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space of Lorentz metrics

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Abstract: The space of Lorentz metrics on a compact manifold is very different from its Riemannian analogue. There are usually many connected components. We show that some of them turn out to be not simply connected. We also show that, in dimension greater than 2, the distance between two components is always 0.

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1. Introduction

It is well known (and trivial) that the space of Riemannian metrics on a compact manifold, endowed with the compact open topology, is contractible. And so is the space of metrics with given total volume, or with given volume element. Furthermore, the Gram–Schmidt process tells us that, if the manifold is parallelisable, there exists for any Riemannian metric a global orthonormal frame field.

In the present article, we shall see that the analogous properties completely fail in the Lorentz case. Namely, we discuss "Lorentz-parallelisability" (cf. Section 3) and study the topology of the space $\mathfrak{L}(M)$ of Lorentz metrics. We will show (cf. Theorem 3.6) that, if M is a 2 or 3 dimensional compact orientable Lorentz manifold then $\mathfrak{L}(M)$ has an infinite number of connected components and we will compute the fundamental group of some of them (which will not be trivial). Moreover, for a natural metric on $\mathfrak{L}(M)$ (cf. Section 2.2), we study the distance of these components in the space of metrics with prescribed volume element. We show that, if $\dim M > 2$, this distance is always zero and, if $\dim M = 2$, it is always positive (cf. Theorem 4.6).

2. General facts

First we introduce some notations, M will always be a n-dimensional *compact* manifold and h a Lorentz metric, i.e., of signature $(n-1, 1)$. By a "planefield" we shall always mean a vector subbundle of the tangent bundle; we will call it *line field* if it is 1-dimensional. Finally, the vector fields we consider will be supposed everywhere nonzero.

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2.1. About the metrics

Definition 2.1. A tangent vector $v \neq 0$ will be called *negative*, (respectively, *positive*) with respect to h, if $h(v, v) < 0$ (respectively, $h(v, v) > 0$). We shall call *time bundle* (respectively, *space bundle*) any line field (respectively, hyperplane field) composed of negative (respectively, positive) vectors.

If we consider a decomposition of the tangent bundle into a line field and a hyperplane field, it is easy to construct a metric h such that those spaces are time and space bundles of h. Conversely, we have the following.

Lemma 2.2. 1. *Any couple* (M, h), *possesses time bundles and space bundles.*
2. *Different time bundles* (*respectively, space bundles*) *associated to h are homotopic.*

Proof. This is a property of fibre bundles with contractible fibre. But it is known that the negative grassmannian is contractible. □

Remarks. This lemma gives us the well-known condition for a manifold to admit a Lorentz structure: the Euler class of the manifold has to be zero. Hence, the only 2-dimensional manifolds which possess such a structure are the torus and the Klein bottle and, on the other hand, any odd dimensional compact manifold admits a Lorentz structure.

Furthermore, it is known (cf. [4, p. 29]) that two homotopic fibre bundles are isomorphic (this property will be useful again later), hence time bundles (resp. space bundles) of a Lorentz manifold are isomorphic.

Lemma 2.3. *Two trivial line fields nowhere collinear are homotopic. Two line fields nowhere orthogonal, for any Riemannian metric* g, *are homotopic.*

Proof. The first point is evident. For the second, we translate the problem to the projective tangent. We equip each fibre with the Riemannian metric defined by g, thereby obtaining a distance. The two line fields are represented by a couple of points in each fibre. The hypothesis implies that the distance between them is less than $\frac{1}{2}\pi$. Consequently there exists a unique geodesic between them. These geodesics give us the desired homotopy. □

Definition 2.4. We will say that h is *temporally orientable* if its time bundles are orientable.

Hence, on a manifold M such that $H_1(M, \mathbb{Z}_2) = 0$, there exist only temporally orientable metrics. But, as we will see soon with $\mathbf{P}^3$, there exist manifolds M such that $H_1(M, \mathbb{Z}_2) \neq 0$ and which possess only trivial line fields. Indeed, we are talking about subbundles of the tangent bundle. Anyway, we can easily find non-orientable line fields on $\mathbf{T}^2$ and so non-temporally orientable metrics.

2.2. About the space of metrics

We are going to define a family of distances on the space, $\mathcal{L}(M)$, of Lorentz metrics. Thanks to a Riemannian metric g, we can identify S^2M, the vector space of symmetric 2-tensors with the vector space of endomorphisms of the tangent to M which are symmetric with respect to g. The endomorphism associated to a tensor h will be denoted $\tilde{h}$. If we choose a Riemannian metric g, we have a distance on S^2M (since M is compact) given by

$$d_g^{\infty}(h, h') = \sup_{x \in M} (\mathrm{tr}((\tilde{h}_x - \tilde{h}'_x)^2)^{\frac{1}{2}}).$$

One shows that $\mathcal{L}(M)$ is an open set of S^2M equipped with this distance. Since M is compact, it is well known that if g_1 and g_2 are two Riemannian metrics, there exist positive constants a and b such that $a\, g_1 < g_2 < bg_1$. This means d_{g_1} and d_{g_2} are uniformly equivalent distances. In particular the zero–non zero distance phenomena we study in Section 4.2 do not depend on g. The topology induced on $\mathcal{L}(M)$ is the one we are going to use in the next paragraphs.

3. Lorentz-parallelism

3.1. Definition and examples

Definition 3.1. We will say that the couple (M, h) is *Lorentz-parallelisable* (or L-parallelisable) if there exists, on M, a global h-orthogonal frame field.

It is, of course, necessary for the manifold to be parallelisable. Indeed, a non-temporally orientable metric is not L-parallelisable and there exist such metrics on $\mathbf{T}^2$. We can reinterpret the definition, (M, h) is L-parallelisable if and only if there exist space-time decompositions in trivial subbundles. The Lemma 2.2 then says that all the space-time decompositions are trivial. It is interesting to note that any 3-dimensional compact orientable manifold is parallelisable (cf. [7]). On $S^2 \times S^1$ the metric given by $d_{S^2} \oplus (-d_{S^1})$ is not L-parallelisable but is temporally orientable.

Proposition 3.2. *Any Lorentz metric on S^3 or $\mathbf{P}^3$ is Lorentz-parallelisable.*

Proof. It is well known that any vector bundle on S^3 (see [5]) is trivial. Any tangent subbundle of $T\mathbf{P}^3$ is orientable since the existence of a non-orientable subbundle implies the existence of a map from S^3 to S^2 anti-invariant by antipodal map, and that is in contradiction with Bursuk–Ulam Theorem (see [1, p. 240]). The only case left to prove is the case of orientable plane fields of $T\mathbf{P}^3$. To this purpose, we consider the Euler class of such a vector bundle, if it vanishes, the plane field is going to be trivial. We know that it lies in $H^2(\mathbf{P}^3, \mathbb{Z})$ which is isomorphic to $\mathbb{Z}_2$. On the other hand, we know that any transversal line field of a plane field is trivial and that TS^3 is trivial. This together with the additivity of Stiefel–Whitney classes (cf. [6]) implies that its second Stiefel–Whitney class vanishes. Furthermore we also know that this class is the reduce modulo 2 of the Euler class and therefore the latter must be zero. □

We recall that $L_{p,k}$ (with $(p, k) = 1$) denotes the quotient manifold of S^3 by the action of a primitive p^{th} root of unity, ρ. More precisely we put $\rho \cdot (z, z') = (\rho\, z, \rho^k z')$. Those spaces are called lens spaces.

Proposition 3.3. *There exist non-Lorentz-parallelisable temporally orientable metrics on $L_{n,1}$ for $n > 2$.*

Proof. We consider the following map (which is the classical covering) from S^3 to SO(3). If we interpret its column vectors as coordinates in a left invariant trivialisation of TS^3, it defines a frame field on S^3,

$$(x, y, z, t) \longmapsto \begin{pmatrix} x^2+y^2-\frac{1}{2} & yz+xt & yt-xz \\ yz-xt & x^2+z^2-\frac{1}{2} & xy+zt \\ xz+yt & zt-xy & x^2+t^2-\frac{1}{2} \end{pmatrix}.$$

This map defines an orientable plane field on all $L_{n,1}$ since the first column vector (which is the Hopf map) is invariant under the action of roots of unity. We want to prove that this plane field is not trivial, i.e., that the map from $L_{n,1}$ to S^2 given by the first column vector can not be lifted to SO(3) via the Hopf map. If n is odd, we use the fact that a map from $L_{n,1}$ to S^2 can be lifted to SO(3) if and only if it can be lifted to S^3(since the fundamental group of $L_{n,1}$ is $\mathbb{Z}_n$). The first column vector can be lifted to $L_{n,1}$ by the identity. If this map could be lifted to S^3 it would imply that the identity on $L_{n,1}$ could be lifted to S^3. But, in return, this would imply that the covering of S^3 on $L_{n,1}$ is trivial. If n is even, we can use the nontrivial covering of $\mathbf{P}^3$ on $L_{n,1}$ and repeat the above proof. We have an orientable nontrivial plane field and therefore we can find temporally orientable non-Lorentz-parallelisable metrics. □

3.2. Lorentz-parallelism and homotopy

Proposition 3.4. *The map which associates to a Lorentz metric its negative eigenspace* (*relatively to a Riemannian metric g*) *is continuous.*

Proof. It is standard. □

Lemma 3.5. *Two Lorentz metrics on M are homotopic if and only if their time bundles are homotopic into TM.*

Proof. If h_1 and h_2 are two homotopic metrics Proposition 3.4 says that their time bundles are homotopic. Let us show the converse.

Let g be a Riemannian metric. With g and a nowhere vanishing vector field X we define the 2-tensor $X^\delta = X^\flat \otimes X^\flat$. We remark that $X^\delta = (-X)^\delta$. Hence we can define ζ^δ for any line field ζ, thanks to its elements of norm 1 (calculated with g). Next we compare h_1 and $g - 2\,\zeta_1^\delta$ where ζ_1 is the eigenspace field associated to the negative eigenvalue of h_1. They are both Lorentz metrics (that is why we chose elements of norm 1). The orthogonal bundle of ζ_1 is the same for both metrics, we denote it by ξ_1. The set of symmetric definite positive 2-tensors on ξ_1 is convex. Therefore $h_1|_{\xi_1}$ is homotopic to $(g - 2\,\zeta_1^\delta)|_{\xi_1}$. We rewrite the two metrics

$$h_1 = h_1|_{\xi_1} \oplus \lambda_1\, \zeta_1^\delta,$$

where λ_1 is the eigenvalue function associated to ζ_1, and

$$g - 2\,\zeta_1^\delta = g|_{\xi_1} \oplus (-\zeta_1^\delta).$$

Now those metrics are clearly homotopic (λ_1 is negative). We proceed in the same way for h_2. It only remains to compare $g - 2\,\zeta_1^\delta$ and $g - 2\,\zeta_2^\delta$. By hypothesis, ζ_1 is homotopic to ζ_2 and so $g - 2\,\zeta_1^\delta$ is homotopic to $g - 2\,\zeta_2^\delta$. □

The above lemma shows us that the time and space bundles of two homotopic metrics are isomorphic. In particular, the Lorentz-parallelism is invariant by homotopy. In fact this lemma give us the arcwise connected components of $\mathcal{L}(M)$, moreover the Proposition 3.4 and the Lemma 2.3 imply that the connected components are equal to the arcwise connected components. By evaluating the number of line fields homotopy classes we thus get the number of connected components of $\mathcal{L}(M)$.

Theorem 3.6. *Let M be a compact, orientable 3-dimensional manifold and $\mathcal{L}(M)$ its space of Lorentz metrics on M. Then $\mathcal{L}(M)$ possesses an infinite number of connected components.*

Proof. Those manifolds are parallelisable. Hence, given a parallelism and an auxiliary Riemannian metric, we can consider vector fields as maps from M to S^2 (or rather $\mathbb{R}^3 - \{0\}$ but we work up to homotopy). First we restrict ourself to connected components which are temporally orientable (this means that any metric in it is temporally orientable). From the previous lemma, we see that homotopy classes of such metrics are in one-to-one correspondence with homotopy classes of orientable line fields. There is no such correspondence with the homotopy classes of nonzero vector fields. Indeed, non-homotopic vector fields may be homotopic considered as line fields. Consequently, we restrict ourself once more: we consider only L-parallelisable metrics. We easily see that the situation we have to suppress is the case where a vector field is not homotopic to its opposite. Since this is never the case for a negative vector field of a L-parallelisable metric, we can work with homotopy classes of vector fields that can be extended to a frame field of M, i.e., of maps from M to S^2 that can be lifted to SO(3). We consider the maps from M to S^2 which can be lifted to S^3 via the Hopf map (this is a particular case of the latter one). Composing two non-homotopic maps from M to S^3 with the Hopf map yields two non-homotopic maps from M to S^2 (cf. [3, p. 69]). Hence to show the result it suffices to have an infinite set of maps from M to S^3 pairwise non-homotopic. Hopf Classification theorem (cf. [1, p. 300]) says that the homotopy classes of maps from M to S^3 are in one-to-one correspondence with $H^3(M, \mathbb{Z})$. The manifolds being orientable, this group is infinite. □

Remarks. 1. In fact we have bounded from below the number of connected components composed of L-parallelisable metrics. The theorem is false in greater dimension. Hence, knowing that $\pi_7(S^6) \simeq \mathbb{Z}_2$ and that S^7 is simply connected and parallelisable, we see that $\mathcal{L}(S^7)$ has only *two* connected components. Nevertheless the result is still true for $\mathbf{T}^2$.

2. The referee pointed out that the correspondence between homotopy classes of Lorentz metrics and homotopy classes of nowhere vanishing vector fields was known to Geroch (see [2]). In this article, he posed the question: “which global properties of space times are invariants of the homotopy class?”

4. The space of Lorentz metrics

4.1. Topology of the connected components

We denote by $\mathcal{C}(h)$ the connected component of the metric h in $\mathcal{L}(M)$.

Lemma 4.1. *If h and h' have isomorphic space and time bundles then $\mathcal{C}(h)$ is homeomorphic to $\mathcal{C}(h')$.*

Proof. Those isomorphisms extend to an automorphism A of the tangent space. We can transport metrics thanks to this morphism by putting $A^*(h)(u, v) = h(A(u), A(v))$, Hence obtaining a homeomorphism of $\mathcal{L}(M)$. It is clear that $A^*(h)$ is homotopic to h' and therefore $A^*(\mathcal{C}(h)) = \mathcal{C}(A^*(h))$. □

In the particular case of L-parallelisable connected components, this lemma shows us that two of them are always homeomorphic. Hence from Proposition 3.2 any two connected components of $\mathcal{L}(S^3)$ or of $\mathcal{L}(\mathbf{P}^3)$ are homeomorphic.

Proposition 4.2. 1. *For any Lorentz-parallelisable connected component, $\mathcal{C}_T$, of $\mathcal{L}(\mathbf{T}^2)$, we have $\pi_1(\mathcal{C}_T) \simeq \mathbb{Z}$.*

2. *For any connected component, $\mathcal{C}_S$, of the space $\mathcal{L}(S^3)$, we have: $\pi_1(\mathcal{C}_S) \simeq \mathbb{Z}_2$.*

Proof. Always from Lemma 3.5, and knowing that in the cases we are interested in, any vector field will be homotopic to its opposite, we can work with loops of vector fields. According to Proposition 4.1, it is sufficient to establish the result for one connected component.

Let us begin with the sphere. Let h be a metric in $\mathcal{L}(S^3)$ and X a h-negative vector field. We complete this vector field into a frame field (this is always possible on S^3). We then identify TS^3 to $S^3 \times \mathbb{R}^3$ by means of this trivialisation, the expression of the field X is then a constant map. This implies that the homotopy classes of loops of maps from S^3 to S^2, in the connected component of the constant map, are in one-to-one correspondence with the homotopy classes of loops of vector fields on S^3 in the homotopy class of X. On the other hand, they are in one-to-one correspondence with the homotopy classes of maps from "the" suspension of S^3 to S^2. We have to be careful about what we call suspension. Since the maps we study are not pointed, the space we should consider as suspension should be the space $S^3 \times [0, 1]$ with *both* ends quotiented in one point. Nevertheless, S^2 is simply connected and therefore we can consider the usual suspension (unreduced), i.e., $S^3 \times [0, 1]$ with *each* end quotiented in a point. In this case the suspension of S^3 is S^4. The homotopy classes of maps from S^4 to S^2 are well known, they define $\pi_4(S^2)$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

For the torus we proceed in the same way. However, the maps take values in S^1 which is not simply connected, and therefore we can not use the classical suspension (the double cone). Consequently we are looking for homotopy classes of maps from $(\mathbf{T}^2 \times [0, 1])/(\mathbf{T}^2 \times \{0, 1\})$ to S^1. The Hopf Classification theorem tells us that they are in one-to-one correspondence with the H^1 of this space (cf. [1]). We compute the latter, thanks to a Mayer–Vietoris sequence, and we find it isomorphic to $\mathbb{Z}$. □

We wish to express a non-contractible loop of maps from S^3 to S^2 in order to see if it goes down to some quotients of S^3. We know a generator of $\pi_4(S^2)$: the suspension of the Hopf map composed with the Hopf map (noted H)(cf. [1, p. 464]). Let us find back the loop of maps whose quotient gives this map. Let P_2 be a map from $S^2 \times [0, 1]$ to S^3 which, when we quotient, gives an homeomorphism between the suspension $\mathcal{S}S^2$ and S^3 (cf. [8]). We can write $P_2(x_0, x_1, x_2, t) = (\cos^2 \pi t + x_0 \sin^2 \pi t,\ x_1 \sin^2 \pi t,\ x_2 \sin^2 \pi t,\ \sqrt{(1 - x_0)/2}\, \sin 2\pi t)$.

The map $H \circ P_2 \circ (H \times \mathrm{Id})$ from $S^3 \times [0,1]$ to S^2 has the desired properties. So, from any trivialisation of the tangent of S^3, we realize a loop of vector fields X_t. Now, let g be a Riemannian metric, we can realize a loop of Lorentz metrics: $h_t = g - 2\,X_t^\delta$ (with the condition $g(X_t, X_t) = 1$). This loop is non-contractible.

Corollary 4.3. *In any Lorentz-parallelisable connected component of $\mathcal{L}(L_{p,k})$, there exists a non-contractible loop of Lorentz metrics.*

Proof. We are looking for a map from S^3 to S^2 which is well defined on $\mathcal{L}(L_{p,k})$ and whose suspension (from S^4 to S^3) is non-homotopic to 0. The latter is true if its Hopf invariant is odd (cf. [1, p. 126]). Indeed, if we have a trivialisation of the tangent which is invariant under this action, the path of vector fields defined thanks to the suspension of this map and the Hopf map is also invariant. This path goes down on the lens space and is of course non-contractible. The lens spaces are parallelisable, so there does exist an invariant basis. On the other hand, there exists an odd integer l (by Bezout theorem) such that $k \cdot l \equiv 1 \bmod p$. The degree of the map f_l defined by $f(z, z') = (z, z'^l)$ is l. Knowing that the Hopf invariant of $H \circ f_l$ is the Hopf invariant of H times the degree of f_l (see [3]) we can say that it is l. Finally, we check that it is still well defined on $\mathcal{L}(L_{p,k})$. We conclude with 4.1. □

Instead of a left invariant trivialisation, we can take the trivialisation given by the matrix of Proposition 3.3 and replace $H \circ P_2 \circ (H \times \mathrm{Id})$ by $H \circ P_2 \circ ((H \circ f_3) \times \mathrm{Id})$. We thereby obtain a non-contractible loop of line fields on $L_{4,1}$ and so a non-contractible loop of Lorentz metrics through non-L-parallelisable metrics.

Proposition 4.4. *Let N be a 2-dimensional compact orientable manifold and $M = N \times S^1$. Any Lorentz-parallelisable connected component of $\mathcal{L}(M)$ has infinite fundamental group and we can find a non-contractible loop through non-temporally orientable metrics.*

Proof. The tangent being trivial, we suppose we have a global frame bundle on M. We are going to construct maps from $M \times S^1$, i.e., $N \times \mathbf{T}^2$ to S^2. There exists an infinite number of pairwise non-homotopic maps from $\mathbf{T}^2$ to S^2. Considering them as maps from $N \times \mathbf{T}^2$ to S^2 they remain non-homotopic. At a given time, the vector fields they define are homotopic to each component of the tangent trivialisation. Therefore the metrics defined thanks to them are going to be L-parallelisable and all in the same connected component of $\mathcal{L}(M)$ (which depends only on the choice of the trivialisation). Any two of those loops are clearly non-homotopic. That proves the first part. To finish the proof it suffices to find a map from $\mathbf{T}^2$ to $\mathbf{P}^2$ whose restriction to S^1 cannot be lifted to $\mathbf{T}^2$ and which is not homotopic to this restriction (seen as a map from $\mathbf{T}^2$). We have a natural set of maps from S^1 to $\mathbf{T}^2$ which are given by the parameterization of half great circles of S^2 joining two given antipodal points (of course all those maps are homotopic). We parameterize this set by the angle between the planes defining those loops. We thus obtain a map from $\mathbf{T}^2$ to S^2 which satisfies evidently the first property. It satisfies also the second one because the restriction to S^1 is not onto and the map above is one-to-one and therefore these maps do not have the same degree modulo 2 and so are not homotopic. □

4.2. Distances

Now, we are interested in distances between connected components of the space $\mathcal{L}(M)$. First, we remark that in $\mathcal{L}(M)$ these distances will always be zero. Indeed, the zero tensor will be in the closure of all connected components.

Definition 4.5. We denote by $\mathcal{L}_\omega(M)$, the subset of $\mathcal{L}(M)$ consisting of metrics which all induce the same volume element, ω.

Remark. During the study of $\mathcal{L}_\omega(M)$, the Riemannian metric g, we use to define the distance d_g (see 2.2), will also be with associated volume element ω. Consequently, the condition "h belongs to $\mathcal{L}_\omega(M)$" will be equivalent to $\tilde{h}_x$ (the associated endomorphism to h_x via g) has determinant 1 for all x in M. It is interesting to note that when we go from $\mathcal{L}(M)$ to $\mathcal{L}_\omega(M)$ we do not change the topology. Thanks to a Riemannian metric g, we can define a projection of $\mathcal{L}(M)$ on $\mathcal{L}_\omega(M)$ by dividing $\tilde{h}$ by some power of the determinant. Actually, this projection defines a deformation retract of $\mathcal{L}(M)$ on $\mathcal{L}_\omega(M)$. All this explains why we choose this space.

Theorem 4.6. 1. *If M is 2-dimensional, any two connected components of $\mathcal{L}_\omega(M)$ are always at nonzero distance.*

2. *If M is n-dimensional, with $n > 2$, then any two connected components of $\mathcal{L}_\omega(M)$ are always at zero distance.*

Proof. Let us take h and h' two non-homotopic metrics. There is at least one point x such that their negative eigenspaces (relatively to the Riemannian metric g used to define d_g) are orthogonal. At this point, we choose an h-orthogonal and g-orthonormal trivialisation of the tangent. In such a basis the matrix of $\tilde{h}(x)$ is diagonal and of determinant -1. The matrix of $\tilde{h}'(x)$ is orthogonally conjugated to such a matrix. We denote by a and $-1/a$ ($a > 0$) the coefficients of the first one and by b and $-1/b$ ($b > 0$) those of the second one. The matrix of conjugacy must be

$$\begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix},$$

because it lies in O(2). We have

$$\mathrm{tr}((\tilde{h}(x) - \tilde{h}'(x))^2) = (1/b + a)^2 + (1/a + b)^2 \geqslant 8.$$

This is sufficient to show the first part, actually we can go further and show that the distance between $\mathcal{C}(h)$ and $\mathcal{C}(h')$ is effectively 8. Therefore any two connected components are equally distant.

2. Before showing this result there is some work to do and some notations to be introduced.

Notations. Let g be a Riemannian metric with induced volume ω on a 3-dimensional manifold. Let X and Y be vector fields such that $g(X, X) = g(Y, Y) = 1$. We denote by $\pi(Y)$ (resp. $\pi(X)$) the vector field obtained by orthogonal projection of Y (resp. X) on the orthogonal of X (resp. Y). That is $\pi(Y) = Y - g(X, Y)X$. Let Z be the vector field obtained from $\pi(Y)$ by a $\pi/2$ rotation in the orthogonal of X. As in Proposition 3.5, we denote by X^δ the symmetric 2-form given by $X^\delta(u, v) = g(X, u)\, g(X, v)$. Finally, Π^2 will denote $g(\pi(Y), \pi(Y)) = g(Z, Z)$. Now we can state the following

Proposition 4.7. *For any $n \in \mathbb{N}$, we define*

$$h_n = ng - \left(n + \frac{1}{n^2}\right) X^\delta - n\left(1 - \frac{1}{n^2}\right) \phi_n (\pi(Y))^\delta + \frac{n(1 - 1/n^2)\, \phi_n}{1 - (1 - 1/n^2)\, \phi_n\, \Pi^2}\, Z^\delta,$$

where ϕ_n is a positive function on M such that $\phi_n\, \Pi^2 \leqslant 1$. Then we have $h_n \in \mathcal{L}_\omega(M)$.

Proof. Clearly, h_n is a smooth symmetric 2-tensor. Therefore there exists a g-symmetric endomorphism $\tilde{h}_n$ associated to it. We are going to find its eigenvalues at a point m of M in order to compute its determinant. There are two cases: $\pi(Y)(m) = 0$ or $\pi(Y)(m) \neq 0$. The first case is obvious: the set of eigenvalues is $\{-1/n^2, n, n\}$ and their product is -1. If $\pi(Y)(m) \neq 0$, we have

$$\tilde{h}_n(X) = -(1/n^2)\, X,$$

$$\tilde{h}_n(\pi(Y)) = n(1 - (1 - 1/n^2)\, \phi_n \Pi^2)\, \pi(Y),$$

$$\tilde{h}_n(Z) = n\left(1 + \frac{(1 - 1/n^2)\, \phi_n\, \Pi^2}{1 - (1 - 1/n^2)\, \phi_n\, \Pi^2}\right) Z.$$

The determinant is still equal to -1. We see also that h_n is Lorentzian. □

The vector field Z is actually orthogonal to X and Y and then to $\pi(X)$. We deduce from this that the tensor k_n obtained by permuting X and Y in h_n is also in $\mathcal{L}_\omega(M)$.

$$k_n = ng - \left(n + \frac{1}{n^2}\right) Y^\delta - n\left(1 - \frac{1}{n^2}\right) \phi_n (\pi(X))^\delta + \frac{n(1 - 1/n^2)\, \phi_n}{1 - (1 - 1/n^2)\, \phi_n\, \Pi^2}\, Z^\delta.$$

Lemma 4.8. *Let ϕ_n be such that if $\Pi \geqslant 1/n^2$ then $\phi_n = 1/\Pi^2$ Then*

$$0 \leqslant n(1 - (1 - 1/n^2)\, \phi_n\, \Pi^2)\Pi \leqslant 1/n.$$

Proof. It comes directly from $\Pi \leqslant 1$, $0 \leqslant \phi_n\, \Pi^2 \leqslant 1$ and the above condition on ϕ_n. □

Proposition 4.9. *If we choose ϕ_n as in Lemma* 4.8 *then $d_g(h_n, k_n)$ goes to* 0 *when n goes to infinity.*

Proof. First it is obvious that if $\pi(Y)(m) = 0$ then $\mathrm{tr}((\tilde{h}_n(m) - \tilde{k}_n(m)) = 0$. Let m be a point such that $\pi(Y)(m) \neq 0$. We have two g-orthonormal bases of $T_m M$: $(X, \pi(Y)/\Pi, Z/\Pi)$ and $(Y, \pi(X)/\Pi, Z/\Pi)$. The matrix which expresses the second basis in the first one is

$$\begin{pmatrix} (X, Y) & \Pi & 0 \\ \Pi & -g(X, Y) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We compute thanks to it the matrix of $\tilde{k}_n(m)$ in the basis $(X, \pi(Y)/\Pi, Z/\Pi)$:

$$\begin{pmatrix} -1/n^2 g(X,Y)^2 + nW_n\Pi^2 & \Pi g(X,Y)(-1/n^2 - nW_n) & 0 \\ \Pi g(X,Y)(-1/n^2 - nW_n) & -1/n^2 + ng(X,Y)^2 W_n & 0 \\ 0 & 0 & \Delta_n \end{pmatrix}$$

where $W_n = 1 - (1 - 1/n^2)\phi_n\Pi^2$ and $\Delta_n = n(1 + ((1 - 1/n^2)\phi_n\Pi^2)/W_n)$.

Now, we are able to compute

$$\begin{aligned} \operatorname{tr}((\tilde{h}_n(m) - \tilde{k}_n(m))^2) &= (1/n^2(1 - g(X,Y)^2) + n(1 - (1 - 1/n^2)\phi_n\Pi^2)\Pi^2)^2 \\ &\quad + 2(\Pi g(X,Y)(-1/n^2 - n(1 - (1 - 1/n^2)\phi_n\Pi^2)))^2 \\ &\quad + (1/n^2 + n(1 - g(X,Y)^2)(1 - (1 - 1/n^2)\phi_n\Pi^2))^2 + (\Delta_n - \Delta_n)^2. \end{aligned}$$

We remark that $(1 - g(X,Y)^2) = \Pi^2$ and then we replace the first by the second in the third line and use Lemma 4.8 to have an upper bound. We have

$$\operatorname{tr}((\tilde{h}_n(m) - \tilde{k}_n(m))^2) \leqslant (1/n^2 + 1/n)^2 + 2(1/n^2 + 1/n)^2 + (1/n^2 + 1/n)^2.$$

This obviously implies that $d_g(h_n, k_n) = \sup_{m\in M}(\operatorname{tr}((\tilde{h}_n(m) - \tilde{k}_n(m))^2))$ goes to 0 when n goes to infinity. □

Proof of Theorem 4.6. We first give the proof for the dimension 3. Let us consider two temporally orientable connected components of $\mathcal{L}_\omega(M)$: $\mathcal{C}_1$ and $\mathcal{C}_2$. Let X (resp. Y) be a negative vector field of a metric in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$). We know that they are not homotopic. Anyway thanks to them we can define the metrics h_n and k_n as before. It is clear that the h_n belong to $\mathcal{C}_1$ and the k_n to $\mathcal{C}_2$. Furthermore, from 4.9, we have that $d(h_n, k_n)$ goes to 0. We conclude that $d(\mathcal{C}_1, \mathcal{C}_2) = 0$ for any two temporally orientable connected components. There is no difficulty to extend this result to any connected component. In fact, as we have seen in 3.5, $X^\delta = (-X)^\delta$ and so our construction can be done in the same way for non-orientable line fields. Consequently, the distance between two connected components is always 0.

The last thing to do is to say how we can manage in greater dimension. The main difference is that there is no vector field to take place of Z. We define a kind of projection P on the orthogonal of Vect(X, Y). It is defined by $P(u) = \Pi^2 u - g(u, X)\Pi^2 X - g(u, \pi(Y))\pi(Y)$. Thanks to P, we define the following symmetric 2-tensor $\beta_{X,Y}(u, v) = g(P(u), P(v))$. For a 3-dimensional manifold, we have $\beta_{X,Y} = Z^\delta$. And, which is more important, $\beta_{X,Y} = \beta_{\pm X,\pm Y}$, and therefore it can be defined for any line field. Now we put

$$\begin{aligned} h_n &= ng - \left(n + \frac{1}{n^{l-1}}\right)X^\delta - n\left(1 - \frac{1}{n^2}\right)\psi_n\Pi^{2(l-3)}\pi(Y)^\delta \\ &\quad + n\sqrt[l-2]{\frac{(1 - 1/n^2)\,\psi_n}{1 - (1 - 1/n^2)\psi_n\Pi^{2(l-2)}}}\,\beta \end{aligned}$$

where $l = \dim M$ and ψ_n is a positive function verifying $\psi_n\,\Pi^{2(l-2)} \leqslant 1$ and $\psi_n\,\Pi^{2(l-2)} = 1$ if $\Pi^2 \leqslant 1/n^4$. We define k_n in the same way. Those metrics belong to $\mathcal{L}_\omega(M)$. And, with the same computations as above, we have $d(h_n, k_n) \to 0$. □

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