(\omega) \sim \omega \to +\infty \sqrt{-\frac{2\pi}{\omega f''(x_0)}} u(x_0) e^{\omega f(x_0)}. \]  

(1.2)

The large deviation theory being a natural stochastic generalization of Laplace principle takes into consideration (the simplest case)

\[ J_n = \int_{\mathbb{R}} e^{\eta f(x)} d\Phi_n(x) \]

and examine a logarithmic asymptotics of this integral.

In general let \( \{P_\lambda\} \) be a family of probability measures on a Polish space \( X \) and let \( I \) be a nonnegative function on \( X \) with compact level sets. Following Bryc [1] and Puhalskii [2] we say that \( \{P_\lambda\} \) obeys the large deviation principle with a rate function \( I \) if and only if

\[ \lim_{\lambda \to \infty} \left[ \int_X (g(x))^\lambda P_\lambda(dx) \right]^{1/\lambda} = \sup_{x \in X} g(x)e^{-I(x)} \]

for all bounded continuous nonnegative functions \( g \) on \( X \).

It means in our case that for all bounded from above continuous functions \( f(x) \) on \( \mathbb{R} \)

\[ \lim_{n \to \infty} \frac{1}{n} \log \int_{\mathbb{R}} e^{\eta f(x)} d\Phi_n(x) = \sup_{x \in \mathbb{R}} [f(x) - I(x)] \]  

(1.3)

with some non-negative function \( I(x) \).

We say that in this sense \( \{P_\lambda\} \) converges to an idempotent measure or Choquet’s capacity

\[ \prod(A) = \sup_{x \in A} \exp\{-I(x)\} \]  

(see e.g. [3–5]). This idempotent measure specifies rough logarithmic asymptotics of Laplace method and determines so called max-integral \( \sup_{x \in X} g(x)e^{-I(x)} \) as the object of idempotent analysis.

Recall that more usual Varadhan’s definition [11] involves measures instead of integrals. More precisely a family \( \{P_\lambda\} \) obeys the large deviation principle if for all closed sets \( F \subset X \)

\[ \limsup_{\lambda \to \infty} \frac{1}{\lambda} \ln P_\lambda(F) \leq -\inf_{x \in F} I(x) \]

and for all open sets \( G \subset X \)

\[ \liminf_{\lambda \to \infty} \frac{1}{\lambda} \ln P_\lambda(G) \geq -\inf_{x \in G} I(x). \]

However, the equivalent Puhalskii and Bryc definition is more natural in the context of our discussion. We also note that some asymptotic problems of quantum mechanics lead to a more general non-commutative analog of large deviations without evident probabilistic representation behind where only the definition in terms of functionals can be applied [5].
It is well known (see, e.g. [6]) that Maslov dequantization gives the natural passage from classical to max-plus algebra. Generalizations of Maslov dequantization give rise to mathematical constructions of tropical mathematics in different areas of application. Large deviations, Laplace and stationary phase methods are a way to go from classical algebra to tropical algebra and these are indeed motivations to study tropical algebra for itself.

The stationary phase method is applied to Fourier’s integral

$$J(\omega) = \int_a^b u(x)e^{i\omega f(x)}dx,$$

where \(\omega > 0\) is a large real parameter, \(f(x), u(x)\) are real valued smooth functions. Since \(e^{i\omega f(x)}\) is a fast oscillating function for a large \(\omega\), integrals \(J(\omega)\) take small values in contrast to the asymptotics of Laplace’s integral. Nevertheless the result (see e.g. [8,9])

$$J(\omega) = \int_{\mathbb{R}} u(x)e^{i\omega f(x)}dx \sim_{\omega \to +\infty} \sqrt{\frac{2\pi}{\omega |f''(x_0)|}} \exp \left(i\omega f(x_0) + \frac{1}{4} \frac{\omega}{|f''(x_0)|} \right) u(x_0) \quad (1.4)$$

for \(x_0\) such that \(f'(x_0) = 0\) looks similar to (1.2) except for the phase shift \(\frac{1}{4} \frac{\omega}{|f''(x_0)|}\) where \(\nu = \text{sign} f''(x_0)\), \(f''(x_0) \neq 0\) (note that this term does not depend on parameter \(\omega\)).

Let \(f(x) = \text{Re} f(x) + i\text{Im} f(x) := U(x) + iV(x)\) be a complex valued function, \(V(x) = \text{Im} f(x) \geq 0\). Hörmander [7] considers (we present here only one dimensional version of his result)

$$J(\omega) = \int_{\mathbb{R}} u(x)e^{i\omega f(x)}dx = \int_{\mathbb{R}} e^{i\omega U(x)}u(x)e^{-\omega V(x)}dx \quad (1.5)$$

and gives the precise formulation of the stationary phase method.

Set \(g_{x_0}(x) = f(x) - f(x_0) - \frac{1}{2}f''(x_0)(x-x_0)^2\).

**Theorem 1.1** ([7], Theorem 7.5.5). Let \(K \subset \subset \mathbb{R}\) be a compact set and \(X \supset K\) be an open neighborhood of \(K\). Assume that \(u \in C^0_0(K), f \in C^{k+1}(X), \text{Im} f \geq 0\) in \(X, \text{Im} f(x_0) = 0, f'(x_0) = 0, f''(x_0) \neq 0, f'(x) \neq 0\) in \(K \setminus \{x_0\}\). Then for all positive integers \(k\)

$$\left| \int_{\mathbb{R}} u(x)e^{i\omega f(x)}dx - \sqrt{\frac{2\pi}{\omega |f''(x_0)|}} \exp \left(i\omega f(x_0) - \frac{1}{2} \arctan \left[ \frac{-U''(x_0)}{V''(x_0)} \right] \right) \sum_{j<k} \omega^{-j}L_j u \right| \leq \omega^{-k}C \sum_{\alpha \leq 2k} \sup_{\alpha} \left| \frac{d^\alpha}{dx^\alpha} u \right|,$$

where \(L_j u = \sum_{(\nu - \mu = j), (2\nu \geq 3\mu)} (-1)^{j-1} y^{j-\nu} \left[ f''(x_0)^{-1} \frac{d^2}{dx^2} \right]^{\nu} (g_{x_0}^{\nu}\mu)(x_0) \frac{1}{\mu! \nu!}.\)

Notice that if \(\Phi_n\) in (1.1) is the distribution function of the \((0,1/n)\)-Gaussian random variable then

$$\int_{\mathbb{R}} u(x)e^{inP(x)}d\Phi_n(x) = \sqrt{\frac{n}{2\pi}} \int_{\mathbb{R}} u(x)e^{inP(x)}e^{-\frac{n}{2}x^2}dx.$$

It is clear that in this case we are in the framework of Theorem 1.1. Hence our setting is a natural stochastic generalization of the classical problem (similarly to large deviations) and we can expect a kind of exponential asymptotics for the problem (1.1). However in contrast to large deviations we cannot consider a rough logarithmic asymptotics for this model since a phase correction \(\frac{1}{2} \arctan(\cdot)\) would disappear.
It is appropriate to note that it is possible to study sharp large deviations as well. This group of methods is based on an expansion of the tail probability $P_\lambda(A), A \subset X$ (see e.g. [12–14]).

2. Problem formulation and preliminaries

Let $(\xi_k)_{k \geq 1}$ be a sequence of i.i.d. of random variables with $E\xi_1 = 0, E\xi_1^2 = 1$. Set $\zeta_n = \frac{1}{n} \sum_{k=1}^{n} \xi_k$ and denote $\Phi_n$ the distribution function of $\zeta_n$.

The aim of this paper is to establish an asymptotic behavior in $n \to \infty$ of the integral

$$\int_{\mathbb{R}} \exp \left[ \frac{1}{n} P(x) \right] d\Phi_n(x) = E \exp \left[ \frac{1}{n} P(\zeta_n) \right], \quad \iota = \sqrt{-1}.$$  \hspace{1cm} (2.1)

where $P(x)$ is sufficiently smooth function with $P'(0) = 0$.

Set $S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \xi_k$, denote $F_n$ the distribution function of $S_n$ and remark that

$$\int_{\mathbb{R}} \exp \left[ \frac{1}{n} P(x) \right] d\Phi_n(x) = \int_{\mathbb{R}} \exp \left[ \frac{1}{\sqrt{n}} \frac{P'(0)}{2} \right] dF_n(x).$$  \hspace{1cm} (2.2)

Hence we can study asymptotic behavior of the integral

$$\int_{\mathbb{R}} \exp \left[ \frac{1}{\sqrt{n}} \frac{P'(0)}{2} \right] dF_n(x)$$  \hspace{1cm} (2.3)

instead of (2.1). By the central limit theorem $F_n \Rightarrow \Phi$ as $n \to \infty$, where $\Phi(x)$ denotes the distribution function of the $(0,1)$-Gaussian random variable $\zeta$.

Assume for a moment that $P(0) = 0$. By Taylor’s expansion

$$\ln P \left( \frac{x}{\sqrt{n}} \right) = \iota \frac{1}{2} P''(0)x^2 \quad \text{and} \quad \frac{1}{\sqrt{n}} \frac{P''(0)}{2} \to -\frac{1}{n}.$$  \hspace{1cm} (2.4)

Consider

$$\int_{\mathbb{R}} \exp \left[ \frac{1}{2} P''(0)x^2 \right] dF_n(x)$$  \hspace{1cm} (2.5)

and note that by definition of weak convergence of probability measures, taking into account that $v(x) = \exp \left[ \frac{1}{2} P''(0)x^2 \right]$ is bounded continuous function, we can write

$$\lim_{n \to \infty} \int_{\mathbb{R}} \exp \left( \frac{1}{2} P''(0)x^2 \right) dF_n(x) = \int_{\mathbb{R}} \exp \left( \frac{1}{2} P''(0)x^2 \right) d\Phi(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left( -\frac{x^2}{2} \left[ 1 - \iota P''(0) \right] \right) dx = \frac{1}{\sqrt{1 - \iota P''(0)}},$$  \hspace{1cm} (2.6)

where

$$\left( 1 - \iota P''(0) \right)^{-1/2} = \left| 1 - \iota P''(0) \right|^{-1/2} \exp \left( -\frac{1}{2} \arg(1 - \iota P''(0)) \right),$$  \hspace{1cm} (2.7)

with $|\arg(1 - \iota P''(0))| \leq \frac{\pi}{2}$. General definition of the square root for symmetrical complex matrix $A$ with $((\text{Re} A)x, x) \geq 0$ could be found in [7, Section 3.4].
In the framework of the stationary phase method the goal is to find conditions for the function \( P(x) \) and random variables \( \xi_k \) such that (2.3) obeys similar asymptotic behavior as the integral with quadratic phase function.

Since \( |\exp[inP(x)]| \leq 1 \), explicit asymptotic behavior of (2.3) is determined by refinements of the central limit theorem (CLT) and in particular by the asymptotic expansion in CLT given below for a reader convenience.

**Theorem 2.1** ([10], XVI.4, Theorem 1. Expansion related to CLT). Assume \( \mu_3 = E\xi_1^3 \) exists. Then

\[
F_n(x) = \Phi(x) + \frac{\mu_3}{6\sqrt{n}} \left(1 - x^2\right) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} + \Delta_n(x)
\]

with

\[
\sup_x |\Delta_n(x)| = o\left(\frac{1}{\sqrt{n}}\right).
\]

**3. Reducing the problem and main results**

First we show that the integral (2.3) can be replaced to within \( O\left(\frac{1}{\sqrt{n}}\right) \) by the integral over the compact set \([-O(\sqrt{n}), O(\sqrt{n})]\). To formulate this result we need some preparations.

Introduce a nonincreasing function \( \varphi(t) \in C^\infty \) such that

\[
\varphi(t) = 1, \quad \text{if} \quad t \leq 0,
\]
\[
\varphi(t) = 0, \quad \text{if} \quad t \geq 1,
\]
\[
0 \leq \varphi(t) \leq 1, \quad \text{if} \quad 0 \leq t \leq 1.
\]

Denote \( r(x, F) \) the distance between a point \( x \in \mathbb{R} \) and a set \( F \subset \mathbb{R} \) and define the function \( \varrho(x) = \varphi(r(x, [-1, 1])) \). It is clear that

\[
\varrho(x) = 1, \quad \text{if} \quad x \in [-1, 1],
\]
\[
\varrho(x) = 0, \quad \text{if} \quad r(x, [-1, 1]) \geq 1,
\]
\[
0 \leq \varrho(x) \leq 1, \quad \text{for all} \quad x \in \mathbb{R},
\]

and \( \varrho(x) \) is a smooth function with compact support \([-2, 2]\).

Next consider \( \varrho\left(\frac{x}{\sqrt{n}}\right) = \varphi\left(r\left(\frac{x}{\sqrt{n}}, [-1, 1]\right)\right) \) and remark that

\[
\varrho\left(\frac{x}{\sqrt{n}}\right) = 1, \quad \text{if} \quad x \in [-\sqrt{n}, \sqrt{n}],
\]
\[
\varrho\left(\frac{x}{\sqrt{n}}\right) = 0, \quad \text{if} \quad r(x, [-\sqrt{n}, \sqrt{n}) \geq \sqrt{n},
\]
\[
0 \leq \varrho\left(\frac{x}{\sqrt{n}}\right) \leq 1, \quad \text{for all} \quad x \in \mathbb{R},
\]

and \( \varrho\left(\frac{x}{\sqrt{n}}\right) \) is a smooth function with compact support \([-2\sqrt{n}, 2\sqrt{n}]\).
Then we can rewrite (2.3) as follows:

\[ \int_{\mathbb{R}} \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) = \int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) + \int_{\mathbb{R}} \left( 1 - \varrho \left( \frac{x}{\sqrt{n}} \right) \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x). \]

**Lemma 3.1.** Assume \( E\xi_1^2 = 1 \). Then

\[ \left| \int_{\mathbb{R}} \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) - \int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) \right| = O \left( \frac{1}{n} \right). \]

**Proof.** Write

\[
\left| \int_{\mathbb{R}} \left( 1 - \varrho \left( \frac{x}{\sqrt{n}} \right) \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) \right| \\
\leq \left| \int_{0}^{\infty} \left( 1 - \varrho \left( \frac{x}{\sqrt{n}} \right) \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) \right| + \left| \int_{-\infty}^{0} \left( 1 - \varrho \left( \frac{x}{\sqrt{n}} \right) \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) \right|.
\]

Since for all \( x \) here \( I \) is the indicator function

\[ 1 \geq 1 - I_{[-\sqrt{n}, \sqrt{n}]}(x) \geq 1 - \varrho \left( \frac{x}{\sqrt{n}} \right) \geq 0 \]

and \( |\exp(it)| \leq 1 \), by the Chebyshev inequality

\[ L_1 \leq \int_{\sqrt{n}}^{\infty} \left( 1 - \varrho \left( \frac{x}{\sqrt{n}} \right) \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) \leq 1 - F_n(\sqrt{n}) = O \left( \frac{1}{n} \right). \]

By symmetry \( L_2 \) is also bounded by \( O(1/n) \).

Lemma 3.1 is proved. \( \square \)

By Lemma 3.1 in what follows we can consider the integral

\[ \int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x). \tag{3.4} \]

We show that within \( O \left( \frac{1}{\sqrt{n}} \right) \) the distribution function \( F_n \) in (3.4) can be replaced by the Gaussian distribution function \( \Phi \).

**Theorem 3.1.** Assume that \( P(x) \) is four times continuously differentiable function with \( P'(0) = 0 \). Assume also that assumptions of Theorem 2.1 are fulfilled. Then

\[ \left| \int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) - \int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ii np \left( \frac{x}{\sqrt{n}} \right) \right] d\Phi(x) \right| \leq \frac{1}{\sqrt{n}} C. \]
Corollary 1

\[
\left| \int_{\mathbb{R}} \exp \left[ i n P(x) \right] d\Phi_n(x) - \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\mathbb{R}} \varrho(x) \exp \left[ i n P(x) \right] \exp \left[ - \frac{nx^2}{2} \right] dx \right| \leq \frac{1}{\sqrt{n}} C.
\]

Proof is given in Section 4. Hence to fulfill our program it remains to consider

\[
\frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\mathbb{R}} \varrho(x) \exp \left[ i n P(x) \right] \exp \left[ - \frac{nx^2}{2} \right] dx
\]

and apply Theorem 1.1 with \(k = 1\) and \(f(x) = P(x) + \frac{\psi(x)}{2}\). Taking into account that \(\varrho(0) = 1\) we get the following result.

**Theorem 3.2.** Assume that \(P(x)\) is four times continuously differentiable function with \(P'(0) = 0\). Assume also that assumptions of Theorem 2.1 are fulfilled. Then

\[
\left| \int_{\mathbb{R}} \exp \left[ i n P(x) \right] d\Phi_n(x) - \left( \left[ P''(0) \right]^2 + 1 \right)^{-\frac{1}{4}} \exp \left[ i n P(0) - \frac{1}{2} \arctan \left( -P''(0) \right) \right] \right| \leq \frac{1}{\sqrt{n}} C.
\]

**Remark 1.** Since Theorem 7.5.5 in [7] is stated in any finite dimension and expansions related to CLT are valid for the random variables \(\xi_k\) that take their values in \(\mathbb{R}^d\) it is not more difficult to formulate and prove our results in any finite dimension. However we prefer to stay in the simple one dimension case just to make clear the approach.

4. **Proof of Theorem 3.1**

To prove this result we need some calculations and Lemmas.

Applying the integrating by parts formula and taking into account that \(\varrho \left( \frac{x}{\sqrt{n}} \right)\) is a smooth function with compact support we have

\[
\int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ i n P\left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) = - \int_{\mathbb{R}} \left( \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ i n P\left( \frac{x}{\sqrt{n}} \right) \right] \right)' F_n(x) dx. \tag{4.1}
\]

Using the asymptotic expansion in CLT (Theorem 2.1) we rewrite (4.1)

\[
\int_{\mathbb{R}} \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ i n P\left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) = - \int_{\mathbb{R}} \left( \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ i n P\left( \frac{x}{\sqrt{n}} \right) \right] \right)' \Phi(x) dx
\]

\[
- \int_{\mathbb{R}} \left( \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ i n P\left( \frac{x}{\sqrt{n}} \right) \right] \right)' \frac{\mu_{3}^{3}}{6\sqrt{n}} (1 - x^2) \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{x^2}{2} \right] dx
\]

\[
- \int_{\mathbb{R}} \left( \varrho \left( \frac{x}{\sqrt{n}} \right) \exp \left[ i n P\left( \frac{x}{\sqrt{n}} \right) \right] \right)' \Delta_n(x) dx.
\]
Applying again the integrating by parts formula we get

\[
\left| \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] dF_n(x) - \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] d\Phi(x) \right| 
\leq \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

\[
+ \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

\[
+ \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

\[
+ \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

Note that

\[
I_1 \leq \frac{\mu_3}{6n \sqrt{2\pi}} \int_\mathbb{R} (1 - x^2) \varphi \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

\[
+ \frac{\mu_3}{6 \sqrt{2\pi}} \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) P' \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

\[
+ \frac{\mu_3}{6 \sqrt{2\pi}} \int_\mathbb{R} x^2 \varphi \left( \frac{x}{\sqrt{n}} \right) P' \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx
\]

To estimate \( I_{12} \) and \( I_{13} \) we use Theorem 1.1.

**Lemma 4.1.** Assume that conditions of Theorem 3.1 are fulfilled. Then

\[
I_{12} = \frac{\mu_3}{6} \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \varphi \left( \frac{x}{\sqrt{n}} \right) P' \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx \leq C \frac{1}{\sqrt{n}}
\]

\[
I_{13} = \frac{\mu_3}{6} \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} x^2 \varphi \left( \frac{x}{\sqrt{n}} \right) P' \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \ln P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left( \frac{x^2}{2} \right) dx \leq C \frac{1}{\sqrt{n}}
\]

**Proof.** We first change the variables \( x/\sqrt{n} = y, x^2 = ny^2, dx = \sqrt{n}dy \) in \( I_{12} \) and \( I_{13} \)

\[
I_{12} = \sqrt{n} \mu_3 \frac{1}{6} \frac{1}{\sqrt{2\pi}} \left| \int_\mathbb{R} \varphi(y) P'(y) \exp \left[ \ln P(y) \right] \exp \left( \frac{ny^2}{2} \right) dy \right|
\]

\[
I_{13} = n \sqrt{n} \mu_3 \frac{1}{6} \frac{1}{\sqrt{2\pi}} \left| \int_\mathbb{R} y^2 \varphi(y) P'(y) \exp \left[ \ln P(y) \right] \exp \left( \frac{ny^2}{2} \right) dy \right|
\]

To estimate \( I_{12} \) we apply Theorem 1.1 with \( f(y) = P(y) + \frac{1}{2} y^2, u(y) = \varphi(y) P'(y) \) and \( k = 1 \). We get the bound provided \( I_{12} = 0 \) for \( j < k = 1 \). Note that in this case \( \nu = \mu = 0 \) and hence \( L_0 u \) does not contain any derivative. Since \( P'(0) = 0 \) we have \( u(0) = 0 \) and hence the necessary result.

Similarly, for \( I_{13} \) we take \( u(y) = y^2 \varphi(y) P'(y), k = 2 \) and the same function \( f \). In this case we have to show that the \( 2\nu \) derivative of \( (g_0^k u)(y) \), where \( g_0 = P(y) - P(0) - \frac{1}{2} P''(0) y^2, (2\nu \geq 3\mu, \)


\( \nu - \mu = j, j < k = 2 \) is equal to 0 in \( y = 0 \). To this end we remark that \((g^\mu_0 u)(y)\) is in the order of \( y^{3\mu+3} \) around \( y = 0 \). Its \( 2\nu \) derivative is in the order of \( y^\alpha \) where \( \alpha = 3\mu + 3 - 2\nu = \mu + 3 - 2j \geq 1 \) when \( \nu - \mu = j \) and \( j < k = 2 \).

Hence \( L_j u = 0 \) for \( j < k = 2 \). This observation yields the bound on \( I_{13} \).

Lemma is proved. □

To prove the estimation for \( I_{11} \) in Lemma 4.2 below we use only the property of the tails of Gaussian distribution.

**Lemma 4.2**

\[
I_{11} = \frac{\mu_3}{6n} \left| \frac{1}{\sqrt{2\pi}} \right| \int_{\mathbb{R}} (1 - x^2) \varphi' \left( \frac{x}{\sqrt{n}} \right) \exp \left[ \frac{1}{n} P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left\{ -\frac{x^2}{2} \right\} dx = o \left( \frac{1}{\sqrt{n}} \right)
\]

**Proof.** To estimate \( I_{11} \) remark that

\[
\varphi' \left( \frac{x}{\sqrt{n}} \right) = 0, \text{ if } x \in [ -\sqrt{n}, \sqrt{n} ] \quad \text{and} \quad r(x, [ -\sqrt{n}, \sqrt{n} ]) \geq \sqrt{n}
\]

and

\[
\varphi' \left( \frac{x}{\sqrt{n}} \right) \leq 0, \text{ if } 0 \leq r(x, [ -\sqrt{n}, \sqrt{n} ]) \leq \sqrt{n}
\]

Hence \( \varphi' \left( \frac{x}{\sqrt{n}} \right) (1 - x^2) \) has compact support \( \{ x : r(x, [ -\sqrt{n}, \sqrt{n} ]) \leq 1 \} \) and

\[
\left| \varphi' \left( \frac{x}{\sqrt{n}} \right) (1 - x^2) \right| \leq C.
\]

Recall the estimation of the tails of the Gaussian distribution

\[
\left| (1 - \Phi(x)) - \frac{1}{x\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \right| < \frac{1}{x^3\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right).
\]

In particular if \( x = \sqrt{n} \)

\[
\left| (1 - \Phi(\sqrt{n})) - \frac{1}{\sqrt{n}\sqrt{2\pi}} \exp \left( -\frac{n}{2} \right) \right| < \frac{1}{n^{3/2}\sqrt{2\pi}} \exp \left( -\frac{n}{2} \right).
\] (4.2)

Thus by (4.2)

\[
I_{11} \leq 2C \frac{\mu_3}{6n} \left| \frac{1}{\sqrt{2\pi}} \right| \int_{\sqrt{n}}^{\infty} \exp \left[ \frac{1}{n} P \left( \frac{x}{\sqrt{n}} \right) \right] \exp \left\{ -\frac{x^2}{2} \right\} dx
\]

\[
\leq 2C \frac{\mu_3}{6n} \left| \frac{1}{\sqrt{2\pi}} \right| \int_{\sqrt{n}}^{\infty} \exp \left\{ -\frac{x^2}{2} \right\} dx = o \left( \frac{1}{\sqrt{n}} \right).
\]

Lemma 4.2 is proved. □
In fact we proved the following result.

\[
\left| \int_{\mathbb{R}} \exp \left[ \iota nP(x) \right] d\Phi_n(x) - \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\mathbb{R}} \varrho(x) \exp \left[ \iota nP(x) \right] \exp \left[ -\frac{nu^2}{2} \right] dx \right| \leq \frac{1}{\sqrt{n}} C.
\]

Theorem is proved.

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References