Tournaments as strong subcontractions

Chris Jagger*

Department of Pure Mathematics and Mathematical Statistics, 16 Mill Lane, Cambridge CB2 1SB, UK

Received 9 October 1995; revised 23 April 1996

Abstract

We determine good bounds for the maximum size of a digraph which has no strong subcontraction to a tournament $T_p$ of order $p$. In particular, we shall show that for a transitive tournament, denoted $TT_p$, then given any $p$ and $\varepsilon > 0$, there exists $n_0$ such that for all $n \geq n_0$, if digraph $D$ has order $n$ and at least $\binom{n}{2}(1 - 1/(p - 1) + \varepsilon)$ edges, then $D \succneq TT_p$, where $\succneq$ denotes strong subcontraction. This uses a Turán type of argument. We also get some exact results for strong subcontraction of complete digraphs.

1. Introduction

A graph $H$ with vertex set $\{v_1, \ldots, v_r\}$ is a subcontraction (or minor) of graph $G$ if there exists a partition of the vertices of $G$ into subsets $V_0, V_1, \ldots, V_r$ with $V_i$ being connected for each $1 \leq i \leq r$, such that whenever $v_i v_j \in E(H)$, there exist vertices $x \in V_i$ and $y \in V_j$ with $xy \in E(G)$. Alternatively, if we are given a graph $G$ and an edge $uv$ of $G$, the result of contracting the edge $uv$ is the graph obtained from $G - \{u, v\}$ by adding a vertex $z$ and edges $zw$ for each $w \in \Gamma(u) \cup \Gamma(v) - \{u, v\}$. (Throughout we shall use the notation of Bollobás [1], thus $\Gamma(u)$ is the neighbourhood of vertex $u$.) Then a subcontraction is a graph $H$ obtained from $G$ by a sequence of edge deletions, vertex deletions and edge contractions of $G$, written $H \prec G$.

It turns out that the smallest number of edges in a graph $G$ that will guarantee that $G \succ K_p$, where $K_p$ is the complete graph of order $p$, is linear in the order of $G$, so it is convenient to define

$$c(p) = \inf \{c; e(G) \geq c|G| \text{ implies } G \succ K_p\}.$$}

* E-mail: C.N. Jagger@dpmms.cam.ac.uk.
with one fewer edge which does not subcontract to \( K_p \); namely, the graph consisting of \( p - 2 \) vertices connected to all vertices. Dirac [2] showed this for \( p \leq 5 \), and Mader for \( p = 6, 7 \). However, recently Jorgensen [6] showed that there exist graphs of order \( n \) with more than \( 6n - 21 \) (i.e. \( (p - 2)n - (\binom{p-1}{2}) \)) edges which do not subcontract to \( K_8 \), although the only such graphs have order \( 6n - 20 \), and are a specified set of graphs of order \( 5m \), for integer \( m \). Thus, \( c(p) = p - 2 \) for \( p = 8 \) also.

More generally, Kostochka [7] showed that \( c(p) \leq 324p\sqrt{\log_2 p} \), and Thomason [9] showed that \( c(p) \leq 2.68p\sqrt{\log_2 p} \) for large \( p \). Several authors ([4, 7, 9]) independently observed that this bound for \( c(p) \) is essentially best possible, since random graphs provide examples showing that \( c(p) > 0.265p\sqrt{\log_2 p}(1 + o(1)) \).

We now turn our attention to digraph subcontractions. If \( D \) is a digraph with vertex set \( \{v_1, \ldots, v_p\} \), then \( H \) is a strong subcontraction of \( D \) (or we shall say \( D \) strongly subcontracts \( H \), or sometimes simply \( D \) subcontracts \( H \)), written \( D \succ_s H \), if and only if there exists a partition of the vertices of \( D \) into subsets \( V_0, V_1, \ldots, V_p \) such that \( D[V_i] \) is strongly connected for \( 1 \leq i \leq p \), and, for every edge \( v_iv_j \in E(H) \), there exists a corresponding edge in \( D \) from \( V_i \) to \( V_j \). We say \( H \) is a weak subcontraction of \( D \), written \( D \succ_w H \), if similar sets \( V_0, \ldots, V_p \) exist but for which each \( D[V_i] \) is weakly connected (\( 1 \leq i \leq p \)). Both these definitions correspond to the first definition of subcontraction for ordinary (undirected) graphs, but there appears to be no simple description of strong subcontraction in terms of a sequence of edge operations. Which of weak and strong subcontraction is the more natural analogue of the undirected case is a matter of debate, and not an issue I wish to get into. However, in this paper we shall be dealing with strong subcontraction. Results about weak subcontraction (which turn out to be similar to those about subcontraction in the undirected case) can be found in [5].

The first thing to notice about strong subcontraction is that it is easy to find digraphs with many edges which do not strongly subcontract \( D K_p \), the complete digraph of order \( p \) (\( p > 1 \)). For, a transitive tournament (that is, a tournament \( T \) such that for all vertices \( u, v, w \), edges \( uv, vw \in E(T) \) imply \( uw \in E(T) \)), has no non-trivial strongly connected components, and therefore certainly will not subcontract any \( D K_p \) (\( p > 1 \)). Thus, we define

\[
s(p) = \inf \left\{ c; \; e(D) \geq c|D| + \left( \frac{|D|}{2} \right) \; \text{implies} \; D \succ_s D K_p \right\}.
\]

Defining \( \alpha \) to be the real number satisfying \( \alpha = 1 + \log 2\alpha \) (so that \( \alpha \approx 2.68 \)), it was shown in [5] that

\[
\frac{c(p)}{2\alpha} - 2 < s(p) \leq c(p).
\]

In Section 2 of this paper we shall get exact results for \( s(p) \) for small values of \( p \), and in Section 3 we shall get results for strongly subcontracting tournaments. More specifically, we shall get bounds for the number of edges required to strongly subcontract a tournament of order \( p \) that are very much dependent on the strong connectivity of that tournament, and are similar to that required to strongly subcontract
a DK_p, unless the strong connectivity is one (and hence the tournament is transitive), when we obtain a very different result.

2. Exact results for strong subcontractions

We start by observing that if \( p = 2 \), then \( D \) can have \( \binom{n}{2} \) edges with \( D \not\cong DK_2 \) (simply by taking \( D \) to be the transitive tournament mentioned before), but if \( D \) has one extra edge then \( D \cong DK_2 \), as there must be a double edge and that itself will be a DK_2. The extremal graphs are tournaments each of whose strong components has order one or three. In the following proposition we prove the corresponding result for \( p \) taking values three to seven.

**Proposition 1.** If \( D \) is a digraph of order \( n \) with the maximum possible number of edges such that \( D \not\cong DK_p \) then

1. \( e(D) = \binom{n}{2} + n - 1 \), for \( p = 3 \),
2. \( e(D) = \binom{n}{2} + 2n - 3 \), for \( p = 4 \),
3. \( e(D) = \binom{n}{2} + 3n - 6 \), for \( p = 5 \),
4. \( e(D) = \binom{n}{2} + 4n - 10 \), for \( p = 6 \),
5. \( e(D) = \binom{n}{2} + 5n - 15 \), for \( p = 7 \).

Moreover, in each case \( D \) is a tournament with the addition of extra edges. If we regard these extra edges as an undirected graph, then these do not subcontract to \( K_p \), and are extremal for this.

**Proof.** To establish the upper bound for (1) take a digraph \( D \) with \( \binom{n}{2} + n \) edges, and form the corresponding simple graph \( G \) which has an edge between two vertices only if there was an edge both ways between the corresponding vertices in \( D \). Then \( G \) has \( n \) edges, and so \( G \cong K_3 \). Hence \( D \cong DK_3 \). For (2) we use the same argument but observe that any (undirected) graph \( G \) with \( 2n - 2 \) edges subcontracts \( K_4 \) (by standard results, due to Mader [8]). Similarly, all the other upper bounds are established. (Notice that any counterexample must have no more than the number of double edges we are claiming, and thus the undirected graph induced by the double edges must be an extrernal graph for the undirected case, as we claimed.)

To establish the lower bound for (1), take an extremal digraph \( D \) which does not subcontract DK_2, and add a vertex \( v \) with an edge both ways to every vertex of \( D \). This forms a digraph \( D \cup \{v\} \) with the appropriate number of edges which does not subcontract DK_3 (since if \( D \cup \{v\} \cong DK_3 \) then \( v \) must be in one of the classes to be contracted, and the other three classes would therefore be entirely contained in \( D \) and so \( D \) would subcontract DK_2, giving us a contradiction). We do the other cases in a similar manner. \( \square \)
Our next aim shall be to get good results for (strong) subcontraction of tournaments, but first we shall digress for a moment and consider what happens if instead of directed edges one has undirected edges that can have different colours. We shall assume that the edges of a multigraph $M$ with at most $s$ different edges between each pair of vertices, can be coloured with a *spectrum* of $s$ different colours, with each colour being used at most once for any pair of vertices (so that each colour induces an ordinary graph). The aim is to find out how many edges are needed to ensure that $M \succ SK_p$ (where our subcontraction must consist of subsets connected in every colour), where $SK_p$ is the complete multigraph on $p$ vertices such that every pair of vertices has every colour of edge between them. In particular, the restriction of $M$ to any particular colour class subcontracts $K_p$. (The case $s = 1$ is obviously the same as the undirected case—there is no direct link between the case $s = 2$ and the directed case.) In the case of this multicoloured subcontraction, if $M$ has

$$\left( s - 1 \right) \binom{n}{2} + c(p)n$$

edges then $M \succ SK_p$, and this bound is the best possible. The proof of this is essentially trivial. If $M$ has fewer edges it may have fewer than $c(p)n$ edges of a particular colour, and thus $M \sim SK_p$. If $M$ has this number of edges then there are $c(p)n$ edges common to all colours, and thus the result is true.

3. Strong subcontracting tournaments

The results we present here for strongly subcontracting tournaments depend on the size of the largest strongly connected component of the tournament. (Note that for a tournament, having largest strongly connected component of order $k$ is equivalent to having the longest directed cycle of the tournament of length $k$.) With this in mind, for $k \geq 2$, we let $T_{p,k}$ be a tournament of order $p$ and largest strongly connected component of order $k$ (which we shall often refer to simply as $T_p$), and define

$$st(p, k) = \inf \left\{ c; e(D) \geq c|D| + \frac{|D|}{2} \right\}$$

implies $D \succ T_{p,k}$ for every $T_{p,k}$.

First we shall need to borrow a lemma from Thomason [9] (recall that $\alpha \approx 2.68$). The details are a bit complicated, but the basic idea is that any graph with enough edges has a subcontraction with large minimum degree (and therefore, for our purposes, is a very dense subcontraction).

**Lemma 2.** Let $r$ be a real number such that $\alpha r \geq 3$ is an integer. Let $C_r = \{ G: |G| \leq \alpha r \text{ and } \delta(G) \geq (|G| + |r| - 3)/2 \}$. Then any graph in the set $\{ G: e(G) \geq \alpha r|G| \}$ has a subcontraction in the class $C_r$.

**Proof.** Given any graph in the set $\{ G: e(G) \geq \alpha r|G| \}$, we can find a graph $G_1$ with $G \succ G_1$ which is minimal with respect to subcontraction within this class of graphs.
Then $e(G_1) = \alpha r |G_1|$, so $\delta(G_1) \leq 2\alpha r$. Now choose $x \in G_1$ with $d(x) = \delta(G_1)$, and let $G_2 = G_1[I(x)]$. If $y \in G_2$, then $d(y) \geq \alpha r$, or else $e(G_1/xy) \geq e(G_1) - \alpha r = \alpha r |G_1/xy|$, contradicting the minimality of $G_1$. Hence $\delta(G_2) \geq \alpha r$, and $|G_2| \leq 2\alpha r$.

Define $f(G) = r|G|[(\log(|G|/r) + 1)/2$, and let $\mathcal{F} = \{G: |G| \geq r$ and $e(G) \geq f(G)\}$. Notice that $G_2 \in \mathcal{F}$, since $e(G_2) \geq \alpha r|G_2|/2 = r|G_2|[(\log 2\alpha + 1)/2 \geq f(G_2)$. Now let $G_3$ be a subcontraction of $G_2$ in $\mathcal{F}$ which is minimal with respect to subcontraction. Since $e(K_{r\alpha}) < r|K_{r\alpha}|/2 < f(K_{r\alpha})$, it follows that $|G_3| \geq r + 1$. Minimality of $G_3$ therefore implies that for any edge $uv \in E(G_3)$,

$$e(G_3) = \left\lceil f(G_3) \right\rceil \quad \text{and} \quad e(G_3/uv) < f(G_3/uv).$$

Choosing $u$ with $d(u) = \delta(G_3)$, and letting $G_4 = G_3[I(u)]$, then

$$|G_4| \leq \left\lceil (2/|G_1|) f(G_3) \right\rceil \leq \left\lceil (2/|G_3|) f(G_3) \right\rceil = \left\lceil r(\log(|G_3|/r) + 1)\right\rceil.$$

Now, for an appropriate vertex $v$ we have

$$\delta(G_4) = e(G_3) - e(G_3/uv) - 1 \geq \left\lceil f(G_3) \right\rceil - \left\lceil f(G_3/uv) \right\rceil - 1.$$

Thus,

$$2\delta(G_4) - |G_4| \geq r |G_3| \left( \log \left| \frac{G_3}{r} \right| + 1 \right) - r(|G_3| - 1) \left( \log \left| \frac{G_3}{r} \right| - 1 \right)$$

$$- r \left( \log \left| \frac{G_3}{r} \right| + 1 \right) - 3$$

$$= r(|G_3| - 1) \left( \log \left| \frac{G_3}{G_3} \right| - 1 \right) - 3$$

$$\geq r \left( 1 - \frac{1}{|G_3| - 1} \right) - 3 \geq r - 4,$$

since $|G_3| \geq r + 1$, and $\log(1 + 1/\lambda) > 1 - 1/\lambda$ for all $\lambda > 1$. Hence, $\delta(G_4) \geq (|G_4| + |r| - 3)/2$. Moreover,

$$|G_4| \leq \left\lceil r \left( \log \left| \frac{G_2}{r} \right| + 1 \right) \right\rceil \leq \left\lceil \alpha r \right\rceil = \alpha r.$$

Thus, $G_4$ is in $\mathcal{C}_r$ as required. \qed

**Theorem 3.** Given any integers $p \geq 1$, and $k \geq 2$,

$$\max \left( \frac{c(k)}{2\alpha} - 2, \frac{k - 2}{2} \right) \leq \text{st}(p, k) \leq c(p).$$

**Proof.** Given a digraph $D$ with vertices $v_1, \ldots, v_n$, we form an undirected graph $G$ with vertices $u_1, \ldots, u_n$ such that $u_i u_j \in E(G)$ if and only if both $v_i v_j \in E(D)$ and
If \( D \) has at least \( c(p)n + \binom{n}{2} \) edges then \( G \) will have at least \( c(p)n \) edges; hence \( G \cong K_p \) and therefore \( D \cong_s D K_p \), and so certainly \( D \cong_s T_p \), for any tournament \( T_p \).

To show that \( st(p,k) > (c(k)/2x) - 2 \), we need to find a tournament \( T_{p,k} \), \((k \geq 2)\), for which there is a digraph \( D \) with \( c|D| + \binom{|D|}{2} \) edges which does not subcontract \( T_{p,k} \). Take any \( T_{p,k} \), with a strongly connected component \( T'_k \) or order \( k \). Let \( 0 < x \leq 1 \) be such that \( c(k) - x \) is an integer. Take a graph \( G' \) with \((c(k) - x)|G'| \) edges such that \( G' \not\cong K_k \). Let \( G \) be a subcontraction of \( G' \) which is in the class \( C \), found in Lemma 2, where \( x r = c(k) - x \). Clearly, \( G \not\cong K_k \), \(|G| \leq c(k) - x \), and \( \delta(G) \geq (|G| + |r| - 3)/2 \). Let \(|G| = t \), and let \( D' \) be the digraph formed from \( G \) with edges \( uv \) and \( vu \) for all \( uv \in E(G) \), so that \( D' \not\cong_s D K_k \). For \( n \) a multiple of \( t \), let \( D \) be formed from \( n/t \) copies of \( D' \), say \( D_1, \ldots, D_n/t \), such that \( v_iv_j \in E(D) \) for all \( v_i \in V(D_i), \ v_j \in V(D_j) \), with \( 1 \leq i < j \leq n/t \). Since \( D_i \not\cong_s D_k \) for \( 1 \leq i \leq n/t \), then \( D_i \not\cong T'_k \) (since all edges are double edges, so if it strongly subcontracted \( T'_k \) then it would also strongly subcontract \( D K_k \)). Thus \( D \not\cong s T_{p,k} \), for clearly no strong subcontraction for \( T'_k \) can contain vertices from two different subgraphs \( D_i, D_j \) (since any subgraph containing vertices from \( D_i \) and \( D_j \) cannot be strongly connected, but the subgraph subcontracting \( T'_k \) needs to be strongly connected). Moreover, \( D \) has size at least

\[
\binom{n}{2} - \frac{n}{i} \binom{t}{2} + (n/t)t \frac{t + |r| - 3}{2} = \binom{n}{2} + n \left\{ \frac{t}{2} \left( \frac{c(k) - x}{x} - 1 \right) - 1 \right\}
\geq \binom{n}{2} + n \left\{ \frac{c(k) - x}{2x} - 2 \right\}
\]

as required.

Finally, we wish to show that \( st(p,k) > (k - 2)/2 \). Take \( n \) to be a multiple of \( k - 1 \), and construct a digraph \( D \) in the following way. Fix an ordering on the vertices and form a transitive tournament. Then we take the first \( k - 1 \) vertices and add in all remaining edges between these vertices. We then take the next \( k - 1 \) vertices and add in all their edges, and so on. The strongly connected component of order \( k \) in the tournament cannot have vertices from different \((k - 1)\)-blocks, but equally clearly cannot have all \( k \) vertices from one block of order \( k - 1 \). This produces a graph with \( \binom{k}{2} + (k - 2)n/2 \) edges, as required, and so the theorem is proved. \( \square \)

Notice that if one makes a small change, and instead of considering \( st(p,k) \) considers \( st(T_{p,k}) \), defined by

\[
st(T_{p,k}) = \inf \left\{ c; e(D) \geq c|D| + \binom{|D|}{2} \implies D \cong_s T_{p,k} \text{ for a given } T_{p,k} \right\},
\]

then Theorem 3 holds for \( st(p,k) \) too.
The case \( k = 1 \) is when \( T_p \) is a transitive tournament, and now there is no reason to believe that strongly connected components are necessary at all, and thus it is quite possible that fewer than \( \binom{p}{2} \) edges will ensure that a digraph strongly subcontracts to \( TT_p \) (where \( TT_p = T_{p,1} \) is the transitive tournament of order \( p \)). As the next theorem shows, this is indeed the case, but first we need a result due to Erdős and Stone [3]. In order to follow standard notation, we shall slightly abuse notation and use \( T_p(n) \) for the \( p \)-partite Turán graph of order \( n \), and \( T_p \) as a tournament of order \( p \).

**Lemma 4.** Given any integers \( p \) and \( m \), and \( \varepsilon > 0 \), then there exists \( n_0 \) such that for any \( n \geq n_0 \), any graph \( G \) of order \( n \), with \( e(G) \geq e(T_{p-1}(n)) + \varepsilon n^2 \) contains a complete \( p \)-partite graph with vertex classes of size at least \( m \).

**Theorem 5.** Given \( p \), and any \( \varepsilon > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \), if digraph \( D \) has order \( n \) and at least \( \binom{p}{2}(1 - 1/(p - 1)) + \varepsilon \) edges then \( D \nrightarrow_s TT_p \). Furthermore, there are digraphs \( D' \) of order \( n \) with \( \binom{p}{2}(1 - 1/(p - 1)) \) edges that do not strongly subcontrct to \( TT_p \).

**Proof.** Take \( D' \) to be the oriented Turán graph of order \( n \) with \( p - 1 \) classes, with all edges oriented so that the digraph is transitive (that is, the digraph formed from vertex classes \( V_1, \ldots, V_{p-1} \) with all edges \( x_i x_j \) with \( x_i \in V_i, x_j \in V_j, 1 \leq i < j \leq p - 1 \)), which we shall denote \( DT_{p-1}(n) \), then it is clear that \( D' \nrightarrow_s TT_p \).

For large enough \( n \), if digraph \( D \) has order \( n \) and \( c(p)n \) double edges then \( D \nrightarrow_s DK_p \), and so certainly \( D \nrightarrow_s TT_p \). Hence, we may assume \( D \) has fewer than \( c(p)n \) double edges. Delete one edge in each double edge to get a subgraph \( D_1 \) of \( D \), so that \( D_1 \) is in fact an oriented graph. If \( e(D) \geq e(T_{p-1}(n)) + \varepsilon n^2/2 \), then, provided \( n \) is large enough, \( e(D_1) \geq e(T_{p-1}(n)) + \varepsilon n^2/3 \). By the previous lemma (again, provided \( n \) is large enough), if \( e(D_1) \geq e(T_{p-1}(n)) + \varepsilon n^2/3 \), then \( D_1 \) contains a complete oriented \( p \)-partite graph with vertex classes of size \( t \). By a Ramsey-like argument, provided \( t \) is large enough there is a \( p \)-partite subgraph \( D_2 \) with \( p \) vertices in each class such between any two classes any two edges go the same way. (For example, by the usual Zarankiewicz bound, between any two classes there must be a \( (2 \)-partite\) subgraph with all edges from one side to the other with order approximately \( \log_2 t \).

Now, noting that there is a \( p \)-colouring of the edges of \( K_p \), we can pair off classes and find these \( 2 \)-partite subgraphs for each pair of order \( \log_2 t \). Next we pair off in a new way and find new \( 2 \)-partite subgraphs of order \( \log_2 \log_2 t \). We carry on in this way and provided that \( p \) iterations of the log function on \( t \) is at least \( p \), we have \( D_2 \) as claimed.)

If \( D_2 \) is transitive then it is clear that \( D_2 \nrightarrow_s TT_p \). If not, it contains three classes which form a triangle, that is, \( V_1, V_2, V_3 \) such that if \( x \in V_i \) and \( y \in V_j \), \( i \neq j \), then \( xy \in E(D_2) \) if \( i \equiv j - 1 \mod 3 \). To form our disjoint classes \( C_1, \ldots, C_p \), which form our \( TT_p \), we let \( C_i \) \( (1 \leq i \leq p) \) consist of a vertex from each class. It is clear that for all \( 1 \leq i, j \leq p \) \( (i \neq j) \), \( C_i \) is strongly connected, and there exist \( x \in C_i \), \( y \in C_j \) with \( xy \in E(D_2) \). Hence, \( D \nrightarrow_s TT_p \), as required. \( \Box \)
References